Nonassociative algebras and nonlinear PDEs

Vladimir G. Tkachev

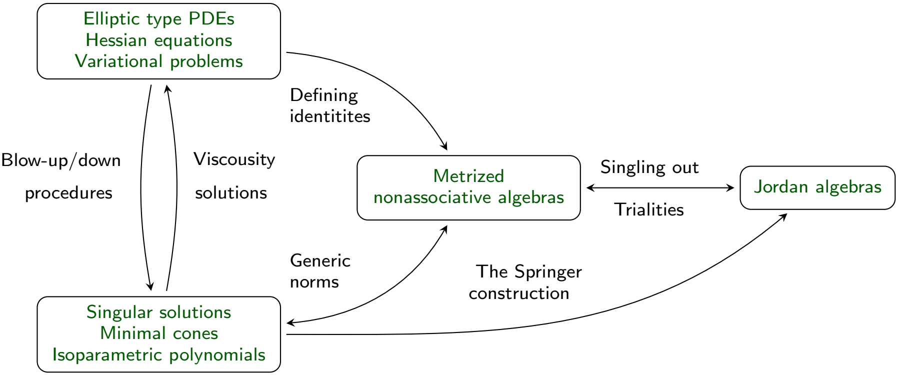
Linköping University

Motivations

Among others:

- Bernstein's problem on entire minimal graphs
- Hsiang problem on minimal cubic cones
- Non-classical and singular solutions to nonlinear elliptic PDEs [NTV14]
- Classification of isoparametric hypersurfaces (Yau's Problem 34)
- Some further problems having importance in an algebro-geometric context, such as homaloidal polynomials, prehomogeneous varieties, cubic hypersurfaces with vanishing hessian (Gordan-Noether problem)

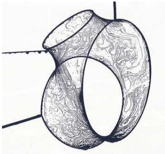
How it works


Elliptic type PDEs
Hessian equations
Variational problems
Defining
identitites
Blow-up/down
procedures
Viscousity
solutions
Metrized
nonassociative algebras
Singling out
Trialities
Jordan algebras
Generic
norms
The Springer
construction
Singular solutions
Minimal cones
Isoparametric polynomials

Minimal surface equation

The following conditions are equivalent:

- A critical point of the area functional
- The mean curvature $= 0$
- If $x_{n+1} = u(x)$ is a graph over $\mathbb{R}^n$ then $\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$



Theorem (S. Bernstein, 1915)

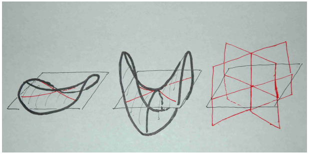
If u is an **entire** solution on $\mathbb{R}^2$ then $u = ax + by + c$.

Notice that if we remove a tiny disk from the plane, there is a function defined everywhere outside that disk whose graph is a minimal surface!

The Bernstein result holds true for $3 \leq n \leq 7$: Fleming (1962), De Giorgi (1965), Almgren (1966), Simons (1968). For $n \geq 8$ there exist non-linear solutions: Bombieri-De Giorgi-Giusti (1969), L. Simon (1986).

How do minimal cones enter?

- A submanifold $M \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is minimal iff the cone $CM \subset \mathbb{R}^n$ is so.
- Blowing-down entire graphs yields area minimizing cones, FLEMING, DE GIORGI.



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Theorem (Bombieri-De Giorgi-Giusti, 1969)

The Clifford-Simons cone

$$\{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 - |y|^2 = 0\}$$

is area-minimizing in $\mathbb{R}^8$. In particular, Bernstein theorem fails for $n \geq 8$

HSIANG's Problem

In 1967, W.-Y. HSIANG publishes a paper in the 1st issue of *J. of Differential Geometry*. He remarks

is essentially larger than that of homogeneous ones. It is then quite interesting to classify real algebraic minimal submanifolds of degree higher than two up to equivalence under the orthogonal transformations. It turns out that the algebraic difficulties involved in such a problem are rather formidable.

What we have?

