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On the Bore Radius for Minimal Surfaces

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ABSTRACT. A least upper bound for the inner radius R of an opening in a complete minimal hypersurface contained in a parallel layer is given. Namely, if Δ is the width of this layer, then $R \le \Delta/(2c_p)$, where c_p is an absolute constant depending only on the dimension p of the minimal hypersurface.

Recently Hoffman and Meeks [1] announced a theorem "on half-space" according to which the only possible minimal surface properly immersed in $\mathbb{R}^3$ and contained in a certain half-space $\mathbb{R}^3_+$ is the plane. However, it is well known that for any greater dimension, i.e., in $\mathbb{R}^{p+1}$, for $p \ge 3$, there exist nontrivial properly immersed hypersurfaces contained in the layer between two parallel hyperplanes (examples can be found in [1, 2]). In [3–6], it was shown that for $p \ge 3$ this property holds for any minimal surface of arbitrary codimension all of whose sections by a sheaf of parallel hyperplanes are compact sets. Furthermore, in [3, 5, 6] the width Δ of the layer was estimated in terms of the minimal radius r of the balls circumscribed about these sections:

$$\Delta \le 2c_p r, \tag{1}$$

where

$$c_p = \int_1^{+\infty} \frac{dt}{\sqrt{t^{2p}-1}}. \tag{2}$$

In a sense, the theorem below can be regarded as a reverse estimate.

Theorem. *Let $\mathcal{M}$ be a p-dimensional properly immersed connected minimal hypersurface lying in a parallel hyperlayer of width Δ. Suppose that an open ball of radius R can go through the projection of $\mathcal{M}$ on the boundary hyperplanes of the layer. Then*

$$R \le \frac{\Delta}{2c_p}, \tag{3}$$

where c_p is the constant defined by (2).

Remark. The constant in inequality (3) is unimprovable, as the examples of minimal surfaces of revolution used in the proof below will show.

This estimate can be interpreted as a restriction on holes in surfaces of zero average curvature that are "too wide." However, it is not difficult to construct examples of minimal surfaces enclosed in a layer whose projections on its boundary are unbounded.

Proof. Denote by $x\colon M \to \mathbb{R}^{p+1}$ the isometric immersion of a p-dimensional manifold M that realizes the given surface $\mathcal{M}$. Since the class of surfaces that we consider includes self-intersecting surfaces, we shall always distinguish a point $m \in M$ on the manifold from its image $x(m) \in \mathcal{M}$ on the surface.

Suppose that inequality (3) is not valid, that is,

$$k^4 \equiv \frac{2Rc_p}{\Delta} > 1.$$

Then, taking into account the fact that the minimality condition and inequality (3) are invariant under dilations and translations in the space $\mathbb{R}^{p+1}$, we can assume without loss of generality that $\mathcal{M}$ lies in the hyperlayer $|x_{p+1}| < \Delta/2$, where

$$\Delta = \frac{2c_p}{k^2} < 2c_p, \tag{4}$$

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and that the projection of $\mathcal{M}$ on the hyperplane $x_{p+1} = 0$ lets through a ball of radius $R \equiv k^2 > 1$ centered at the coordinate origin. Denote this ball by $B(R)$ and consider the special auxiliary minimal hypersurface of revolution C^+ given by the equation [3, 2]

$$x_{p+1} = \Phi_\beta\left(\sqrt{x_1^2 + \cdots + x_p^2}\right), \qquad x_{p+1} > 0, \tag{5}$$

whose boundary in the hyperplane $x_{p+1} = 0$ is the sphere $\partial B(1)$. Here and subsequently,

$$\Phi_p(t) = \int_1^t \frac{d\tau}{\sqrt{\tau^{2p} - 1}}.$$

It will be convenient to use the following natural terminology. Suppose we have two surfaces $\mathcal{M}$ and $\mathcal{N}$ immersed in $\mathbb{R}^{p+1}$. We shall say that $\mathcal{N}$ *lies strictly above* (*above*) *the surface* $\mathcal{M}$ if any two points

$$m = (x_1, \dots, x_p, x_{p+1}) \in \mathcal{M} \qquad \text{and} \qquad n = (x_1, \dots, x_p, y_{p+1}) \in \mathcal{N}$$

with the same first p coordinates satisfy the inequality $y_{p+1} > x_{p+1}$ (the nonstrict inequality $y_{p+1} \geq x_{p+1}$), respectively.

