Nonassociative algebras of cubic forms

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(Pseudo-) Composition algebras

We always assume that $\mathrm{char} K \neq 2, 3$

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Let N be a **cubic form** on a vector space V. Then a K-trilinear form $N(x, y, z)$ is called the **full linearization** of N if

$$N(x, x, x) = 6N(x).$$

If $N(x, y, z) = 0$ for all $y, z \in V$ implies $x = 0$ then N is called **nondegenerate**.

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- Schafer (1960): a classification of finite-dimensional nonassociative algebras with a nondegenerate cubic form permitting composition: $N(xy) = N(x)N(y)$
- Ocubo (1978): division algebras without unit element satisfying a cubic relation
- Gradl, Meyberg, Walcher (1992), Meyberg, Osborn (1993): commutative pseudo-composition algebras with $x^3 = \phi(x, x)x$
- Elduque, Ocubo (2000): algebras satisfying $x^2x^2 = N(x)x$ (cf. with the **adjoint identity** below)

Jordan algebras of cubic forms

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The Springer Construction (McCrimmon, 1969)

A cubic form $N : V \to K$ with a distinguished element $e \in V$: $N(e) = 1$ is **admissible** if

- the bilinear form $T(x;y) = N(e;x)N(e;y) - N(e;x;y)$ is a nondegenerate, where $N(x,y) := \frac{1}{2}N(x,x,y)$ is quadratic in x and linear in y;
- the map $\#$ uniquely determined by $T(x^{\#};y) = N(x;y)$ satisfies the **adjoint identity**
$$(x^{\#})^{\#} = N(x)x.$$

Let $x\#y = (x+y)^{\#} - x^{\#} - y^{\#}$. Then

$$x \bullet y = \frac{1}{2}(x\#y + N(e;x)y + N(e;y)x - N(e;x;y)e)$$

defines a Jordan algebra structure on V and

$$x^{\bullet 3} - N(e;x)x^{\bullet 2} + N(x;e)x - N(x)e = 0, \quad \forall x \in V.$$

Metrised algebras

Let A be a K-algebra on V. A K-bilinear symmetric form Q on a vector space V is called **associative** (or **invariant**) if

$$Q(x,y)=0 \quad \forall y\in V \quad \Rightarrow \quad x=0,$$
$$Q(xy,z)=Q(x,yz) \qquad \forall x,y,z\in V.$$

In other words:

$$R_y^*=L_y \quad \text{for all } y\in V.$$

Definition

An algebra (A,Q) is called **metrised** if Q is associative.

Examples:

- a full matrix algebra with its trace $Q(x,y)=\operatorname{tr} xy$
- a real semisimple Lie algebra with its Killing form $Q(a,b)=\operatorname{tr}\operatorname{ad}(a)\operatorname{ad}(b)$
- a real semisimple Jordan algebra with its trace form $Q(a,b)=\operatorname{tr} ab$

In what follows $\langle x,y\rangle$ denote a nondegenerate symmetric bilinear form (an inner product).

Commutative metrised algebras

We are interested in **commutative metrised algebras** (=:CMA) V, i.e. those satisfying

- $xy = yx$
- $\langle xy, z\rangle = \langle x, yz\rangle$

In this setting, the study of V is essentially equivalent to study of the associated cubic form $N(x)$ on a vector space with a distinguished nondegenerate bilinear form. Namely, if

$$N(x) := \tfrac{1}{6}\langle xx, x\rangle = \tfrac{1}{6}\langle x^2, x\rangle$$

then the multiplication structure is recovered by linearization:

$$\langle xy, z\rangle = N(x, y, z). \qquad (*)$$

Conversely, if $N(x)$ is a cubic form on an inner product space $(V, \langle,\rangle)$ then the multiplication is uniquely determined by $(*)$ and turns V into a CMA.

