Nonassociative algebras of cubic minimal cones

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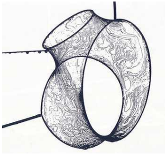
What is this all about?

A **minimal surface** is a critical point of the area functional. It minimizes the surface energy (in a wider sense, a string). Geometrically, this means that the **mean curvature vanishes**.

A **minimal cone** is a typical singularity of a minimal surface.

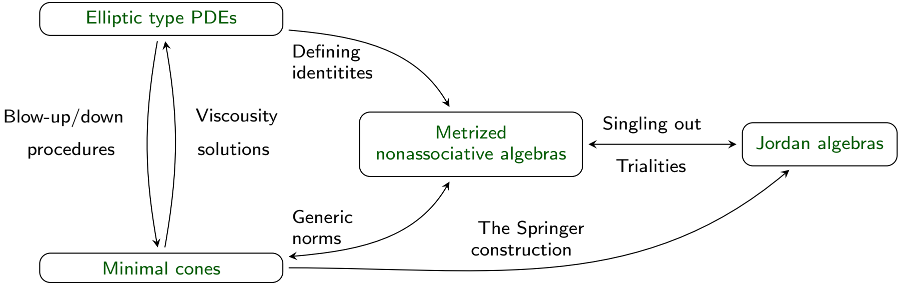


All known minimal cones are algebraic, i.e. zero level sets of a homogenepus polynomial $u \in \mathbb{R}[x_1, \ldots, x_n]$:

- the Clifford-Simons cone, $u(x) := (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$ played a crucial role in the solution of the famous Bernstein prolem; notice that u is **the norm for split octonions**.
- The **triality polynomials** $\mathrm{Re}((z_1 z_2) z_3), z_i \in \mathbb{K}_d, d = 1, 2, 4, 8$ are examples of **cubic minimal cones** in $\mathbb{R}^{3d}$.
- Another example of a cubic minimal cone is the **generic norm** on the trace free subspace of the **cubic Jordan algebra** $\mathscr{H}_3'(\mathbb{K}_d)$
- The determinant varieties are examples of minimal cones of higher degree.

The Main Problem: How to characterize algebraic minimal cones?

How it works



- **N. Nadirashvili, V.T., S. Vlăduț**, Nonlinear Elliptic Equations and Nonassociative Algebras Vol.200, Mathematical Surveys and Monographs, AMS, 2015

A short introduction into Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

- Any Jordan algebra is power associative.

Examples.

- The Jordan algebra $\mathscr{H}_n(\mathbb{F}_d)$ of Hermitian matrices of order n over a real division algebra $\mathbb{K}_d$, $d = 1, 2, 4$ with Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$
- $\mathscr{H}_3(\mathbb{F}_8)$ the Albert exceptional algebra.

Some notation.

- $\mathrm{rk}\,(V) = \max\{\dim V(x) : x \in V\}$, $V(x) =$ a subalgebra generated by x.
- Any $x \in V$ satisfies the **minimum polynomial** equation $m_x(x) = 0$, with

$$m_x(\lambda) = \lambda^r - \sigma_1(x)\lambda^{r-1} + \ldots + (-1)^r \sigma_r(x).$$

where $\sigma_1(x)$ is the **generic trace** of x and $\sigma_n(x) = N(x)$ is the **generic norm** of x.

A short introduction into Jordan algebras

En algebra is called **formally real** if $\sum x_i^2 = 0 \;\; \Rightarrow x_i = 0 \quad \forall i.$

Theorem (JORDAN-VON NEUMANN-WIGNER, 1934)

Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle; \, x_0 y + y_0 x)$
- the Jordan algebras $\mathscr{H}_n(\mathbb{F}_d)$, $n \geq 3$, $d = 1, 2, 4$;
- the Albert algebra $\mathscr{H}_3(\mathbb{F}_8)$.

