Traces of multipliers in pairs of weighted Sobolev spaces

Vladimir Maz’ya, Tatyana Shaposhnikova*

Department of Mathematics, University of Linköping
SE-581 83 Linköping, Sweden

Abstract. We prove that the pointwise multipliers acting in a pair of fractional Sobolev spaces form the space of boundary traces of multipliers in a pair of weighted Sobolev space of functions in a domain.

AMS Subject Classifications: 46E35, 46E25

Key words: multipliers, weighted Sobolev spaces, fractional Sobolev spaces

1 Introduction

By a multiplier acting from one Banach function space $S_1$ into another $S_2$ we call a function $\gamma$ such that $\gamma u \in S_2$ for any $u \in S_1$. By $M(S_1 \to S_2)$ we denote the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$
\|\gamma\|_{M(S_1 \to S_2)} = \sup \{\|\gamma u\|_{S_2} : \|u\|_{S_1} \leq 1\}.
$$

We write $MS$ instead of $M(S \to S)$, where $S$ is a Banach function space. We shall use the same notation both for spaces of scalar and vector-valued multipliers.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. It is well known that the fractional Sobolev space $W^{l,p}_p(\partial \Omega)$ is the space of traces of the weighted Sobolev space $W^{s,\alpha}_p(\Omega)$ endowed with the norm

$$
\left( \int_{\Omega} \left( \text{dist}(x, \partial \Omega) \right)^{p\alpha} \sum_{\tau, 0 \leq |\tau| \leq s} |D^\tau u|^p dx \right)^{1/p},
$$

where $\alpha = 1 - \{l\} - 1/p$, $s = [l] + 1$ and $p \in (1, \infty)$ (see [5]). It is straightforward to deduce from this fact that the trace $\gamma$ of the function

$$
\Gamma \in M(W^{t,\beta}_p(\Omega) \to W^{s,\alpha}_p(\Omega))
$$

belongs to $M(W^{m}_p(\partial \Omega) \to W^{l}_p(\partial \Omega))$. Here $m$ and $l$ are nonintegers, $m \geq l > 0$, $s$ and $\alpha$ are given above, $t = [m] + 1$, $\beta = 1 - \{m\} - 1/p$.

In the present paper we prove that the converse assertion is also true showing that there exists an extension $\Gamma$ of $\gamma \in M(W^{m}_p(\partial \Omega) \to W^{m}_p(\partial \Omega))$ subject to (1).

*The authors were supported by grants of the Swedish National Science Foundation.
2 The space $M(W^m_p(R^{n-1}) \rightarrow W^l_p(R^{n-1}))$

By $B^{n-1}_r(x)$ we mean the ball $\{\xi \in R^{n-1} : |\xi - x| < r\}$ and write $B^{n-1}_r(0)$ instead of $B^{n-1}_r(0)$.

We need the spaces $S_{loc}$ and $S_{unif}$ of functions on $R^{n-1}$ defined as follows. By $S_{loc}$ we denote the space

$$\{u : \eta u \in S \text{ for all } \eta \in C_0^\infty(R^{n-1})\}$$

and by $S_{unif}$ we mean the space

$$\{u : \sup_{\xi} \|\eta_\xi u\|_S < \infty\},$$

where $\eta_\xi(x) = \eta(x - \xi)$, $\eta \in C_0^\infty(R^{n-1})$, $\eta = 1$ on $B_1^{n-1}$. The space $S_{unif}$ is endowed with the norm

$$\|u\|_{S_{unif}} = \sup_{\xi} \|\eta_\xi u\|_S.$$

Let $W^l_p(R^{n-1})$ denote the fractional Sobolev space with the norm

$$\|D_p;\gamma; R^{n-1}\|_{L_p} + \|\eta; R^{n-1}\|_{L_p},$$

where

$$(D_p;\gamma;\eta)(x) = \left(\int_{R^{n-1}} |\nabla[p]\eta u(x + h) - \nabla[p]\eta u(x)|^p |h|^{1 - p(l)} dh\right)^{1/p}, \quad (2)$$

with $\nabla[p]$ being the gradient of order $[l]$, i.e. $\nabla[p] = \{\partial_{x_1}^{\tau_1}, \ldots, \partial_{x_{n-1}}^{\tau_{n-1}}\}$, $\tau_1 + \ldots + \tau_{n-1} = [l]$.

In this section we collect some known properties of multipliers between fractional Sobolev spaces $W^m_p(R^{n-1})$ and $W^l_p(R^{n-1})$, $m \geq l \geq 0$. The equivalence $a \sim b$ means that $a/b$ is bounded and separated from zero by positive constants depending on $n$, $p$, $m$, and $l$.

**Proposition 1** [3] Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$.

(i) There holds

$$\|\gamma; R^{n-1}\|_{M(W^m_p \rightarrow W^l_p)} \sim \|D_p;\gamma; R^{n-1}\|_{M(W^m_p \rightarrow L_p)} + \|\gamma; R^{n-1}\|_{M(W^{m-1}_p \rightarrow L_p)}.$$

(ii) If $\gamma \in M(W^m_p(R^{n-1}) \rightarrow W^l_p(R^{n-1}))$ then for any multi-index $\sigma$, $|\sigma| \leq [l]$,

$$D^\sigma;\gamma \in M(W^m_p(R^{n-1}) \rightarrow W^{l-|\sigma|}(R^{n-1})).$$

(iii) Let $0 < \lambda < \mu$. Then

$$\|\gamma^{\lambda/\mu}; R^{n-1}\|_{M(W^\lambda_p \rightarrow L_p)} \leq c \|\gamma; R^{n-1}\|^{\lambda/\mu}_{M(W^\mu_p \rightarrow L_p)}.$$

**Proposition 2** [3] Let $m$ and $l$ be nonintegers, $m \geq l \geq 0$, and let $p \in (1, \infty)$. Then

$$\|\gamma; R^{n-1}\|_{M(W^m_p \rightarrow W^l_p)} \sim \sup_{e \subset R^{n-1} \text{ diam}(e) \leq 1} \frac{\|D_p;\gamma; e\|_{L_p}}{(\text{cap}_{p,m}(e))^{1/p}}.$$
where \( e \) is a compact set in \( \mathbb{R}^{n-1} \) and \( \text{cap}_{p,m}(e) \) is the \((p, m)\)-capacity of \( e \) defined by
\[
\text{cap}_{p,m}(e) = \inf\{\|u; \mathbb{R}^{n-1}\|_{W^p_m} : u \in C_0^\infty(\mathbb{R}^{n-1}), \ u \geq 1 \text{ on } e\}
\]
For \( l = 0 \) one should replace \( D_{p,1}\gamma \) by \( \gamma \).

Upper estimates for the norm in \( M(W^m_p(\mathbb{R}^{n-1}) \to W^l_p(\mathbb{R}^{n-1})) \) are given in the following assertion. By \( \text{mes}_{n-1} \) we mean the \((n - 1)\)-dimensional Lebesgue measure of a compact set \( e \).

**Proposition 3** [3] Let \( m \) and \( l \) be nonintegers, \( m \geq l \geq 0 \), and let \( p \in (1, \infty) \).

(i) If \( mp < n - 1 \), then
\[
c\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \leq \sup_{c < n-1} \frac{\|D_{p,1}\gamma; c\|_{L_p}}{(\text{mes}_{n-1}(e))^{1/p - m/(n-1)}}
\]
\[
+ \left\{ \sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{m-1}_1(x)\|_{L_1} \quad \text{for } m > l,
\right. \]
\[
\left. \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} \quad \text{for } m = l. \right\}
\]

(ii) If \( mp = n - 1 \), then
\[
c\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \leq \sup_{c < n-1} (\log \frac{2^{n-1}}{\text{mes}_{n-1}(e)})^{1/p} \|D_{p,1}\gamma; c\|_{L_p}
\]
\[
+ \left\{ \sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{m-1}_1(x)\|_{L_1} \quad \text{for } m > l,
\right. \]
\[
\left. \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} \quad \text{for } m = l. \right\}
\]

Now we list lower estimates for the norm in \( M(W^m_p \to W^l_p) \).

