# Traces of multipliers in pairs of weighted Sobolev spaces

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*Abstract.* We prove that the pointwise multipliers acting in a pair of fractional Sobolev spaces form the space of boundary traces of multipliers in a pair of weighted Sobolev space of functions in a domain.

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*Key words*: multipliers, weighted Sobolev spaces, fractional Sobolev spaces

## 1 Introduction

By a multiplier acting from one Banach function space $S_1$ into another $S_2$ we call a function $\gamma$ such that $\gamma u \in S_2$ for any $u \in S_1$. By $M(S_1 \to S_2)$ we denote the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$\|\gamma\|_{M(S_1 \to S_2)} = \sup\{\|\gamma u\|_{S_2} : \ \|u\|_{S_1} \le 1\}.$$

We write $MS$ instead of $M(S \to S)$, where $S$ is a Banach function space. We shall use the same notation both for spaces of scalar and vector-valued multipliers.

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. It is well known that the fractional Sobolev space $W_p^l(\partial\Omega)$ is the space of traces of the weighted Sobolev space $W_p^{s,\alpha}(\Omega)$ endowed with the norm

$$\Big(\int_\Omega \big(\mathrm{dist}(x, \partial\Omega)\big)^{p\alpha} \sum_{\{\tau: 0 \le |\tau| \le s\}} |D^\tau u|^p dx\Big)^{1/p},$$

where $\alpha = 1 - \{l\} - 1/p$, $s = [l] + 1$ and $p \in (1, \infty)$ (see [5]). It is straightforward to deduce from this fact that the trace $\gamma$ of the function

$$\Gamma \in M(W_p^{t,\beta}(\Omega) \to W_p^{s,\alpha}(\Omega)) \qquad (1)$$

belongs to $M(W_p^m(\partial\Omega) \to W_p^l(\partial\Omega))$. Here $m$ and $l$ are nonintegers, $m \ge l > 0$, $s$ and $\alpha$ are given above, $t = [m] + 1$, $\beta = 1 - \{m\} - 1/p$.

In the present paper we prove that the converse assertion is also true showing that there exists an extension $\Gamma$ of $\gamma \in M(W_p^m(\partial\Omega) \to W_p^m(\partial\Omega))$ subject to (1).

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## 2 The space $M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$

By $\mathcal{B}_r^{n-1}(x)$ we mean the ball $\{\xi \in \mathbf{R}^{n-1} : |\xi - x| < r\}$ and write $\mathcal{B}_r^{n-1}$ instead of $\mathcal{B}_r^{n-1}(0)$.

We need the spaces $S_{loc}$ and $S_{unif}$ of functions on $\mathbf{R}^{n-1}$ defined as follows. By $S_{loc}$ we denote the space

$$\{u : \ \eta u \in S \ \text{ for all } \ \eta \in C_0^\infty(\mathbf{R}^{n-1})\}$$

and by $S_{unif}$ we mean the space

$$\{u : \ \sup_{\xi} \ \|\eta_\xi u\|_S < \infty\},$$

where $\eta_\xi(x) = \eta(x - \xi)$, $\eta \in C_0^\infty(\mathbf{R}^{n-1})$, $\eta = 1$ on $\mathcal{B}_1^{n-1}$. The space $S_{unif}$ is endowed with the norm

$$\|u\|_{S_{unif}} = \sup_{\xi} \|\eta_\xi u\|_S.$$

Let $W_p^l(\mathbf{R}^{n-1})$ denote the fractional Sobolev space with the norm

$$\|D_{p,l} u; \mathbf{R}^{n-1}\|_{L_p} + \|u; \mathbf{R}^{n-1}\|_{L_p},$$

where

$$(D_{p,l} u)(x) = \Big( \int_{\mathbf{R}^{n-1}} |\nabla_{[l]} u(x+h) - \nabla_{[l]} u(x)|^p |h|^{1-n-p\{l\}} dh \Big)^{1/p}, \tag{2}$$

with $\nabla_{[l]}$ being the gradient of order $[l]$, i.e. $\nabla_{[l]} = \{\partial_{x_1}^{\tau_1}, \dots, \partial_{x_{n-1}}^{\tau_{n-1}}\}$, $\tau_1 + \ldots + \tau_{n-1} = [l]$.

In this section we collect some known properties of multipliers between fractional Sobolev spaces $W_p^m(\mathbf{R}^{n-1})$ and $W_p^l(\mathbf{R}^{n-1})$, $m \geq l \geq 0$. The equivalence $a \sim b$ means that $a/b$ is bounded and separated from zero by positive constants depending on $n$, $p$, $m$, and $l$.

**Proposition 1** [3] *Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.*

(i) *There holds*

$$\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \sim \|D_{p,l}\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to L_p)} + \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}.$$

(ii) *If $\gamma \in M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$ then for any multi-index $\sigma$, $|\sigma| \leq [l]$,*

$$D^\sigma \gamma \in M(W_p^m(\mathbf{R}^{n-1}) \to W_p^{l-|\sigma|}(\mathbf{R}^{n-1})).$$

(iii) *Let $0 < \lambda < \mu$. Then*

$$\|\gamma^{\lambda/\mu}; \mathbf{R}^{n-1}\|_{M(W_p^\lambda \to L_p)} \leq c \, \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^\mu \to L_p)}^{\lambda/\mu}.$$

**Proposition 2** [3] *Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$. Then*

$$\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \sim \sup_{\substack{e \subset \mathbf{R}^{n-1} \\ \mathrm{diam}(e) \leq 1}} \frac{\|D_{p,l}\gamma; e\|_{L_p}}{(\mathrm{cap}_{p,m}(e))^{1/p}}$$

$$+\begin{cases} \sup\limits_{x\in\mathbf{R}^{n-1}} \|\gamma;\mathcal{B}_1^{n-1}(x)\|_{L_1} & \textit{for } m>l,\\ \|\gamma;\mathbf{R}^{n-1}\|_{L_\infty} & \textit{for } m=l, \end{cases}$$

*where $e$ is a compact set in $\mathbf{R}^{n-1}$ and $\mathrm{cap}_{p,m}(e)$ is the $(p,m)$-capacity of $e$ defined by*

$$\mathrm{cap}_{p,m}(e)=\inf\{\|u;\mathbf{R}^{n-1}\|^p_{W_p^m}: u\in C_0^\infty(\mathbf{R}^{n-1}),\ u\ge 1 \text{ on } e\}$$

*For $l=0$ one should replace $D_{p,l}\gamma$ by $\gamma$.*

Upper estimates for the norm in $M(W_p^m(\mathbf{R}^{n-1})\to W_p^l(\mathbf{R}^{n-1}))$ are given in the following assertion. By $\mathrm{mes}_{n-1}$ we mean the $(n-1)$-dimensional Lebesgue measure of a compact set $e$.

**Proposition 3** [3] *Let $m$ and $l$ be nonintegers, $m\ge l\ge 0$, and let $p\in(1,\infty)$.*
(i) *If $mp<n-1$, then*

$$c\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^m\to W_p^l)}\le \sup_{\substack{e\subset\mathbf{R}^n\\ \mathrm{diam}(e)\le 1}} \frac{\|D_{p,l}\gamma;e\|_{L_p}}{(\mathrm{mes}_{n-1}(e))^{1/p-m/(n-1)}}$$

$$+\begin{cases} \sup\limits_{x\in\mathbf{R}^{n-1}} \|\gamma;\mathcal{B}_1^{n-1}(x)\|_{L_1} & \textit{for } m>l,\\ \|\gamma;\mathbf{R}^{n-1}\|_{L_\infty} & \textit{for } m=l. \end{cases}$$

(ii) *If $mp=n-1$, then*

$$c\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^m\to W_p^l)}\le \sup_{\substack{e\subset\mathbf{R}^{n-1}\\ \mathrm{diam}(e)\le 1}} \big(\log\frac{2^{n-1}}{\mathrm{mes}_{n-1}(e)}\big)^{1-1/p}\|D_{p,l}\gamma;e\|_{L_p}$$

$$+\begin{cases} \sup\limits_{x\in\mathbf{R}^{n-1}} \|\gamma;\mathcal{B}_1^{n-1}(x)\|_{L_1} & \textit{for } m>l,\\ \|\gamma;\mathbf{R}^{n-1}\|_{L_\infty} & \textit{for } m=l. \end{cases}$$

Now we list lower estimates for the norm in $M(W_p^m\to W_p^l)$.