Jordan algebras of cubic forms

Given a cubic form $u : V \to \mathbb{K}$, consider its linearizations

- $u(x,y,z) = u(x+y+z) - u(x+y) - u(x+z) - u(y+z) + u(x) + u(y) + u(z)$
- $\partial_y u(x) = u(x;y) = \frac{1}{2}u(x,x,y)$

The Springer Construction (McCrimmon, 1969)

A cubic form $N : V \to \mathbb{K}$, $N(e) = 1$, is called a admissible if the bilinear form

$$T(x;y) = N(e;x)N(e;y) - N(e;x;y)$$

is a *nondegenerate* and the map $\# : V \to V$ uniquely determined by $T(x^{\#};y) = N(x;y)$ satisfies the adjoint identity

$$(x^{\#})^{\#} = N(x)x.$$

If N is Jordan and $x\#y = (x+y)^{\#} - x^{\#} - y^{\#}$ then

$$x \bullet y = \frac{1}{2}(x\#y + N(e;x)y + N(e;y)x - N(e;x;y)e)$$

defines a Jordan algebra structure on V and

$$x^{\bullet 3} - N(e;x)x^{\bullet 2} + N(x;e)x - N(x)e = 0, \quad \forall x \in V.$$

Jordan algebras of cubic forms

A *cubic Jordan algebra* V is the Jordan algebra of an admissible cubic form.

[Tka14]: There is a natural correspondence between the following categories:

$$\text{cubic solutions of } |Du(x)|^2 = 9|x|^4 \quad \leftrightarrow \quad \text{cubic formally real Jordan algebras}$$

$$u(x) \;=\; \tfrac{1}{\sqrt{2}}N(x), \quad x \in 1^{\perp}$$

where $N(x)$ is the generic norm of V.

A similar result for pseudo-composition algebras is given by by Meyberg-Osborn (1993): a commutative algebra V over $\mathbb{K}$ with an associative form τ such that

$$x^3 = \tau(x,x)x.$$

Then V is either a quadratic algebra or it may be constructed from an alternative quadratic algebra.

Freudenthal-Springer algebras

Definition

Given a cubic form u on an inner product space $(V, \langle , \rangle)$, define $(x, y) \to xy$ by

$$\langle xy, z\rangle = u(x, y, z)$$

Thus obtained algebra $V^{\mathrm{FS}}(u)$ is said to be the **Freudenthal-Springer algebra** of u.

Definition

Algebras $(V_1, \langle , \rangle_1)$ and $(V_2, \langle , \rangle_2)$ are called **similar** if there exists an isometry $\phi : V_1 \to V_2$ and a constant $c \in \mathbb{K}^\times$ such that

$$\phi(xy) = c\phi(x)\phi(y), \quad \forall x, y \in V_1.$$

Proposition

Two Freudenthal-Springer algebras $V^{\mathrm{FS}}(u_1)$ and $V^{\mathrm{FS}}(u_2)$ are similar iff the cubic forms u_1 and u_2 are congruent.

Freudenthal-Springer algebras

Proposition

- $V^{\mathrm{FS}}(u)$ is **commutative** and **metrised**, i.e. $\langle xy, z\rangle = \langle x, zy\rangle$
- $u(x) = \frac{1}{6}\langle x, x^2\rangle$
- $x^2 = 2Du(x)$, i.e. the square of x is proportional to the *gradient* of u at x
- $L_x = D^2u(x)$, i.e. the multiplication operator by x is the *Hessian* of u at x
- If $(V, \langle,\rangle)$ is Euclidean then there are nonzero idempotents: $\mathscr{I}(V^{\mathrm{FS}}(u)) \neq \emptyset$.

If u is a solution of a PDE then $V^{\mathrm{FS}}(u)$ possesses an identity.

Hsiang's Problem

W.-Y. Hsiang (1967): Given a homogeneous polynomial $u \in \mathbb{R}[x_1, \dots, x_n]$, the cone $u^{-1}(0)$ is minimal iff

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u \tag{1}$$

- If $\deg = 2$ then $u(x) = (m-1)|y|^2 - (k-1)|z|^2$, $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^m$.
- The first non-trivial case is $\deg u = 3$ when

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \tag{2}$$

- In fact, all known irreducible cubic minimal cones satisfy a very special equation:

$$\Delta_1 u = \lambda |x|^2 \cdot u(x) \tag{3}$$

Hsiang's Problem: Classify all cubic minimal cones, or at least all solutions of (3).

How to construct minimal cubic cones?