**Proposition 4** [3] Let \( m \) and \( l \) be nonintegers, \( m \geq l \geq 0 \), and let \( p \in (1, \infty) \).

(i) If \( mp < n - 1 \), then
\[
c\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \geq \sup_{x \in (0,1)} \frac{\|D_{p,1}\gamma; B^{m-1}_1(x)\|_{L_p}}{r^{(n-1)/p - m}}
\]
\[
+ \left\{ \sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{m-1}_1(x)\|_{L_1} \quad \text{for } m > l,
\right. \]
\[
\left. \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} \quad \text{for } m = l. \right\}
\]

(ii) If \( mp = n - 1 \), then
\[
c\|\gamma; \mathbb{R}^{n-1}\|_{M(W^m_p \to W^l_p)} \geq \sup_{c \in (0,1)} (\log \frac{2}{r})^{1/p} \|D_{p,1}\gamma; B^{m-1}_1(x)\|_{L_p}
\]
\[
+ \left\{ \sup_{x \in \mathbb{R}^{n-1}} \|\gamma; B^{m-1}_1(x)\|_{L_1} \quad \text{for } m > l,
\right. \]
\[
\left. \|\gamma; \mathbb{R}^{n-1}\|_{L_\infty} \quad \text{for } m = l. \right\}
3 Multipliers in pairs of weighted Sobolev spaces in $\mathbb{R}^n_+$

3.1 Preliminary facts

Let $\mathbb{R}^n_+$ denote the upper half-space $\{z = (x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$. We introduce the weighted Sobolev space $W_p^{s,\alpha}(\mathbb{R}^n_+)$ with the norm

$$\|(\min\{1, y\})^\alpha \nabla s U; \mathbb{R}^n_+\|_{L_p} + \|(\min\{1, y\})^\alpha U; \mathbb{R}^n_+\|_{L_p},$$

(3)

where $s$ is nonnegative integer. We always assume that $-1 < \alpha p < p - 1$.

It is well known that the fractional Sobolev space $W_p^s(\mathbb{R}^{n-1})$, is the space of traces on $\mathbb{R}^{n-1}$ of functions in the space $W_p^{s,\alpha}(\mathbb{R}^n_+)$, where $s = \lfloor l \rfloor + 1$, $\alpha = 1 - \lfloor l \rfloor - 1/p$, and $p \in (1, \infty)$ (see [5]). We show that a similar result holds for spaces of pointwise multipliers acting in a pair of fractional Sobolev spaces. To be precise, we prove that for all noninteger $m$ and $l$, $m \geq l > 0$, the multiplier space $M(W_p^m(\mathbb{R}^{n-1}) \to W_p^l(\mathbb{R}^{n-1}))$ is the space of traces on $\mathbb{R}^{n-1}$ of functions in $M(W_p^{m,\alpha}(\mathbb{R}^n_+) \to W_p^{l,\alpha}(\mathbb{R}^n_+))$, where $s$ and $\alpha$ are as above and $\beta = 1 - \lfloor m \rfloor - 1/p$, $t = \lceil m \rceil + 1$. Different positive constants depending on $n, p, l, m, s, t$ will be denoted by $c$. We shall omit $\mathbb{R}^n_+$ in notations of norms.

We introduce the notion of $(p, s, \alpha)$-capacity of a compact set $e \subset \mathbb{R}^n_+$:

$$\text{cap}_{p, s, \alpha}(e) = \inf\{\|U; \mathbb{R}^n_+\|_{W_p^{s,\alpha}}^p : U \in C_0^\infty(\mathbb{R}^n_+), U \geq 1 \text{ on } e\}.$$

The following result is essentially known (see [2], Sect. 8.1, 8.2).

**Proposition 5** Let $k$ be a nonnegative integer, $-1 < \beta p < p - 1$, and let $1 < p < \infty$. Then $\Gamma \in M(W_p^{k,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))$ if and only if

$$\sup_{e \subset \mathbb{R}^n_+} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{\text{cap}_{p, k, \beta}(e)^{1/p}} < \infty.$$

The equivalence relation

$$\|\Gamma\|_{M(W_p^{k,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))} \sim \sup_{e \subset \mathbb{R}^n_+} \frac{\|(\min\{1, y\})^\alpha \Gamma; e\|_{L_p}}{\text{cap}_{p, k, \beta}(e)^{1/p}}$$

(4)

is valid.

We shall use some general properties of multipliers. We start with the inequality

$$\|\Gamma\|_{M(W_p^{s-i,\beta}(\mathbb{R}^n_+) \to W_p^{s-i,\alpha}(\mathbb{R}^n_+))} \leq c \|\Gamma\|_{M(W_p^{s,\beta}(\mathbb{R}^n_+) \to W_p^{s,\alpha}(\mathbb{R}^n_+))}^{(s-j)/s} \|\Gamma\|_{M(W_p^{s-j,\beta}(\mathbb{R}^n_+) \to W_p^{s-j,\alpha}(\mathbb{R}^n_+))}^{j/s},$$

(5)

where $0 \leq j \leq s$, $-1 < \alpha p < p - 1$, $-1 < \beta p < p - 1$, which follows from the interpolation property of weighted Sobolev spaces (see [4], Sect.3.4.2).

The next assertion contains inequalities between multipliers and their mollifiers in variables $x$. 

4
Lemma 1 Let $\Gamma_\rho$ denote a mollifier of a function $\Gamma$ defined by
\[
\Gamma_\rho(x, y) = \rho^{-n+1} \int_{\mathbb{R}^{n-1}} K(\rho^{-1}(x - \xi))\Gamma(\xi, y)d\xi,
\]
where $K \in C_0^\infty(\mathbb{R}_1^{n-1})$, $K \geq 0$, and $\|K; \mathbb{R}_{n-1}\|_{L_1} = 1$. Then
\[
\|\Gamma_\rho\|_{M(W_p^{s,\alpha} - W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{s,\alpha} - W_p^{s,\alpha})} \leq \lim_{\rho \to 0} \|\Gamma_\rho\|_{M(W_p^{s,\alpha} - W_p^{s,\alpha})},
\]
(6)

Proof. Let $U \in C_0^\infty$. By Minkowski’s inequality
\[
\left( \int_{\mathbb{R}^{n-1}_+} (\min\{1, y\})^{p\alpha} |\nabla_j U|dz\right)^{1/p} \leq \int_{\mathbb{R}^{n-1}_+} \rho^{-n} K(\xi/\rho)\Gamma(x - \xi, y)U(x, y)d\xi \leq \|\Gamma_\rho\|_{M(W_p^{s,\alpha} - W_p^{s,\alpha})} \|U\|_{W_p^{s,\beta}}.
\]
This gives the left inequality (6). The right inequality (6) follows from
\[
\|\Gamma_\rho\|_{W_p^{s,\alpha}} \leq \lim_{\rho \to 0} \|\Gamma_\rho U\|_{W_p^{s,\alpha}} \leq \lim_{\rho \to 0} \|\Gamma_\rho\|_{M(W_p^{s,\alpha} - W_p^{s,\alpha})} \|U\|_{W_p^{s,\beta}}.
\]
The proof is complete.

Lemma 2 Let $\Gamma \in L_{p, loc}$, $p \in (1, \infty)$, $-1 < \beta p < p - 1$, and let $U$ be an arbitrary function in $C_0^\infty(\mathbb{R}_+^n)$. The best constant in the inequality
\[
\|(\min\{1, y\})^{\alpha} \Gamma \nabla_x U\|_{L_p} + \|(\min\{1, y\})^{\alpha} \Gamma U\|_{L_p} \leq C \|U\|_{W_p^{1,\beta}},
\]
(7)
is equivalent to the norm $\|\Gamma\|_{M(W_p^{1-\beta,\alpha} - W_p^{1-\beta,\alpha})}$.