First, we show that C^+ lies strictly above $\mathcal{M}$. To this end, we assume the converse, i.e., that $\mathcal{M}$ and the interior of C^+ have a common point. Consider an auxiliary family of surfaces $C^+(\varepsilon)$ obtained from C^+ under translations by $\varepsilon \geq 0$ along the $(p+1)$st coordinate. Notice that $C^+(\varepsilon)$ is a minimal surface again and that it is disjoint from $\mathcal{M}$ for $\varepsilon > \Delta/2$.

Let

$$\varepsilon_0 = \sup\{\varepsilon \geq 0 : \mathcal{M} \cap C^+(\varepsilon) \neq \varnothing\}, \qquad \text{where} \quad C^+(0) \equiv C^+;$$

ε_0 is well defined by virtue of the above remark. If $\varepsilon_0 > 0$, then we can choose a sequence $\varepsilon_k \uparrow \varepsilon_0$; denote by m_k the common point of the surfaces $\mathcal{M}$ and $C^+(\varepsilon_k)$. Notice that

$$\sqrt{x_1^2(m_k) + \cdots + x_p^2(m_k)} = \Psi_p\left(x_{p+1}(m_k) - \varepsilon_k\right) < \Psi_p\left(\frac{\Delta}{2}\right),$$

where $\Psi_p(t)$ is the function inverse to $\Phi_p(\rho)$. It follows from assumption (4) that

$$\Psi_p(\Delta/2) < \Psi_p(c_p) = +\infty,$$

so all the points m_k lie in the bounded cylinder

$$\left\{x \in \mathbb{R}^p : \ |x_{p+1}| < \frac{\Delta}{2}, \ \sqrt{x_1^2 + \cdots + x_p^2} < \Psi_p\left(\frac{\Delta}{2}\right) < +\infty\right\}.$$

Since $\mathcal{M}$ is given by its immersion, the sequence $\{m_k\}$ has a limit point $m_0 \in M$. Clearly,

$$x(m_0) \in \mathcal{M} \cap C^+(\varepsilon_0);$$

so, by the definition of ε_0, the surface $C^+(\varepsilon_0)$ lies above $\mathcal{M}$.

Notice that when $\varepsilon_0 = 0$ or $\varepsilon_0 > 0$, the common point $x(m_0)$ of the surfaces $\mathcal{M}$ and $C^+(\varepsilon_0)$ belongs to the interior of $C^+(\varepsilon_0)$. However, the above reasoning shows that $x(m_0)$ is a point of contact of $\mathcal{M}$ and $C^+(\varepsilon_0)$.

The surface $C^+(\varepsilon_0)$ is defined nonparametrically, as a graph over the perforated hyperplane $\mathbb{R}^p \setminus \overline{B(1)}$. So the common tangent space

$$T_{m_0}\mathcal{M} \equiv T_{m_0}C^+(\varepsilon_0)$$

of both surfaces makes a nonzero angle with the x_{p+1}-axis. It follows that the point $m_0 \in M$ has a neighborhood $\mathcal{O}$ such that the corresponding part of the surface $\mathcal{M}$ is also a graph over the hyperplane

