Some results on interval edge colorings of (α, β)-biregular bipartite graphs

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Abstract. A bipartite graph G is called (α, β)-biregular if all vertices in one part of G have the degree α and all vertices in the other part have the degree β. An edge coloring of a graph G with colors $1, 2, 3, \dots, t$ is called an interval t-coloring if the colors received by the edges incident with each vertex of G are distinct and form an interval of integers and at least one edge of G is colored i, for $i = 1, \dots, t$. We show that the problem to determine whether an (α, β)-biregular bipartite graph G has an interval t-coloring is $\mathcal{NP}$-complete in the case when $\alpha > \beta \geq 3$ and β is a divisor of α. It is known that if an (α, β)-biregular bipartite graph G on m vertices has an interval t-coloring then $\alpha + \beta - \gcd(\alpha, \beta) \leq t \leq m - 1$, where $\gcd(\alpha, \beta)$ is the greatest common divisor of α and β. We prove that if an (α, β)-biregular bipartite graph has $m \geq 2(\alpha + \beta)$ vertices then the upper bound can be improved to $m - 3$. We also show that this bound is tight by constructing, for every integer $n \geq 1$, a connected (α, β)-biregular bipartite graph G which has $m = n(\alpha + \beta)$ vertices and admits an interval t-coloring for every t satisfying $\alpha + \beta - \gcd(\alpha, \beta) \leq t \leq m - 3$.

1 Introduction

We use [5] and [3] for terminology and notation not defined here and consider simple finite graphs only. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively.

An edge coloring of a graph G with colors $1, 2, 3, \dots$ is called a *proper coloring* if no two edges incident with the same vertex of G receive the same color. A proper coloring is called an *interval* (or *consecutive*) *coloring* if the colors received by the edges incident with each vertex of G form an interval of integers. An interval coloring of G with colors $1, 2, \ldots, t$ is called an *interval t-coloring* if at least one edge is colored i, for $i = 1, \ldots, t$.

The notion of interval colorings was introduced in 1987 by Asratian and Kamalian [2][1]. Later the theory of interval (consecutive) colorings was developed in e.g. [4-6, 8-18]. Generally, it is an $\mathcal{NP}$-complete problem to determine whether a bipartite graph has an interval coloring [17]. However, some classes of bipartite graphs have been proved to admit interval colorings. It is known, for example, that trees, regular and complete bipartite graphs (see [1], [11]), doubly convex bipartite graphs [14], grids [9] and outerplanar bipartite graphs [8] have interval colorings.

Some results were obtained for (α, β)-*biregular* bipartite graphs. A simple bipartite graph with bipartition (X, Y) is called (α, β)-*biregular* if every vertex in X has degree α and every vertex in Y has degree β. Hansen proved in [11] that every $(\alpha, 2)$-biregular bipartite graph admits an interval coloring if α is an even integer. A similar result for $(\alpha, 2)$-biregular bipartite graphs for odd α was given by Hanson, Loten and Toft [12] and independently by Kostochka [15]. Very little is known for (α, β)-biregular bipartite graphs where $\alpha > \beta \geq 3$. Toft conjectured [18] that every (α, β)-biregular bipartite graph always has an interval coloring. Kamalian [14] showed that the complete bipartite graph $K_{\alpha,\beta}$ has an interval t-coloring if and only if $\alpha + \beta - \gcd(\alpha, \beta) \leq t \leq \alpha + \beta - 1$, where $\gcd(\alpha, \beta)$ is the greatest common divisor of α and β. Hanson and Loten [13] showed that if an (α, β)-biregular bipartite graph has an interval t-coloring, then $t \geq \alpha + \beta - \gcd(\alpha, \beta)$. Two different sufficient conditions for the existence of an interval 6-coloring of a $(4, 3)$-biregular bipartite graph were found by Pyatkin [16] and Casselgren [6].

In this paper we show that the problem to determine whether a $(6, 3)$-biregular bipartite graph G has an interval 6-coloring is $\mathcal{NP}$-complete. We deduce from this result that the problem to determine whether an (α, β)-biregular bipartite graph G has an interval t-coloring is $\mathcal{NP}$-complete for any pair of integers α, β such that $\alpha > \beta \geq 3$ and β is a divisor of α. We also prove that if an (α, β)-biregular bipartite graph on $m \geq 2(\alpha + \beta)$ vertices admits an interval coloring then the number of used colors is at most $m - 3$. This bound is better than the bound $m - 1$ obtained in [1] for an arbitrary bipartite graph on m vertices. We show that the bound $m - 3$ is tight and construct, for every integer $n \geq 1$, a connected (α, β)-biregular bipartite graph with $m = n(\alpha + \beta)$ vertices which admits an interval t-coloring for every t satisfying $\alpha + \beta - \gcd(\alpha, \beta) \leq t \leq m - 3$.