While a CMA is not power associative in general ($x^2x^2 \neq x^3x$), the first moments are well defined:

$$\langle x^2, x^2\rangle = \langle x, x^3\rangle,$$
$$\langle x, x^2x^2\rangle = \langle x^2, x^3\rangle.$$

Commutative metrised algebras

In CMA

$$L_x = R_x = L_x^* = R_x^*$$

Example. Any power associative CMA is a Jordan algebra ($\mathrm{char} K \neq 5$).
Indeed,

$$x^2 x^2 = x x^3$$

implies (by commutativity) that

$$4(xy)(x^2) = yx^3 + x(x^2 y + 2x(xy)) \quad \Rightarrow \quad 4L_x L_{x^2} - L_{x^2} L_x = 2L_x^3 + L_{x^3}.$$

Applying conjugation yields by $L_x^* = L_x$ that

$$L_{x^2} L_x - 4L_x L_{x^2} = 2L_x^3 + L_{x^3} \quad \Rightarrow$$

$$\Rightarrow \quad [L_x, L_{x^2}] = L_{x^2} L_x - L_x L_{x^2} = 0.$$

Polar algebras

Recall that a section $A : X \to \mathrm{End}^S(Y)$ is a *symmetric Clifford system* (or $A \in \mathrm{Cliff}(X, Y)$) if

$$A(x)^2 = |x|^2 \mathbf{1}_Y \quad \forall x \in X.$$

Definition. A commutative metrised $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is called **polar** if

$$V_0 V_0 = \{0\} \qquad \text{and} \qquad L_x^2 = |x|^2 \text{ on } V_1, \quad \forall x \in V_0.$$

Proposition (the correspondence)

If $A \in \mathrm{Cliff}(X, Y)$ then $(X \times Y)^{\mathrm{FS}}(u)$, where $u(x, y) = \frac{1}{2}\langle y, A(x)y\rangle$, is a polar algebra with $V_0 = X \times \{0\}$, $V_1 = \{0\} \times Y$.

Conversely, if $V = V_0 \oplus V_1$ is a polar algebra then $L_x \in \mathrm{Cliff}(V_0, V_1)$.

It is well-known that

$$\mathrm{Cliff}(X, Y) \neq \emptyset \quad \Leftrightarrow \quad \dim X \leq 1 + \rho(\tfrac{1}{2} \dim Y),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the Hurwitz-Radon function.

Commutative metrised algebras

Let $K = \mathbb{R}$, $(V, \langle\cdot,\cdot\rangle)$ be an inner product vector space and $u(x)$ be a cubic form V. Denote by $V = \mathrm{CMA}(u)$ the corresponding metrised algebra, i.e.

$$u(x, y, z) = \langle xy, z\rangle, \qquad \forall z \in V.$$

In this setting,

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a cubic form u + a PDE $=$ a metrised algebra $V(u)$ with a defining identity

Three basic examples

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(A) The Cartan-Münzner equations (describe isoparametric hypersurfaces with $g = 3$ distinct principal curvatures):

$$\begin{cases} |Du(x)|^2 & = 9|x|^4 \\ \Delta u(x) & = 0 \end{cases}$$

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(B) Hsiang (1967) asked to classify all cubic homogeneous solutions of

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$$\implies \begin{cases} \langle x^2, x^3\rangle & = -\tfrac{4}{3}\lambda \langle x, x^2\rangle |x|^2 \\ \operatorname{tr} L_x & = 0 \qquad \text{(a nontrivial implication)} \end{cases}$$

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(C) **Isoparametric hypersurfaces** $=$ the hypersurfaces $M \subset S^{n-1} \subset \mathbb{R}^n$ of the Euclidean sphere S^{n-1} whose principal curvatures are constant along M.

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By Codazzi equations, the trilinear form

$$S(X, Y, Z) := \langle (\nabla_X A)Y, Z \rangle$$

is symmetric, where $A : TM \to TM$ is the Weingarten map. Define a CMA structure on V by

$$X \cdot Y := (\nabla_X A)Y.$$

(A) Cartan-Münzner equations

Appears as a defining equation in É. Cartan's study on isoparametric hypersurfaces with $g = 3$ distinct principal curvatures and also in cubic minimal cones (Hsiang's problem).

