Nonassociative Algebras and Cubic Minimal Cones

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(joint work with N. Nadirashvili and S. Vlăduț)

Motivations

Among others:

- Bernstein's problem on entire minimal graphs: e.g., explicit solutions, polynomials solutions (L. Simon, 1994), polynomial growth conjecture (Bombieri-Giusti, 1972), (problems 103, 104 in Yau's list, 1982).
- Classification of cubic minimal cones (Hsiang, 1967)
- Non-classical and singular solutions to nonlinear elliptic PDEs [NTV14] (Nikolai will talking on Thursday)
- Classification of isoparametric hypersurfaces (Problem 34 in Yau's list, 1992)
- Construction of embedded minimal hypersurfaces with isolated singularities in $\mathbb{R}^n$, $n \geq 4$ (Caffarelli-Nirenberg-Spruck, 1985), (N. Smale, 1989)
- Implicitly, in construction and study of entire solutions to certain PDEs like the Allen-Cahn equation, the Ginzburg-Landau system etc.

Minimal surface equation

Theorem (S. Bernstein, 1915)

If $u(x)$ is an **entire** solution to

$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0, \qquad x \in \mathbb{R}^2,$$

on $\mathbb{R}^2$ then $u = ax + by + c$.

1904 solved Hilbert's 19th problem (a C^3-solution of a nonlinear elliptic analytic equation in 2 variables is analytic)

1910s introduced a priori estimates for Dirichlet's boundary problem for non-linear equations of elliptic type

1912 the foundations of constructive function theory (Bernstein's theorem in approximation theory, Bernstein's polynomial).

1915 'Bernstein's Theorem' on entire solutions of minimal surface equation.

1917 the first axiomatic foundation of probability theory, based on the underlying algebraic structure (superseded by Kolmogorov's measure-theoretic approach)

1924 limit theorems for sums of dependent random variables

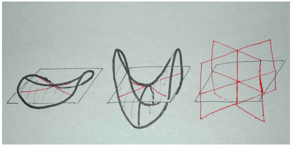
1923 axiomatic foundation of a theory of heredity: genetic algebras (Bernstein algebras)

How do minimal cones enter?

Theorem(Fleming, De Giorgi, Almgren, Simons, 1962-1968)

The Bernstein result holds true for $3 \leq n \leq 7$.

- $M \subset \mathbb{S}^{n-1}$ is minimal iff the cone $CM \subset \mathbb{R}^n$ is so.
- Blowing-down entire graphs yields area minimizing cones (Fleming, De Giorgi)



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Theorem (Bombieri-De Giorgi-Giusti, 1969)

The Clifford-Simons cone $\{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 - |y|^2 = 0\}$ *is area-minimizing in* $\mathbb{R}^8$*. In particular, Bernstein theorem fails for* $n \geq 8$

HSIANG's Problem

What we have?

- All minimal cones known so far are algebraic
- In 1967, W.-Y. HSIANG remarks in the 1st issue of J. of Differential Geometry:

> *It is quite interesting to classify real algebraic minimal submanifolds of degree higher than two up to equivalence under the orthogonal transformations. It turns out that the algebraic difficulties involved in such a problem are rather formidable.*

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- Hsiang proves that the only minimal cones of $\deg \le 2$ are defined by

$$\{(x, y) \in \mathbb{R}^k \times \mathbb{R}^m \cong \mathbb{R}^{n+m} : (m-1)|x|^2 - (k-1)|y|^2 = 0\}$$

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- ...and that any algebraic minimal cone is a polynomial solution u of

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u \qquad (1)$$

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- In particular, if $\deg u = 3$ then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \qquad (2)$$

Hsiang's Problem

In particular, Hsiang suggests the following problem:

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}^r_n$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t = \pm(x_1^2 + \cdots + x_n^2)\cdot F\,.$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