Proof. The estimate $C \leq c\|\Gamma\|_{M(W_p^{1-\beta,\alpha} - W_p^{1-\beta,\alpha})}$ is obvious. To derive the converse estimate, we introduce a function $x \mapsto \sigma$ which is positive on $[0, \infty)$ and is equal to $x$ for $x > 1$. For any $U \in C_0^\infty(\mathbb{R}_+^n)$ there holds
\[
U = (-\Delta)^s(\sigma(-\Delta))^{-[n-1]/2}u + T(-\Delta)u,
\]
where $T$ is a function in $C_0^\infty([0, \infty))$. Since
\[
(-\Delta)^s = (-1)^s \sum_{|\beta|=s} \frac{s!}{\beta!}D_\beta^2,
\]

it follows from (7) and the theorem on the boundedness of convolution operators in weighted $L_p$ spaces (see [1]) that

$$
\int_{\mathbb{R}^n_+} (\min\{1,y\})^{p\alpha} |\Gamma(z)U(z)|^p dz
\leq c C \left( \|\nabla_s (\sigma(-\Delta))^{-s} U\|_{W^{p,\alpha}_p}^{p} + \|TU\|_{W^{p,\alpha}_p}^{p} \right) \leq c C \|U\|_{W^{p,\alpha-\delta}_p}^{p}.
$$

The proof is complete.

4 Characterisation of the space $M(W^{t,\beta}_p \to W^{s,\alpha}_p)$

Here we derive necessary and sufficient conditions for a function to belong to the space $M(W^{t,\beta}_p \to W^{s,\alpha}_p)$ for $p \in (1,\infty)$ with $\alpha$ and $\beta$ satisfying

$$
-1 < \alpha p < p-1, \quad -1 < \beta p < p-1, \quad t \geq s.
$$

These inequalities will be assumed everywhere. We start with an assertion on derivatives of multipliers. We shall omit $\mathbb{R}^n_{++}$ in notations of spaces, norms, and integrals.

**Lemma 3** Suppose

$$
\Gamma \in M(W^{t,\beta}_p \to W^{s,\alpha}_p) \cap M(W^{t-s,\beta}_p \to W^{0,\alpha}_p), \quad p \in (1,\infty).
$$

Then $D^\sigma \Gamma \in M(W^{t,\beta}_p \to W^{s-|\sigma|,\alpha}_p)$ for any multiindex $\sigma$ of order $|\sigma| \leq s$ and

$$
\|D^\sigma \Gamma\|_{M(W^{t,\beta}_p \to W^{s-|\sigma|,\alpha}_p)} \leq \epsilon \|\Gamma\|_{M(W^{t-s,\beta}_p \to W^{0,\alpha}_p)} + c(\epsilon) \|\Gamma\|_{M(W^{t,\beta}_p \to W^{s,\alpha}_p)}, \quad (9)
$$

where $\epsilon$ is an arbitrary positive number.

**Proof.** Let $U \in W^{s,\alpha}_p$ and let $\varphi$ be an arbitrary function in $C_0^\infty$. Applying Leibniz formula

$$
D^\sigma (\varphi U) = \sum_{\{\tau, \sigma \geq \tau \geq 0\}} \frac{\sigma!}{\tau!(\sigma - \tau)!} D^\tau \varphi D^{\sigma-\tau} U,
$$

we find

$$
\int \varphi U(-D)^\sigma \Gamma dz = \int \Gamma D^\sigma (\varphi U) dz = \sum_{\{\tau, \sigma \geq \tau \geq 0\}} \frac{\sigma!}{\tau!(\sigma - \tau)!} \Gamma D^\tau \varphi D^{\sigma-\tau} U dz
\leq \int \varphi \sum_{\{\beta, \sigma \geq \tau \geq 0\}} \frac{\sigma!}{\tau!(\sigma - \tau)!} (-D)^\tau (\Gamma D^{\sigma-\tau} U) dz.
$$

Therefore,

$$
UD^\sigma \Gamma = \sum_{\{\tau, \sigma \geq \tau \geq 0\}} \frac{\sigma!}{\tau!(\sigma - \tau)!} (D)^\tau (\Gamma (-D)^{\sigma-\tau} U),
$$

which implies the estimate

$$
\|UD^\sigma \Gamma\|_{W^{s-|\sigma|,\alpha}_p} \leq c \sum_{\{\tau, \sigma \geq \tau \geq 0\}} \|\Gamma D^{\sigma-\tau} U\|_{W^{s-|\sigma|+|\tau|,\alpha}_p}.
$$

6
Hence, it suffices to prove (9) for \(|\sigma| = 1\). We have

\[
\|U\nabla \Gamma\|_{W_p^{1,s,\alpha}} \leq \|U\Gamma\|_{W_p^{1,s,\alpha}} + \|\Gamma \nabla U\|_{W_p^{1,s,\alpha}}
\]

\[
\leq (\|\Gamma\|_{M(W_p^{1,s,\alpha} \rightarrow W_p^{s,\alpha})} + \|\Gamma\|_{M(W_p^{1,s,\alpha} \rightarrow W_p^{s,\alpha})})\|U\|_{W_p^{1,s,\alpha}}.
\]

Estimating the norm \(\|\Gamma\|_{M(W_p^{1,s,\alpha} \rightarrow W_p^{s,\alpha})}\) by (5) we arrive at (9).

We pass now to two-sided estimates of norms in \(M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha}), \ p \in (1, \infty)\), given in terms of the spaces \(M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha})\). We start with lower estimates.

**Lemma 4** Let \(\Gamma \in M(W_p^{t,\beta} \rightarrow W_p^{s,\alpha})\). Then

\[
\|\nabla \Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})} + \|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})} \leq c \|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})}.
\]

**Proof.** Suppose first that \(\Gamma \in M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})\). We have

\[
\|\nabla \Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})} + c \sum_{j=1}^{s} \|\nabla \Gamma\|_{M(W_p^{t-j,s,\alpha} \rightarrow W_p^{s,\alpha})} \|U\|_{W_p^{1,s,\alpha}}.
\]

By Lemma 3,

\[
\|\nabla \Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})} + c \|\Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})}.
\]

Estimating the last norm by (5) we obtain

\[
\|\nabla \Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})} + c(\varepsilon) \|\Gamma\|_{M(W_p^{t-j,\alpha} \rightarrow W_p^{s,\alpha})}.
\]

Substitution of this into (11) gives

\[
\|\nabla \Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})} \|U\|_{W_p^{1,s,\alpha}}.
\]

Besides,

\[
\|\Gamma U\|_{W_p^{s,\alpha}} \leq \|\Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})} \|U\|_{W_p^{1,s,\alpha}}.
\]

Summing up two last estimates and applying Lemma 2 we arrive at

\[
\|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})} \leq \|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})} + c(\varepsilon) \|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})}.
\]

Hence,

\[
\|\Gamma\|_{M(W_p^{t-s,\alpha} \rightarrow W_p^{s,\alpha})} \leq c \|\Gamma\|_{M(W_p^{t,s,\alpha} \rightarrow W_p^{s,\alpha})}.
\]
Now, we are going to remove the assumption $\Gamma \in M(W^{s,\alpha}_p \to W^{s,\alpha}_p)$. Since $\Gamma \in M(W^{r,\beta}_p \to W^{s,\alpha}_p)$, then
\[
\|\Gamma\|_{W^{s,\alpha}_p} \leq c \|\eta\|_{W^{r,\beta}_p},
\]
where $\eta \in C_{0,0}^\infty(B^n_0(z))$, $\eta = 1$ on $B^n_0(z)$, and $z$ is an arbitrary point in $\mathbb{R}^n$. Hence $\Gamma \in W^{s,\alpha}_{r,\beta,\text{uni}}(\mathbb{R}^n_0)$ which implies that for any $(n-1)$-dimensional multiindex $\tau$ the derivative $D^\tau_s \Gamma$ belongs to $W^{s,\alpha}_{r,\beta,\text{uni}}(\mathbb{R}^n_+)$. Therefore, $\Gamma_{r,\beta} \in L_\infty(\mathbb{R}^n_0)$ which in its turn guarantees that $\Gamma_{r,\beta} \in M(W^{s,\alpha}_p \to W^{s,\alpha}_p)$. Thus, we may put $\Gamma_{r,\beta}$ into (15) in order to obtain
\[
\|\Gamma\|_{M(W^{s,\alpha}_p \to W^{s,\alpha}_p)} \leq c \|\Gamma\|_{M(W^{s,\alpha}_p \to W^{s,\alpha}_p)}.
\]
Letting $\rho \to 0$ and using Lemma 1 we arrive at (15) for all $\Gamma \in M(W^{s,\alpha}_p \to W^{s,\alpha}_p)$.

