The resultant on compact Riemann surfaces and the exponential transform

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The exponential transform of QD's

- Ω := a *quadrature domain* (for analytic functions) (or an *algebraic domain*) if

$$\int_\Omega h\,dxdy = \sum_{i=1}^{n} c_i h(z_i), \quad (z_i \in \Omega, \;\; c_i \in \mathbb{C})$$

for every integrable analytic function h in Ω.

- Let Ω be a bounded closed set. The moments of Ω:

$$a_{m,n} = \iint_\Omega \zeta^m \bar{\zeta}^n dA(z).$$

- The exponential transform:

$$E_\Omega(z,w) = \exp\left(-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right) = 1 - \sum_{m,n=0}^{\infty} \frac{b_{m,n}}{z^{m+1}\bar{w}^{n+1}},$$

where the correspondence (on the level of generating functions) between $a_{m,n}$ and $b_{m,n}$ is given by

$$\sum_{m,n=0}^{\infty} \frac{b_{m,n}}{z^{m+1}\bar{w}^{n+1}} = 1 - \exp\left(-\sum_{m,n=0}^{\infty} a_{m,n} z^{m+1} \bar{w}^{n+1}\right).$$

- $E_\Omega(z,w) = 1 - \frac{1}{\bar{w}} C_\Omega(z) + \mathcal{O}(\frac{1}{|w|^2})$, as $|w| \to \infty$,, where $C_\Omega(z) = \frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{z - \zeta}$ is the Cauchy transform of Ω.

The following conditions are equivalent (Aharonov, Shapiro, 1976; Gustafsson, 1983, Putinar, 1996):

- $C_\Omega(z)$ is rational ($:= R(z)$) outside Ω;

- $E_\Omega(z,w)$ is rational $= \frac{Q(z,w)}{P(z)\overline{P(w)}}$, $|z|, |w| \gg 1$;

- Ω is an algebraic domain (quadrature domain);

- $\exists S(z)$ meromorphic in Ω: $S(z) = \bar{z}$ on $\partial\Omega$,

$$S(z) = \bar{z} - C_\Omega(z) + R(z), \qquad z \in \Omega.$$

- Ω is determined by finitely many moments a_{jk};

- $\det(b_{jk})_0^N = 0$ for some N.

- There is a bounded linear operator T acting on a Hilbert space, with spectrum equal to Ω, with rank one self commutator $[T^*, T] = \xi \oplus \xi$ and such that the linear span $(T^{*k}\xi)_{k\geq 0}$ is finite dimensional.

But: $\partial\Omega$ is algebraic $\not\Rightarrow$ Ω is a QD.

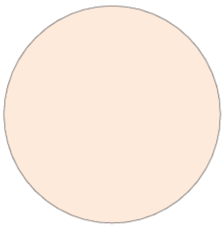
- $E(z,z)$ can be viewed as equation of the boundary:

$$E(z,z) = \exp\left[-\frac{1}{\pi}\int_{\Omega}\frac{dA(\zeta)}{|\zeta - z|^2}\right] = \begin{cases} 0, & \text{on} \quad \partial\Omega; \\ > 0, & \text{outside } \Omega, \end{cases}$$

- if Ω_1 and Ω_2 are disjoint then

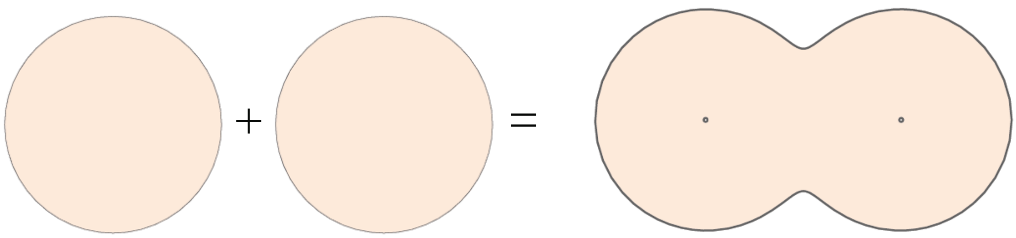
$$E_{\Omega_1 \cup \Omega_2} = E_{\Omega_1} E_{\Omega_2}.$$

Example 1: the unit disk:



$C_{\mathbb{D}} = \frac{1}{z}$, $\quad S(z) = \frac{1}{z}$, $\quad E_{\mathbb{D}} = 1 - \frac{1}{z\bar{w}}$.