**Proposition 4** [3] *Let $m$ and $l$ be nonintegers, $m\ge l\ge 0$, and let $p\in(1,\infty)$.*
(i) *If $mp<n-1$, then*

$$c\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^m\to W_p^l)}\ge \sup_{\substack{x\in\mathbf{R}^{n-1}\\ r\in(0,1)}} \frac{\|D_{p,l}\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}}{r^{(n-1)/p-m}}$$

$$+\begin{cases} \sup\limits_{x\in\mathbf{R}^{n-1}} \|\gamma;\mathcal{B}_1^{n-1}(x)\|_{L_1} & \textit{for } m>l,\\ \|\gamma;\mathbf{R}^{n-1}\|_{L_\infty} & \textit{for } m=l. \end{cases}$$

(ii) *If $mp=n-1$, then*

$$c\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^m\to W_p^l)}\ge \sup_{\substack{x\in\mathbf{R}^{n}\\ r\in(0,1)}} \big(\log\frac{2}{r}\big)^{1-1/p}\|D_{p,l}\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}$$

$$+\begin{cases} \sup\limits_{x\in\mathbf{R}^{n-1}} \|\gamma;\mathcal{B}_1^{n-1}(x)\|_{L_1} & \textit{for } m>l,\\ \|\gamma;\mathbf{R}^{n-1}\|_{L_\infty} & \textit{for } m=l. \end{cases}$$

# 3 Multipliers in pairs of weighted Sobolev spaces in $\mathbf{R}_+^n$

## 3.1 Preliminary facts

Let $\mathbf{R}_+^n$ denote the upper half-space $\{z = (x, y) : x \in \mathbf{R}^{n-1},\ y > 0\}$. We introduce the weighted Sobolev space $W_p^{s,\alpha}(\mathbf{R}_+^n)$ with the norm

$$\|(\min\{1, y\})^\alpha \nabla_s U; \mathbf{R}_+^n\|_{L_p} + \|(\min\{1, y\})^\alpha U; \mathbf{R}_+^n\|_{L_p}, \tag{3}$$

where $s$ is nonnegative integer. We always asume that $-1 < \alpha p < p - 1$.

It is well known that the fractional Sobolev space $W_p^l(\mathbf{R}^{n-1})$, is the space of traces on $\mathbf{R}^{n-1}$ of functions in the space $W_p^{s,\alpha}(\mathbf{R}_+^n)$, where $s = [l]+1$, $\alpha = 1-\{l\}-1/p$, and $p \in (1, \infty)$ (see [5]). We show that a similar result holds for spaces of pointwise multipliers acting in a pair of fractional Sobolev spaces. To be precise, we prove that for all noninteger $m$ and $l$, $m \geq l > 0$, the multiplier space $M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$ is the space of traces on $\mathbf{R}^{n-1}$ of functions in $M(W_p^{t,\beta}(\mathbf{R}_+^n) \to W_p^{s,\alpha}(\mathbf{R}_+^n))$, where $s$ and $\alpha$ are as above and $\beta = 1 - \{m\} - 1/p$, $t = [m] + 1$. Different positive constants depending on $n, p, l, m, s, t$ will be denoted by $c$. We shall omit $\mathbf{R}_+^n$ in notations of norms.

We introduce the notion of $(p, s, \alpha)$-capacity of a compact set $e \subset \mathbf{R}_+^n$:

$$\mathrm{cap}_{p,s,\alpha}(e) = \inf\{\|U; \mathbf{R}_+^n\|^p_{W_p^{s,\alpha}} :\ \ U \in C_0^\infty(\mathbf{R}_+^n),\ \ U \geq 1 \ \text{ on } \ e\}.$$

The following result is essentially known (see [2], Sect. 8.1, 8.2).

**Proposition 5** *Let $k$ be a nonnegative integer, $-1 < \beta p < p-1$, and let $1 < p < \infty$. Then $\Gamma \in M(W_p^{k,\beta}(\mathbf{R}_+^n) \to W_p^{0,\alpha}(\mathbf{R}_+^n))$ if and only if*

$$\sup_{\substack{e \subset \mathbf{R}_+^n \\ \mathrm{diam}(e) \leq 1}} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,k,\beta}(e))^{1/p}} < \infty.$$

*The equivalence relation*

$$\|\Gamma\|_{M(W_p^{k,\beta} \to W_p^{0,\alpha})} \sim \sup_{\substack{e \subset \mathbf{R}_+^n \\ \mathrm{diam}(e) \leq 1}} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,k,\beta}(e))^{1/p}} \tag{4}$$

*is valid.*

We shall use some general properties of multipliers. We start with the inequality

$$\|\Gamma\|_{M(W_p^{t-j,\beta} \to W_p^{s-j,\alpha})}$$

$$\leq c\, \|\Gamma\|^{(s-j)/s}_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \|\Gamma\|^{j/s}_{M(W_p^{t-s,\beta} \to W_p^{0,\alpha})}, \tag{5}$$

where $0 \leq j \leq s$, $-1 < \alpha p < p - 1$, $-1 < \beta p < p - 1$, which follows from the interpolation property of weighted Sobolev spaces (see [4], Sect.3.4.2).

The next assertion contains inequalities between multipliers and their mollifiers in variables $x$.

**Lemma 1** *Let $\Gamma_\rho$ denote a mollifier of a function $\Gamma$ defined by*

$$\Gamma_\rho(x,y) = \rho^{-n+1} \int_{\mathbf{R}^{n-1}} K(\rho^{-1}(x-\xi))\Gamma(\xi,y)d\xi,$$

*where $K \in C_0^\infty(\mathcal{B}_1^{n-1})$, $K \geq 0$, and $\|K;\mathbf{R}^{n-1}\|_{L_1} = 1$. Then*

$$\|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \leq \liminf_{\rho\to 0} \|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}. \qquad (6)$$

**Proof.** Let $U \in C_0^\infty$. By Minkowski's inequality

$$\Big(\int_{\mathbf{R}_+^n} (\min\{1,y\})^{p\alpha} \big|\nabla_{j,z} \int_{\mathbf{R}^{n-1}} \rho^{-n} K(\xi/\rho)\Gamma(x-\xi,y)U(x,y)d\xi\big|^p dz\Big)^{1/p}$$

$$\leq \int_{\mathbf{R}^{n-1}} \rho^{-n}K(\xi/\rho)\Big(\int_{\mathbf{R}_+^n} (\min\{1,y\})^{p\alpha}|\nabla_{j,z}\big(\Gamma(x,y)U(x+\xi,y)\big)|^p dz\Big)^{1/p} d\xi,$$

where $j = 0, s$. Therefore,

$$\|\Gamma_\rho u\|_{W_p^{s,\alpha}} \leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}$$

$$\times \int_{\mathbf{R}^{n-1}} \rho^{-n}K(\xi/\rho)\Big\{\Big(\int_{\mathbf{R}_+^n} (\min\{1,y\})^{p\beta}|\nabla_{t,z}U(x+\xi,y)|^p dz\Big)^{1/p}$$

$$+\Big(\int_{\mathbf{R}_+^n} (\min\{1,y\})^{p\beta}|U(x+\xi,y)|^p dz\Big)^{1/p}\Big\}d\xi$$

$$\leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|U\|_{W_p^{t,\beta}}.$$

This gives the left inequality (6). The right inequality (6) follows from

$$\|\Gamma u\|_{W_p^{s,\alpha}} = \liminf_{\rho\to 0} \|\Gamma_\rho U\|_{W_p^{s,\alpha}} \leq \liminf_{\rho\to 0} \|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|U\|_{W_p^{t,\beta}}.$$

The proof is complete.