- All minimal cones known so far are algebraic
- Hsiang classified all cones of $\deg \leq 2$ in all dimensions.
- Any algebraic cone comes from a polynomial solution $u \in \mathbb{R}[x_1, \ldots, x_n]$ of

$$\Delta_1 u := |Du|^2 \Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u \tag{1}$$

  $\Delta_p u$ being the p-Laplace operator.
- In particular, if $\deg u = 3$ then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \tag{2}$$

Hsiang's Problem

In fact, all known so far cubic minimal cones satisfy

$$\Delta_1 u = \lambda |x|^2 u(x), \quad \lambda \in \mathbb{R}. \tag{3}$$

Hsiang suggests the following problem:

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t = \pm(x_1^2 + \cdots + x_n^2)\cdot F\,.$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

... Which four solutions?

HSIANG's trick

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $u(X) := \operatorname{tr} X^3$, where

$$X \in \mathrm{Herm}'_k(\mathbb{F}),$$

the real vector space of trace free hermitian matrices of order k with the inner product $\operatorname{tr} XY$. Hsiang shows that Δ_1 is an $O(n)$-invariant operator, which implies

$$\Delta_1 u \in \mathbb{F}[\operatorname{tr} X^2, \ldots, \operatorname{tr} X^k].$$

Further, $N = \deg \Delta_1 u = 3 \deg u - 4 = 5$ hence

$$\Delta_1 u = c_1 \operatorname{tr} X^2 \operatorname{tr} X^3 + c_2 \operatorname{tr} X^5$$

If additionally $k = 3, 4$ then $c_2 = 0$, thus

$$\Delta_1 u(X) = c_1 \operatorname{tr} X^2 \operatorname{tr} X^3 = c_1 |X|^2 u(X).$$

This yields the four Hsiang examples in $\mathbb{R}^{(k-1)(2+k \dim \mathbb{F})/2}$, i.e.

$$\begin{aligned} k = 3: &\quad \mathrm{Herm}'_3(\mathbb{R}) \cong \mathbb{R}^5, \quad \mathrm{Herm}'_3(\mathbb{C}) \cong \mathbb{R}^8 \\ k = 4: &\quad \mathrm{Herm}'_4(\mathbb{R}) \cong \mathbb{R}^9, \quad \mathrm{Herm}'_4(\mathbb{C}) \cong \mathbb{R}^{15} \end{aligned}$$

Isoparametric hypersurfaces of the unit sphere

A hypersurface is called **isoparametric** if its principal curvatures are constant. Origins in geometrical optics and wavefronts Segre, Levi-Civita in the 1920's.

É. Cartan, 1937-1939:

- Among $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$, the latter case is the most mysterious.
- If $m = 3$ then there are exactly four solutions of (4) given by

$$u(x) = \operatorname{tr} X^3, \qquad X \in \mathscr{H}_3'(\mathbb{F}_d) \cong \mathbb{R}^{3d+2}, \quad d \in \{1, 2, 4, 8\},$$

  where $\mathscr{H}_3(\mathbb{F}_d)$ is a hermitian rank 3 Jordan algebra over a real division algebra $\mathbb{F}_d$.
- For $d = 1$ and 2, one actually has $\mathscr{H}_3(\mathbb{F}_d) \approx \mathrm{Herm}_3(\mathbb{F}_d)$. E.g. for $d = 1$

$$u(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

Theorem (Münzner) The number of distinct principal curvatures $m \in \{1, 2, 3, 4, 6\}$ and $M = u^{-1}(t) \cap \mathbb{S}^n$, where u is a homogeneous degree m polynomial solution of

$$\text{(a)} \ |Df(x)|^2 = m^2|x|^{2m-2}, \qquad \text{(b)} \ \Delta f(x) = C|x|^{m-2}. \tag{4}$$

A short introduction into Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

P. JORDAN (1932): a program to discover a new algebraic setting for quantum mechanics by capture intrinsic algebraic properties of Hermitian matrices.