To estimate the first term in the right-hand side of (10), we combine (15) with (12) for $j = s$.

The estimate converse to (10) is contained in the following lemma.

**Lemma 5** Let $\Gamma \in M(W^{s,\beta}_p \to W^{0,\alpha}_p)$ and let $\nabla_s \Gamma \in M(W^{r,\beta}_p \to W^{0,\alpha}_p)$. Then $\Gamma \in M(W^{s,\alpha}_p \to W^{s,\alpha}_p)$ and the estimate
\[
\|\Gamma\|_{M(W^{r,\beta}_p \to W^{s,\alpha}_p)} \leq c \left(\|\nabla_s \Gamma\|_{M(W^{r,\beta}_p \to W^{0,\alpha}_p)} + \|\Gamma\|_{M(W^{s,\beta}_p \to W^{s,\alpha}_p)}\right)
\]
(16) is valid.

**Proof.** By Lemma 4 and (5) we have
\[
\|\nabla_j \Gamma\|_{M(W^{r-s+j,\beta}_p \to W^{0,\alpha}_p)} \leq c \|\Gamma\|_{M(W^{r-s+j,\beta}_p \to W^{0,\alpha}_p)}
\]
\[
\leq c \|\nabla_j \Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)}\|\nabla_j \Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)}(1/\gamma)
\]
(17)
where $j = 1, \ldots, s$. For any $U \in C_{0,0}^\infty$,
\[
\|\min\{1, y\}\|_{\gamma} \nabla_s (\Gamma U)\|_{L_p} \leq c \sum_{j=0}^s \|\min\{1, y\}\|_{\gamma} \nabla_s \Gamma\|_{\gamma} \|\nabla_s \nabla_j U\|_{L_p}
\]
\[
\leq c \left(\|\nabla_s \Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)} + \|\Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)}\right)\|\nabla_s \Gamma\|_{\gamma}\|\nabla_j U\|_{L_p}.
\]
This and (17) imply
\[
\|\min\{1, y\}\|_{\gamma} \nabla_s (\Gamma U)\|_{L_p} \leq c \left(\|\nabla_s \Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)} + \|\Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)}\right)\|\nabla_s \Gamma\|_{\gamma}\|\nabla_j U\|_{L_p}.
\]
It remains to note that
\[
\|\min\{1, y\}\|_{\gamma} U\|_{L_p} \leq \|\Gamma\|_{M(W^{s,\beta}_p \to W^{0,\alpha}_p)}\|\nabla_s \Gamma\|_{\gamma}\|U\|_{W^{s,\beta}_p}.
\]
The proof is complete.

Using Lemmas 4 and 5 we arrive at the following description of the space $M(W^{s,\alpha}_p(\mathbb{R}^n_+) \to W^{s,\alpha}_p(\mathbb{R}^n_+))$.
Corollary 1
Moreover, for any compact set $e \in \Gamma$ and the estimate

$$\|\Gamma\|_{M(W^s_p,-W^0_p)} \leq \|\nabla_s \Gamma\|_{M(W^s_p,-W^0_p)} + \|\Gamma\|_{M(W^{t-s}_p,-W^0_p)}.$$  

The equivalence relation (4) enables one to reformulate Theorem 1.

Theorem 2
A function $\Gamma$ belongs to the space $M(W^t_p, -W^s_p)$ if and only if $\Gamma \in W^s_p, \Gamma \in M(W^{t-s}_p, -W^0_p)$, and $\nabla_s \Gamma \in M(W^t_p, -W^0_p)$. Moreover,

$$\|\Gamma\|_{M(W^t_p, -W^s_p)} \sim \|\nabla_s \Gamma\|_{M(W^t_p, -W^0_p)} + \|\Gamma\|_{M(W^{t-s}_p, -W^0_p)}.$$  

Theorem 3
Let $m$ and $l$ be positive noninteger, $m \geq l$ and let

$$\Gamma \in M(W^t_m, \Gamma^{n+1}_p, R^n) - W^s_p, \Gamma^{n+1}_p, R^n)$$

where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. If $\gamma$ is the trace of $\Gamma$ on $R^{n-1}$, then

$$\gamma \in M(W^m_p, R^{n-1}) - W^l_p, R^{n-1})$$

and the estimate

$$\|\gamma; R^{n-1}\|_{M(W^m_p, -W^l_p)} \leq c\|\Gamma; R^n\|_{M(W^t_m, -W^s_p)}$$  

holds.

5 Trace theorems for multipliers in weighted Sobolev spaces

We start with the following simple fact concerning traces of multipliers.

Theorem 4
A function $\Gamma$ belongs to the space $MW^{t,\alpha}_p$ if and only if $\Gamma \in W^{\alpha,\alpha}_p$, and for any compact set $e \subset R^n_p$

$$\|\Gamma\|_{MW^{t,\alpha}_p} \sim \sup_{e \subset R^n_p, \text{diam}(e) \leq \gamma} \|\nabla \Gamma\|_{L^p} + \|\Gamma\|_{L^\infty}.$$  

An important particular case of Theorem 2 is $t = s$. 

Corollary 1
A function $\Gamma$ belongs to the space $MW^{t,\alpha}_p$ if and only if $\Gamma \in W^{\alpha,\alpha}_p$, and for any compact set $e \subset R^n_p$

$$\|\nabla \Gamma\|_{L^p} \leq c\|\Gamma\|_{L^\infty}.$$  

Moreover,

$$\|\Gamma\|_{MW^{t,\alpha}_p} \sim \sup_{e \subset R^n_p, \text{diam}(e) \leq \gamma} \|\nabla \Gamma\|_{L^p} + \|\Gamma\|_{L^\infty}.$$  

5 Trace theorems for multipliers in weighted Sobolev spaces

We start with the following simple fact concerning traces of multipliers.

Theorem 3
Let $m$ and $l$ be positive noninteger, $m \geq l$ and let

$$\Gamma \in M(W^t_m, \Gamma^{n+1}_p, R^n) - W^s_p, \Gamma^{n+1}_p, R^n)$$

where $t = [m] + 1$, $s = [l] + 1$, $\beta = 1 - \{m\} - 1/p$, and $\alpha = 1 - \{l\} - 1/p$. If $\gamma$ is the trace of $\Gamma$ on $R^{n-1}$, then

$$\gamma \in M(W^m_p, R^{n-1}) - W^l_p, R^{n-1})$$

and the estimate

$$\|\gamma; R^{n-1}\|_{M(W^m_p, -W^l_p)} \leq c\|\Gamma; R^n\|_{M(W^t_m, -W^s_p)}$$  

holds.
Proof. Let $U \in W^{t,\beta}(R^n_+)$ and let $u$ be the trace of $U$ on $R^{n-1}$. By setting $\Gamma U$ and $\gamma u$ instead of $U$ and $u$, respectively, in the inequality
\[
\|u; R^{n-1}\|_{W^p} \leq c\|U; R^n_+\|_{W^{t,\alpha}_+}
\]
we arrive at the estimate
\[
\|\gamma u; R^{n-1}\|_{W^p} \leq c \|\Gamma; R^n_+\|_{M(W^{t,\beta}_{n-1}\rightarrow W^{t,\alpha}_{n+1})}\|U; R^n_+\|_{W^{t,\alpha}_+}.
\]
Minimizing the right-hand side over all extensions $U$ of $u$ we obtain
\[
\|\gamma u; R^{n-1}\|_{W^p} \leq c \|\Gamma; R^n_+\|_{M(W^{t,\beta}_{n-1}\rightarrow W^{t,\alpha}_{n+1})}\|u; R^{n-1}\|_{W^p},
\]
which gives (20).