Definition

We call $u(x)$ a *radial eigencubic*, or *Hsiang eigencubic* if it solves

$$\Delta_1 u = \lambda |x|^2 u(x), \quad \lambda \in \mathbb{R}. \tag{3}$$

Some other related problems

- W.-Y. HSIANG: Classify all cubic minimal cones, i.e. cubics solutions of

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle = Q(x)u(x).$$

Remark. (i) All known *irreducible* cubic cones come from Hsiang eigencubics. However, there are *reducible* cubic solutions, e.g. $u = x_1(2x_2^2 + 2x_3^2 - x_4^2 - x_5^2 - x_6^2)$ with

$$Q = -16x_1^2 - 28(x_2^2 + x_3^2) - 10(x_4^2 + x_5^2 + x_6^2).$$

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- J.L. LEWIS (1980): Do there exist homogeneous polynomial solutions in $\mathbb{R}^n$, $n \geq 3$ of the general p-Laplace equation

$$\Delta_p u := |Du|^2 \Delta u + \tfrac{p-2}{2}\langle Du, D|Du|^2\rangle = 0\,?$$

Remark. (i) Answer is "no" for $n = 2$ (Lewis, 1980) and $n \geq 2$ and $\deg u = 3$ [Tka15b]
(ii) There is, however, a plenty of quasi-polynomial solutions for any $n \geq 2$ (Krol'-Maz'ya 1973, Arronsson 1974, Véron 1996, [Tka06])

What are the four Hsiang solutions?

- Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathrm{Herm}'_k(\mathbb{A})$ be the real vector space of trace free hermitian matrices of order k with the inner product $\langle X, Y\rangle := \mathrm{tr}\, XY$, and define

$$u(X) := \mathrm{tr}\, X^3, \qquad X \in \mathrm{Herm}'_k(\mathbb{A}).$$

- Δ_1 is an $O(n)$-invariant operator, hence $\Delta_1 u \in \mathbb{A}[\mathrm{tr}\, X^2, \ldots, \mathrm{tr}\, X^k]$.
- Since $N = \deg \Delta_1 u = 5$ then

$$\Delta_1 u = c_1 \mathrm{tr}\, X^2\, \mathrm{tr}\, X^3 + c_2 \mathrm{tr}\, X^5$$

therefore if $k \le 4$ then $c_2 = 0$ and

$$\Delta_1 u(X) = c_1\, \mathrm{tr}\, X^2\, \mathrm{tr}\, X^3 = c_1 |X|^2 u(X).$$

This yields the four Hsiang examples in $\mathbb{R}^{(k-1)(2+k \dim \mathbb{A})/2}$, i.e.

$$k = 3: \quad \mathrm{Herm}'_3(\mathbb{R}) \cong \mathbb{R}^5 \quad \text{and} \quad \mathrm{Herm}'_3(\mathbb{C}) \cong \mathbb{R}^8$$
$$k = 4: \quad \mathrm{Herm}'_4(\mathbb{R}) \cong \mathbb{R}^9 \quad \text{and} \quad \mathrm{Herm}'_4(\mathbb{C}) \cong \mathbb{R}^{15}$$

Examples

The **Lawson cubic cone** in $\mathbb{R}^4$ with the defining polynomial

$$u(z) = 2x_1x_2y_1 + (x_1^2 - x_2^2)y_2 = \langle x, A_1x\rangle y_1 + \langle x, A_2x\rangle y_1, \qquad z = (x, y) \in \mathbb{R}^4$$

where $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a *symmetric Clifford system*, i.e.

$$A_i^t = A_i \quad \text{and} \quad A_iA_j + A_jA_i = 2I\delta_{ij}, \ \forall i, j.$$

Proposition, [Tka10b]

In general, given is symmetric Clifford system $A_i : Y \to Y$, $1 \le i \le q$,

$$u_A(z) = \sum_{i=1}^{q} \langle x, A_ix\rangle y_i, \quad z = (x, y) \in \mathbb{R}^p \times \mathbb{R}^{2q}$$

is a radial eigencubic. Moreover, u_A exists iff

$$q - 1 \le \rho(p),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \text{odd}$, $0 \le b \le 3$, is the Hurwitz-Radon function

Dichotomy of Hisang eigencubics

Definition 1

A radial eigencubic u is said to be of **Clifford type** if the cone $u^{-1}(0)$ coincides with some $u_A^{-1}(0)$ up to an orthogonal transformation; otherwise, it is called **exceptional**.

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The four Hsiang eigencubics are exceptional, e.g.

- In $\mathbb{R}^5$ (a Cartan isoparametric cubic)

$$u(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

- In $\mathbb{R}^9$: $u(x) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$

Another example of an exceptional eigencubic:

- In $\mathbb{R}^{21}$: $u(x) = \mathrm{Re}(t_1t_2)t_3, \qquad t_i \in \mathbb{O} \cong \mathbb{R}^8.$

- Exceptional Hsiang eigencubics are distinguished in many relations.
- It turns out that all known exceptional eigencubics come from rank 3 Jordan algebras!