- Cartan (1938) originally approach: in some coordinates $x = (x_1, x_2, y, z, v)$

$$u(x) = x_1^3 + \tfrac{3}{2}x_1(|y|^2 + |z|^2 - 2|v|^2 - 2x_2^2) + \tfrac{3\sqrt{3}}{2}x_2(|y|^2 - |z|^2) + \underbrace{\sum v_i Q_i(y, z)}_{\text{triality term}}$$

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- G. Karcher (1988) simplifies arguments of Cartan by using invariant froms
- N. Knarr, L. Kramer (1995) any isoparametric hypersurface with $g = 3$ is naturally identified with the flag space of a compact connected Moufang plane
- S. Console, C. Olmos (1998) using properties of Clifford systems give another proof
- V.T. (2011) dropped the harmonictiy condition
- S. Console, A. Fino, G. Thorbergsson (2013) "Composition algebras and Cartans isoparametric hypersurfaces"

How to connect Cartan-Münzner eqs with Jordan algebras

Let us drop the second (harmonictiy) equation. Then

Theorem (V.T., *J. of Algebra*, 2014). There is a natural correspondence between

- cubic solutions of $|\nabla u(x)|^2 = 9|x|^4$, and
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$$N(\boldsymbol{x}) = x_0^3 - \frac{3}{2}x_0|x|^2 + \frac{1}{6\sqrt{2}}\langle x^2, x\rangle, \quad \boldsymbol{x} = (x_0, x).$$

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$$N(\boldsymbol{x}; \boldsymbol{y}) \quad = 3x_0^2 y_0 - 3x_0\langle x, y\rangle - \tfrac{3}{2}|x|^2 y_0 + \tfrac{1}{2\sqrt{2}}\langle x^2, y\rangle$$

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\Rightarrow \quad & N(\boldsymbol{x};\boldsymbol{e}) && = 3x_0^2 - \tfrac{3}{2}|x|^2 \qquad \text{and} \qquad N(\boldsymbol{e};\boldsymbol{x}) = 3x_0 \\
\Rightarrow \quad & T(\boldsymbol{x},\boldsymbol{y}) && = N(\boldsymbol{e};\boldsymbol{x})N(\boldsymbol{e};\boldsymbol{y}) - N(\boldsymbol{x};\boldsymbol{y};\boldsymbol{e}) = 3(x_0y_0 + \langle x, y\rangle) = 3\langle \boldsymbol{x},\boldsymbol{y}\rangle \\
\Rightarrow \quad & \boldsymbol{x}^{\#} && = (x_0^2 - \tfrac{1}{2}|x|^2, \tfrac{1}{6\sqrt{2}}x^2 - x_0x\,) \\
\Rightarrow \quad & (\boldsymbol{x}^{\#})^{\#} && = N(\boldsymbol{x})\boldsymbol{x} \quad \Rightarrow \quad N(\boldsymbol{x}) \text{ is admissible} \quad \square
\end{aligned}$$

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$$\sigma(L_c) \subset \{-1, \tfrac{1}{2}, 1\} \quad \Rightarrow \quad \text{the Peirce decomposition: } \; V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(\tfrac{1}{2}),$$

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The multiplication table of $V_c(t_i)$:

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- $L_x : V_c(\frac{1}{2}) \to V_c(\frac{1}{2})$ and $L_x^2 = \frac{3}{4}|x|^2$ for any $x \in V_c(-1)$
- Hence $(L_x, V_c(-1), V_c(\frac{1}{2}))$ is a symmetric Clifford system, implying that

$$d \le \rho(d) \quad \Rightarrow \quad d \in \{1, 2, 4, 8\}!$$

(B) CMA of Hsiang minimal cones

W.-Y. Hsiang (*J. Diff. Geometry*, **1**, 1967): Let u be a homogeneous polynomial in $\mathbb{R}^n$. Then $u^{-1}(0)$ is a minimal cone iff