**Lemma 2** *Let $\Gamma \in L_{p,loc}$, $p \in (1,\infty)$, $-1 < \beta p < p-1$, and let $U$ be an arbitrary function in $C_0^\infty(\mathbf{R}_+^n)$. The best constant in the inequality*

$$\|(\min\{1,y\})^\alpha \Gamma \nabla_s U\|_{L_p} + \|(\min\{1,y\})^\alpha \Gamma U\|_{L_p} \leq C\,\|U\|_{W_p^{t,\beta}} \qquad (7)$$

*is equivalent to the norm $\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}$.*

**Proof.** The estimate $C \leq c\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}$ is obvious. To derive the converse estimate, we introduce a function $x \to \sigma$ which is positive on $[0,\infty)$ and is equal to $x$ for $x > 1$. For any $U \in C_0^\infty(\mathbf{R}_+^n)$ there holds

$$U = (-\Delta)^s\big(\sigma(-\Delta)\big)^{-[l]-1}u + T(-\Delta)u,$$

where $T$ is a function in $C_0^\infty([0,\infty))$. Since

$$(-\Delta)^s = (-1)^s \sum_{|\tau|=s} \frac{s!}{\tau!} D^{2\tau},$$

it follows from (7) and the theorem on the boundedness of convolution operators in weighted $L_p$ spaces (see [1]) that

$$\int_{\mathbf{R}^n_+} (\min\{1, y\})^{p\alpha} |\Gamma(z)U(z)|^p dz$$

$$\leq c\, C\big(\|\nabla_s(\sigma(-\Delta))^{-s}U\|^p_{W^{t,\beta}_p} + \|TU\|^p_{W^{t,\beta}_p}\big) \leq c\, C\|U\|^p_{W^{t-s,\beta}_p}.$$

The proof is complete.

# 4 Characterisation of the space $M(W^{t,\beta}_p \to W^{s,\alpha}_p)$

Here we derive necessary and sufficient conditions for a function to belong to the space $M(W^{t,\beta}_p \to W^{s,\alpha}_p)$ for $p \in (1, \infty)$ with $\alpha$ and $\beta$ satisfying

$$-1 < \alpha p < p - 1, \quad -1 < \beta p < p - 1, \quad t \geq s. \tag{8}$$

These inequalities will be assumed everywhere. We start with an assertion on derivatives of multipliers. We shall omit $\mathbf{R}^{n+1}_+$ in notations of spaces, norms, and integrals.

**Lemma 3** *Suppose*

$$\Gamma \in M(W^{t,\beta}_p \to W^{s,\alpha}_p) \cap M(W^{t-s,\beta}_p \to W^{0,\alpha}_p), \quad p \in (1, \infty).$$

*Then* $D^\sigma\Gamma \in M(W^{t,\beta}_p \to W^{s-|\sigma|,\alpha}_p)$ *for any multiindex* $\sigma$ *of order* $|\sigma| \leq s$ *and*

$$\|D^\sigma\Gamma\|_{M(W^{t,\beta}_p \to W^{s-|\sigma|,\alpha}_p)}$$

$$\leq \varepsilon\, \|\Gamma\|_{M(W^{t-s,\beta}_p \to W^{0,\alpha}_p)} + c(\varepsilon)\, \|\Gamma\|_{M(W^{t,\beta}_p \to W^{s,\alpha}_p)}, \tag{9}$$

*where* $\varepsilon$ *is an arbitrary positive number.*

**Proof.** Let $U \in W^{s,\alpha}_p$ and let $\varphi$ be an arbitrary function in $C^\infty_0$. Applying Leibniz formula

$$D^\sigma(\varphi U) = \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} D^\tau\varphi D^{\sigma-\tau}U,$$

we find

$$\int \varphi U(-D)^\sigma\Gamma dz = \int \Gamma D^\sigma(\varphi U)dz = \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} \Gamma D^\tau\varphi D^{\sigma-\tau}U dz$$

$$= \int \varphi \sum_{\{\beta:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} (-D)^\tau(\Gamma D^{\sigma-\tau}U)dz.$$

Therefore,

$$UD^\sigma\Gamma = \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \frac{\sigma!}{\tau!(\sigma-\tau)!} (D)^\tau(\Gamma(-D)^{\sigma-\tau}U),$$

which implies the estimate

$$\|UD^\sigma\Gamma\|_{W^{s-|\sigma|,\alpha}_p} \leq c \sum_{\{\tau:\sigma\geq\tau\geq 0\}} \|\Gamma D^{\sigma-\tau}U\|_{W^{s-|\sigma|+|\tau|,\alpha}_p}.$$

Hence, it suffices to prove (9) for $|\sigma| = 1$. We have

$$\|U\nabla\Gamma\|_{W_p^{s-1,\alpha}} \leq \|U\Gamma\|_{W_p^{s,\alpha}} + \|\Gamma\nabla U\|_{W_p^{s-1,\alpha}}$$
$$\leq \big(\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} + \|\Gamma\|_{M(W_p^{t-1,\beta}\to W_p^{s-1,\alpha})}\big)\|U\|_{W_p^{t,\beta}}.$$

Estimating the norm $\|\Gamma\|_{M(W_p^{t-1,\beta}\to W_p^{s-1,\alpha})}$ by (5) we arrive at (9).

We pass now to two-sided estimates of norms in $M(W_p^{t,\beta}\to W_p^{s,\alpha})$, $p \in (1,\infty)$, given in terms of the spaces $M(W_p^{k,\beta}\to W_p^{0,\alpha})$. We start with lower estimates.

**Lemma 4** *Let* $\Gamma \in M(W_p^{t,\beta}\to W_p^{s,\alpha})$. *Then*

$$\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \leq c\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}. \tag{10}$$

**Proof.** Suppose first that $\Gamma \in M(W_p^{t-s,\beta}\to W_p^{0,\alpha})$. We have

$$\|\Gamma\nabla_s U\|_{W_p^{0,\alpha}} \leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|U\|_{W_p^{t,\beta}} + c\sum_{\substack{|\sigma|+|\tau|=s,\\ \tau\neq 0}} \|D^\sigma U D^\tau\Gamma\|_{W_p^{0,\alpha}}$$

$$\leq \Big(\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} + c\sum_{j=1}^{s}\|\nabla_j\Gamma\|_{M(W_p^{t-s+j}\to W_p^{0,\alpha})}\Big)\|U\|_{W_p^{t,\beta}}. \tag{11}$$

By Lemma 3,

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}$$

$$\leq \varepsilon\,\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\,\|\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{j,\alpha})}. \tag{12}$$

Estimating the last norm by (5) we obtain

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}$$
$$\leq \varepsilon\,\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\,\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Substitution of this into (11) gives

$$\|\Gamma\nabla_s U\|_{W_p^{0,\alpha}} \leq \Big(\varepsilon\,\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\,\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\Big)\|U\|_{W_p^{t,\beta}}. \tag{13}$$

Besides,

$$\|\Gamma U\|_{W_p^{0,\alpha}} \leq \|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|U\|_{W_p^{t,\beta}}. \tag{14}$$

Summing up two last estimates and applying Lemma 2 we arrive at

$$\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \leq \varepsilon\,\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} + c(\varepsilon)\,\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Hence,

$$\|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \leq c\,\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}. \tag{15}$$

Now, we are going to remove the assumption $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$. Since $\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha})$, then

$$\|\Gamma\eta\|_{W_p^{s,\alpha}} \le c\,\|\eta\|_{W_p^{t,\beta}}\ ,$$

where $\eta \in C_0^\infty(\mathcal{B}_2^n(z))$, $\eta = 1$ on $\mathcal{B}_1^n(z)$, and $z$ is an arbitrary point in $\mathbf{R}_+^n$. Hence $\Gamma \in W_{p,\mathrm{unif}}^{s,\alpha}(\mathbf{R}_+^n)$ which implies that for any $(n-1)$-dimensional multiindex $\tau$ the derivative $D_x^\tau \Gamma_\rho$ belongs to $W_{p,\mathrm{unif}}^{s,\alpha}(\mathbf{R}_+^n)$. Therefore, $\Gamma_\rho \in L_\infty(\mathbf{R}_+^n)$ which in its turn guarantees that $\Gamma_\rho \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$. Thus, we may put $\Gamma_\rho$ into (15) in order to obtain

$$\|\Gamma_\rho\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})} \le c\,\|\Gamma_\rho\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}.$$

Letting $\rho \to 0$ and using Lemma 1 we arrive at (15) for all $\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha})$.

