Sobolev inequalities and embedding theorems

The simplest Sobolev imbedding theorem is the following (trivial) inclusion

\[ H_0^{1,p}(U) \hookrightarrow L^p(U) \tag{1} \]

which follows immediately from general Poincare-Friedrichs inequality

\[ \|v\|_{1,p} \leq C_p \|\nabla v\|_p \]

It turns out that even this information can be made more precise if one takes into account the dimension of the ambient space. There two distinguished cases: \( p < n \) and \( p > n \). The case \( p = n \) is also called critical.

We start with the sub-critical case:

**Theorem 1** (Sobolev inequality: \( p < n \)) Let \( U \) be a bounded domain in \( \mathbb{R}^n \). Then

\[ \|v\|_{p^*} \leq C_p \|\nabla v\|_p , \tag{2} \]

Here

\[ p^* = \frac{pn}{n-p} \]

is the so-called critical Sobolev’s exponent and \( C_p \) depends only on \( p \) and \( n \).

- The crucial step is to prove the Sobolev inequality for

**The first case** \( p = 1 \).

Notice that it suffices only to prove (2) for test functions, that is \( v \in C_0^\infty(U) \). We extend any given \( v \in C_0^\infty(U) \) by zero to the whole \( \mathbb{R}^n \) and shall denote for any index \( 1 \leq i \leq n \) and a point \( y \in \mathbb{R}^n \)

\[ L_i(y) = \{ x \in \mathbb{R}^n | x_i = y_i \} \]

\[ L_{ij}(y) = \{ x \in \mathbb{R}^n | x_i = y_i, x_j = y_j \} \]

etc.
Since \( v \in C_0^\infty(U) \) we have for any index \( i \):

\[
|v(y)|^{1/(n-1)} = \left| \int_{-\infty}^{y_i} v_i'(y_1, \ldots, x_i, \ldots, y_n) \, dx_i \right|^{1/(n-1)} \leq \left( \int_{L_i(y)} |\nabla v| \right)^{1/(n-1)} \equiv h_i^{\frac{n}{n-1}}
\]

Here and in what follows in the proof we use the shorthand

\[
L_i(y) = L_i, \quad h_i = \int_{L_i} |\nabla v|, \quad h_{ij} = \int_{L_i} h_j \, dx_i \equiv \int_{L_i \cap L_j} |\nabla v|, \quad \text{etc.}
\]

(the latter integrals should be understood as line, surface integrals with respect to the corresponding measure). Notice also that \( h_i \) does not depend on \( y_i \), \( h_{ij} \) does not depend on \( y_i, y_j \) and so on.

And after multiplication over all \( i = 1, 2, \ldots, n \):

\[
|v(y)|^{n/(n-1)} \leq h_1^{\frac{n}{n-1}} \cdot h_2^{\frac{n}{n-1}} \cdot \cdots \cdot h_n^{\frac{n}{n-1}}
\]  

(3)

Thus, integrating (3) over \( L_1(y) \) and applying the Hölder inequality gives

\[
\int_{L_1} |v|^{\frac{n}{n-1}} \leq h_1^{\frac{n}{n-1}} \cdot \prod_{i=2}^{n} h_i^{\frac{n}{n-1}} \leq h_1^{\frac{n}{n-1}} \cdot \prod_{i=2}^{n} \left( \int_{L_1} h_i \, dx_1 \right)^{\frac{n}{n-1}} = h_1^{\frac{n}{n-1}} \cdot \prod_{i=2}^{n} h_i^{\frac{n}{n-1}}
\]

Writing the last product as

\[
h_1^{\frac{n}{n-1}} \cdot \prod_{i=2}^{n} h_i^{\frac{n}{n-1}} = h_{12}^{\frac{n}{n-1}} \cdot h_1 \cdot \prod_{i=3}^{n} h_i^{\frac{n}{n-1}}
\]

and integrating over \( L_2(y) \) (recall that \( h_{12} \) does not depend on \( y_1 \) and \( y_2 \)) with application the Hölder inequality yields

\[
\int_{L_1 L_2} |v|^{\frac{n}{n-1}} \equiv \int_{L_2} \int_{L_1} |v|^{\frac{n}{n-1}} \leq h_{12}^{\frac{n}{n-1}} \cdot \int_{L_2} \left( h_1^{\frac{n}{n-1}} \cdot \prod_{i=3}^{n} h_i^{\frac{n}{n-1}} \right) \leq h_{12}^{\frac{n}{n-1}} \cdot \prod_{i=3}^{n} h_i^{\frac{n}{n-1}}
\]