Example 2: $\Omega = \mathbb{D}(-1,r) \oplus \mathbb{D}(1,r)$, $r > 1$:



$C_{\Omega}(z) = \frac{r^2}{1+z} + \frac{r^2}{z-1}$,

$E_{\Omega} = 1 - \frac{1+A(r)z\bar{w}}{(\bar{w}^2-1)(z^2-1)}$.

- The Schottky double $\widehat{\Omega} = \Omega \cup \partial\Omega \cup \widetilde{\Omega}$,

Let Ω be a quadrature domain. Then

$$f(\zeta) = (\zeta, \overline{S(\zeta)}), \quad g(\zeta) = (S(\zeta), \bar{\zeta})$$

are meromorphic on the Schottky double $\widehat{\Omega}$, where S is the Schwarz function of Ω.

Main Theorem.

The exponential transform $E_\Omega(z,w)$ of a QD can be viewed as the resultant (an *elimination function*) on the Schottky double $\widehat{\Omega}$:

$$E_\Omega(z,w) = \mathcal{R}(f - z, g - \bar{w}) \equiv \mathcal{E}_{f,g}(z, \bar{w}).$$

Corollary. Let $\Omega = P(\mathbb{D})$, where $P(\zeta) = a_1\zeta + \ldots + a_n\zeta^n$, then for all small enough u, v:

$$\begin{aligned} E_\Omega(z,w) &= \frac{1}{z^n \bar{w}^n} \mathcal{R}_{\mathrm{pol}}(P - z, P^* - \bar{w}\zeta^n) \\ &= \det \begin{pmatrix} -\frac{1}{z} & & & \bar{a}_n & & \\ a_1 & \ddots & & \vdots & \ddots & \\ \vdots & & -\frac{1}{z} & \bar{a}_1 & & \bar{a}_n \\ a_n & & a_1 & -\frac{1}{\bar{w}} & & \vdots \\ & \ddots & \vdots & & \ddots & \bar{a}_1 \\ & & a_n & & & -\frac{1}{\bar{w}} \end{pmatrix}, \end{aligned}$$

and $P^*(\zeta) = \bar{a}_n + \bar{a}_{n-1}\zeta + \ldots + \bar{a}_1\zeta^{n-1}$.

Polynomial resultant

$A = A_0 + A_1 z + \ldots + A_m z^m$,

$B = B_0 + B_1 z + \ldots + B_n z^n$,

The (polynomial) **resultant** is defined by

$$\mathcal{R}_{\mathsf{pol}}(A, B) = A_m^n B_n^m \prod_{i,j} (a_i - b_j)$$

$$= \prod_{i=1}^{m} B(a_i) = (-1)^{mn} \prod_{j=1}^{n} A(b_j)$$

$$= \det \begin{pmatrix} A_0 & A_1 & \ldots & & A_m & & & \\ & A_0 & A_1 & \ldots & & A_m & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & A_0 & A_1 & \ldots & \ldots & A_m \\ B_0 & B_1 & \ldots & & B_n & & & \\ & B_0 & B_1 & \ldots & & B_n & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & B_0 & B_1 & \ldots & & B_n \end{pmatrix},$$

Algebraic

- **Elimination property**: $\mathcal{R}_{\mathsf{pol}}(A,B) = 0 \Leftrightarrow A$ and B have a common zero.

- **Anti-commutativity**:

$$\mathcal{R}_{\mathsf{pol}}(P,Q) = (-1)^{mn}\,\mathcal{R}_{\mathsf{pol}}(Q,P),$$

(the defect multiplier $(-1)^{mn}$ comes from non-compactness of the complex plane $\mathbb{C}$ and geometrically means that polynomials have a unique common pole).