To estimate the first term in the right-hand side of (10), we combine (15) with (12) for $j = s$.

The estimate converse to (10) is contained in the following lemma.

**Lemma 5** *Let $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$ and let $\nabla_s\Gamma \in M(W_p^{t,\beta} \to W_p^{0,\alpha})$. Then $\Gamma \in M(W_p^{t,\beta} \to W_p^{s,\alpha})$ and the estimate*

$$\|\Gamma\|_{M(W_p^{t,\beta}\to W_p^{s,\alpha})} \le c\Big(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\Big) \tag{16}$$

*is valid.*

**Proof.** By Lemma 4 and (5) we have

$$\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})} \le c\,\|\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{j,\alpha})}$$

$$\le c\,\|\Gamma\|^{j/s}_{M(W_p^{t,\beta}\to W_p^{s,\alpha})}\|\Gamma\|^{1-j/s}_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}, \tag{17}$$

where $j = 1, \dots, s$. For any $U \in C_0^\infty$,

$$\|(\min\{1,y\})^\alpha \nabla_s(\Gamma U)\|_{L_p} \le c\sum_{j=0}^{s}\|(\min\{1,y\})^\alpha|\nabla_j\Gamma|\,|\nabla_{s-j}U|\,\|_{L_p}$$

$$\le c\Big(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}$$
$$+\sum_{j=1}^{s-1}\|\nabla_j\Gamma\|_{M(W_p^{t-s+j,\beta}\to W_p^{0,\alpha})}\Big)\|U\|_{W_p^{t,\beta}}.$$

This and (17) imply

$$\|(\min\{1,y\})^\alpha \nabla_s(\Gamma U)\|_{L_p}$$
$$\le c\,\Big(\|\nabla_s\Gamma\|_{M(W_p^{t,\beta}\to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\Big)\|U\|_{W_p^{t,\beta}}.$$

It remains to note that

$$\|(\min\{1,y\})^\alpha \Gamma U\|_{L_p} \le \|\Gamma\|_{M(W_p^{t-s,\beta}\to W_p^{0,\alpha})}\|U\|_{W_p^{t-s,\beta}}.$$

The proof is complete.

Using Lemmas 4 and 5 we arrive at the following description of the space $M(W_p^{t,\beta}(\mathbf{R}_+^n) \to W_p^{s,\alpha}(\mathbf{R}_+^n))$.

**Theorem 1** *A function* $\Gamma$ *belongs to the space* $M(W_p^{t,\beta} \to W_p^{s,\alpha})$ *if and only if* $\Gamma \in W_{p,\mathrm{loc}}^{s,\alpha}$, $\Gamma \in M(W_p^{t-s,\beta} \to W_p^{0,\alpha})$, *and* $\nabla_s \Gamma \in M(W_p^{t,\beta} \to W_p^{0,\alpha})$. *Moreover,*

$$\|\Gamma\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \sim \|\nabla_s \Gamma\|_{M(W_p^{t,\beta} \to W_p^{0,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\beta} \to W_p^{0,\alpha})}.$$

The equivalence relation (4) enables one to reformulate Theorem 1.

**Theorem 2** *A function* $\Gamma$ *belongs to the space* $M(W_p^{t,\beta} \to W_p^{s,\alpha})$ *if and only if* $\Gamma \in W_{p,\mathrm{loc}}^{s,\alpha}$ *and for any compact set* $e \subset \mathbf{R}_+^n$

$$\|(\min\{1, y\})^\alpha \nabla_s \Gamma; e\|_{L_p}^p \le c \,\mathrm{cap}_{p,t,\beta}(e),$$

$$\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}^p \le c \,\mathrm{cap}_{p,t-s,\beta}(e).$$

*Moreover,*

$$\|\Gamma\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})}$$

$$\sim \sup_{\substack{e \subset \mathbf{R}_+^n \\ \mathrm{diam}(e) \le 1}} \Big( \frac{\|(\min\{1, y\})^\alpha \nabla_s \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,t,\beta}(e))^{1/p}} + \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,t-s,\beta}(e))^{1/p}} \Big). \tag{18}$$

An important particular case of Theorem 2 is $t = s$.

**Corollary 1** *A function* $\Gamma$ *belongs to the space* $MW_p^{s,\alpha}$ *if and only if* $\Gamma \in W_{p,\mathrm{loc}}^{s,\alpha}$ *and for any compact set* $e \subset \mathbf{R}_+^n$

$$\|(\min\{1, y\})^\alpha \nabla_s \Gamma; e\|_{L_p}^p \le c \,\mathrm{cap}_{p,s,\alpha}(e),$$

*Moreover,*

$$\|\Gamma\|_{MW_p^{s,\alpha}} \sim \sup_{\substack{e \subset \mathbf{R}_+^n \\ \mathrm{diam}(e) \le 1}} \frac{\|(\min\{1, y\})^\alpha \nabla_s \Gamma; e\|_{L_p}}{(\mathrm{cap}_{p,s,\alpha}(e))^{1/p}} + \|\Gamma\|_{L_\infty}. \tag{19}$$

# 5 Trace theorems for multipliers in weighted Sobolev spaces

We start with the following simple fact concerning traces of multipliers.

**Theorem 3** *Let* $m$ *and* $l$ *be positive noninteger,* $m \ge l$ *and let*

$$\Gamma \in M(W_p^{t,\beta}(\mathbf{R}_+^n) \;\to\; W_p^{s,\alpha}(\mathbf{R}_+^n))$$

*where* $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, *and* $\alpha = 1 - \{l\} - 1/p$. *If* $\gamma$ *is the trace of* $\Gamma$ *on* $\mathbf{R}^{n-1}$, *then*

$$\gamma \in M(W_p^m(\mathbf{R}^{n-1}) \;\to\; W_p^l(\mathbf{R}^{n-1}))$$

*and the estimate*

$$\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to W_p^l)} \le c \|\Gamma; \mathbf{R}_+^n\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \tag{20}$$

*holds.*

**Proof.** Let $U \in W_p^{t,\beta}(\mathbf{R}^n_+)$ and let $u$ be the trace of $U$ on $\mathbf{R}^{n-1}$. By setting $\Gamma U$ and $\gamma u$ instead of $U$ and $u$, respectively, in the inequality

$$\|u; \mathbf{R}^{n-1}\|_{W_p^l} \le c\|U; \mathbf{R}^n_+\|_{W_p^{s,\alpha}}$$

we arrive at the estimate

$$\|\gamma u; \mathbf{R}^{n-1}\|_{W_p^l} \le c\, \|\Gamma; \mathbf{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \|U; \mathbf{R}^n_+\|_{W_p^{t,\beta}}.$$

Minimizing the right-hand side over all extensions $U$ of $u$ we obtain

$$\|\gamma u; \mathbf{R}^{n-1}\|_{W_p^l} \le c\, \|\Gamma; \mathbf{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \|u; \mathbf{R}^{n-1}\|_{W_p^m}$$

which gives (20).

We state an extension theorem for functions in $M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$ to be proved in Sect. 7.