We state an extension theorem for functions in $M(W^m_p(R^{n-1}) \rightarrow W^l_p(R^{n-1}))$ to be proved in Sect. 7.

**Theorem 4** Let $m$ and $l$ be positive nonintegers, $m \geq l$, $p \in (1, \infty)$, and let
\[
\gamma \in M(W^m_p(R^{n-1}) \rightarrow W^l_p(R^{n-1})).
\]
Then the Dirichlet problem
\[
\Delta \Gamma = 0 \text{ on } R^n_+, \quad \Gamma|_{R^{n-1}} = \gamma
\]
has a unique solution in $M(W^{t,\beta}_{n} \rightarrow W^{s,\alpha}_{n} (R^n_+))$, where $t = [m] + 1, s = [l] + 1, \beta = 1 - \{m\}/p$, and $\alpha = 1 - \{l\}/p$. There holds the estimate
\[
\|\Gamma; R^n_+\|_{M(W^{t,\beta}_{n-1}\rightarrow W^{t,\alpha}_{n+1})} \leq c \|\gamma; R^{n-1}\|_{M(W^{m-\{l\}}_{p}\rightarrow W^l_p)}.\]

6 Auxiliary estimates for an extension operator

6.1 Pointwise estimate for $T\gamma$ and $\nabla T\gamma$

For functions $\gamma \in L^1_{\text{unif}}(R^{n-1})$, we introduce the operator
\[
(T\gamma)(x, y) = y^{1-n} \int_{R^{n-1}} \zeta \left(\frac{x - \xi}{y}\right) \gamma(\xi) d\xi, \quad (x, y) \in R^n_+,
\]
where $\zeta$ is a continuously differentiable function defined on $R^n_+$ outside the origin. We assume that
\[
(|z| + 1)|\nabla \zeta(z)| + |\zeta(z)| \leq C (|z| + 1)^{-n}.
\]

**Lemma 6** Let $\gamma \in M(W^{m-\{l\}}_{p}(R^{n-1}) \rightarrow L^p(R^{n-1}))$, where $m \geq l$ and $1 < p < \infty$. Then
\[
|T\gamma(z)| + y|\nabla (T\gamma(z))| \leq c(1 + y^{1-l})\|\gamma; R^{n-1}\|_{M(W^{m-\{l\}}_{p}\rightarrow L^p)}.\]

**Proof.** In view of (24)
\[
|T\gamma(z)| + y|\nabla (T\gamma(z))|
\]
\[
\leq cy^{1-n} \left(\int_{B^{n-1}_y(z)} |\gamma(\xi)| d\xi + y^n \int_{R^{n-1}\setminus B^{n-1}_y(z)} \frac{|\gamma(\xi)| d\xi}{|\xi - x|^n}\right).
\]
By Hölder’s inequality,
\[
\int_{\mathcal{B}^{n-1}_y(x)} |\gamma(\xi)| d\xi \leq cy^{(n-1)(p-1)/p} \|\gamma; \mathcal{B}^{n-1}_y(x)\|_{L_p}.
\]  
(26)

Let \( y \in (0, 1) \). The right-hand side in (26) does not exceed
\[
cy^{-m+l+n-1} \sup_{r \in (0, 1)} \sup_{x \in \mathbb{R}^{n-1}} (1 + r^{m-l-n-1}) \|\gamma; \mathcal{B}^{n-1}_r(x)\|_{L_p}
\]
\[
\leq cy^{-m+l+n-1} \sup_{r \in (0, 1)} \sup_{x \in \mathbb{R}^{n-1}} (\text{cap}_{p,m-l}(\mathcal{B}^{n-1}_r(x)))^{-1/p} \|\gamma; \mathcal{B}^{n-1}_r(x)\|_{L_p}.
\]  
(27)

This and Proposition 2 show that for \( y \in (0, 1) \)
\[
\int_{\mathcal{B}^{n-1}_y(x)} |\gamma(\xi)| d\xi \leq cy^{-m+l+n-1} \|\gamma; \mathcal{R}^{n-1}\|_{M(W^{m-1}_p \to L_p)}.
\]  
(28)

Suppose \( y > 1 \). Since
\[
\text{cap}_{p,m-l}(\mathcal{B}^{n-1}_r(x)) \sim r^{n-1} \text{ for } r > 1,
\]  
(29)

it follows that the right-hand side of (26) is dominated by
\[
cy^{n-1} (\text{cap}_{p,m-l}(\mathcal{B}^{n-1}_r(x)))^{-1/p} \|\gamma; \mathcal{B}^{n-1}_r(x)\|_{L_p}.
\]

Combining this with (27) and Proposition 2 we conclude that
\[
\int_{\mathcal{B}^{n-1}_y(x)} |\gamma(\xi)| d\xi \leq cy^{n-1} (1 + y^{l-m}) \|\gamma; \mathcal{R}^{n-1}\|_{M(W^{m-1}_p \to L_p)}.
\]  
(30)

We now estimate the second integral in the right-hand side of (25). Clearly,
\[
\int_{\mathbb{R}^{n-1} \setminus \mathcal{B}^{n-1}_y(x)} |\gamma(\xi)| d\xi \leq n \int_y^{\infty} \frac{dp}{\rho^{n+1}} \int_{\mathcal{B}^{n-1}_y(x)} |\gamma(\xi)| d\xi.
\]  
(31)

By Hölder’s inequality the right-hand side of (31) admits the majorant
\[
c \int_y^{\infty} \rho^{2-\frac{n-1}{p}} \|\gamma; \mathcal{B}^{n-1}_\rho(x)\|_{L_p} d\rho.
\]  
(32)

Using (29) we see that the function (32), for \( y > 1 \), does not exceed
\[
cy^{-1} \sup_{r \in (0, 1)} (\text{cap}_{p,m-l}(\mathcal{B}^{n-1}_r(x)))^{-1/p} \|\gamma; \mathcal{B}^{n-1}_r(x)\|_{L_p}
\]

which in view of Proposition 2 is dominated by
\[
cy^{-1} \|\gamma; \mathcal{R}^{n-1}\|_{M(W^{m-1}_p \to L_p)}.
\]  
(33)

Let \( y < 1 \). Then
\[
\int_y^1 \rho^{2-\frac{n-1}{p}} \|\gamma; \mathcal{B}^{n-1}_\rho(x)\|_{L_p} d\rho
\]
Lemma 7

\[ \text{This, together with (33), imply that for all } \]

\[ \int_0^1 y^{p(k-l)-1} (|T\gamma(z)| + y |\nabla(T\gamma)(z)|)^p dy = \int_0^\delta \ldots dy + \int_\delta^1 \ldots dy. \]

In view of (25)

\[ \int_0^\delta \ldots dy \leq c \int_0^\delta y^{p(k+1-l-n)-1} \left( \int_{B_{\rho}^{n-1}(x)} |\gamma(\xi)| d\xi \right)^p dy \]

\[ + c \int_0^\delta y^{p(k+1-l)-1} \left( \int_{\mathbb{R}^{n-1}\setminus B_{\rho}^{n-1}(x)} \frac{|\gamma(\xi)| d\xi}{|\xi-x|^n} \right)^p dy. \]
By the definition of $M$,
\begin{equation}
\int_0^\delta y^{p(k+l-1-n)-1} \left( \int_{B^{p-1}(x)} |\gamma(\xi)| d\xi \right)^p dy \leq c[(M\gamma)(x)]^p \delta^{p(k-l)}.
\end{equation}
(36)