How to make a Jordan algebra structure inside a Hsiang eigencubic visible?

- Introduction
- Cubic minimal cones and Hsiang eigencubics
- Metrized algebras and Jordan algebras
- The main results
- A model example: Cubic eiconals

Metrized algebras

The key principle:

$$\boxed{\text{a cubic form}} + \boxed{\text{inner product vector space}} = \boxed{\text{a metrized algebra}}$$

Recall that an (nonassociative) algebra is a vector space endowed with a bilinear map

$$xy : V \times V \to V.$$

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A commutative nonassociative algebra V with an inner product $\langle , \rangle$ is called **metrized** if the multiplication operator $L_x y := xy$ is **self-adjoint**, i.e.

$$\langle xy, z\rangle = \langle x, yz\rangle, \qquad \forall x, y, z \in V.$$

The cubic form $\frac{1}{6}\langle x, x^2\rangle$ is called the **norm** of $x \in V$.

Metrized algebras

In the converse direction: given a cubic form $u : V \to \mathbb{A}$, consider its *linearizations*

- $u(x, y, z) = u(x + y + z) - u(x + y) - u(x + z) - u(y + z) + u(x) + u(y) + u(z)$ is symmetric trilinear form.
- In the presence of an inner product,

$$u(x, y, z) = \langle x, D^2 u(y) z \rangle$$

- $u(x; y) = \frac{1}{2} u(x, x, y) \equiv \partial_y u(x)$ — quadratic in x and linear in y

Given a cubic form u on an inner product space $(V, \langle , \rangle)$ define $(x, y) \to xy$ by

$$\langle xy, z \rangle = u(x; y; z)$$

Thus obtained algebra $V^{\mathrm{FS}}(u)$ is said to be the *Freudenthal-Springer algebra* of u.

Metrized algebras

A key feature:

u is a solution of a PDE $\Rightarrow$ $V^{\mathrm{FS}}(u)$ possesses an identity.

Proposition

- $V^{\mathrm{FS}}(u)$ is commutative and metrised
- $u(x) = \frac{1}{6}\langle x, x^2\rangle$
- $x^2 = 2Du(x)$, i.e. the square of x is proportional to the *gradient* of u at x
- $L_x = D^2u(x)$, i.e. the multiplication operator by x is the *Hessian* of u at x

Proof. For instance, we have

$$\partial_y u(x) = \langle Du(x), y\rangle.$$

On the other hand, by Euler's homogeneous function theorem

$$\partial_y u(x) = \tfrac{1}{2}u(x;x;y) = \tfrac{1}{2}\langle xx, y\rangle$$

implying $Du(x) = \frac{1}{2}xx = \frac{1}{2}x^2$.

A short introduction into Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

P. JORDAN (1932): a program to discover a new algebraic setting for quantum mechanics by capture intrinsic algebraic properties of Hermitian matrices.

Example

The Jordan algebra $\mathscr{H}_n(\mathbb{A}_d)$ of Hermitian matrices of order n over a real division algebra $\mathbb{A}_d$, $d = 1, 2, 4$ with Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$

- Being nonassociative, any Jordan algebras is **power associative**, i.e. the subalgebra $V(x)$ generated by x is associative.
- $\operatorname{rk} V = \max\{\dim V(x) : x \in V\}$. Clearly, $\operatorname{rk} V \leq \dim V$
- Any $x \in V$ satisfies a polynomial equation

$$x^{\bullet r} - \sigma_1(x)\, x^{\bullet(r-1)} + \ldots + (-1)^r \sigma_r(x) e = 0.$$

where $\sigma_1(x)$ is the **generic trace** of x and $\sigma_n(x) = N(x)$ is the **generic norm** of x.

A short introduction into Jordan algebras

En algebra is called **formally real** if $\sum x_i^2 = 0 \ \Rightarrow x_i = 0 \quad \forall i$.

Theorem (JORDAN-VON NEUMANN-WIGNER, 1934)

Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y \rangle; \, x_0 y + y_0 x)$
- the Jordan algebras $\mathscr{H}_n(\mathbb{A}_d)$, $n \geq 3$, $d = 1, 2, 4$;
- the Albert algebra $\mathscr{H}_3(\mathbb{A}_8)$.