Example. The Jordan algebra W^+ obtained from an associative algebra W replacing the product xy by $x \bullet y = \frac{1}{2}(xy + yx)$ (a so-called **special** J. algebra)

- $V(x)$ (= a subalgebra generated by x) is associative for any $x \in V$.
- $\operatorname{rk}(V) = \max\{\dim V(x) : x \in V\}$
- Any $x \in V$ satisfies the **minimum polynomial** equation $m_x(x) = 0$, with

$$m_x(\lambda) = \lambda^r - \sigma_1(x)\lambda^{r-1} + \ldots + (-1)^r \sigma_r(x).$$

- $\sigma_1(x) = \operatorname{Tr} x =$ the **generic trace** of x,
- $\sigma_n(x) = N(x) =$ the **generic norm** (or generic determinant) of x.

A short introduction into Jordan algebras

JORDAN-VON NEUMANN-WIGNER (1934): Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle; x_0 y + y_0 x)$
- the Jordan algebras $\mathscr{H}_n(\mathbb{F}_1)$, $\mathscr{H}_n(\mathbb{F}_2)$, $\mathscr{H}_n(\mathbb{F}_4)$ of Hermitian matrices of order $n \geq 3$ over the reals, complexes and quaternions, resp.;
- $\mathscr{H}_3(\mathbb{F}_8)$, the Albert exceptional algebra.

In particular, the only formally real Jordan rank three algebras are

- $\mathscr{H}_3(\mathbb{F}_d),\ d = 1, 2, 4, 8$
- $\mathbb{R} \oplus \mathscr{S}(\mathbb{R}^{n+1})$

Theorem 1 (Eiconal ≈ cubic Jordan, [Tka14])

There is a natural correspondence

$$\text{cubic solutions of } |Du(x)|^2 = 9|x|^4 \quad \leftrightarrow \quad \text{rank 3 formally real Jordan algebras}$$

In this picture,

$$u(x) = \frac{1}{\sqrt{2}} N(x), \quad x \in 1^{\perp}$$

and congruent solutions corresponds to isomorphic Jordan algebras.

The Springer Construction

Recall that a function $u : V \to \mathbb{F}$ is called a cubic form if the linearization

$$u(x,y,z) = u(x+y+z) - u(x+y) - u(x+z) - u(y+z) + u(x) + u(y) + u(z)$$

is a trilinear form.

(Springer, 1962; McCrimmon, 1969) A cubic form $N : V \to \mathbb{F}$, $N(e) = 1$, is called a Jordan cubic form if the bilinear form

$$T(x;y) = N(e;x)N(e;y) - N(e;x;y)$$

is a *nondegenerate* and the map $\# : V \to V$ uniquely determined by $T(x^{\#};y) = N(x;y)$ satisfies the adjoint identity

$$(x^{\#})^{\#} = N(x)x.$$

If N is Jordan and $x\#y = (x+y)^{\#} - x^{\#} - y^{\#}$ then

$$x \bullet y = \tfrac{1}{2}(x\#y + N(e;x)y + N(e;y)x - N(e;x;y)e)$$

defines a Jordan algebra structure on V and

$$x^{\bullet 3} - N(e;x)x^{\bullet 2} + N(x;e)x - N(x)e = 0, \quad \forall x \in V.$$

Freudenthal-Springer algebras

The main idea (inspired by Springer's *Characterization of a class of cubic forms, 1961)*:

a cubic form + an inner product space = a metrised algebra

Given a cubic form u on an inner product space $(V, \langle , \rangle)$ define the multiplication by

$$\langle xy, z\rangle = u(x, y, z)$$

Thus obtained algebra $V^{\mathrm{FS}}(u)$ is called the *Freudenthal-Springer algebra* of u.

Proposition 1

- $V^{\mathrm{FS}}(u)$ *is* **commutative** *and* **metrised**, *i.e.* $\langle xy, z\rangle = \langle x, yz\rangle$
- $u(x) = \frac{1}{6}\langle x, x^2\rangle$
- $x^2 = 2Du(x)$, *i.e. the square of* x *is proportional to the gradient of* u *at* x
- $L_x = D^2u(x)$, *i.e. the multiplication operator by* x *is the Hessian of* u *at* x
- *If* V *is Euclidean then there are nonzero idempotents:* $\mathscr{I}(V^{\mathrm{FS}}(u)) \neq \emptyset$.