Hsiang's trick: use the invariant theory. Let $X \in \mathscr{H}'_k(\mathbb{K})$ = trace free hermitian $k \times k$-matrices over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

- Δ_1 is an $O(n)$-invariant $\Rightarrow \Delta_1(\operatorname{tr} X^3) =$ is a polynomial in $\operatorname{tr} X^2, \ldots, \operatorname{tr} X^k$
- $\deg(\Delta_1 \operatorname{tr} X^3) = 5$
- if $3 \le k \le 4$ then $\Delta_1 u(X) = c_1 \operatorname{tr} X^2 \operatorname{tr} X^3 = c_1 |X|^2 u(X)$.

$$\Rightarrow u(X) = \operatorname{tr} X^3 \text{ is a Hsiang cubic!}$$

This yields the four Hsiang examples u in

$$\mathscr{H}'_3(\mathbb{R}) \cong \mathbb{R}^5, \quad \mathscr{H}'_3(\mathbb{C}) \cong \mathbb{R}^8, \quad \mathscr{H}'_4(\mathbb{R}) \cong \mathbb{R}^9, \quad \mathscr{H}'_4(\mathbb{C}) \cong \mathbb{R}^{15}$$

Since $\deg u = 3$ we also have the following Hessian identities:

$\operatorname{tr}(D^2 u) = 0$	the harmonicity
$\operatorname{tr}(D^2 u)^2 = C_1 \lvert x \rvert^2$	the quadratic trace identity
$\operatorname{tr}(D^2 u)^3 = C_2 u$	the cubic trace identity

Hsiang algebras

The Hsiang problem on classifying of cubic polynomial solution to

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda|x|^2 u(x)$$

becomes equivalent to the classification of all **commutative Euclidean metrized algebras** V with the defining identity

$$\boxed{\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda\langle x, x\rangle\langle x^2, x\rangle, \quad \lambda\in\mathbb{R}.} \tag{4}$$

Definition

A metrised Euclidean commutative algebra with (4) is called a **Hsiang algebra**.

V is a Hsiang algebra $\Leftrightarrow$ $u(x) = \langle x, x^2\rangle$ is a Hsiang eigencubic.

Examples

Definition. A *commutative metrised* algebra satisfying

$$x^3 = |x|^2 x, \qquad \operatorname{tr} L_x = 0$$

is called a **Cartan algebra**.

Examples

Definition. A commutative metrised $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is called polar if

$$V_0 V_0 = \{0\} \qquad \text{and} \qquad L_x^2 = |x|^2 \text{ on } V_1, \quad \forall x \in V_0.$$

Definition. A section $A : X \to \mathrm{End}^S(Y)$ is a *symmetric Clifford system* (or $A \in \mathrm{Cliff}(X, Y)$) if

$$A(x)^2 = |x|^2 \mathbf{1}_Y \quad \forall x \in X.$$

Proposition (the correspondence)

- If $A \in \mathrm{Cliff}(X, Y)$ then $(X \times Y)^{\mathrm{FS}}(u)$, where $u(x, y) = \frac{1}{2}\langle y, A(x)y\rangle$, is a polar algebra with $V_0 = X \times \{0\}$, $V_1 = \{0\} \times Y$.
- Conversely, if $V = V_0 \oplus V_1$ is a polar algebra then $L_x \in \mathrm{Cliff}(V_0, V_1)$.

It is well-known that $\mathrm{Cliff}(X, Y) \neq \emptyset \Leftrightarrow \dim X \leq 1 + \rho(\frac{1}{2}\dim Y)$, where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the Hurwitz-Radon function.

Examples of Hsiang algebras

Proposition 1

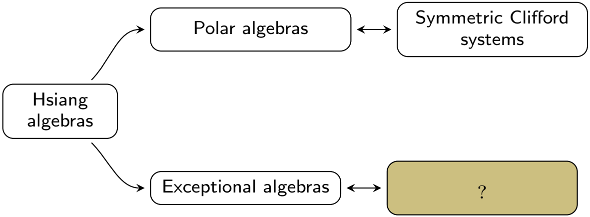
(a) *Any rank 1 metrised algebra (i.e.* $\dim VV = 1$*) is a Hsiang algebra.*

(b) *Any Cartan algebra is a Hsiang algebra.*

(c) *Any polar algebra is a Hsiang algebra.*

How to classify?

Definition. A Hsiang algebra V similar to a polar algebra is said to be of **Clifford type**; otherwise it is called **exceptional**.



Proposition. Any Cartan algebra V is exceptional.

Proof. In a Cartan algebra $\langle x^2, x^2\rangle = \langle x^3, x\rangle = |x|^4 \neq 0$ for $x \neq 0$. On the other hand, if V is a Clifford type algebra then $x^2 = 0$ on a nontrivial subspace $\cong V_0$, a contradiction.