Using (31) we obtain
\begin{equation}
\int_0^\delta y^{p(k+l-1)} \left( \int_{R^{n-1}\setminus B^{p-1}(x)} |\gamma(\xi)| d\xi \right)^p dy \leq c[(M\gamma)(x)]^p \delta^{p(k-l)}.
\end{equation}
(37)

Combining (36) and (37) we conclude that
\begin{equation}
\int_0^\delta \ldots dy \leq c[(M\gamma)(x)]^p \delta^{p(k-l)}.
\end{equation}
(38)

By Lemma 6,
\begin{equation}
\int_{\delta}^1 y^{p(k-l)-1} (|T\gamma| + y|\nabla(T\gamma)(z)|)^p dy
\leq c \|\gamma; R^{n-1}\|_M(W^{m-1}_{p \to L_p}) \delta^{p(k-m)}.
\end{equation}
(39)

Summing up (38) and (39) we find
\begin{equation}
\int_0^1 y^{p(k-l)-1} (|T\gamma| + y|\nabla(T\gamma)(z)|)^p dy
\leq c[(M\gamma)(x)]^p \delta^{p(k-l)} + \|\gamma; R^{n-1}\|_M(W^{m-1}_{p \to L_p}) \delta^{p(k-m)}).
\end{equation}

The right-hand side in this inequality attains its minimum value for
\begin{equation}
\delta = \left( \|\gamma; R^{n-1}\|_M(W^{m-1}_{p \to L_p}) \right)^{1/(m-l)}.
\end{equation}
The proof is complete.

**Lemma 8** Let the operators $T$ and $D_{p,l}$ be defined by (23) and (2). Then
\begin{equation}
\int_0^1 y^{p(1-l)-1} |\nabla(T\gamma)(z)|^p dy \leq c \left((D_{p,l}\gamma)(x)\right)^p.
\end{equation}

**Proof.** Let $R(\xi, x) = \gamma(\xi) - \gamma(x)$. Using the identity
\[ y^{-n+1} \int_{R^{n-1}} \zeta \left( \frac{\xi - x}{y} \right) d\xi = \text{const} \]
we have
\begin{equation}
\frac{\partial T\gamma}{\partial y}(x, y) = \frac{\partial}{\partial y} \left( y^{-n+1} \int_{R^{n-1}} \zeta \left( \frac{\xi - x}{y} \right) R(\xi, x) d\xi \right).
\end{equation}
(40)

Furthermore, it is clear that
\begin{equation}
\frac{\partial T\gamma}{\partial x_j}(x, y) = y^{-n+1} \int_{R^{n-1}} R(\xi, x) \frac{\partial}{\partial x_j} \zeta \left( \frac{\xi - x}{y} \right) d\xi.
\end{equation}
Therefore,

\[ |\nabla (T\gamma)(x, y)| \leq cy^{-n} \sum_{k=0}^{1} \int_{\mathbb{R}^{n-1}} \left| \nabla_k \zeta \left( \frac{\xi - x}{y} \right) \right| \left( 1 + \frac{|\xi - x|}{y} \right)^{k-n} |R(\xi, x)| d\xi. \]

This estimate and (24) imply

\[ |\nabla (T\gamma)(x, y)| \leq cy^{-n} \int_{\mathbb{R}^{n-1}} \left( 1 + \frac{|\xi - x|}{y} \right)^{-n} |R(\xi, x)| d\xi \]

\[ = cy^{-1/p} \int_{\mathbb{R}^{n-1}} \left( \frac{|\xi - x|}{y} \right)^{-n/p} \left( 1 + \frac{|\xi - x|}{y} \right)^{-n} |R(\xi, x)| d\xi. \]

Consequently,

\[ \int_0^1 y^{p(1-l)-1} |\nabla (T\gamma)(x, y)|^p dy \]

\[ \leq c \int_0^1 \left( \int_{\mathbb{R}^{n-1}} f \left( \frac{|\xi - x|}{y} \right) \frac{|R(\xi, x)|}{|\xi - x|^{n-1/p}} d\xi \right)^p y^{p(1-l)-1} dy / y, \]

where \( f(\eta) = \eta^{n-1/p} (1 + \eta)^{-n} \). We write the last integral over \((0, 1)\) as

\[ \int_0^1 \left( \int_0^\infty f(s) g(s, y, x) ds \right)^p y^{p(1-l)-1} dy / y \]

\[ = \int_0^1 \left( \int_0^\infty f(s) g(s, y, x) \frac{ds}{s} \right)^p y^{p(1-l)-1} dy / y, \quad (41) \]

with

\[ g(t, x) = t^{1/p-1} \int_{\partial B^n_{t-1}} |R(t\theta + x, x)| d\theta. \]

By Minkowski’s inequality, the right-hand side of (41) does not exceed

\[ \left( \int_0^\infty \left( \int_0^1 f(s)^p g(s, y, x) \frac{dy}{y} \right)^{1/p} \frac{ds}{s} \right)^p \]

\[ = \left( \int_0^\infty f(s) \left( \int_0^s (g(t, x))^{p-1} \frac{dt}{\tau} \right)^{1/p} \frac{ds}{s^{2-1/p}} \right)^p \]

\[ \leq \left( \int_0^\infty f(s) \frac{ds}{s^{2-1/p}} \right)^p \int_0^\infty (g(t, x))^{p-1} \frac{dt}{\tau}. \quad (42) \]

Therefore,

\[ \int_0^1 y^{p(1-l)-1} |\nabla (T\gamma)(x, y)|^p dy \leq c \int_0^\infty (g(t, x))^{p-1} \frac{dt}{\tau}. \]

It remains to note that

\[ \int_0^\infty (g(\tau, x))^{p-1} \frac{dt}{\tau} = \int_0^\infty \tau^{p-1} \left( \int_{\partial B^n_{\tau-1}} |\gamma(\tau\theta + x) - \gamma(x)|^p d\theta \right)^{1/p} \frac{d\tau}{\tau} \]

\[ \leq c \int_0^\infty \int_{\partial B^n_{\tau-1}} |\gamma(\tau\theta + x) - \gamma(x)|^p d\theta \frac{d\tau}{\tau^{p+1}} \leq c \int_{\mathbb{R}^{n-1}} \frac{|\gamma(x + h) - \gamma(x)|^p}{|h|^{p+n-1}} dh \]

\[ = c \left( (D_p, \gamma)(x) \right)^p. \]

The result follows.
7 Proof of Theorem 4

7.1 The case $l < 1$

Our aim now is to prove that for $l < 1$ and $s = 1$ the operator $T$ defined by (23) maps $M(W^m_p(R^{n-1})) \rightarrow W^{1,\alpha}_p(R^n)$ with $\alpha = 1 - l - 1/p$, $\beta = 1 - \{m\} - 1/p$ and there holds the estimate

$$\|T\gamma; R^n\|_{M(W^{1,\beta}_p \rightarrow W^{1,\alpha}_p)} \leq c C \|\gamma; R^{n-1}\|_{M(W^m_p \rightarrow W^l_p)}, \quad (43)$$

where $C$ is the constant in (24).