**Theorem 4** *Let $m$ and $l$ be positive nonintegers, $m \ge l$, $p \in (1, \infty)$, and let*

$$\gamma \in M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1})).$$

*Then the Dirichlet problem*

$$\Delta \Gamma = 0 \ \ on\ \ \mathbf{R}^n_+, \ \ \Gamma|_{\mathbf{R}^{n-1}} = \gamma \tag{21}$$

*has a unique solution in $M(W_p^{t,\beta}(\mathbf{R}^n_+) \to W_p^{s,\alpha}(\mathbf{R}^n_+))$, where $t = [m]+1$, $s = [l]+1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. There holds the estimate*

$$\|\Gamma; \mathbf{R}^n_+\|_{M(W_p^{t,\beta} \to W_p^{s,\alpha})} \le c\, \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to W_p^l)}. \tag{22}$$

# 6 Auxiliary estimates for an extension operator

## 6.1 Pointwise estimate for $T\gamma$ and $\nabla T\gamma$

For functions $\gamma \in L_{1,unif}(\mathbf{R}^{n-1})$, we introduce the operator

$$(T\gamma)(x,y) = y^{1-n} \int_{\mathbf{R}^{n-1}} \zeta\Big(\frac{x-\xi}{y}\Big)\gamma(\xi)d\xi, \quad (x,y) \in \mathbf{R}^n_+, \tag{23}$$

where $\zeta$ is a continuously differentiable function defined on $\overline{\mathbf{R}^n_+}$ outside the origin. We assume that

$$(|z|+1)|\nabla\zeta(z)| + |\zeta(z)| \le C\, (|z|+1)^{-n}. \tag{24}$$

**Lemma 6** *Let $\gamma \in M(W_p^{m-l}(\mathbf{R}^{n-1}) \to L_p(\mathbf{R}^{n-1}))$, where $m \ge l$ and $1 < p < \infty$. Then*

$$|T\gamma(z)| + y|\nabla(T\gamma(z))| \ \le\ c(1 + y^{l-m})\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}.$$

**Proof**. In view of (24)

$$|T\gamma(z)| + y|\nabla(T\gamma(z))|$$

$$\le cy^{1-n}\Big(\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)|d\xi + y^n \int_{\mathbf{R}^{n-1}\setminus\mathcal{B}_y^{n-1}(x)} \frac{|\gamma(\xi)|d\xi}{|\xi - x|^n}\Big). \tag{25}$$

By Hölder's inequality,

$$\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)| d\xi \le c y^{(n-1)(p-1)/p} \|\gamma; \mathcal{B}_y^{n-1}(x)\|_{L_p}. \tag{26}$$

Let $y \in (0,1)$. The right-hand side in (26) does not exceed

$$c y^{-m+l+n-1} \sup_{\substack{r\in(0,1)\\ x\in\mathbf{R}^{n-1}}} (1 + r^{m-l-\frac{n-1}{p}})\, \|\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p}$$

$$\le c y^{-m+l+n-1} \sup_{\substack{r\in(0,1)\\ x\in\mathbf{R}^{n-1}}} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p}\, \|\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p}. \tag{27}$$

This and Proposition 2 show that for $y \in (0,1)$

$$\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)| d\xi \le c y^{-m+l+n-1} \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{28}$$

Suppose $y > 1$. Since

$$\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x)) \;\sim\; r^{n-1} \;\text{ for }\; r > 1, \tag{29}$$

it follows that the right-hand side of (26) is dominated by

$$c y^{n-1} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_y^{n-1}(x))\big)^{-1/p} \|\gamma; \mathcal{B}_y^{n-1}(x)\|_{L_p}.$$

Combining this with (27) and Proposition 2 we conclude that

$$\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)| d\xi \le c y^{n-1}(1 + y^{l-m}) \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{30}$$

We now estimate the second integral in the right-hand side of (25). Clearly,

$$\int_{\mathbf{R}^{n-1}\setminus\mathcal{B}_y^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^n} \le n \int_y^\infty \frac{d\rho}{\rho^{n+1}} \int_{\mathcal{B}_\rho^{n-1}(x)} |\gamma(\xi)| d\xi. \tag{31}$$

By Hölder's inequality the right-hand side of (31) admits the majorant

$$c \int_y^\infty \rho^{-2-\frac{n-1}{p}} \|\gamma; \mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho. \tag{32}$$

Using (29) we see that the function (32), for $y > 1$, does not exceed

$$c y^{-1} \sup_{\substack{r\in(0,1)\\ x\in\mathbf{R}^{n-1}}} \big(\mathrm{cap}_{p,m-1}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p} \|\gamma; \mathcal{B}_r^{n-1}(x)\|_{L_p}$$

which in view of Proposition 2 is dominated by

$$c y^{-1} \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{33}$$

Let $y < 1$. Then

$$\int_y^1 \rho^{-2-\frac{n-1}{p}} \|\gamma; \mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho$$

$$\leq cy^{-m+l-1} \sup_{\substack{r\in(0,1)\\ x\in\mathbf{R}^{n-1}}} (1+r^{m-1-\frac{n-1}{p}})\, \|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}$$

$$\leq cy^{-m+l-1} \sup_{\substack{r\in(0,1)\\ x\in\mathbf{R}^{n-1}}} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p}\, \|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}. \tag{34}$$

Furthermore, by (29)

$$\int_1^\infty \rho^{-2-\frac{n-1}{p}} \|\gamma;\mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho$$

$$\leq c\int_1^\infty \rho^{-2}\big(\mathrm{cap}_{p,m-l}(\mathcal{B}_\rho^{n-1}(x))\big)^{-1/p}\, \|\gamma;\mathcal{B}_\rho^{n-1}(x)\|_{L_p} d\rho$$

$$\leq c \sup_{\substack{r>0\\ x\in\mathbf{R}^{n-1}}} \big(\mathrm{cap}_{p,m-l}(\mathcal{B}_r^{n-1}(x))\big)^{-1/p}\, \|\gamma;\mathcal{B}_r^{n-1}(x)\|_{L_p}.$$

Summing up this inequality and (34), and using Proposition 2 we conclude that the integral (32) is majorized, for $y<1$, by

$$cy^{-m+l-1}\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}.$$

This, together with (33), imply that for all $y>0$ the integral (32) does not exceed

$$cy^{-1}(1+y^{l-m})\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}.$$

Hence, the result follows from (30), (31), and (25).

## 6.2 Weighted $L_p$-estimates for $T\gamma$ and $\nabla T\gamma$

**Lemma 7** *Let the extension operator $T$ be defined by* (23) *and suppose that $\gamma \in M(W_p^{m-l}(\mathbf{R}^{n-1}) \to L_p(\mathbf{R}^{n-1}))$, where $l\in(0,1)$, $[m]\geq 1$, $1<p<\infty$. Then, for $k=1,\dots,[m]$,*

$$\Big(\int_0^1 y^{p(k-l)-1}\big(|T\gamma(z)|+y|\nabla(T\gamma)(z)|\big)^p dy\Big)^{1/p}$$

$$\leq c\|\gamma;\mathbf{R}^{n-1}\|^{\frac{k-l}{m-l}}_{M(W_p^{m-l}\to L_p)}[(\mathcal{M}\gamma)(x)]^{\frac{m-k}{m-l}}, \tag{35}$$

*where $\mathcal{M}$ is the Hardy-Littlewood maximal operator in $\mathbf{R}^{n-1}$.*

**Proof**. Let $\delta$ be a number in (0,1] to be chosen later. We set

$$\int_0^1 y^{p(k-l)-1}\big(|T\gamma|+y|\nabla(T\gamma)(z)|\big)^p dy \;=\; \int_0^\delta \dots dy \;+\; \int_\delta^1 \dots dy.$$

In view of (25)

$$\int_0^\delta \dots dy \leq c\int_0^\delta y^{p(k+1-l-n)-1}\Big(\int_{\mathcal{B}_y^{n-1}(x)}|\gamma(\xi)|d\xi\Big)^p dy$$

$$+c\int_0^\delta y^{p(k+1-l)-1}\Big(\int_{\mathbf{R}^{n-1}\setminus\mathcal{B}_y^{n-1}(x)}\frac{|\gamma(\xi)|d\xi}{|\xi-x|^n}\Big)^p dy.$$

By the definition of $\mathcal{M}$,

$$\int_0^\delta y^{p(k+1-l-n)-1}\Big(\int_{\mathcal{B}_y^{n-1}(x)} |\gamma(\xi)|d\xi\Big)^p dy \le c[(\mathcal{M}\gamma)(x)]^p\delta^{p(k-l)}. \tag{36}$$