The harmonicity

Theorem 1

Any non-trivial Hsiang algebra V is harmonic, i.e. $\operatorname{tr} L_x = 0$ *for all* $x \in V$*. In particular,*

- *In any Hsiang algebra* $\langle x^2, x^3\rangle = -\frac{2}{3}\lambda\langle x, x^2\rangle|x|^2$ *for some* $\lambda < 0$*.*
- *All idempotents* $\mathscr{I}(V)$ *have* **the same length** $\sqrt{-\frac{3}{2\lambda}}$.

Definition

A Hsiang algebra is called **normalized** if $\lambda = -2$ (i.e. $|c|^2 = \frac{3}{4}$). Then

$$\langle x^2, x^3\rangle = \tfrac{4}{3}\langle x, x\rangle\langle x, x^2\rangle,$$
$$xx^3 + \tfrac{1}{4}x^2x^2 - |x|^2x^2 - \tfrac{2}{3}\langle x^2, x\rangle x = 0.$$

Remark. Hsiang algebras are unique in the class of metrized commutative algebras with

$$Axx^3 + Bx^2x^2 + C|x|^2x^2 + D\langle x^2, x\rangle x = 0.$$

Cf. with algebras satisfying identities of $\deg \leq 4$ (Walcher, Meyberg, Osborn, Okubo, Elduque, Labra)

The Peirce decomposition

- Let $c \in \mathscr{I}(V)$ and $V_c(t) = \ker(L_c - tI)$, then $V_c(1) = \mathbb{R}c$ and

$$V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2})$$

- The **Peirce dimensions**

$$\boxed{n_1(c) = \dim V_c(-1), \quad n_2(c) = \dim V_c(-\tfrac{1}{2}), \quad n_3(c) = \dim V_c(\tfrac{1}{2})}$$

satisfy

$$\begin{aligned} n_3(c) &= 2n_1(c) + n_2(c) - 2 \\ 3n_1(c) + 2n_2(c) - 1 &= \dim V = n. \end{aligned}$$

In particular, any of $n_i(c)$ completely determines two others.

Examples.

- If V is a *polar algebra* then $(n_1(c), n_2(c)) = (\dim V_0 - 1,\ \frac{1}{2}\dim V_1 - \dim V_0 + 2)$.
- If V is a *Cartan algebra* then $(n_1(c), n_2(c)) = (\frac{1+\dim V}{3}, 0)$.

The Peirce decomposition

Proposition 2

Setting $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ *we have*

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ *and* $V_0 \oplus V_2$ *are subalgebras of* V*. Notice however that these subalgebras may be Hsiang subalgebras or not.*

The cubic trace identity

Traces of (powers of) multiplication operators in an algebra is an important tool to study invariant properties. We already have $\operatorname{tr} L_x = 0$ for any $x \in V$. The following property provides an effective tool to determine the Peirce dimensions.

Theorem 2

Any normalized Hsiang algebra satisfies the cubic trace identity

$$\operatorname{tr} L_x^3 = (1 - n_1(c))\langle x, x^2\rangle, \qquad \forall c \in \mathscr{I}(V), x \in V. \tag{5}$$

In particular, the Peirce dimensions $(n_1(c), n_2(c))$ are similarity invariants of a general Hsiang algebra and do not depend on a particular choice of an idempotent c.

In what follows, we write $(n_1(V), n_2(V))$, or just (n_1, n_2).

A 'rough' classification of Hsiang algebras

Theorem 3 (A hidden Clifford algebra structure)

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1),$$

where ρ is the Hurwitz-Radon function.

Proof. One can prove that $A(x) = \sqrt{3}L_x - (1+\sqrt{3})(L_x L_c + L_c L_x)$, $x \in V_1$ satisfies

$$A(x)^2 = |x|^2 \quad \text{on } V_2 \oplus V_3$$

which implies $A \in \mathrm{Cliff}(V_1, V_2 \oplus V_3)$ and the desired obstruction. $\square$

Corollary

Given $n_2 \geq 0$, there are finitely many admissible Peirce dimensions (n_1, n_2).

A ‘rough’ classification of Hsiang algebras

Theorem 4 (A hidden Jordan algebra structure)

Given $c \in \mathscr{I}(V)$, let us define the new algebra structure on $\Lambda_c = (V_0 \oplus V_2, \bullet)$ with the multiplication

$$x \bullet y = \tfrac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c. \tag{6}$$

Then Λ_c is a Euclidean Jordan algebra with unit $c^* = 2c$, the associative trace form $T(x; y) = \langle x, y\rangle$ and

$$\operatorname{rk}\Lambda_c = \min\{3, n_2(V) + 1\} \le 3.$$

Idea of the **Proof**: to verify that the cubic form $N(x) = \frac{1}{6}\langle x, x^2\rangle$ on $V_0 \oplus V_2$ with a basepoint $c^* = 2c$ is Jordan for any $c \in \mathscr{I}(V)$ and apply the Springer-McCrimmon construction. To get the rank property requires a finer analysis of the cubic identity on Λ_c together with the defining identity on V.