Examples: Polar algebras

Definition. A commutative metrised algebra $V = V_0 \oplus V_1$ is called polar if

(i) $V_0 V_0 = \{0\}$ and $V_i V_j \subset V_{i+j \bmod 2}$

(iii) $\forall x \in V_0 : L_x^2 = |x|^2$ on V_1.

Example. $u(x) = x_1 x_2 x_3$ on $\mathbb{R}^3 = V_0 \oplus V_1$ with $V_1 = \{x_1 = 0\}$ and $V_0 = V_1^\perp$.

Definition. A pencil of symmetric endomorphisms $A : X \to \mathrm{End}^S(Y)$ is called a *symmetric Clifford system*, denoted $A \in \mathrm{Cliff}(X, Y)$, if $A(x)^2 = |x|^2 \mathbf{1}_Y$ for all $x \in X$.

Proposition 2

If $A \in \mathrm{Cliff}(X, Y)$ then $(X \times Y)^{\mathrm{FS}}(u)$, where $u(x, y) = \frac{1}{2}\langle y, A(x)y\rangle$, is a polar algebra with $V_0 = X$, $V_1 = Y$. Conversely, if $V = V_0 \oplus V_1$ is a polar algebra then $L_x \in \mathrm{Cliff}(V_0, V_1)$. Furthermore,

$$\mathrm{Cliff}(X, Y) \neq \emptyset \Leftrightarrow \dim X \leq 1 + \rho(\tfrac{1}{2}\dim Y),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the Hurwitz-Radon function.

Examples: Cartan algebras

Definition. A commutative metrised algebra satisfying $x^3 = |x|^2 x$ and $\operatorname{tr} L_x = 0$ is said to be a *Cartan* algebra.

Example. Let u be a cubic solution of (4). Then $\Delta u(x) = \operatorname{tr} L_x = 0$ and

$$|Du(x)|^2 = 9|x|^4 \overset{\text{F.-S. algebra}}{\Rightarrow} \langle x^2, x^2\rangle = |x|^4$$

$$\overset{\text{polarization}}{\Rightarrow} x^3 = |x|^2 x$$

Remark. Using Theorem 1 above, any Cartan algebra V is the trace free subspace in an Euclidean rank 3 Jordan algebra $V \times \mathbb{R}$ with unit $e = (1, 0)$ and the multiplication

$$\xi_1 \bullet \xi_2 = (t_1 t_2 + \langle x_1, x_2\rangle, \, t_1 x_2 + t_2 x_1 + \tfrac{1}{\sqrt{2}} x_1 x_2)$$

Conversely, if W is a unital Euclidean rank 3 Jordan algebra then $V = 1^{\perp}$ with the induced FS-multiplication is a Cartan algebra.

Hsiang algebras

The definition of a Hsiang eigencubic, i.e. a solution of (3), is translated to

$$\boxed{\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda\langle x, x\rangle\langle x^2, x\rangle, \quad \lambda \in \mathbb{R}.} \tag{5}$$

Definition

A metrised Euclidean commutative algebra with (5) is called a **Hsiang algebra**.

Remarks.

(a) V is a Hsiang algebra $\Leftrightarrow$ $u(x) = \frac{1}{6}\langle x, x^2\rangle$ is a Hsiang eigencubic.

(b) The classification of Hsiang algebras is much more transparent conceptually.

(c) Similar (congruent) cones correspond to *similar* Hsiang algebras:

Metrised algebras V and V' are called *similar* if there are $k \in \mathbb{F}^\times$ and $F \in O(V, V')$:

$$k\, F(xy) = F(x)F(y), \qquad \forall x, y \in V.$$

Examples of Hsiang algebras

Proposition 3

(a) Any rank 1 metrised algebra (i.e. $\dim VV = 1$*) is a Hsiang algebra.*

(b) Any Cartan algebra is a Hsiang algebra.

(c) Any polar algebra is a Hsiang algebra.