[1] A revised version of that paper in English was published in [1].

2 Auxiliary results

Lemma 2.1. *Let G be an (α,β)-biregular bipartite graph with bipartition (X,Y). We denote by d the greatest common divisor of α and β. Then $|X| = k\frac{\beta}{d}$ and $|Y| = k\frac{\alpha}{d}$, for some $k \in \mathbf{N}$.*

Proof. Dividing the equality $d_G(X) = d_G(Y)$ by d we obtain

$$\frac{\alpha}{d}|X| = \frac{\beta}{d}|Y|. \tag{1}$$

The greatest common divisor of $\frac{\alpha}{d}$ and $\frac{\beta}{d}$ is 1. This implies that $\frac{\beta}{d}$ divides $|X|$ and thus $|X| = k\frac{\beta}{d}$, for some $k \in \mathbf{N}$. Combining this with (1) gives us $|Y| = k\frac{\alpha}{d}$, which completes the proof of the lemma. □

Lemma 2.2. *The complete bipartite graph $K_{d,d}$ has an interval t-coloring, for every integer t satisfying $d \leq t \leq 2d-1$.*

Proof. Let (X,Y) be the bipartition of $K_{d,d}$ where $X = \{x_1,\dots,x_d\}$, $Y = \{y_1,\dots,y_d\}$. The graph $K_{d,d}$ has an interval $(2d-1)$-coloring where the edge x_iy_j has color $i+j-1$, for each $1 \leq i,j \leq d$. Furthermore, if $K_{d,d}$ has an interval $(t+1)$-coloring, $d < t < 2d-1$, then it has also an interval t-coloring which can be obtained by recoloring all edges of color $t+1$ in color $t+1-d$. Thus $K_{d,d}$ has an interval t-coloring, for every integer t, $d \leq t \leq 2d-1$. □

The following result was obtained by Kamalian [14].

Lemma 2.3. *The complete bipartite graph $K_{\alpha,\beta}$ has an interval t-coloring, for every integer t satisfying $\alpha+\beta-\gcd(\alpha,\beta) \leq t \leq \alpha+\beta-1$.*

Proof. Let (X,Y) be the bipartition of $K_{\alpha,\beta}$ where $X = \{x_1,\dots,x_\alpha\}$ and $Y = \{y_1,\dots,y_\beta\}$. Furthermore, let $t = \alpha+\beta-d+k$ where $d = \gcd(\alpha,\beta)$, $0 \leq k < d$. Let G_1 be the subgraph induced by the vertices $x_1,x_2,\dots,x_d$ and $y_1,y_2,\dots,y_d$. Clearly, $G_1 = K_{d,d}$. By Lemma 2.2 G_1 has an interval $(d+k)$-coloring f. Now using the coloring f we define an interval t-coloring of the graph $K_{\alpha,\beta}$: an edge $x_{i+sd}y_{j+rd}$ we color with the color $(s+r)d + f(x_iy_j)$, where $1 \leq i,j \leq d$, $0 \leq s \leq \frac{\alpha}{d}-1$, $0 \leq r \leq \frac{\beta}{d}-1$. One can verify that the resulting coloring is an interval t-coloring of $K_{\alpha,\beta}$. □

Lemma 2.4. *A $(6,3)$-biregular bipartite graph H has an interval 6-coloring if and only if H has a cubic subgraph covering all vertices of degree 6 in H.*

Proof. Suppose that H has a cubic subgraph F which covers all vertices of degree 6 in H. Then the subgraph F_1 induced by the set $E(H) \setminus E(F)$ is a cubic subgraph of H covering all vertices of degree 6 in H as well. Let f_1 be a proper coloring of F_1 using colors $1, 2, 3$ and let f_2 be a proper coloring of F using colors $4, 5, 6$. The edge colorings f_1 and f_2 clearly constitute an interval 6-coloring of H.

Conversely, suppose that H has an interval 6-coloring f. Put $E' = \{e \in E(H) : f(e) = 1, 2 \text{ or } 3\}$. The subgraph induced by the set E' is easily verified to be a cubic subgraph of H covering all vertices of degree 6. □

3 Complexity results

In this section we will show that the problem of determining whether an (α, β)-biregular bipartite graph has an interval t-coloring is $\mathcal{NP}$-complete when $\alpha > \beta \geq 3$ and β is a divisor of α. Pyatkin [16] proved that the following problem is $\mathcal{NP}$-complete.