$\square$

A 'rough' classification of Hsiang algebras

Theorem 5 (The dichotomy of Hsiang algebras)

The following conditions are equivalent:

1. *A Hsiang algebra V is exceptional*
2. *The Jordan algebra Λ_c is simple for some c*
3. *The Jordan algebra Λ_c is simple for all c*
4. *The quadratic form $x \to \operatorname{tr} L_x^2$ has a single eigenvalue and $n_2(V) \neq 2$*

A key role in the proof play structural properties of nilpotent elements

$$\mathscr{N}_0(V) = \{w : w^2 = 0, |w| = 1\}$$

In particular, the principal idempotents of the Jordan algebra $\Lambda_c(V)$ are characterized by

$$\mathscr{I}_{prim}(\Lambda_c(V)) = \{w \in \mathscr{N}_0(V) : \langle w, c \rangle = \tfrac{1}{2}\}.$$

A 'rough' classification of Hsiang algebras

Combining Theorem 3 and Theorem 5, one obtains

Corollary

There are at most 24 classes of exceptional Hsiang algebras. For any such an algebras $n_2 \in \{0, 5, 8, 14, 26\}$ and the possible corresponding Peirce dimensions are

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

The cells in gray color represent non-realizable Peirce dimensions and the cells in gold color represent unsettled cases

A key question: *Which Peirce dimensions in the above table are indeed realizable*?

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n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

The cells in blue color represent non-realizable Peirce dimensions and the cells in gold color represent unsettled cases

A key question: *Which Peirce dimensions in the above table are indeed realizable*?

A ‘rough’ classification of Hsiang algebras: the existence

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

- If $n_2 = 0$ then $n_2 \in \{2, 5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$, $V = \mathscr{H}_3(\mathbb{K}_d) \ominus \mathbb{R}e$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $n_2 \in \{5, 8, 14\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $\frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, where $z \to \bar{z}$ is the natural involution on $V = \mathscr{H}_3(\mathbb{K}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $n_2 \in \{5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u(z) = \mathrm{Re}\langle z, z^2\rangle$, where $z \in V = \mathscr{H}_3(\mathbb{K}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $V = V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$ on $\mathscr{H}_3(\mathbb{K}_8) \ominus \mathscr{H}_3(\mathbb{K}_1)$

Towards a finer classification: a tetrad construction

Proposition

Let V be a Hisang algebra. If $w^2 = 0$ and $|w| = 1$ then

$$V = V_w(-1) \oplus V_w(0) \oplus V_w(1)$$

$$V_w(0)V_w(0) \subset V_w(0)^{\perp}, \qquad V_w(0)^{\perp}V_w(0)^{\perp} \subset V_w(0).$$

Furthermore,

$$x^2 = \epsilon |x|^2 w, \qquad \forall x \in V_w(\epsilon), \quad \epsilon^2 = 1.$$

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Furthermore,

$$x^2 = \epsilon |x|^2 w, \qquad \forall x \in V_w(\epsilon), \quad \epsilon^2 = 1.$$

(w_1, w_2, w_3) is a **Hsiang triple** if $w_i^2 = 0$ and $w_i w_j = w_k$ for any $\{1, 2, 3\} = \{i, j, k\}$.

Proposition

Let (w_1, w_2, w_3) be a Hsiang triple. Then

- $c_\epsilon = \frac{1}{2}(\epsilon_1 w_1 + \epsilon_2 w_2 + \epsilon_3 w_3)$ is an idempotent in V, $\epsilon_1\epsilon_2\epsilon_3 = 1$, $\epsilon_i = \pm 1$;
- $c_0 + c_1 + c_2 + c_3 = 0$;
- (w_1, w_2, w_3) is a **Jordan frame** in the Jordan algebra Λ_{c_ϵ}.

The inverse is also true.