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u.$$

- In $\deg = 2$: $\{(x,y) \in \mathbb{R}^{k+m} : (m-1)|x|^2 = (k-1)|y|^2\}$

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- In $\deg = 2$: $\{(x, y) \in \mathbb{R}^{k+m} : (m-1)|x|^2 = (k-1)|y|^2\}$
- The first non-trivial case: $\deg u = 3$ and then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \qquad (1)$$

- In fact, all known irreducible cubic minimal cones satisfy very special equation:

$$\Delta_1 u = \lambda |x|^2 \cdot u(x) \qquad (2)$$

Hsiang problem: Classify all cubic polynomial solutions of (2).

A homogeneous cubic solution of (1) is called a **Hsiang cubic**.

(B) CMA of Hsiang minimal cones

Some examples of Hsiang cubics:

- $u = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathbb{A}_d$, $d = 1, 2, 4, 8$, the triality polynomials in $\mathbb{R}^{3d}$ where

  $$\mathbb{A}_1 = \mathbb{R},\ \ \mathbb{A}_2 = \mathbb{C},\ \ \mathbb{A}_4 = \mathbb{H},\ \ \mathbb{A}_8 = \mathbb{O}$$

  are the classical Hurwitz algebras.

- $u(x) = \begin{vmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_2 & \frac{-2}{\sqrt{3}}x_1 & x_5 \\ x_4 & x_5 & \frac{1}{\sqrt{3}}x_1 - x_2 \end{vmatrix}$ = a Cartan isoparametric cubic in $\mathbb{R}^5$

- $u(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$

Thus, Hsiang cubics are nicely encoded by certain algebraic structures. Which ones?

Hsiang cubics of Clifford type

Theorem (V.T., 2010) Let $\{A_i\}_{1\le 1\le q}$ be a symmetric Clifford system, i.e.

$$A_iA_j + A_jA_i = 2\delta_{ij}I.$$

Then

$$u_A(z) = \sum\nolimits_{i=1}^{q} \langle x, A_i x\rangle y_i, \quad z = (x, y) \in \mathbb{R}^{2p} \times \mathbb{R}^q$$

is a Hsiang cubic.

The existence of a symmetric Clifford system in $\mathbb{R}^{2p}$ is equivalent to

$$q - 1 \le \rho(p),$$

$\rho(p) =$ Hurwitz-Radon function $= 1 + \#$(of independent vector fields on $\mathbb{S}^{p-1}$)

Example. The **Lawson cubic cone** in $\mathbb{R}^4$ with the defining polynomial

$$u(z) = (x_1^2 - x_2^2)y_1 + 2x_1x_2y_2 = \langle x, A_1x\rangle y_1 + \langle x, A_2x\rangle y_1, \qquad z = (x, y) \in \mathbb{R}^4$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The dichotomy of Hisang cubics

A Hsiang cubic u is of **Clifford type** if $u \cong u_A$, otherwise it is called **exceptional**.

Theorem (V.T., 2010) *Hsiang cubics of Clifford type are congruent if and only if the corresponding symmetric Clifford systems are geometrically equivalent.*

Representation theory of Clifford algebras $\Rightarrow$ a classification for the Clifford type.

Main Problem: How to determine all exceptional Hsiang cubics?

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Main Problem: How to determine all exceptional Hsiang cubics?

If $\operatorname{tr}(D^2u)^2 = c|x|^2$ and $n \neq 3d$ $(d = 1, 2, 4, 8)$ then u is an exceptional Hsiang cubic.

Proof. Indeed, if u is of the Clifford type then

$$u \cong u_A = \sum_{i=1}^{q} \langle x, A_i x\rangle y_i, \quad \Rightarrow \quad \operatorname{tr}(D^2 u_A)^2 = 2q|x|^2 + 2p|y|^2,$$

thus $q = p$, and therefore $p - 1 \leq \rho(p) \Rightarrow p \in \{1, 2, 4, 8\}$. This yields $n = 3p$, a contradiction.