Using (31) we obtain

$$\int_0^\delta y^{p(k+1-l)-1}\Big(\int_{\mathbf{R}^{n-1}\backslash\mathcal{B}_y^{n-1}(x)} \frac{|\gamma(\xi)|d\xi}{|\xi-x|^n}\Big)^p dy \le c[(\mathcal{M}\gamma)(x)]^p\delta^{p(k-l)}. \tag{37}$$

Combining (36) and (37) we conclude that

$$\int_0^\delta \dots dy \ \le c[(\mathcal{M}\gamma)(x)]^p\delta^{p(k-l)}. \tag{38}$$

By Lemma 6,

$$\int_\delta^1 y^{p(k-l)-1}\big(|T\gamma| + y|\nabla(T\gamma)(z)|\big)^p dy$$

$$\le c\ \|\gamma;\mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\delta^{p(k-m)}. \tag{39}$$

Summing up (38) and (39) we find

$$\int_0^1 y^{p(k-l)-1}\big(|T\gamma| + y|\nabla(T\gamma)(z)|\big)^p dy$$

$$\le c\big([(\mathcal{M}\gamma)(x)]^p\delta^{p(k-l)} + \|\gamma;\mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\delta^{p(k-m)}\big).$$

The right-hand side in this inequality attains its minimum value for

$$\delta \ = \ \Big(\frac{\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}}{(\mathcal{M}\gamma)(x)}\Big)^{1/(m-l)}.$$

The proof is complete.

**Lemma 8** *Let the operators $T$ and $D_{p,l}$ be defined by* (23) *and* (2). *Then*

$$\int_0^1 y^{p(1-l)-1}|\nabla(T\gamma)(z)|^p dy \ \le \ c\ \big((D_{p,l}\gamma)(x)\big)^p.$$

**Proof.** Let $R(\xi,x)=\gamma(\xi)-\gamma(x)$. Using the identity

$$y^{-n+1}\int_{\mathbf{R}^{n-1}}\zeta\Big(\frac{\xi-x}{y}\Big)d\xi = const$$

we have

$$\frac{\partial T\gamma}{\partial y}(x,y) = \frac{\partial}{\partial y}\Big(y^{-n+1}\int_{\mathbf{R}^{n-1}}\zeta\Big(\frac{\xi-x}{y}\Big)R(\xi,x)d\xi\Big). \tag{40}$$

Furthermore, it is clear that

$$\frac{\partial T\gamma}{\partial x_j}(x,y) = y^{-n+1}\int_{\mathbf{R}^{n-1}} R(\xi,x)\frac{\partial}{\partial x_j}\zeta\Big(\frac{\xi-x}{y}\Big)d\xi.$$

Therefore,

$$|\nabla(T\gamma)(x,y)| \leq cy^{-n}\sum_{k=0}^{1}\int_{\mathbf{R}^{n-1}}\Big|\nabla_k\zeta\Big(\frac{\xi-x}{y}\Big)\Big|\Big(1+\frac{|\xi-x|}{y}\Big)^k|R(\xi,x)|d\xi.$$

This estimate and (24) imply

$$|\nabla(T\gamma)(x,y)| \leq cy^{-n}\int_{\mathbf{R}^{n-1}}\Big(1+\frac{|\xi-x|}{y}\Big)^{-n}|R(\xi,x)|d\xi$$

$$= cy^{-1/p}\int_{\mathbf{R}^{n-1}}\Big(\frac{|\xi-x|}{y}\Big)^{n-1/p}\Big(1+\frac{|\xi-x|}{y}\Big)^{-n}\frac{|R(\xi,x)|}{|\xi-x|^{n-1/p}}d\xi.$$

Consequently,

$$\int_0^1 y^{p(1-l)-1}|\nabla(T\gamma)(x,y)|^p dy$$

$$\leq c\int_0^1\Big(\int_{\mathbf{R}^{n-1}}f\Big(\frac{|\xi-x|}{y}\Big)\frac{|R(\xi,x)|}{|\xi-x|^{n-1/p}}d\xi\Big)^p y^{p(1-l)-1}\frac{dy}{y},$$

where $f(\eta)=\eta^{n-1/p}(1+\eta)^{-n}$. We write the last integral over $(0,1)$ as

$$\int_0^1\Big(\int_0^\infty f(\frac{t}{y})g(t,x)\frac{dt}{t}\Big)^p y^{p(1-l)-1}\frac{dy}{y}$$

$$=\int_0^1\Big(\int_0^\infty f(s)g(sy,x)\frac{ds}{s}\Big)^p y^{p(1-l)-1}\frac{dy}{y}, \tag{41}$$

with

$$g(t,x)=t^{1/p-1}\int_{\partial\mathcal{B}_1^{n-1}}|R(t\theta+x,x)|d\theta.$$

By Minkowski's inequality, the right-hand side of (41) does not exceed

$$\Big(\int_0^\infty\Big(\int_0^1(f(s))^p(g(sy,x))^p y^{p(1-l)-1}\frac{dy}{y}\Big)^{1/p}\frac{ds}{s}\Big)^p$$

$$=\Big(\int_0^\infty f(s)\Big(\int_0^s(g(\tau,x))^p\tau^{p(1-l)-1}\frac{d\tau}{\tau}\Big)^{1/p}\frac{ds}{s^{2-l-1/p}}\Big)^p$$

$$\leq\Big(\int_0^\infty f(s)\frac{ds}{s^{2-l-1/p}}\Big)^p\int_0^\infty(g(\tau,x))^p\tau^{p(1-l)-1}\frac{d\tau}{\tau}. \tag{42}$$

Therefore,

$$\int_0^1 y^{p(1-l)-1}|\nabla(T\gamma)(x,y)|^p dy \leq c\int_0^\infty(g(\tau,x))^p\tau^{p(1-l)-1}\frac{d\tau}{\tau}.$$

It remains to note that

$$\int_0^\infty(g(\tau,x))^p\tau^{p(1-l)-1}\frac{d\tau}{\tau}=\int_0^\infty\tau^{-pl}\Big(\int_{\partial\mathcal{B}_1^{n-1}}|\gamma(\tau\theta+x)-\gamma(x)|d\theta\Big)^p\frac{d\tau}{\tau}$$

$$\leq c\int_0^\infty\int_{\partial\mathcal{B}_1^{n-1}}|\gamma(\tau\theta+x)-\gamma(x)|^p d\theta\frac{d\tau}{\tau^{pl+1}}\leq c\int_{\mathbf{R}^{n-1}}\frac{|\gamma(x+h)-\gamma(x)|^p}{|h|^{pl+n-1}}dh$$

$$=c\Big(\big(D_{p,l}\gamma\big)(x)\Big)^p.$$

The result follows.

## 7 Proof of Theorem 4

### 7.1 The case $l < 1$

Our aim now is to prove that for $l < 1$ and $s = 1$ the operator $T$ defined by (23) maps $M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$ into $M(W_p^{[m]+1,\beta}(\mathbf{R}_+^n) \to W_p^{1,\alpha}(\mathbf{R}_+^n))$ with $\alpha = 1 - l - 1/p$, $\beta = 1 - \{m\} - 1/p$ and there holds the estimate

$$\|T\gamma; \mathbf{R}_+^n\|_{M(W_p^{t,\beta} \to W_p^{1,\alpha})} \le c\, C \|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^m \to W_p^l)}, \tag{43}$$

where $C$ is the constant in (24).

We have

$$\|(\min\{1, y\})^\alpha \nabla(UT\gamma); \mathbf{R}_+^n\|_{L_p}^p \le c \int_0^1 y^{p\alpha} \int_{\mathbf{R}^{n-1}} \big(|\nabla(T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p\big) dz$$

$$+c \int_1^\infty \int_{\mathbf{R}^{n-1}} \big(|\nabla(T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p\big) dz$$

$$= c \int_{0<y<1} \dots dz + c \int_{y>1} \dots dz. \tag{44}$$

By Lemma 1, for $y > 1$

$$y|\nabla(T\gamma)(z)| + |(T\gamma)(z)| \le c\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}.$$

Hence,

$$\int_{y>1} \dots dz \le c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \|U; \mathbf{R}_+^n\|^p_{W_p^{1,\alpha}}. \tag{45}$$

It remains to refer to the estimate

$$\|U; \mathbf{R}_+^n\|_{W_p^{1,\alpha}} \le c\|U; \mathbf{R}_+^n\|_{W_p^{t,\beta}}$$

which follows from the one dimensional Hardy inequality.