Proof of (b). Then $x^3 = |x|^2 x$ yields

$$L_{x^2} + 2L_x^2 = 2x \otimes x + |x|^2$$

therefore $x^2 x^2 + |x|^2 x^2 = 2\langle x, x^2 \rangle x$, implying

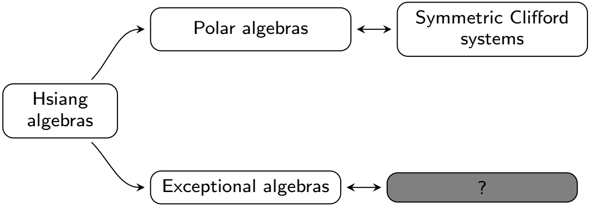
$$\langle x^2, x^3 \rangle = \langle x^2 x^2, x \rangle = \langle x, x^2 \rangle |x|^2$$

therefore, in virtue of $\operatorname{tr} L_x = 0$, V satisfies (5) with $\lambda = -\frac{3}{2}$.

$\square$

How to classify?

Definition. A Hsiang algebra V is said to be of **Clifford type** if it is similar to a polar algebra; otherwise it is called **exceptional**.



Proposition 4

Any Cartan algebra V is exceptional.

Proof. Indeed, in a Cartan algebra $\langle x^2, x^2\rangle = \langle x^3, x\rangle = |x|^4 \neq 0$ for $x \neq 0$. On the other hand, if V is a Clifford type algebra then $x^2 = 0$ on a nontrivial subspace ($\cong V_0$) implying a contradiction.

The harmonicity

Theorem 2

Any non-trivial Hsiang algebra V is harmonic, i.e. $\operatorname{tr} L_x = 0$ for all $x \in V$. Equivalently, any Hsiang eigencubic in $\mathbb{R}^n$, $n \geq 2$, is a harmonic function.

Corollaries:

- A Hsiang algebra possesses a simpler identity $\langle x^2, x^3 \rangle = -\frac{2}{3}\lambda \langle x, x^2 \rangle |x|^2$.
- If $c \in \mathscr{I}(V)$ then $|c|^2 = -\frac{3}{2\lambda}$, i.e. all idempotents have the same norm.
- A Hsiang algebra is called **normalized** if $|c|^2 = \frac{3}{4}$, equivalently $\lambda = -2$.
- Let V be a normalized Hsiang algebra. Then

$$\langle x^2, x^3 \rangle = \tfrac{4}{3} \langle x, x \rangle \langle x, x^2 \rangle,$$

$$4xx^3 + x^2x^2 - 4|x|^2x^2 - \tfrac{8}{3}\langle x^2, x \rangle x = 0.$$

Remark. Cf. with baric algebras and algebras satisfying identities of $\deg \leq 4$ (Walcher, Meyberg, Osborn, Okubo, Elduque, Labra)

The Peirce decomposition

- Let $c \in \mathscr{I}(V)$ and $V_c(t) = \ker(L_c - tI)$, then $V_c(1) = \mathbb{R}c$ and

$$V = \mathbb{R}c \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2})$$

- The **Peirce dimensions**

$$\boxed{n_1(c) = \dim V_c(-1), \quad n_2(c) = \dim V_c(-\tfrac{1}{2}), \quad n_3(c) = \dim V_c(\tfrac{1}{2})}$$

satisfy

$$\begin{aligned} n_3(c) &= 2n_1(c) + n_2(c) - 2 \\ 3n_1(c) + 2n_2(c) - 1 &= \dim V = n. \end{aligned}$$

In particular, any of $n_i(c)$ completely determines two others.

Examples.

- If V is a *polar algebra* then $(n_1(c), n_2(c)) = (\dim V_0 - 1,\ \frac{1}{2}\dim V_1 - \dim V_0 + 2)$.
- If V is a *Cartan algebra* then $(n_1(c), n_2(c)) = (\frac{1+\dim V}{3}, 0)$.

The Peirce decomposition

Proposition 5

Setting $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ *we have*

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ *and* $V_0 \oplus V_2$ *are subalgebras of* V*. Notice however that these subalgebras may be Hsiang subalgebras or not.*

The cubic trace identity

We already know that $\operatorname{tr} L_x = 0$ for any $x \in V$. The following property provides an effective and simple recovering tool for the Peirce dimensions.