$\square$

CMA of Hsiang cubics

Let $u(x)$ be a Hsiang cubic, i.e.

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda|x|^2 u(x).$$

Then the corresponding CMA obeys the (Hsiang) identity

$$\boxed{\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda\langle x, x\rangle\langle x^2, x\rangle}$$

We have

$$\boxed{\text{Hsiang cubics} \quad \leftrightsquigarrow \quad \text{Hsiang algebras}}$$

Key steps of the proof

A Hsiang algebra := **trivial** if $\dim(VV) = 1$ (or $u = (a_1x_1 + \ldots + a_nx_n)^3$).

Theorem A (The traceless property)

Any nontrivial Hsiang algebra is harmonic: $\operatorname{tr} L_x = 0$ and $\lambda \neq 0$. In particular,

$$\langle x^2, x^3\rangle = \frac{4}{3}\langle x^2, x\rangle |x|^2.$$

- an important role plays the notion of **maximal idempotent**

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Theorem B (The dichotomy)

u is a Hsiang cubic of Clifford type if and only is polar, i.e. if $V(u)$ admits a non-trivial $\mathbb{Z}_2$-grading

$$V = V_0 \oplus V_1, \quad V_0V_0 = 0$$

and $\forall x \in V_0$: $L_x^2 = |x|^2$ on V_1.

Key steps of the proof

Theorem C (The hidden Clifford algebra structure)

Let V be a Hisang algebra. Then

(i) $\forall c \in \mathrm{Ide}(V)$, the associated Peirce decomposition is

$$V = V_c(1) \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2}) \quad \text{and} \quad \dim V_c(1) = 1;$$

(ii) The Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ and $n_3 = \dim V_c(\frac{1}{2})$ do not depend on a particular choice of c and

$$n_3 = 2n_1 + n_2 - 2;$$

(iii) If ρ is the Hurwitz-Radon function then

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1).$$

In particular, for each n_2 there exist only finitely many possible values of n_1.

Key steps of the proof

Theorem D (The Multiplication Table)

If $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ then

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ and $V_0 \oplus V_2$ are subalgebras of V.

Key steps of the proof

Theorem E (The hidden Jordan algebra structure)

Let V be a Hisang algebra and $c \in \mathrm{Ide}(V)$. Then the subspace

$$J_c := V_c(1) \oplus V_c(-\tfrac{1}{2})$$

carries a structure of a formally real rank 3 Jordan algebra, and the following conditions are equivalent:

(i) the Hsiang algebra V is *exceptional*;

(ii) J_c is a *simple* Jordan algebra;

(iii) $n_2 \neq 2$ and the *quadratic trace identity* $\mathrm{tr}\, L_x^2 = k|x|^2$ holds for some $k \in \mathbb{R}$.

The proof of the first part of the theorem is heavily based on the McCrimmon-Springer construction of a cubic Jordan algebra.

The Finiteness of Exceptional Hsiang Algebras

If V is an exceptional Hsiang algebra then

$$J_c = V_c(1) \oplus V_c(-\tfrac{1}{2})$$

is a simple formally real Jordan algebra of rank≤ 3 and $\dim J_c = 1 + n_2$.

The Jordan-von Neumann-Wigner classification implies that

either $\dim J_c = 1$ or $\dim J_c = 3d + 3$, where $d \in \{1, 2, 4, 8\}$.

Thus, $n_2 = 0$ or $n_2 = 3d + 2$.

Using the obstruction

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1)$$

implies the finiteness.

The Finiteness of Exceptional Hsiang Algebras

Theorem (The finiteness)

There exists finitely many isomorphy classes of exceptional Hsiang algebras.

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

In the realizable cases (uncolored):

If $n_2 = 0$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3'(\mathbb{A}_d)$, $d = 0, 1, 2, 4, 8$.

If $n_1 = 0$ then $u(z) = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d)$, $d = 2, 4, 8$.

If $n_1 = 1$ then $u(z) = \mathrm{Re}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.