- **Multiplicativity**:

$$\mathcal{R}_{\mathsf{pol}}(P_1P_2,Q) = \mathcal{R}_{\mathsf{pol}}(P_1,Q)\,\mathcal{R}_{\mathsf{pol}}(P_2,Q).$$

Analytic

- **The Fisher-Hartwig formula:** let $q_0 \neq 0$ and $\frac{P(z)}{Q(z)} = \sum_{k=0}^{\infty} s_k z^k$. Then for any $N \geq n$

$$\mathcal{R}_{\mathsf{pol}}(P,Q) = p_m^{n-N} q_0^{m+N} \begin{vmatrix} s_m & s_{m-1} & \cdots & s_{m-N+1} \\ s_{m+1} & s_m & \cdots & s_{m-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m+N-1} & s_{m+N-2} & \cdots & s_m \end{vmatrix}$$

- Residues etc.

Recent development:

- elimination algebra Jouanolou (1991);

- A-resultants: Gelfand, Kapranov Zelevinsky (1994);

- resultants via Koszul complex by Chardin (1993);

- A "differential resultant" due to E. Previato (1991);

- toric geometry, resultants and residues, Cattani, Cox, Dickenstein, (1995).

Meromorphic resultant

Let M be a compact Riemann surface and f, g be meromorphic on M. Their divisors: $(f) = \sum a_i - \sum b_i$, $(g) = \sum c_j - \sum d_j$.

If $x \to \text{ord}_x f \cdot \text{ord}_x g$ is semi-definite on M then the (**meromorphic**) **resultant** is defined by

$$\mathcal{R}(f,g) := g((f)) = \prod_{x \in M} g(x)^{\text{ord}_x(f)}$$

$$= \prod_{i=1}^{m} \frac{g(a_i)}{g(b_i)} = \frac{g(f^{-1}(0))}{g(f^{-1}(\infty))}$$

• If $M = \mathbb{P}^1$ and $f(z) = \lambda \prod_{i=1}^{m} \frac{z-a_i}{z-b_i}$, $g(z) = \mu \prod_{j=1}^{n} \frac{z-c_j}{z-d_j}$, then

$$\mathcal{R}(f,g) = \frac{b_i - d_j}{b_i - c_j} = \prod_{i=1}^{m} \prod_{j=1}^{n} (a_i, b_i, c_j, d_j)$$

where $(a,b,c,d) := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}$.

• In general $\mathcal{R}(f,g)$ depends only on divisors (homogeneous of degree 0);

• multiplicative: $\mathcal{R}(f_1 f_2, g) = \mathcal{R}(f_1, g)\mathcal{R}(f_2, g)$;

• Elimination property: if f and g are admissible on M then $\mathcal{R}(f,g) = 0$ iff f and g have a common zero or a common pole.

• In particular, $\mathcal{R}(f,g) = 0$ if f and g are polynomials.

Weil reciprocity law (1940):

$$\prod_{i=1}^{m} \frac{g(a_i)}{g(b_i)} = \prod_{j=1}^{n} \frac{f(c_j)}{f(d_j)}$$

yields symmetry

$$\mathcal{R}(f,g) = \mathcal{R}(g,f).$$

An integral representations:

$$\mathcal{R}(f,g) = \exp[\frac{1}{2\pi \mathrm{i}} \int_M \frac{df}{f} \wedge d\,\mathrm{Log}\ g],$$

$$|\,\mathcal{R}(f,g)|^2 = \exp[\frac{1}{2\pi \mathrm{i}} \int_M \frac{df}{f} \wedge \frac{d\bar{g}}{\bar{g}}].$$

The latter integrand is a true two-dimensional integral, whereas in the former integral over M the integrand vanishes outside a one-dimensional set of singularities.

• Potential theoretic interpretations: if $d\mu = \delta_{(f)} dx \wedge dy$ is regarded as a charge distribution then, up to constant factors,

$$\mathcal{E}(\mu,\nu) = -\log|\,\mathcal{R}(f,g)|$$

is the mutual energy between μ and ν, $d\nu = \delta_{(g)} dx \wedge dy$.

Elimination function

With $z, w \in \mathbb{C}$ free variables (simultanously regarded as constant maps $M \to \mathbb{C}$), consider the **elimination function**:

$$\mathcal{E}_{f,g}(z,w) := \mathcal{R}(f-z, g-w) = \frac{(g-w)(f^{-1}(z))}{(g-w)(f^{-1}(\infty))}$$
$$= \frac{\prod_{i=1}^{m}(g(f_i^{-1}(z)) - w)}{\prod_{i=1}^{m}(g(f_i^{-1}(\infty)) - w)}$$

Theorem. *Let f and g be meromorphic and have no common poles. Then the elimination function is a rational function of the form*

$$\mathcal{E}_{f,g}(z,w) = \frac{Q(z,w)}{P(z)R(w)}, \qquad (1)$$

where Q, P, R are polynomials.