Problem 1. *Cubic subgraph of a* $(4, 3)$*-biregular bipartite graph.*
Instance: A $(4, 3)$-biregular bipartite graph G with bipartition (X, Y).
Question: Does there exist a cubic subgraph F of G covering X, that is, satisfying $X \subseteq V(F)$?

We will prove that the next problem is also $\mathcal{NP}$- complete.

Problem 2. *Interval* 6*-coloring of a* $(6, 3)$*-biregular bipartite graph.*
Instance: A $(6, 3)$-biregular bipartite graph H.
Question: Does there exist an interval 6-coloring of H?

Theorem 3.1. *Problem 2 is* $\mathcal{NP}$*-complete.*

Proof. We will reduce Problem 1 to Problem 2. For each $(4, 3)$-biregular bipartite graph G we will construct a $(6, 3)$-biregular bipartite graph H. Let (X, Y) be the bipartition of G. By Lemma 2.1 we have $|X| = 3k$ and $|Y| = 4k$ for some $k \in \mathbf{N}$. Let $X = \{v_1, v_2, \dots, v_{3k}\}$. We will construct a new graph $H = H(G, k)$ in the following way:

First we define the graph H_1 by setting

$$V(H_1) = V(G) \cup \{u_1, u_2, \dots, u_{3k}\}, \text{ where } V(G) \cap \{u_1, u_2, \dots, u_{3k}\} = \emptyset \text{ and}$$
$$E(H_1) = E(G) \cup \bigcup_{i=1}^{k} \{u_{3i-2}v_{3i-2}, u_{3i-2}v_{3i-1}, u_{3i-1}v_{3i-2}, u_{3i-1}v_{3i}, u_{3i}v_{3i-1}, u_{3i}v_{3i}\}.$$

Then we put $H_2 = \bigcup_{i=1}^{3k} K_{3,5}^{(i)} \cup H_1$ where $K_{3,5}^{(1)}, K_{3,5}^{(2)}, \dots, K_{3,5}^{(3k)}$ are disjoint copies of the complete bipartite graph $K_{3,5}$ and $K_{3,5}^{(i)}$ has bipartition (A_i, B_i), where $A_i = \{a_1^{(i)}, \dots, a_5^{(i)}\}$, $B_i = \{x_i, y_i, z_i\}$ and $V(H_1) \cap (A_i \cup B_i) = \emptyset$.

Now we define $H = H(G, k)$ by setting

$$V(H) = V(H_2) \cup \{w_1, w_2, \dots, w_{2k}\} \text{ and}$$
$$E(H) = E(H_2) \cup \bigcup_{i=1}^{3k} \{x_i u_i\} \cup \bigcup_{i=1}^{k} M_i,$$

where $V(H_2) \cap \{w_1, w_2, \dots, w_{2k}\} = \emptyset$,

$M_1 = \{w_1 y_1, w_1 z_1, w_1 y_2, w_2 z_2, w_2 y_3, w_2 y_4\}$,
$M_i = \{w_{2i-1} z_{3(i-1)}, w_{2i-1} z_{3i-2}, w_{2i-1} y_{3i-1}, w_{2i} z_{3i-1}, w_{2i} y_{3i}, w_{2i} y_{3i+1}\}$,
for $2 \le i \le k-1$, and
$M_k = \{w_{2k-1} z_{3(k-1)}, w_{2k-1} z_{3k-2}, w_{2k-1} y_{3k-1}, w_{2k} z_{3k-1}, w_{2k} y_{3k}, w_{2k} z_{3k}\}$.

The graph $H = H(G, k)$ is easily verified to be a $(6, 3)$-biregular bipartite graph. (For $H = H(G, 4)$ see Figure 1). We will show that G has a cubic subgraph covering X if and only if the graph $H(G, k)$ has an interval 6-coloring.