We have

$$\|(\min\{1, y\})^\alpha \nabla(U T\gamma); R^n\|_{L_p} \leq c \int_0^1 y^{p\alpha} \int_{R^{n-1}} (|\nabla(T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p) \, dz$$

$$+ c \int_1^\infty \int_{R^{n-1}} (|\nabla(T\gamma)|^p |U|^p + |T\gamma|^p |\nabla U|^p) \, dz$$

$$= c \int_{0<y<1} \ldots \, dz + c \int_{y>1} \ldots \, dz. \quad (44)$$

By Lemma 1, for $y > 1$

$$y |\nabla(T\gamma)(z)| + |(T\gamma)(z)| \leq c \|\gamma; R^{n-1}\|_{M(W^m_p \rightarrow L_p)}.$$

Hence,

$$\int_{y>1} \ldots \, dz \leq c \|\gamma; R^{n-1}\|_{M(W^m_p \rightarrow L_p)} \|U; R^n\|_{W^{1,\alpha}_p}. \quad (45)$$

It remains to refer to the estimate

$$\|U; R^n\|_{W^{1,\beta}_p} \leq c \|U; R^n\|_{W^{1,\alpha}_p}$$

which follows from the one dimensional Hardy inequality.

Introducing the notation

$$R_0 U(z) = U(z) - \sum_{k=0}^{[m]} \frac{\partial^k}{\partial y^k} U(x, 0) \frac{y^k}{k!},$$

$$R_1 U(z) = \begin{cases} \nabla U(z) - \sum_{k=0}^{[m]-1} \frac{\partial^k}{\partial y^k} \nabla U(x, 0) \frac{y^k}{k!} & \text{for } m > 1 \\ \nabla U(z) & \text{for } m < 1 \end{cases}$$

we have

$$\int_{0<y<1} \ldots \, dz \leq c \int_{0<y<1} y^{p(1-l)-1} \sum_{j=0}^1 |\nabla_j(T\gamma)|^p |R_{1-j} U(z)|^p \, dz$$

$$+ c \int_{0<y<1} y^{-pl-1} (|T\gamma(z)| + y|\nabla(T\gamma)(z)|)^p \sum_{k=1}^{[m]} |\nabla_k U(x, 0)|^p y^k \, dz$$

15
for \( m > 1 \). In case \( m < 1 \) the second integral in the right hand side of (46) should be omitted.

By Lemma 6, for \( 0 < y < 1 \)

\[
|T\gamma(z)| + y|\nabla(T\gamma)(z)| \leq c y^{m-1} \|\gamma; R^{n-1}\|_{M(W_p^{m-1} \to L_p)}.
\]

Since for \( j = 0, 1 \)

\[
|R_{1-j}U(z)| \leq y^{m+j-1} \int_0^y |\nabla_j U(x,t)| dt,
\]

we have

\[
\int_{0<y<1} y^{p(1-(m)-1)}|R_{1-j}U(z)|^p dz 
\leq c \int_{0<y<1} y^{-p[m]-1} \int_0^y |\nabla_j U(x,t)| dt \, dz.
\]

By Hardy’s inequality the right-hand side does not exceed \( c \|U; R^n_p\|_{W^{m+1}_{p+1}} \). Combining this with (47) we obtain that the first integral in the right-hand side of (46) does not exceed

\[
\|\gamma; R^{n-1}\|^p_{M(W_p^{m-1} \to L_p)} \|U; R^n_p\|^p_{W^{m+1}_{p+1}}.
\]

We now pass to the estimate of the second integral in the right-hand side of (46) for \( k = 1, \ldots, [m], m > 1 \). Applying Lemma 7, we find

\[
\int_{0<y<1} y^{p(k-l)-1} |T\gamma(z)| \leq c \|\gamma; R^{n-1}\|_{M(W_p^{m-1} \to L_p)} \int_{R^n} (M\gamma(x))^p \|\nabla_k U(x,0)\|^p dx.
\]

The last integral is not greater than

\[
\|\gamma; R^{n-1}\|_{M(W_p^{m-1} \to L_p)} \|\nabla_k U(\cdot,0); R^{n-1}\|_{W^{m-k}_p}.
\]

Using Proposition 2 with \( \lambda = m-k, \mu = m-l \) and Verbitsky’s theorem on the boundedness of the maximal operator \( \mathcal{M} \) in the space \( M(W_p^{m-l}(\mathbb{R}^n) \to L_p(\mathbb{R}^{n-1})) \) (see [3], Ch.2), we find that (51) is dominated by

\[
c \|\gamma; R^{n-1}\|_{M(W_p^{m-1} \to L_p)} \|U(\cdot,0); R^{n-1}\|_{W^{m}_p}.
\]

Hence and by (50)

\[
\int_{0<y<1} y^{p(k-l)-1} |T\gamma(z)| \leq c \|\gamma; R^{n-1}\|_{M(W_p^{m-1} \to L_p)} \|U; R^n_p\|_{W^{m+1}_{p+1}}.
\]
By Lemma 8, the integral
\[ \int_{0<y<1} y^{p(1-l)-1} |\nabla (T\gamma)(z)|^p |U(x, 0)|^p \, dz \]  
(53)
does not exceed
\[ c \int_{\mathbb{R}^n} (D_{p,l} \gamma(x))^p |U(x, 0)|^p \, dx \]
\[ \leq c \| D_{p,l} \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \| U(.; 0); R^{n-1}_{p} \|^p_{W_{p}^m} \]
\[ \leq c \| \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \| U; R^{n}_{+} \|^p_{W_{p}^{(m)+, \alpha}}. \]
(54)
Thus we arrive at the inequality
\[ \int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |U(z)|^p \, dz \leq c \| \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \| U; R^{n}_{+} \|^p_{W_{p}^{(m)+, \alpha}}. \]
It remains to estimate the integral
\[ \int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |U(z)|^p \, dz. \]
Clearly,
\[ \int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |U(z)|^p \, dz \leq \int_{0<y<1} y^{p(1-l)-1} |(T\gamma)(z)|^p |R_{0} U(z)|^p \, dz \]
\[ + \sum_{k=0}^{[m]} \int_{0<y<1} y^{p(k-l)-1} |(T\gamma)(z)|^p |\nabla_k U(x, 0)|^p \, dz. \]
(55)
By (47) and (48) with \( j = 1 \) we have
\[ \int_{0<y<1} y^{p(l-1)-1} |(T\gamma)(z)|^p |R_{0} U(z)|^p \, dz \]
\[ \leq c \| \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \int_{0<y<1} y^{p(l-1)-1} (\int_{0}^{y} |\nabla_{[m]+1} U(x, t)| \, dt)^p \, dz \]
which by Hardy’s inequality is dominated by (49). In view of (52)
\[ \int_{0<y<1} y^{p(k-l)-1} |(T\gamma)(z)|^p |\nabla_k U(x, 0)|^p \, dz \]
\[ \leq c \| \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \| U; R^{n}_{+} \|^p_{W_{p}^{(m)+, \alpha}}. \]
Thus we arrive at the estimate
\[ \int_{0<y<1} \ldots \, dz \leq c \| \gamma; R^{n-1}_{(p)} \|^p_{M(W_{p}^m \rightarrow W_{p}^l)} \| U; R^{n}_{+} \|^p_{W_{p}^{(m)+, \alpha}}. \]
Since the Poisson kernel satisfies condition (24), Theorem 4 with \( l < 1 \) follows.
7.2 The case \( l > 1 \)

Lemma 9 Let \( m \) and \( l \) be nonintegers, \( m \geq l > 0 \), and let \( T \) be the extension operator (23). Suppose that \( \gamma \in M(W_p^{m-1}(\mathbb{R}_+^n) \to L_p(\mathbb{R}^n)) \). Then
\[
T\gamma \in M(W_p^{[m]-[l],\beta}(\mathbb{R}_+^n) \to W_p^{0,\alpha}(\mathbb{R}_+^n))
\]
and
\[
\|T\gamma; R^n_+\|_{M(W_p^{m-1}; L_p)} \leq c\|\gamma; R^{n-1}\|_{M(W_p^{m-1}; L_p)}.
\]  

Proof. To begin with, let \( [m] = l \). Then by (47)
\[
\int_{0<y<1} y^{p(1-[l]) - 1}|U(z)(T\gamma)(z)|^p dz
\]
\[
\leq c\|\gamma; R^{n-1}\|_{M(W_p^{m-1}; L_p)} \int_{0<y<1} y^{p(1-[m]) - 1}|U(z)|^p dz
\]
which gives the result.