Applying this argument, we obtain easily by induction that for any \( k \leq n - 1 \)

\[
\int_{L_{12 \ldots k}} |v|^{\frac{n}{n-1}} \leq h_{12 \ldots k}^{\frac{k}{n-1}} \cdot \prod_{i=k+1}^{n} h_{12 \ldots ki}^{\frac{n}{n-1}}
\]

Hence for \( k = n - 1 \) we have

\[
\int_{L_{12 \ldots n-1}} |v|^{\frac{n}{n-1}} \leq h_{12 \ldots n-1}^{\frac{1}{n-1}} \cdot h_{12 \ldots n-1, n}^{\frac{n}{n-1}}
\]

Integrating this inequality over \( L_n \), and taking into account that \( h_{12 \ldots n-1, n}^{\frac{n}{n-1}} \) does not depend on \( y_n \) and that \( \mathbb{R}^n = L_{12 \ldots n} \), we find

\[
\int_{L_1} |v|^{\frac{n}{n-1}} \leq h_1^{\frac{n}{n-1}} \cdot \prod_{i=2}^{n} h_i^{\frac{n}{n-1}} \cdot h_{12 \ldots n-1, n}^{\frac{n}{n-1}}
\]
\[
\int_{\mathbb{R}^n} |v|^\frac{n}{n-1} \equiv \int_{L_{12}^{n-1}} |v|^\frac{n}{n-1} \leq h_{12}^{n-1,n-1} \int_{L_n} h_{12} \equiv h_{12}^{n-1,n-1} \equiv \left( \int_{\mathbb{R}^n} |\nabla v|^p \right)^{\frac{n}{n-p}}
\]

So we have proved the Sobolev inequality for \( p = 1 \).

**The second case: \( 1 < p < n \).**

Now, let us denote by \( w = |v|^s \), where \( s > 0 \) and \( v \in C_0^\infty(\mathbb{R}^n) \). It is easy to see that \( w \in C_0^\infty(\mathbb{R}^n) \). Hence applying the Sobolev inequality for \( p = 1 \) to \( w \) and then the Hölder inequality, we get

\[
\left( \int_{\mathbb{R}^n} |\nabla w|^\frac{p}{p-1} \right)^{\frac{p-1}{p}} \leq \left( \int_{\mathbb{R}^n} |\nabla v|^p \right)^{\frac{1}{p}}
\]

(by Hölder's inequality)

\[
\leq s \left( \int_{\mathbb{R}^n} |v|^\frac{(s-1)p}{p-1} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^n} |\nabla v|^p \right)^{\frac{1}{p}}
\]

Let us choose \( s \) so that

\[
\frac{sn}{n-1} = \frac{(s-1)p}{p-1},
\]

that is \( s = \frac{(n-1)p}{n-p} \). Then we find for this value of \( s \):

\[
\left( \int_{\mathbb{R}^n} |v|^\frac{pn}{n-p} \right)^{\frac{1}{p}} \leq s \left( \int_{\mathbb{R}^n} |\nabla v|^p \right)^{\frac{1}{p}}
\]

which is equivalent to the required inequality

\[
\|v\|_{\frac{pn}{n-p}} \leq s \|\nabla v\|_p
\]

The theorem is proved. ■

**Corollary 1** (Sobolev embedding theorem for \( p < n \)). Let \( U \) be a bounded domain in \( \mathbb{R}^n \). Then for \( p < n \)

\[
H^{1,p}_0(U) \hookrightarrow L^q(U), \quad \text{if} \ 1 \leq p \leq q \leq p^* \equiv \frac{np}{n-p}.
\]

and the embedding continuous in the sense that the following inequality true:

\[
\|v\|_q \leq C_p \|v\|_{1,p}, \quad p \leq q < p^*.
\]
Proof: apply the Hölder inequality.

In other words, taking into account inequality \( p^* \equiv \frac{p}{1 - \frac{p}{n}} > p \), we have the following diagram (recall that \( U \) is a bounded domain):

\[
\cdots \subset C^0(\overline{U}) \subset L^\infty(U) \subset H_0^{1,p}(U) \subset L^p(U) \subset L^q(U) \subset L^q(U) \subset L^1(U) \quad (q \geq p).
\]

\[
p^* - p = \frac{p^2}{n-p} \sim \frac{p^2}{n}.
\]

Finally we prove the super-critical case of the Sobolev inequality.