Theorem 3

Any normalized Hsiang algebra satisfies the cubic trace identity

$$\operatorname{tr} L_x^3 = (1 - n_1(c))\langle x, x^2 \rangle, \qquad \forall c \in \mathscr{I}(V), x \in V. \tag{6}$$

In particular, the Peirce dimensions $(n_1(c), n_2(c))$ *are similarity invariants of a general Hsiang algebra and do not depend on a particular choice of an idempotent* c.

In what follows, we write $(n_1(V), n_2(V))$, or just (n_1, n_2).

Classification of Hsiang algebras, I

Theorem 4 (A hidden Clifford algebra structure)

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1),$$

where ρ is the Hurwitz-Radon function.

Proof. One can prove that $A(x) = \sqrt{3}L_x - (1+\sqrt{3})(L_x L_c + L_c L_x)$, $x \in V_1$ satisfies

$$A(x)^2 = |x|^2 \quad \text{on } V_2 \oplus V_3$$

which implies $A \in \mathrm{Cliff}(V_1, V_2 \oplus V_3)$ and the desired obstruction. □

Example 5

If $n_2(V) = 0$ then $n_1(V) \leq \rho(n_1(V) - 1)$, hence $n_1(V) \in \{0, 1, 2, 3, 5, 9\}$.

Corollary 1

Given $n_2 \geq 0$, there are finitely many admissible Peirce dimensions (n_1, n_2).

Classification of Hsiang algebras, II

Theorem 6 (A hidden Jordan algebra structure)

Given $c \in \mathscr{I}(V)$, *let us define the new algebra structure on* $\Lambda_c = (V_0 \oplus V_2, \bullet)$ *with the multiplication*

$$x \bullet y = \tfrac{1}{2}xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c. \tag{7}$$

Then Λ_c *is a Euclidean Jordan algebra with unit* $c^* = 2c$, *the associative trace form* $T(x; y) = \langle x, y\rangle$ *and*

$$\operatorname{rk}\Lambda_c = \min\{3, n_2(V) + 1\} \leq 3.$$

Idea of the **Proof**: to verify that the cubic form $N(x) = \frac{1}{6}\langle x, x^2\rangle$ on $V_0 \oplus V_2$ with a basepoint $c^* = 2c$ is Jordan for any $c \in \mathscr{I}(V)$ and apply the Springer-McCrimmon construction. To get the rank property requires a finer analysis of the cubic identity on Λ_c together with the defining identity on V.

□

Classification of Hsiang algebras, II

Theorem 7 (The dichotomy of Hsiang algebras)

The following conditions are equivalent:

(i) *A Hsiang algebra V is Clifford*

(ii) *The Jordan algebra Λ_c is reducible for some c*

Prove, for example, (ii)⇒(i). Define $\mathscr{N}_0(V) = \{w : w^2 = 0, |w| = 1\}$.

(A) If Λ_c is reducible then $\frac{1}{2} \notin \operatorname{Spect} L^\bullet_w$ for some $w \in \mathscr{I}(\Lambda_c)$.

(A') Replacing w by $w' = 2c - w$ one still has (A).

(B) One of the w and w' is in $\mathscr{N}_0(V)$; denote it by w. Then

(C) $V = V_w(0) \oplus V_w(-1) \oplus V_w(1)$, and $\Lambda_c(w) := [c, w]^\perp \cap \Lambda_c \subset V_w(0) \oplus V_w(-1)$.

(D) On the other hand, $L^\bullet_w = \frac{1}{2}(L_w + 1)$ on $\Lambda_c(w)$, implying $\Lambda_c(w) \subset V_w(-1)$

(F) $(L_w + 1)(V_c(-1))$ is a zero algebra

(G) $V_w(0) = \mathbb{R}w \oplus (L_w + 1)(V_c(-1)) \Rightarrow V_w(0)V_w(0) = \{0\}$.