$x_{p+1} = 0$, and in this neighborhood the surface $C^+(\varepsilon_0)$ lies above $\mathcal{M}$ (except for the point m_0). Recall that the equation of minimal surfaces in explicit form is uniformly elliptic near any point m_0 with a nonzero angle $\alpha(m_0)$ between the tangent plane and the vector e_{p+1}. Therefore, we can apply the strong maximum principle [7, Lemma 3.4, p. 41 of the Russian translation] to the $(p+1)$st coordinate functions of the surfaces $C^+(\varepsilon_0)$ and $\mathcal{M}$ in the neighborhood of m_0 to conclude that in this neighborhood $\mathcal{M} \equiv C^+(\varepsilon_0)$. Hence, the set M_0 of the points m_0 for which the last identity is true must be open in M. Indeed, if m_1 is a boundary point for O and an interior point for $C^+(\varepsilon_0)$ at the same time, than the angle $\alpha(m_1)$ is nonzero and we can repeat the above argument to find the desired neighborhood. On the other hand, M_0 must be a closed set as well, because the equality condition extends to boundary points by the continuity of the immersion. Since the surfaces are connected, we conclude that $\overline{C^+(\varepsilon_0)} \in \mathcal{M}$. But this is impossible, because, by our assumption, $\mathcal{M}$ lets through a ball of a radius R strictly greater than one.

In a similar way, we can prove that $\mathcal{M}$ lies everywhere strictly higher than the surface C^- specified by the equation

$$x_{p+1} = -\Phi_p\big(\sqrt{x_1^2 + \cdots + x_p^2}\big).$$

Combining these results, we obtain a complete minimal surface of revolution $C \equiv \overline{C}^- \cup C^+$ such that $\mathcal{M} \cap C = \varnothing$. In particular, we have

$$|x_{p+1}(m)| < \Phi_p\big(\sqrt{x_1^2 + \cdots + x_p^2}\big). \tag{6}$$

Consider another family $C(t)$ of surfaces obtained from C under the dilation by $t \geq 1$:

$$C(t) \sim x_{p+1} = t \cdot \Phi_p\big(t^{-1}\sqrt{x_1^2 + \cdots + x_p^2}\big).$$

In view of (6), the number

$$t_0 = \sup\{t \geq 1 : \ C(t) \cap \mathcal{M} = \varnothing\} < +\infty$$

is well defined. Since $C(t)$ lies in a layer of a width strictly greater than Δ for $t > 1$, the method described above shows that $C(t_0)$ is tangent to $\mathcal{M}$ at a certain point m_0, while the inequality

$$t_0 \cdot \Phi_p\big(t_0^{-1}\sqrt{x_1^2 + \cdots + x_p^2}\big) \geq |x_{p+1}(m)|$$

holds everywhere on $\mathcal{M}$.

The surfaces $\mathcal{M}$ and $C(t_0)$ have a common tangent space at m_0 that makes the angle $\alpha(m_0)$ with the vector e_{p+1}. Consider two cases.

Case 1. Suppose that this angle is not equal to zero. Then we are in the situation considered above, so $C(t_0) \equiv \mathcal{M}$. But the width of the layer for $C(t_0)$ is strictly greater than Δ, and we arrive at a contradiction.

Case 2. Now suppose that $\alpha(m_0) = 0$. From the definition of $C(t_0)$, it follows that the corresponding common point

$$x(m_0) \in \mathcal{M} \cap C(t_0)$$

lies on the waist of the catenoid $C(t_0)$, that is, in the hyperplane $x_{p+1} = 0$.

Taking into account the condition [2], consider the minimal surface $\widetilde{\mathcal{M}}$ obtained from $\mathcal{M}$ under the translation along the vector e_{p+1} such that it still remains in the layer

$$|x_{p+1}| \leq \Delta_1 \leq c_p$$

of width less than $2c_p$. Then, repeating the argument from the beginning, we shall obtain a similar surface $C(t_1)$. Now from the fact that the catenoids $C(t_0)$ and $C(t_1)$ are homothetic, we can derive that $t_1 < t_0$, i.e., the waist radius of the new catenoid $\mathcal{C}(t_1)$ is strictly less than that of $\mathcal{C}(t_0)$. Therefore, the common point $\tilde{m}$ of the surfaces $\widetilde{\mathcal{M}}$ and $\mathcal{C}(t_1)$ cannot lie on the waist of $\mathcal{C}(t_0)$ and so the nondegeneracy condition $\alpha(\tilde{m}) \neq 0$ holds. Therefore, the surfaces $\widetilde{\mathcal{M}}$ and $\mathcal{C}(t_1)$ meet the assumptions of Case 1 as before. The resulting contradiction completes the proof of the theorem. □

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