Introducing the notation

$$\mathcal{R}_0 U(z) = U(z) - \sum_{k=0}^{[m]} \frac{\partial^k}{\partial y^k} U(x, 0) \frac{y^k}{k!},$$

$$\mathcal{R}_1 U(z) = \begin{cases} \nabla U(z) - \displaystyle\sum_{k=0}^{[m]-1} \frac{\partial^k}{\partial y^k} \nabla U(x, 0) \frac{y^k}{k!} & \text{for } m > 1 \\ \nabla U(z) & \text{for } m < 1 \end{cases}$$

we have

$$\int_{0<y<1} \dots dz \le c \int_{0<y<1} y^{p(1-l)-1} \sum_{j=0}^{1} |\nabla_j (T\gamma)|^p |\mathcal{R}_{1-j} U(z)|^p dz$$

$$+c \int_{0<y<1} y^{-pl-1} \big(|T\gamma(z)| + y|\nabla(T\gamma)(z)|\big)^p \sum_{k=1}^{[m]} |\nabla_k U(x, 0)|^p y^{pk} dz$$

$$+c\int_{0<y<1} y^{p(1-l)-1}|\nabla T\gamma(z)|^p|U(x,0)|^p dz \tag{46}$$

for $m>1$. In case $m<1$ the second integral in the right hand side of (46) should be omitted.

By Lemma 6, for $0<y<1$

$$|T\gamma(z)|+y|\nabla(T\gamma)(z)|\le c\, y^{l-m}\|\gamma;\mathbf{R}^{n-1}\|_{M(W_p^{m-l}\to L_p)}. \tag{47}$$

Since for $j=0,1$

$$|\mathcal{R}_{1-j}U(z)|\le \frac{y^{[m]+j-1}}{([m]+j-1)!}\int_0^y |\nabla_t U(x,t)|dt, \tag{48}$$

we have

$$\int_{0<y<1} y^{p(1-\{m\})-1}|\mathcal{R}_{1-j}U(z)|^p dz$$

$$\le c\int_{0<y<1} y^{-p\{m\}-1}\Big(\int_0^y |\nabla_{[m]+1}U(x,t)|dt\Big)^p dz.$$

By Hardy's inequality the right-hand side does not exceed $c\,\|U;\mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}$. Combining this with (47) we obtain that the first integral in the right-hand side of (46) does not exceed

$$\|\gamma;\mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}. \tag{49}$$

We now pass to the estimate of the second integral in the right-hand side of (46) for $k=1,\dots,[m]$, $m>1$. Applying Lemma 7, we find

$$\int_{0<y<1} y^{p(k-l)-1}\big(|T\gamma(z)|+y|\nabla(T\gamma)(z)|\big)^p|\nabla_k U(x,0)|^p dz$$

$$\le c\|\gamma;\mathbf{R}^{n-1}\|^{p\frac{k-l}{m-l}}_{M(W_p^{m-l}\to L_p)}\int_{\mathbf{R}^{n-1}}\big(\mathcal{M}\gamma(x)\big)^{p\frac{m-k}{m-l}}|\nabla_k U(x,0)|^p dx. \tag{50}$$

The last integral is not greater than

$$\|(\mathcal{M}\gamma)^{\frac{m-k}{m-l}};\mathbf{R}^{n-1}\|^p_{M(W_p^{m-k}\to L_p)}\|\nabla_k U(\cdot,0);\mathbf{R}^{n-1}\|^p_{W_p^{m-k}}. \tag{51}$$

Using Proposition 2 with $\lambda=m-k$, $\mu=m-l$ and Verbitsky's theorem on the boundedness of the maximal operator $\mathcal{M}$ in the space $M(W_p^{m-l}(\mathbf{R}^{n-1})\to L_p(\mathbf{R}^{n-1}))$ (see [3], Ch.2), we find that (51) is dominated by

$$c\|\gamma,\mathbf{R}^{n-1}\|^{\frac{p(m-k)}{m-l}}_{M(W_p^{m-l}\to L_p)}\|U(\cdot,0);\mathbf{R}^{n-1}\|^p_{W_p^m}.$$

Hence and by (50)

$$\int_{0<y<1} y^{p(k-l)-1}\big(|T\gamma(z)|+y|\nabla(T\gamma)(z)|\big)^p|\nabla_k U(x,0)|^p dz$$

$$\le c\|\gamma,\mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}. \tag{52}$$

By Lemma 8, the integral

$$\int_{0<y<1} y^{p(1-l)-1} |\nabla (T\gamma)(z)|^p |U(x,0)|^p dz \tag{53}$$

does not exceed

$$c\int_{\mathbf{R}^{n-1}} (D_{p,l}\gamma(x))^p |U(x,0)|^p dx$$

$$\leq c\|D_{p,l}\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^m\to L_p)} \|U(\cdot,0); \mathbf{R}^{n-1}\|^p_{W_p^m}$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)} \|U; \mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}. \tag{54}$$

Thus we arrive at the inequality

$$\int_{0<y<1} y^{p\alpha} |\nabla (UT\gamma)(z)|^p dz \leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)} \|U; \mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

It remains to estimate the integral

$$\int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |U(z)|^p dz.$$

Clearly,

$$\int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |U(z)|^p dz \leq \int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |\mathcal{R}_0 U(z)|^p dz$$

$$+\sum_{k=0}^{[m]} \int_{0<y<1} y^{pk} y^{p(1-l)-1} |(T\gamma)(z)|^p |\nabla_k U(x,0)|^p dz. \tag{55}$$

By (47) and (48) with $j=1$ we have

$$\int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |\mathcal{R}_0 U(z)|^p dz$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)} \int_{0<y<1} y^{p(1-\{m\})-1} \Big(\int_0^y |\nabla_{[m]+1} U(x,t)| dt\Big)^p dz$$

which by Hardy's inequality is dominated by (49). In view of (52)

$$\int_{0<y<1} y^{p(k-l)-1} |(T\gamma)(z)|^p |\nabla_k U(x,0)|^p dz$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)} \|U; \mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

Thus we arrive at the estimate

$$\int_{0<y<1} \dots dz \leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^m\to W_p^l)} \|U; \mathbf{R}^n_+\|^p_{W_p^{[m]+1,\beta}}.$$

Since the Poisson kernel satisfies condition (24), Theorem 4 with $l<1$ follows.

## 7.2 The case $l > 1$

**Lemma 9** *Let $m$ and $l$ be nonintegers, $m \geq l > 0$, and let $T$ be the extension operator* (23)*. Suppose that $\gamma \in M(W_p^{m-l}(\mathbf{R}^{n-1}) \to L_p(\mathbf{R}^{n-1}))$. Then*

$$T\gamma \in M(W_p^{[m]-[l],\beta}(\mathbf{R}_+^n) \to W_p^{0,\alpha}(\mathbf{R}_+^n))$$

*and*

$$\|T\gamma; \mathbf{R}_+^n\|_{M(W_p^{[m]-[l],\beta} \to W_p^{0,\alpha})} \leq c\|\gamma; \mathbf{R}^{n-1}\|_{M(W_p^{m-l} \to L_p)}. \tag{56}$$

**Proof.** To begin with, let $[m] = [l]$. Then by (47)

$$\int_{0<y<1} y^{p(1-\{l\})-1} |U(z)(T\gamma)(z)|^p dz$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \int_{0<y<1} y^{p(1-\{m\})-1} |U(z)|^p dz$$

which gives the result.