The Springer Construction (McCrimmon, 1969)

A cubic form $N : V \to \mathbb{A}$, $N(e) = 1$, is called a Jordan cubic form if the bilinear form

$$T(x; y) = N(e; x)N(e; y) - N(e; x; y)$$

is a *nondegenerate* and the map $\# : V \to V$ uniquely determined by $T(x^{\#}; y) = N(x; y)$ satisfies the adjoint identity

$$(x^{\#})^{\#} = N(x)x.$$

If N is Jordan and

$$x\#y = (x + y)^{\#} - x^{\#} - y^{\#}$$

then

$$x \bullet y = \frac{1}{2}(x\#y + N(e; x)y + N(e; y)x - N(e; x; y)e)$$

defines a Jordan algebra structure on V and

$$x^{\bullet 3} - N(e; x)x^{\bullet 2} + N(x; e)x - N(x)e = 0, \quad \forall x \in V.$$

Some applications of Jordan algebras

- Origins in Quantum mechanics (27-dimensional Albert's exceptional algebra)
- Non-associative algebras (Zelmanov's theory)
- Self-dual homogeneous cones (Vinberg's and Koecher's theory)
- Lie algebras (Lie algebra functor, Exceptional Lie algebras, Freudental's magic square)
- Non-linear PDE's (integrable hierarchies, Generalized KdV equation, regularity of Hessian equations, higherdimensional minimal surface equation)
- Extremal black hole, supergravity
- Operator theory (JB-algebras)
- Differential geometry (symmetric spaces, projective geometry, isoparametric hypersurfaces)
- Statistics (Wishart distributions on Hermitian matrices and on Euclidean Jordan algebras)

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$$\Delta_1 u = -2|x|^2 u(x), \qquad x \in V = \mathbb{R}^n \tag{4}$$

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Theorem A

Given a nontrivial solution $u(x)$ to (4), there exists a commutative metrized such that

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A commutative metrized algebra with (2) and (3) above is called **Hsiang algebra**.

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What about Clifford type Hsiang eigencubics?

A commutative metrised algebra is called polar if

- V is $\mathbb{Z}_2$-graded: $V = V_0 \oplus V_1$ and $V_i V_j \subset V_{i \oplus j}$
- $V_0 V_0 = \{0\}$
- $L_x^2 = |x|^2$ on V_1, $\quad \forall x \in V_0$.

Proposition

- A polar algebra is a Hsiang algebra
- $u(x)$ is of Clifford type **iff** $u(x) = \frac{1}{6}\langle x, x^2 \rangle$, $x \in V$ in a polar algebra V.

Using classical results on Clifford modules due to Atiyah, Bott and Shapiro (1964), one readily gets a complete classification of Clifford type Hsiang eigencubics.

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Theorem B (Hidden Clifford Algebra Structure)

Let V be a arbitrary Hsiang algebra. Then

(i) $\operatorname{tr} L_x^3 = (1 - n_1)u(x)$, where $n_1 \in \mathbb{Z}^+$ (the cubic trace identity)

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Example. If $V = V_0 \oplus V_1$ is a *polar algebra* then

$$n_1 = \dim V_0 - 1, \qquad n_2 = \frac{1}{2}(\dim V_1 - \dim V_0 + 2)$$

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Theorem C (Hidden Jordan Algebra Structure)

There exists a *subalgebra* $\Lambda \subset V$ carrying a *formally real cubic Jordan algebra* structure.

- $\dim \Lambda = n_2 + 1$
- The Jordan multiplication is given explicitly by

$$x \bullet y = \frac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c.$$

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- Λ is reducible (as a Jordan algebra) if and only if V is polar (The Dichotomy Property)
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- Equivalently, Λ is reducible if and only if $u(x) = \frac{1}{6}\langle x, x^2\rangle$ is of Clifford type
- There exist *finitely many exceptional Hsiang algebras* and the only possible dimensions are

$$n \in \{5, 8, 9, 12, 15, 21, \mathbf{24}, 27, 30, \mathbf{42}, 54\}$$

Remark. All dimensions, except for the two marked cases, are realizable and may be explicitly written by means of certain cubic Jordan algebras.

How does it work?