Suppose \( [m] > [l] + 1 \). We introduce the function
\[
RU = U(z) - \sum_{j=0}^{[m] - [l] - 1} \frac{\partial^j U(x, 0)}{j!} y^j
\]
which, clearly, satisfies
\[
|RU(z)| \leq \frac{y^{[m] - [l] - 1}}{([m] - [l] - 1)!} \int_0^y |\nabla_{[m]-[l]} U(x, t)| dt.
\]
This and (47) imply
\[
\int_{0<y<1} y^{p(1-[l]) - 1}|T\gamma(z)|^p |RU(z)|^p dz
\]
\[
\leq c\|\gamma; R^{n-1}\|_{M(W_p^{m-1}; L_p)} \int_{0<y<1} y^{-p[m] - 1} \left( \int_0^y |\nabla_{[m]-[l]} U(x, t)| dt \right)^p dz.
\]
By Hardy’s inequality the right-hand side is dominated by
\[
c\|\gamma; R^{n-1}\|_{M(W_p^{m-1}; L_p)} \|U; R^n_+\|_{W_p^{[m]-[l],\beta}}^p.
\]
Furthermore, by Lemma 7 with \( m \) replaced by \( m - [l] \), \( l \) replaced by \( [l] \) and \( k = j + 1 \) we have for \( j = 0, \ldots, [m] - [l] - 1 \)
\[
\int_{0<y<1} y^{p(j+1-[l]) - 1}|T\gamma(z)|^p |\nabla_j U(x, 0)|^p dz
\]
\[
\leq c\|\gamma; R^{n-1}\|_{M(W_p^{m-1}; L_p)} \int_{\mathbb{R}^n} (M\gamma(x))^{p \frac{m-[l]-1}{m-[l]-j-1}} |\nabla_j U(x, 0)|^p dx.
\]  

The last integral is dominated by
\[
\|(M\gamma)^{\frac{m-[l]-1}{m-[l]-j-1}} R^{n-1}\|_{M(W_p^{m-1}; L_p)} \|U(\cdot, 0); R^{n-1}\|_{W_p^{[l]-1}}^p
\]

18
which by Proposition 3 does not exceed
\[ \|M \gamma; R^{n-1}\|_{M(W_p^{m-1} - L_p)} \|U; R^n_+\|_{W_p^{[m]-[l],\alpha}}. \]

Hence and by (57)
\[ \int_{0<y<1} y^p(j+1-(l))|T \gamma(z)|^p |\nabla_j U(x,0)|^p dz \leq c\|M \gamma; R^{n-1}\|_{M(W_p^{m-1} - L_p)} \|U; R^n_+\|_{W_p^{[m]-[l],\alpha}}. \]

The result follows.

**Proof of Theorem 4 for \( l > 1 \).**

Suppose Theorem has been proved for \( [l] = 1, \ldots, \mathcal{L} - 1 \), where \( \mathcal{L} \geq 2 \). Let \( [l] = \mathcal{L} \) and let
\[ \gamma \in M(W_p^{m}(R^{n-1}) \to W_p^{l}(R^{n-1})), \]

for \( m \geq \mathcal{L} \).

Let \( T \gamma \) denote the Poisson integral. Since by Proposition 1 one has
\[ \gamma \in M(W_p^{m-l}(R^{n-1}) \to L_p(R^{n-1})), \]

it follows from Lemma 9 that
\[ T \gamma \in M(W_p^{[m]-[l],\beta}(R^n_+) \to W_p^{0,\alpha}(R^n_+)) \]

and (56) holds. Next we show that
\[ \nabla_{\mathcal{L}+1}(T \gamma) \in M(W_p^{[m]+1,\beta}(R^n_+) \to W_p^{0,\alpha}(R^n_+)). \]

Using Proposition 1, we obtain
\[ \frac{\partial \gamma}{\partial x_k} \in M(W_p^{m}(R^{n-1}) \to W_p^{l-1}(R^{n-1})), \quad k = 1, \ldots, n - 1. \]

Then, by the induction hypothesis applied to \( \partial \gamma/\partial x_k \),
\[ \frac{\partial}{\partial x_k}(T \gamma) = T \frac{\partial \gamma}{\partial x_k} \in M(W_p^{[m]+1,\beta}(R^n_+) \to W_p^{\mathcal{L},\alpha}(R^n_+)). \]

By Lemma 3,
\[ \nabla_{\mathcal{L}} \frac{\partial}{\partial x_k}(T \gamma) \in M(W_p^{[m]+1,\beta}(R^n_+) \to W_p^{0,\alpha}(R^n_+)). \]

Using the harmonicity of \( T \gamma \) and (61) we find
\[ \frac{\partial^{\mathcal{L}+1}}{\partial y^{\mathcal{L}+1}}(T \gamma) = - \frac{\partial^{\mathcal{L}-1}}{\partial y^{\mathcal{L}-1}}(\Delta_x (T \gamma)) \in M(W_p^{[m]+1,\beta}(R^n_+) \to W_p^{0,\alpha}(R^n_+) \]

which together with (61) implies the inclusion (59). Combining this with (56) we find that \( T \gamma \in M(W_p^{[m]+1,\beta}(R^n_+) \to W_p^{[l]+1,\alpha}(R^n_+)) \). It remains to note that all above inclusions are accompanied by the corresponding estimates. The result follows.
8 Extension of multipliers on $\partial \Omega$

We return to the assertion stated in Introduction.

**Theorem 5** Let $\gamma \in M(W^{m}_{p}(\partial \Omega) \to W^{l}_{p}(\partial \Omega))$, where $m$ and $l$ are nonintegers, $m \geq l > 0$, $p \in (1, \infty)$. There exists a linear extension operator

$$\gamma \to \Gamma \in M(W^{t,\beta}_{m}(\Omega) \to W^{s,\alpha}_{p}(\Omega)),$$

where $t = \lceil m \rceil + 1$, $s = \lceil l \rceil + 1$, $\beta = 1 - \{ m \} - 1/p$, and $\alpha = 1 - \{ l \} - 1/p$.

**Proof.** It suffices to construct an extension $\Gamma$ only for $\gamma$ with sufficiently small support. To be precise, we assume that $\gamma = 0$ outside the ball $B_{\rho}^{n}$ centered at 0 $\in \partial \Omega$, where $\rho$ is small enough. We introduce a cut off function $\varphi \in C^{\infty}_{0}(B_{3\rho}^{n})$, equal to one on $B_{\rho}^{n}$. Let us define cartesian coordinates $\zeta = (\xi, \eta)$ with the origin 0, where $\xi \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}^{1}$. Let $\Omega \cap B_{3\rho}^{n} = \{ \zeta : \xi \in B_{3\rho}^{n-1}, \eta > f(\xi) \}$, where $f$ is a smooth function. We make the standard change of variables $\kappa : \zeta \to (x, y)$, where $x = \xi$, $y = \eta - f(\xi)$. The diffeomorphism $\kappa$ maps $\Omega \cap B_{3\rho}^{n}$ into the half space $\mathbb{R}^{n}_{+} = \{ (x, y) : x \in \mathbb{R}^{n-1}, y > 0 \}$. Clearly, the function $\tilde{\gamma} = \gamma \circ \kappa^{-1}$ belongs to $M(W^{m}_{p}(\mathbb{R}^{n-1}) \to W^{l}_{p}(\mathbb{R}^{n-1}))$. Its harmonic extension to $\mathbb{R}^{n}_{+}$, denoted by $\tilde{\Gamma}$, is in $M(W^{t,\beta}_{m}(\mathbb{R}^{n}_{+}) \to W^{s,\alpha}_{p}(\mathbb{R}^{n}_{+}))$ and satisfies the estimate (22) according to Theorem 4. Hence the function $\gamma = (\tilde{\Gamma} \circ \kappa)\varphi$ is a desired extension. The proof is complete.

**References**