- $E_{f,g}$ may be well-defined even if $\mathcal{R}(f,g)$ is not defined.
- Elimination property: let $\zeta \in \mathcal{M}$ arbitrary and insert $z = f(\zeta)$, $w = g(\zeta)$ into the original expression for $\mathcal{E}(z,w)$ (spelled out as a product). This gives immediately that

$$\mathcal{E}(f(\zeta), g(\zeta)) = 0 \quad (\zeta \in \mathcal{M}).$$

In particular, we have the following implicitization:

$$Q(f,g) = 0,$$

i.e., the classical polynomial relation between two functions on a compact Riemann surface.

Extended elimination function

Now let f and g be *arbitrary* meromorphic functions and let us consider the *extended* elimination function of four complex variables:

$$\mathcal{E}_{f,g}(z,w;z_0,w_0)=\mathcal{R}\left(\frac{f-z}{f-z_0},\frac{g-w}{g-w_0}\right),$$

which is now well-defined and is also rational.

For example, let f be any meromorphic function of order n and $g=f$. Then

$$\mathcal{E}_{f,f}(z,w;z_0,w_0)=(z,w;z_0,w_0)^n,$$

where $(z,w;z_0,w_0)$ is the cross-product, while $\mathcal{E}_{f,f}(z,w)$ is not defined at all.

- cross-ratio like symmetries:
- if $\mathcal{E}_{f,g}(z,w)$ is well-defined then

$$\mathcal{E}(z,w;z_0,w_0)=\frac{\mathcal{E}(z,w)\mathcal{E}(z_0,w_0)}{\mathcal{E}(z,w_0)\mathcal{E}(z_0,w)},$$

and in the other direction,

$$\lim_{z_0,w_0\to\infty}\mathcal{E}(z,w;z_0,w_0)=\mathcal{E}(z,w).$$

Determinantal representations

Let $a \in L^\infty(\mathbb{T})$ and $T(a) : H^2 \to H^2$ be the Toeplitz operator acting on the Hardy space H^2:

$$T(a) : \phi \to P_+(a\phi),$$

($P_+ : L^2 \to H^2$ is the orthogonal projection).

Theorem. Let f and g be rational functions such that $|(f)| \subset \mathbb{D}$, $|(g)| \subset \mathbb{C} \setminus \mathbb{D}$. Then

$$\mathcal{R}(f, g) = \det T(\frac{f}{g})T(\frac{g}{f}) = \det[T(f)^{-1}T(g)T(f)T(g)^{-1}]$$
$$= \lim_{N\to\infty} \left(\frac{g(\infty)g(0)}{f(\infty)}\right)^N \cdot \det t_N(\frac{f}{g}), \tag{2}$$

where

$$\det t_N(a) \equiv \begin{vmatrix} a_0 & a_{-1} & \dots & a_{1-n} \\ a_1 & a_0 & \dots & a_{2-n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{vmatrix},$$

and $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}n\theta}a(e^{\mathrm{i}\theta})d\theta$.

- the Szegö strong limit theorem;

Polynomial vs Meromorphic

- Note that the meromorphic resultant for polynomials degenerates: $\mathcal{R}(A,B)=0$. A naïve way to correct this 'defect' is to assign the "value at infinity":

$$B(\infty)=z^n B(1/z)|_{z=0}=B_m$$

and use the original definition

$$\mathcal{R}(A,B):=\prod_{i=1}^{m}\frac{B(a_i)}{B(\infty)}=\prod_{i=1}^{m}\prod_{j=1}^{n}(a_i-b_j),$$

which is consistent with the new definition.