Towards a finer classification: a tetrad construction

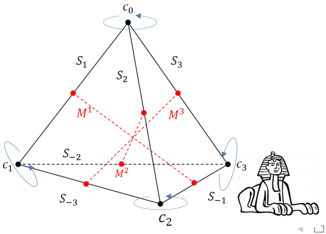
Theorem 6

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3,$$

where $M^i := V_{w_i}(0) \cap V_{w_j}(0)^{\perp} \cap V_{w_k}(0)^{\perp}$ *and* $S^i := V_{w_i}(0)^{\perp} \cap V_{w_j}(0) \cap V_{w_k}(0)$. *If* V *is an* **exceptional** *Hsiang algebra with* $n_2 = 3d + 2$, $d \in \{1, 2, 4, 8\}$ *then*

- M^{α} *is a null-subalgebra*, $\dim M_{\alpha} = n_1 + 1$,
- $S^{\alpha} = S_{\alpha} \oplus S_{-\alpha}$, $\dim S_{\pm\alpha} = d$.
- *any 'vertex-adjacent' triple* $S_{\alpha}, S_{\beta}, S_{\gamma}$ *forms a triality:*

$$S_{\alpha} S_{\beta} = S_{\gamma}, \qquad |x_{\alpha} x_{\beta}|^2 = \tfrac{1}{2} |x_{\alpha}|^2 |x_{\beta}|^2,$$



Towards a finer classification: a tetrad construction

Define $T^\alpha := \mathrm{Span}[S^\alpha S^\alpha]$. Then

Theorem 7

- $T^\alpha \subset M^\alpha$
- $T^\alpha \cong T^\beta$
- T^α *admits a structure of a* commutative *real division algebra, in particular,* $\tau(V) := \dim T^\alpha \in \{1,2\}$
- *If* $d > n_1$ *then* $\tau(V) = n_1$.
- *If* $n_1 \geq 1$ *and* $d \geq \rho(n_1) - 1$ *then* $\tau(V) = 1$.
- *If* $\tau(V) = 1$ *then* $n_1 \equiv 1 \mod 2$.
- *There is no exceptional Hsiang algebras with the blue Peirce dimensions.*

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
d	0	0	0	0	0	1	1	1	1	2	2	2	2	2	2	4	4	4	4	8	8	8	8	8

S.N. Bernstein (1880 – 1968)

A Russian and Soviet mathematician, known for contributions to partial differential equations, differential geometry, probability theory, and approximation theory:

- 1904 solved Hilbert's 19th problem (a C^3-solution of a nonlinear elliptic analytic equation in 2 variables is analytic)
- 1910s introduced a priori estimates for Dirichlet's boundary problem for non-linear equations of elliptic type
- 1912 laid the foundations of constructive function theory (Bernstein's theorem in approximation theory, Bernstein's polynomial).
- 1915 the famous 'Bernstein's Theorem' on entire solutions of minimal surface equation.
- 1917 the first axiomatic foundation of probability theory, based on the underlying algebraic structure (later superseded by the measure-theoretic approach of Kolmogorov)
- 1924 introduced a method for proving limit theorems for sums of dependent random variables
- 1923 axiomatic foundation of a theory of heredity: genetic algebras (Bernstein algebras)

Epilogue: Nonassociative algebras and singular solutions

- Evans, Crandall, Lions, Jensen, Ishii: If $\Omega \subset \mathbb{R}^n$ is bounded with C^1-boundary, ϕ continuous on $\partial\Omega$, F uniformly elliptic operator then the Dirichlet problem

$$F(D^2u) = 0, \text{ in } \Omega$$
$$u = \phi \text{ on } \partial\Omega$$

  has a unique **viscosity solution** $u \in C(\Omega)$;
- Krylov, Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$
- Nirenberg, 50's: if $n = 2$ then u is classical (C^2) solution
- Nadirashvili, Vlăduț, 2007-2011: if $n \geq 12$ then there are solutions which are not C^2.

Theorem (Nadirashvili-V.T.-Vlăduţ, [NTV12], [NTV14])

The function $w(x) := \frac{u_1(x)}{|x|}$ *where*

$$u_1(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

is a singular viscosity solution of the uniformly elliptic Hessian equation

$$(\Delta w)^5 + 2^8 3^2 (\Delta w)^3 + 2^{12} 3^5 \Delta w + 2^{15} \det D^2(w) = 0,$$

This also gives the best possible dimension ($n = 5$) where homogeneous *order 2 real analytic functions in* $\mathbb{R}^n \setminus \{0\}$.

References

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THANK YOU FOR YOUR ATTENTION!