**Theorem 2** (Sobolev inequality: \( p > n \)) Let \( U \) be a bounded domain in \( \mathbb{R}^n \). Then

\[
H_0^{1,p}(U) \subset C^{0,1-\frac{n}{p}}(\overline{U}), \quad p > n.
\]

Moreover, the embedding \( i: H_0^{1,p}(U) \hookrightarrow C^{0,1-\frac{n}{p}}(U) \) is continuous in the sense that the following inequality true:

\[
\frac{|v(x) - v(y)|}{|x - y|^{1-\frac{n}{p}}} \leq C(U, n, p) \|\nabla v\|_p
\]

The latter is called Morrey's inequality.

We consider \( v \in C_0^\infty(U) \) and extend it by zero outside \( U \) so that \( v \in C_0^\infty(\mathbb{R}^n) \). For any fixed pair of points \( x, y \in U \) we denote by \( B \) the ball centered at \( \frac{x+y}{2} \) of radius \( R = \frac{|x-y|}{2} \).

The points of segment \([x, z]\) can be parameterized by: \( x + t(z - x) \), when \( t \in [0,1] \). We have

\[
v(z) - v(x) = \int_0^1 \frac{d}{dt} v(x + t(z - x)) dt \leq \int_0^1 |\nabla v(x + t(z - x))| \cdot |z - x| dt
\]
Integrating the obtained inequality over all points $z \in B$ and dividing by the measure of the ball $|B| = \Omega_n R^n$ (here $\Omega_n$ stands for the $n$-dimensional volume of the $n$-dimensional unit ball) gives

$$
\frac{1}{\Omega_n R^n} \int_B v(z) \, dz - v(x) \leq \frac{1}{\Omega_n R^n} \int_B |z - x| \, dz \int_0^1 |\nabla v(x + t(z - x))| \cdot dt
$$

We have also $|z - x| \leq 2R$ for any $z$ in the ball $B$. Hence, passing to the absolute values and applying Fubini’s theorem we find

$$
\frac{1}{\Omega_n R^n} \int_B |v(z) - v(x)| \, dz \leq \frac{2}{\Omega_n R^{n-1}} \int_0^1 dt \int_B |\nabla v(x + t(z - x))| \, dz \quad (5)
$$

Applying the (linear) change of variables $\xi(z) = x + t(z - x)$ with Jacobian $\frac{dz}{d\xi} = t^{-n}$ we obtain for the inner integral:

$$
\int_B |\nabla v(x + t(z - x))| \, dz = \int_{B'} |\nabla v(\xi)| \frac{dz}{d\xi} \, d\xi = \frac{1}{t^n} \int_{B'} |\nabla v(\xi)| \, d\xi \leq
$$

by the Hölder inequality

$$
\leq \frac{1}{t^n} \left( \int_{B'} |\nabla v(\xi)|^p \, d\xi \right)^{\frac{1}{p}} \left( \int_{B'} 1 \, d\xi \right)^{\frac{p-1}{p}} \leq \frac{1}{t^n} \|\nabla v\|_p \cdot (\Omega_n t^n R^n)^{\frac{p-1}{p}}
$$

Here we used the fact that $B' = \xi(B)$ is a ball of radius $tR$. The substitution of the found relations into (5) implies

$$
\left| \frac{1}{\Omega_n R^n} \int_B v(z) \, dz - v(x) \right| \leq C_1 R^{1-n} \|\nabla v\|_p \int_0^1 t^{-\frac{n}{p}} \, dt
$$

Notice that for $p > n$ the integral $\int_0^1 t^{-\frac{n}{p}} \, dt$ converges, so that we find (recalling that $R = \frac{1}{2} |x - y|$)

$$
|a - v(x)| \leq C_2 |x - y|^{1-n} \|\nabla v\|_p
$$

and changing the roles $x \leftrightarrow y$:

$$
|a - v(y)| \leq C_2 |x - y|^{1-n} \|\nabla v\|_p
$$

where $a = \frac{1}{\Omega_n R^n} \int_B v(z) \, dz$. Applying the triangle inequality to the last two inequalities we arrive at

$$
|v(x) - v(y)| \leq |a - v(x)| + |a - v(y)| \leq C_3 |x - y|^{1-n} \|\nabla v\|_p
$$

The theorem is proved. ■