We consider as a model example the following result:

Theorem ([Tka10a], [NTV14], [Tka14])

There is a natural correspondence between the categories:

- cubic solutions of $|Du(x)|^2 = C|x|^4$,

  and

- rank 3 formally real Jordan algebras,

given explicitly by

$$u(x) = \frac{\sqrt{2C}}{3} N(x), \qquad x \in 1^{\perp}$$

such that congruent solutions corresponds to isomorphic Jordan algebras.

Metrized algebras

Recall the following definitions:

- Given $c \in V$ is called an *idempotent* if $c^2 = c$.
- A metrized algebra is called *Euclidean* if $\langle , \rangle$ is positive definite.

Proposition

If V is Euclidean and $u \not\equiv 0$ then there are nonzero idempotents: $\mathscr{I}(V^{\mathrm{FS}}(u)) \neq \emptyset$.

Proof. Indeed, $u_0 := \max_{|x|=1} u(x) > 0$. Suppose x be the maximum point. Then

$$Du(x) = qx, \quad q \in \mathbb{R},$$

hence $q = \langle Du(x), x\rangle = 3u(x) = u_0 > 0$. On the other hand,

$$Du(x) = \tfrac{1}{2}x^2,$$

hence $x^2 = 2qx$, and therefore $c := x/2q$ is a nonzero idempotent.

Step I: Defining identities

Let us start with a cubic solution of the system

$$|Du|^2 = \tfrac{1}{4}|x|^4, \qquad \Delta u(x) = 0.$$

Translated to the metrized algebraic language, this means

$$\langle \tfrac{1}{2}x^2, \tfrac{1}{2}x^2 \rangle = \tfrac{1}{4}|x|^4, \qquad \mathrm{tr}\, L_x = 0.$$

The first equation yields on polarization:

$$\begin{aligned}
\langle x^2, x^2 \rangle &= |x|^4 && \text{apply } \partial_y \\
4\langle x^2, xy \rangle &= 4|x|^2\langle x, y \rangle && \text{using the weak associativity} \\
\langle x^2 x, y \rangle &= \langle |x|^2 x, y \rangle && \text{the non-degeneracy of } \langle \cdot, y \rangle \\
x^3 &= |x|^2 x
\end{aligned}$$

Definition. A commutative metrised algebra satisfying $x^3 = |x|^2 x$ and $\mathrm{tr}\, L_x = 0$ is said to be a *Cartan* algebra.

Step II: Peirce decomposition

On polarizing $(x^2)x = |x|^2x$ we find $(2xy)x + x^2y = 2\langle x, y\rangle x + |x|^2y$, hence

$$\boxed{2L_x^2 + L_{x^2} = 2x \otimes x + |x|^2} \qquad (a \otimes b)(x) := a\langle b, x\rangle \tag{5}$$

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Now suppose $c \neq 0$ be an idempotent of V, i.e. $c^2 = c$. Then

- $c^3 = c = |c|^2 c$ yields $|c|^2 = 1$,

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- This yields the Peirce decomposition:

$$\boxed{V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(\tfrac{1}{2}), \qquad V_c(t) := \ker(L_c - t)}$$

- where in virtue of $\operatorname{tr} L_c = 0$,

$$\boxed{\dim V_c(-1) = d + 1, \qquad \dim V_c(\tfrac{1}{2}) = 2d, \qquad d \in \mathbb{Z}^+}$$

- hence $\dim V = 3d + 2$

Step III: Multiplicative properties

Polarizing one more time:

$$2L_x^2 + L_{x^2} = 2x \otimes x + |x|^2$$

$$\Downarrow$$

$$(L_x L_c + L_c L_x) + L_{cx} = (x \otimes c + c \otimes x) + \langle x, c \rangle$$

$$\Downarrow \quad \text{... and applying to } y \perp c\text{:}$$

$$x(cy) + c(xy) + (cx)y = c\langle x, y \rangle$$

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Now, if $x_i \in V_c(t_i)$, where $t_i \neq 1$, $i = 1, 2, 3$ then $x_i \perp c$, hence

$$c(x_1 x_2) + (\mu + \lambda) x_1 x_2 = c\langle x_1, x_2 \rangle$$

$$\Rightarrow \quad \boxed{(t_1 + t_2 + t_3)\langle x_1 x_2, x_3 \rangle = 0}$$

Corollary

$V_c(t_1)V_c(t_2) \perp V_c(t_3)$ unless $t_1 + t_2 + t_3 = 0$.