Classification of Hsiang algebras, II

Combining Theorem 4 and Theorem 7, one obtains

Corollary 2

There are at most 24 classes of exceptional Hsiang algebras with the Peirce dimensions

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

The cells in gray color represents non-realizable Peirce dimensions and the cells in gold color represents unsettled cases

Proof. The case $n_2 = 0$ yields Cartan algebras. Suppose that V is exceptional and $n_2 \geq 1$. Then $\operatorname{rk} \Lambda_c = 3$, hence the Jordan-von Neumann-Wigner classification yields $\dim \Lambda_c \in \{6, 9, 15, 27\}$, implying $n_2 \in \{5, 8, 14, 26\}$, and therefore by Theorem 4 the desired Peirce dimensions.

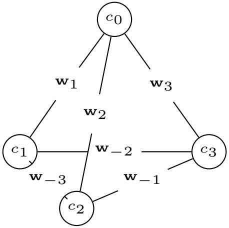
Some existence results

Thus, we have $n_2 \in \{0, 5, 8, 14, 26\}$. Which Peirce dimensions of exceptional algebras in the table above are *actually* realizable?

- For $n_2 = 0$ the are four Cartan algebras with $n_1 \in \{0, 1, 2, 3, 5, 9\}$.
- For $n_1 = 0$ there are Hsiang algebras with $n_2 \in \{5, 8, 14\}$ only. One can show that

$$z = x + y \to \bar{z} = x - y, \qquad x \oplus y \in V = \Lambda_c \oplus \Lambda_c^{\perp},$$

is a self-adjoint involution on V, and the cubic form $N(z) = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$ is Jordan with basepoint $c^* = 2c$. The corresponding Jordan algebra V_N is Euclidean and Λ_c is a Jordan subalgebra of V_N. Furthermore, V is an exceptional Hsiang algebra iff V_N is a simple Jordan algebra.


c_0
$\mathbf{w}_1$
$\mathbf{w}_3$
$\mathbf{w}_2$
c_1
$\mathbf{w}_{-2}$
c_3
$\mathbf{w}_{-3}$
$\mathbf{w}_{-1}$
c_2

Mutants, symmetric algebras and trace identities

Definition. A Hsiang algebra := *mutant* if $n_2 = 2$ for some $c \in \mathscr{I}(V)$.

• Any mutant V is a reducible algebra and

$$V \;\cong\; \mathscr{H}_3(\mathbb{F}_d)/\{\text{diag=0}\} \;\cong\; V^{\mathrm{FS}}(\mathrm{Re}(x_1x_2)x_3),$$

where $x = (x_1, x_2, x_3) \in V = \oplus_{i=1}^{3}\mathbb{F}_d$.

• Formally, a mutant can be thought of as an exceptional Hsiang algebra with $d = 0$. Furthermore, mutants occupy an intermediate place between Clifford type and exceptional algebras by sharing certain characteristic properties of both these classes.

Definition. A Hsiang algebra := *symmetric* if it is either exceptional or mutant. Notice that the second Peirce dimensions of a symmetric algebra is

$$n_2 = 3d + 2, \qquad d \in \{0, 1, 2, 4, 8\}.$$

Proposition 6 (The quadratic trace identity)

If V is a (normalized) symmetric Hsiang algebra then

$$\operatorname{tr} L_x^2 = \tfrac{2}{3}(3n_1 + n_2 + 1)|x|^2 = 2(n_1 + d + 1)|x|^2. \tag{8}$$

Mutants, symmetric algebras and trace identities

Theorem D. The cubic trace identity

Any *normalized Hsiang algebra satisfies the cubic trace identity*

$$\operatorname{tr} L_x^3 = (1-n_1)\langle x, x^2\rangle, \qquad \forall c \in \mathscr{I}(V), x \in V. \tag{9}$$

In particular, the Peirce dimensions $(n_1(c), n_2(c))$ *are similarity invariants of a general Hsiang algebra and do not depend on a particular choice of an idempotent* c.

The latter provides an effective and simple recovering tool for the Peirce dimensions.

Corollary 3

- *A Hsiang algebra is symmetric if and only if it permits a quadratic trace identity.*
- *A Hsiang algebra is exceptional* $\Leftrightarrow$ *it permits a quadratic trace identity and* $n_2 \neq 2$.