If $(n_1, n_2) = (4, 5)$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{O}) \ominus \mathscr{H}_3(\mathbb{R})$

$\mathscr{H}_3(\mathbb{A}_d)$ is the Jordan algebra of 3×3-hermitian matrices over the Hurwitz algebra $\mathbb{A}_d$

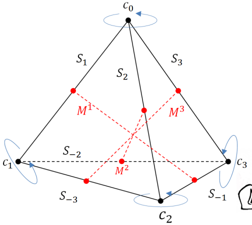
Towards a finer classification

The Tetrad Decomposition

Let V be an exceptional Hsiang algebra, $n_2 = 3d + 2$. Then

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3, \quad S^\alpha = S_\alpha \oplus S_{-\alpha},$$

- M^α are nilpotent;
- $S^\alpha S^\alpha \subset M^\alpha$;
- each S_α is a real division algebra isomorphic to $\mathbb{A}_d$;
- Any 'vertex-adjacent' triple $(S_\alpha, S_\beta, S_\gamma)$ is a triality



(C) Isoparametric algebras

Let V be the $\mathrm{CMA}(S)$, where S is the 3rd fundamental form of an isoparametric hs. Consider the eigen decomposition associated with the Weingarten map A:

$$V := T_{(m)}M = \bigoplus_{\alpha} \Pi_\alpha$$

Then

- $\Pi_\alpha \cdot \Pi_\alpha = 0$
- $\Pi_\alpha \cdot \Pi_\beta \subset \Pi_\alpha^\perp \cap \Pi_\beta^\perp$ for $\alpha \neq \beta$
- Weyl's identities (composition type identities)

$$|X|^2|Y|^2 = \frac{2}{1+\lambda_\alpha\lambda_\beta} \sum_{\gamma} \frac{|(XY)^{\Pi_\gamma}|^2}{(\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma)}, \quad X \in \Pi_\alpha, Y \in \Pi_\beta,\ \alpha \neq \beta.$$

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Karcher (1988): if $g = 3$ then $V = \Pi_1 \oplus \Pi_2 \oplus \Pi_3$ and $\Pi_i \Pi_{i+1} \subset \Pi_{i+2 \mod 3}$, with

$$|X_i|^2|X_{i+1}|^2 = c_{\alpha\beta}|X_i X_{i+1}|^2 \qquad \Rightarrow \quad \text{composition algebra property}$$

It would be very important to find a generalization on $g \geq 4$.

(C) Isoparametric algebras: open questions

It is well-known that the following key properties are true:

- $g \in \{1, 2, 3, 4, 6\}$
- if $\lambda_1 \leq \ldots \leq \lambda_g$ then $\dim \Pi_{i+2} = \dim \Pi_i$ (Münzner, 1978)
- $\dim \Pi_1 = \dim \Pi_2 \in \{1, 2\}$ for $g = 6$ (Abresch 1983 and Dorfmeister and Neher 1985)
- Clifford type isoparametric hs with $g = 4$ (Ferus, Karcher and Münzner 1981)
- classification for $g = 4$ (Cecil, Q.-S. Chi and G. Jensen 2007, Chi 2016) and $g = 6$ (Miyaoka 2014)

Is it possible to deduce these properties purely by NA methods?

N. Nadirashvili, V.G. Tkachev, and S. Vlăduţ, *Nonlinear elliptic equations and nonassociative algebras*, Mathematical Surveys and Monographs, vol. 200, American Mathematical Society, Providence, RI, 2014.

V.G. Tkachev, *A generalization of Cartan's theorem on isoparametric cubics*, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2889–2895.

______, *A Jordan algebra approach to the cubic eiconal equation*, J. of Algebra **419** (2014), 34–51.

______, *On the non-vanishing property for real analytic solutions of the p-Laplace equation*, Proc. Amer. Math. Soc. **144** (2016).

______, *Cubic minimal cones and nonassociative algebras*, in preparation.

THANK YOU FOR YOUR ATTENTION!