Suppose $[m] \geq [l] + 1$. We introduce the function

$$\mathcal{R}U = U(z) - \sum_{j=0}^{[m]-[l]-1} \frac{\partial^j U}{\partial y^j}(x, 0) \frac{y^j}{j!}$$

which, clearly, satisfies

$$|\mathcal{R}U(z)| \leq \frac{y^{[m]-[l]-1}}{([m]-[l]-1)!} \int_0^y |\nabla_{[m]-[l]} U(x,t)| dt.$$

This and (47) imply

$$\int_{0<y<1} y^{p(1-\{l\})-1} |T\gamma(z)|^p |\mathcal{R}U(z)|^p dz$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \int_{0<y<1} y^{-p\{m\}-1} \Big( \int_0^y |\nabla_{[m]-[l]} U(x,t)| dt \Big)^p dz.$$

By Hardy's inequality the right-hand side is dominated by

$$c\|\gamma; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-l} \to L_p)} \|U; \mathbf{R}_+^n\|^p_{W_p^{[m]-[l],\beta}}.$$

Furthermore, by Lemma 7 with $m$ replaced by $m - [l]$, $l$ replaced by $\{l\}$and $k = j + 1$ we have for $j = 0, \dots, [m] - [l] - 1$

$$\int_{0<y<1} y^{p(j+1-\{l\})-1} |T\gamma(z)|^p |\nabla_j U(x, 0)|^p dz$$

$$\leq c\|\gamma; \mathbf{R}^{n-1}\|^{p\frac{j+1-\{l\}}{m-l}}_{M(W_p^{m-l} \to L_p)} \int_{\mathbf{R}^{n-1}} \big(\mathcal{M}\gamma(x)\big)^{p\frac{m-[l]-j-1}{m-l}} |\nabla_j U(x,0)|^p dx. \tag{57}$$

The last integral is dominated by

$$\|\big(\mathcal{M}\gamma\big)^{p\frac{m-[l]-j-1}{m-l}}; \mathbf{R}^{n-1}\|^p_{M(W_p^{m-[l]-j-1} \to L_p)} \|U(\cdot, 0); \mathbf{R}^{n-1}\|^p_{W_p^{m-[l]-1}}$$

which by Proposition 3 does not exceed

$$\|\mathcal{M}\gamma;\mathbf{R}^{n-1}\|^{p\frac{m-[l]-j-1}{m-l}}_{M(W_p^{m-l}\to L_p)}\|U;\mathbf{R}^n_+\|^p_{W_p^{[m]-[l],\beta}}.$$

Hence and by (57)

$$\int_{0<y<1} y^{p(j+1-\{l\})-1}|T\gamma(z)|^p|\nabla_j U(x,0)|^p dz$$

$$\leq c\|\gamma;\mathbf{R}^{n-1}\|^p_{M(W_p^{m-l}\to L_p)}\|U;\mathbf{R}^n_+\|^p_{W_p^{[m]-[l],\beta}}.$$

The result follows.

**Proof of Theorem 4 for $l > 1$.**

Suppose Theorem has been proved for $[l] = 1,\dots,\mathcal{L}-1$, where $\mathcal{L}\geq 2$. Let $[l]=\mathcal{L}$ and let

$$\gamma \in M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1})) \ \text{ for } \ m \geq \mathcal{L}. \tag{58}$$

Let $T\gamma$ denote the Poisson integral. Since by Proposition 1 one has

$$\gamma \in M(W_p^{m-l}(\mathbf{R}^{n-1}) \to L_p(\mathbf{R}^{n-1})),$$

it follows from Lemma 9 that

$$T\gamma \in M(W_p^{[m]-[l],\beta}(\mathbf{R}^n_+) \to W_p^{0,\alpha}(\mathbf{R}^n_+))$$

and (56) holds. Next we show that

$$\nabla_{\mathcal{L}+1}(T\gamma) \in M(W_p^{[m]+1,\beta}(\mathbf{R}^n_+) \to W_p^{0,\alpha}(\mathbf{R}^n_+)). \tag{59}$$

Using Proposition 1, we obtain

$$\frac{\partial\gamma}{\partial x_k} \in M(W_p^m(\mathbf{R}^{n-1}) \to W_p^{l-1}(\mathbf{R}^{n-1})), \quad k=1,\dots,n-1.$$

Then, by the induction hypothesis applied to $\partial\gamma/\partial x_k$,

$$\frac{\partial}{\partial x_k}(T\gamma) = T\frac{\partial\gamma}{\partial x_k} \in M(W_p^{[m]+1,\beta}(\mathbf{R}^n_+) \to W_p^{\mathcal{L},\alpha}(\mathbf{R}^n_+)). \tag{60}$$

By Lemma 3,

$$\nabla_{\mathcal{L}}\frac{\partial}{\partial x_k}(T\gamma) \in M(W_p^{[m]+1,\beta}(\mathbf{R}^n_+) \to W_p^{0,\alpha}(\mathbf{R}^n_+)). \tag{61}$$

Using the harmonicity of $T\gamma$ and (61) we find

$$\frac{\partial^{\mathcal{L}+1}(T\gamma)}{\partial y^{\mathcal{L}+1}} = -\frac{\partial^{\mathcal{L}-1}(\Delta_x(T\gamma))}{\partial y^{\mathcal{L}-1}} \in M(W_p^{[m]+1,\beta}(\mathbf{R}^n_+) \to W_p^{0,\alpha}(\mathbf{R}^n_+))$$

which together with (61) implies the inclusion (59). Combining this with (56) we find that $T\gamma \in M(W_p^{[m]+1,\beta}(\mathbf{R}^n_+) \to W_p^{[l]+1,\alpha}(\mathbf{R}^n_+))$. It remains to note that all above inclusions are accompanied by the corresponding estimates. The result follows.

## 8 Extension of multipliers on $\partial\Omega$

We return to the assertion stated in Introduction.

**Theorem 5** *Let* $\gamma \in M(W_p^m(\partial\Omega) \to W_p^l(\partial\Omega))$*, where* $m$ *and* $l$ *are nonintegers,* $m \geq l > 0$*,* $p \in (1, \infty)$*. There exists a linear extension operator*

$$\gamma \to \Gamma \in M(W_p^{t,\beta}(\Omega) \to W_p^{s,\alpha}(\Omega)),$$

*where* $t = [m] + 1$*,* $s = [l] + 1$*,* $\beta = 1 - \{m\} - 1/p$*, and* $\alpha = 1 - \{l\} - 1/p$*.*

**Proof.** It suffices to construct an extension $\Gamma$ only for $\gamma$ with sufficiently small support. To be precise, we assume that $\gamma = 0$ outside the ball $\mathcal{B}_\rho^n$ centered at $0 \in \partial\Omega$, where $\rho$ is small enough. We introduce a cut off function $\varphi \in C_0^\infty(\mathcal{B}_{3\rho}^n)$, equal to one on $\mathcal{B}_{2\rho}^n$. Let us define cartesian coordinates $\zeta = (\xi, \eta)$ with the origin 0, where $\xi \in \mathbf{R}^{n-1}$ and $\eta \in \mathbf{R}^1$. Let $\Omega \cap \mathcal{B}_{3\rho}^n = \{\zeta : \xi \in \mathcal{B}_{3\rho}^{n-1}, \quad \eta > f(\xi)\}$, where $f$ is a smooth function. We make the standard change of variables $\kappa : \zeta \to (x, y)$, where $x = \xi$, $y = \eta - f(\xi)$. The diffeomorphism $\kappa$ maps $\Omega \cap \mathcal{B}_{3\rho}^n$ into the half space $\mathbf{R}_+^n = \{(x, y) : x \in \mathbf{R}^{n-1}, y > 0\}$. Clearly, the function $\tilde{\gamma} = \gamma \circ \kappa^{-1}$ belongs to $M(W_p^m(\mathbf{R}^{n-1}) \to W_p^l(\mathbf{R}^{n-1}))$. Its harmonic extension to $\mathbf{R}_+^n$, denoted by $\tilde{\Gamma}$, is in $M(W_p^{t,\beta}(\mathbf{R}_+^n) \to W_p^{s,\alpha}(\mathbf{R}_+^n))$ and satisfies the estimate (22) according to Theorem 4. Hence the function $\gamma = (\tilde{\Gamma} \circ \kappa)\varphi$ is a desired extension. The proof is complete.

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