Step III: Hidden Clifford algebra structure

This implies

- The multiplication table of $V_c(t_i)$:

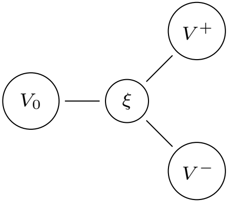
	$V_c(-1)$	$V_c(\frac{1}{2})$
$V_c(-1)$	$\mathbb{R}c$	$V_c(\frac{1}{2})$
$V_c(\frac{1}{2})$	$V_c(\frac{1}{2})$	$\mathbb{R}c \oplus V_c(-1)$

- $x^2 = -|x|^2 c$ for any $x \in V_c(-1)$
- $L_x : V_c(\frac{1}{2}) \to V_c(\frac{1}{2})$ and $L_x^2 = \frac{3}{4}|x|^2$ for any $x \in V_c(-1)$
- Hence $(L_x, V_c(-1), V_c(\frac{1}{2}))$ is a symmetric Clifford system, implying that

$$d \leq \rho(d) \quad \Rightarrow \quad d \in \{1, 2, 4, 8\}!$$

Step IV: Towards a Jordan algebra structure

- Choose some $\xi \in V_c(-1)$ and define $V_0 := \xi^\perp \cap V_c(-1)$,
- Then $V^- \oplus V^+ = V_c(-\frac{1}{2})$ is the eigen-decomposition according to $L_\xi = \pm\frac{\sqrt{3}}{2}$.



- This decomposition is *a triality*:

$$V_iV_i = 0, \qquad V_iV_j = V_k, \quad \{i,j,k\} = \{1,2,3\}.$$

- One can turn each V^* into a real division algebra in a standard way
- A new algebra defined above then $\mathbb{R} \times V$ with unit $e = (1,0)$ and the multiplication

$$(t_1, x_1) \bullet (t_2, x_2) = (\langle x_1, x_2 \rangle + t_1t_2,\ t_1x_2 + t_2x_1 + \tfrac{1}{3\sqrt{2}}x_1x_2),$$

is a formally real Jordan algebra.

The Classification of Hsiang algebras revisited

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All possible Peirce dimensions of *exceptional* Hsiang algebras

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

The cells in blue color represent non-realizable Peirce dimensions and the cells in gold color represent unsettled cases

The following Hsiang algebras are realizable:

- If $n_2 = 0$ then $n_2 \in \{2, 5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$, $V = \mathscr{H}_3(\mathbb{F}_d) \ominus \mathbb{R}e$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $n_2 \in \{5, 8, 14\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $\frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, where $z \to \bar{z}$ is the natural involution on $V = \mathscr{H}_3(\mathbb{F}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $n_2 \in \{5, 8, 14, 26\}$. The corresponding Hsiang algebras are $V^{\mathrm{FS}}(u)$, $u(z) = \mathrm{Re}\langle z, z^2\rangle$, where $z \in V = \mathscr{H}_3(\mathbb{F}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $V = V^{\mathrm{FS}}(u)$, $u = \frac{1}{6}\langle z, z^2\rangle$ on $\mathscr{H}_3(\mathbb{F}_8) \ominus \mathscr{H}_3(\mathbb{F}_1)$

Towards a finer classification

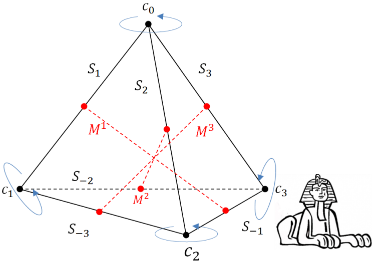
The Tetrad Decomposition

If V is an exceptional Hsiang algebra, $n_2 = 3d + 2$. Then

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3, \quad S^\alpha = S_\alpha \oplus S_{-\alpha},$$

where M^α is a null-subalgebra and each S_α is a real division algebra isomorphic to $\mathbb{A}_d$. Furthermore, and any 'vertex-adjacent' triple $(S_\alpha, S_\beta, S_\gamma)$ forms a triality:

$$S_\alpha S_\beta = S_\gamma, \qquad |x_\alpha x_\beta|^2 = \frac{1}{2}|x_\alpha|^2 |x_\beta|^2,$$



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THANK YOU FOR YOUR ATTENTION!