In practice, the role of such a 'blow-up' plays the local (or tame) symbol which is define as follows. Note that for any $x\in M$, the function

$$z\longrightarrow f(z)^{\mathrm{ord}_x g}g(z)^{-\mathrm{ord}_x f}$$

is well defined at x. The **local symbol** (Serre, 1959; Tate, 1969) of f,g at x is the value

$$\tau_x(f,g):=(-1)^{\mathrm{ord}_x f\,\mathrm{ord}_x g}\cdot\frac{f^{\mathrm{ord}_x g}}{g^{\mathrm{ord}_x f}}(x)\neq 0$$

- $\tau_x(f,g)=1$ a.e. in M;

Weil's reciprocity law: on a *compact* M,

$$\prod_{x\in M}\tau_x(f,g)=1.$$

A pair (f,g) := *admissible* on $A \subset M$ if the function

$$x \to \mathrm{ord}_x(g)\mathrm{ord}_x(f)$$

is sign semi-definite in A. For examples, polynomials on $\mathbb{P}^1 \setminus \{\infty\}$.

Let f and g be admissible on $M \setminus \{\xi\}$ and ω is a local coordinate near ξ, $\omega(\xi) = 0$. The **reduced** resultant:

$$\mathcal{R}_\omega(f,g) = \tau_\xi(\omega, f)^{\mathrm{ord}_\xi g} \prod_{x \neq \xi} g(x)^{\mathrm{ord}_x(f)},$$

Main example: $M = \mathbb{P}^1$, $A = \mathbb{P}^1 \setminus \{\infty\}$, $\xi = \infty$, $\omega(z) = \frac{1}{z}$. Then for any polynomial $A(z) = A_0 + A_1 z + \ldots + A_m z^m$:

$$\tau_\xi(\omega, A) = \lim_{z \to \infty} \frac{z^m}{A(z)} = \frac{1}{A_m},$$

where $m = -\mathrm{ord}_\infty(A) = \deg A$. Hence

$$\mathcal{R}_\omega(A,B) = A_m^n \prod_{x \neq \infty} B(x)^{\mathrm{ord}_x(A)}$$

coincides with the classical definition:

$$\mathcal{R}_{z,\infty}(A,B) = \mathcal{R}_{\mathrm{pol}}(A,B).$$

Appendix 3: Rational morphisms

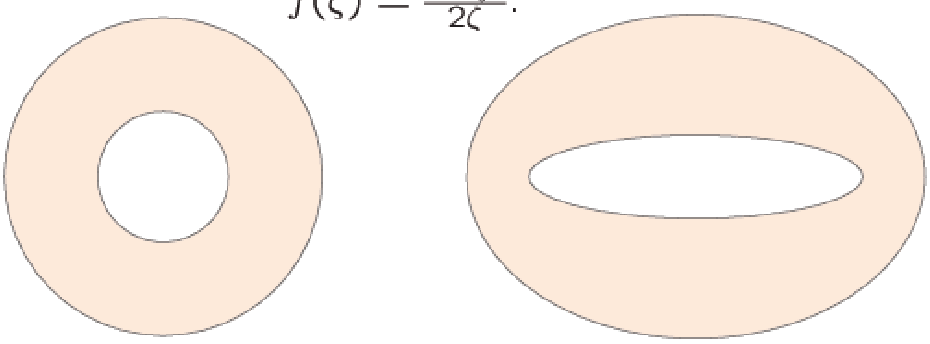
Theorem. *Let* Ω_i, $i = 1, 2$, *be two open sets in the complex plane and* $F(\zeta)$ *be a* p*-valent proper rational mapping which maps* Ω_1 *onto* Ω_2. *Then for all* $z, w \in \Omega_2^e \equiv \mathbb{C} \setminus \overline{\Omega}_2$

$$E^p_{\Omega_2}(z, w) = \mathcal{R}_u(F(u) - z, \mathcal{R}_v(F(v) - w, E_1(u, v)))$$

where $\mathcal{R}_u$ *denote the resultant in* u*-variable. Moreover, if additionally* E_{Ω_1} *is a rational function then* $E^p_{\Omega_2}$ *is also rational.*

Example 3: let $1 < a < b$ and $\Omega = f(\mathbb{D}(a) \setminus \mathbb{D}(b))$ be the confocal elliptic domain,

$f(\zeta) = \frac{1+\zeta^2}{2\zeta}$:



then

$$E_\Omega = \frac{a^4}{b^4} \cdot \frac{(b^4 - 1)^2 - 4b^2(1 + b^4)z\bar{w} + 4b^4(z^2 + \bar{w}^2)}{(a^4 - 1)^2 - 4a^2(1 + a^4)z\bar{w} + 4a^4(z^2 + \bar{w}^2)}.$$