The existence: Hsiang algebras with $n_2(c) = 0$

Theorem D

Given a Hsiang algebra, the following conditions are equivalent:

(i) V *is a Cartan Hsiang algebra (in dimension* $n = 3d + 2$, $d = 1, 2, 4, 8$*).*

(ii) *For any* $c \in \mathscr{I}(V)$ *there holds* $n_2(c) = 0$*.*

(iii) *There exists* $c \in \mathscr{I}(V)$ *such that* $n_2(c) = 0$*.*

(iv) $\mathscr{N}_0(V) = \emptyset$*.*

Since $n_2(c)$ implies $\Lambda_c \cong \mathbb{R}$ we have

Corollary

If $n_2(c) = 0$ *for some* $c \in \mathscr{I}(V)$ *then* V *is exceptional.*

REC algebras

Theorem 1: (1) implies the trace free condition $\operatorname{tr} L_x = 0 \ \forall x \in V$.

The Classification Theorem

Let V be a REC algebra. Then for any idempotent $c \in V$ there exists a subalgebra of

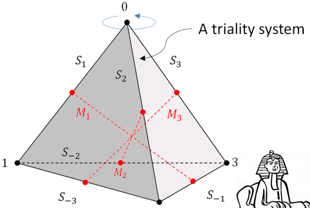
$$J_c \subset V$$

carrying a rank 3 **Jordan algebra** structure such that the following holds:

- $J_c \cong J_{c'}$ for any two idempotents $c, c' \in V$
- V is a Clifford REC algebra $\iff$ J is a reducible Jordan algebra
- There are finitely many REC algebras with **irreducible** J ($=:$ **exceptional REC algebras**)

The Tetrad structure of Exceptional REC algebra

- $V = \bigoplus_{1 \le \alpha \le 3} (S_{-\alpha} \oplus S_{\alpha} \oplus M_{\alpha})$,
- each M_{α} is nilpotent, i.e. $M_{\alpha} M_{\alpha} = \{0\}$
- any 'vertex-adjacent' triple $S_{\alpha} \oplus S_{\beta} \oplus S_{\gamma}$ forms a triality



Isoparametric cubics, J-algebras and singular solutions

For instance, for $d = 1$

$$u_1(x) = x_5^3 + \tfrac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \tfrac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

is a unique (up to isometries) solution of

$$|\nabla u(x)|^2 = 9|x|^4, \qquad \Delta u(x) = 0, \quad x \in \mathbb{R}^5.$$

Theorem (Nadirashvili-V.T.-Vlăduţ, [NTV12]). *The function*

$$w(x) := \frac{u_1(x)}{|x|}$$

is a singular viscosity solution of the uniformly elliptic Hessian equation

$$(\Delta w)^5 + 2^8 3^2 (\Delta w)^3 + 2^{12} 3^5 \Delta w + 2^{15} \det D^2(w) = 0,$$

This give the best possible dimension ($n = 5$) where homogeneous order 2 real analytic functions in $\mathbb{R}^n \setminus \{0\}$

Some very related problems

• Classify all cubics solutions of (2), i.e. cubic minimal cones.

Remark. For all known *irreducible* cubic solutions $Q = \lambda |x|^2$.

• J.L. Lewis (1980): Do there exist homogeneous polynomial solutions in $\mathbb{R}^n$, $n \geq 3$ of the general p-Laplace equation

$$\Delta_p u := |Du|^2 \Delta u + \tfrac{p-2}{2} \langle Du, D|Du|^2 \rangle = 0\,?$$

Answer is "no" for $n = 2$ (Lewis), and $n \geq 2$ and $\deg u = 3$ [Tka15]. There are however quasi-polynomial solutions for any $n \geq 2$ (G. Aransson, Krol'-Maz'ya, [Tka06])

• Cubic hypersurfaces with vanishing Hessian, function dividing their Hessian determinants, Segre varieties (R. Gondim, F. Russo, D. Fox).

THANK YOU FOR YOUR ATTENTION!

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