Partial Differential Equations / Partiella differentialekvationer

- 1TT462 (6.0 hp)
- 1MA054 (5.0 hp)

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Goals

The course aims at developing the theory for hyperbolic, parabolic, and elliptic partial differential equations in connection with physical problems.

Contents

- Characteristics
- 1st order PDE
- Linear second order PDE: the Laplace and Poisson equations, the wave equation and the heat equation
- Sobolev spaces
- Linear elliptic equations
- Systems of conservations laws

Examinationsform (4 poäng)

Skriftligt prov med problem och teoriuppgifter vid kursens slut.

Examinationsform (6 poäng)

Ett skriftligt och i allmänhet ett muntligt prov ges vid kursens slut.

Dessutom förekommer obligatoriska inlämningsuppgifter eller ett teoriprov som redovisas i skriftlig och/eller muntlig form.

Deltagarna förväntas utföra ett projektarbete.

Course material:

- Lawrence C. Evans. Partial Differential Equations
- Robert C. McOwen, Partial Differential Equations, Methods and Applications
Plan

- Introduction, course information

- Review of ordinary differential equations (existence, uniqueness and non-uniqueness)
  - Existence of solutions
  - Exact solutions
    - separable equations
    - homogeneous equations
    - linear equations (higher orders)

- Definition of PDE
  - PDE are (much) more difficult than ODE
  - Systems of ODE ↔ 1st order PDE

- Why does one study PDE?
  - Example of derivation of PDE

- Classes of PDE
  - Variational problems (conservation laws)
  - Kinetics, gas dynamics (evolution)
  - Free-boundary problems/phase transitions
  - Solitons and exactly solvable (integrable) eqns.
  - etc...

- 1st order PDE
  - Characteristic curves for a linear equation
  - Why Cauchy problem?
Some important examples

- Inviscid Burger’s (Hopf) equation
  \[ u_t + uu_x = 0 \]

- Scalar conservation law
  \[ u_t + \text{div}\ F(u) = 0 \]

- Laplace Equation
  \[ \Delta u \equiv u''_{xx} + u''_{yy} = 0 \]

- Poisson’s equation
  \[ \Delta u \equiv u''_{xx} + u''_{yy} = f(x,y) \]

- Heat (or diffusion) equation
  \[ u_t - \Delta u = 0 \]

- Wave equation
  \[ u'''_{tt} - \Delta u = 0 \]

- Minimal surface equation
  \[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \]
Derivation of the heat equation

We consider the flow of heat along a metal insulated rod

\[ q(a,t) \]
\[ q(b,t) \]

Initial temperature distribution \( u(x, 0) = f(x) \)

Energy of an arbitrary piece of rod from \( a \) to \( b \) is

\[ E = \int_{a}^{b} A \cdot \rho \cdot c \cdot u(x, t) \, dx \]

- \( u = u(x, t) \) is temperature at time \( t \) at a given point \( x \),
- \( A \) is the cross sectional area of the rod,
- \( c \) is the specific heat capacity of the rod

The wave heat flow:

\[ R = A(q(a, t) - q(b, t)) = -A \int_{a}^{b} \frac{\partial}{\partial x} q(x, t) \, dx \]

Conservation of energy (in terms of power = time-derivative of energy):

\[ R = \frac{\partial E}{\partial t} \]

implies the integral form of the heat equation:

\[ \int_{a}^{b} \left( \rho \cdot c \cdot \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} q(x, t) \right) dx = 0. \]

By virtue of arbitrariness of \( a \) and \( b \) we get

\[ \rho \cdot c \cdot \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} q(x, t) = 0. \]

Finally, by using the Fourier law \( q(x, t) = -\lambda \frac{\partial}{\partial x} q(x, t) \) we arrive at (the differential form of) the heat equation:

\[ \frac{\partial u}{\partial t} - c \frac{\partial^2 u}{\partial x^2} = 0. \]