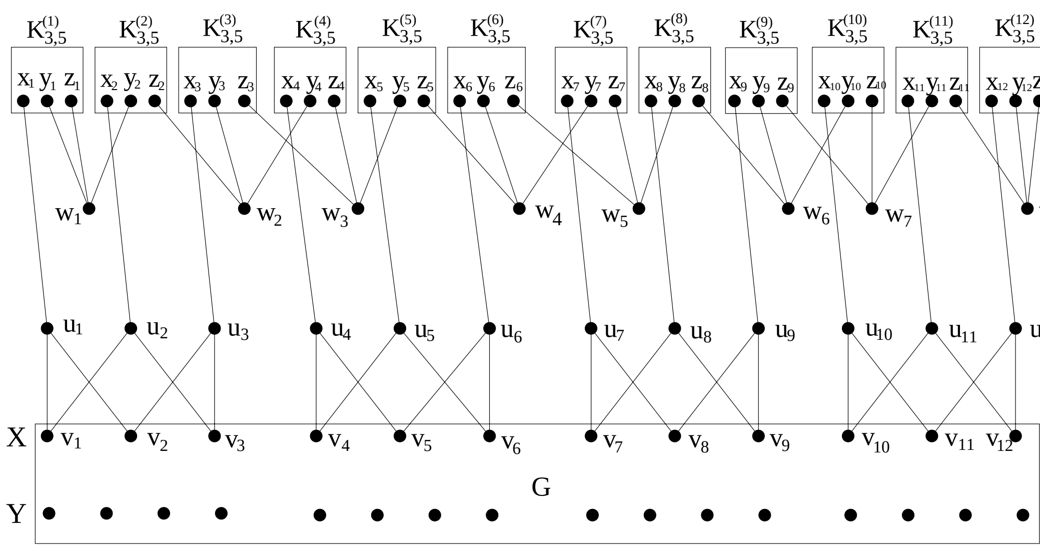

Figure 1: The graph $H(G, 4)$.

Suppose that G has a cubic subgraph F covering X. Consider the graph F' induced by the vertices $\bigcup_{i=1}^{3k} \{a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, x_i, y_i, z_i\}$. This obviously is a cubic subgraph of $H - V(G)$ and $F \cup F'$ clearly constitutes a cubic subgraph of H which covers all vertices of H of degree 6. Then, by Lemma 2.4, $H = H(G, k)$ has an interval 6-coloring.

Conversely, suppose that $H = H(G, k)$ has an interval 6-coloring. Then, by Lemma 2.4, H has a cubic subgraph F covering all vertices of degree 6. We will show that G has a cubic subgraph covering all vertices of degree 4.

First we will prove that if at least one vertex from $U = \{u_1, u_2, \dots, u_{3k}\}$ belongs to $V(F)$, then $U \subseteq V(F)$. Let $W = \{w_1, w_2, \dots, w_{2k}\}$. We need the following properties.

Property 1. *If a vertex $u_i \in U$ belongs to $V(F)$, then at least one vertex from W belongs to $V(F)$.*

Proof. We have that u_i is adjacent to x_i and $u_i \in V(F)$. Then exactly two of the vertices $a_1^{(i)}, \dots, a_5^{(i)}$ belong to $V(F)$. This implies that y_i must be adjacent to a vertex in $W \cap V(F)$, since in F the vertex y_i must have three neighbors. □

Property 2. *The following holds for the vertices $w_1, \dots, w_{2k}$:*

a) If $w_1 \in V(F)$ then $w_2 \in V(F)$.

b) If $w_j \in V(F)$, $1 < j < 2k$, then $w_{j-1}, w_{j+1} \in V(F)$.

c) If $v_{2k} \in V(F)$ then $w_{2k-1} \in V(F)$.

Proof. Let $w_1 \in V(F)$. Then w_1 is adjacent to y_2 and therefore exactly two vertices from A_2 belong to $V(F)$. Since z_2 must have three neighbors in F, the vertex w_2 also must belong to $V(F)$.

Suppose that $w_j \in V(F)$, where $1 < j < 2k$. According to the construction of H, w_j is adjacent to a vertex z_l of some triple B_l and w_{j-1} is adjacent to y_l. Since $w_j \in V(F)$, exactly two vertices from A_l can belong to $V(F)$. This implies that $w_{j-1} \in V(F)$. Furthermore, w_j is also adjacent to a vertex y_m of some triple B_m $(m > l)$ and w_{j+1} is adjacent to z_m. As before, we must have $w_{j+1} \in V(F)$.

Now suppose $w_{2k} \in V(F)$. We have that w_{2k} is adjacent to z_{3k-1} and that w_{2k-1} and y_{3k-1} are adjacent vertices. Arguing in the same way as above we may conclude that $w_{2k-1} \in V(F)$. □

The next property follows immediately from Property 2:

Property 3. *If at least one vertex from the set W belongs to $V(F)$ then $W \subseteq V(F)$.*

Property 4. *If at least one vertex from U belongs to $V(F)$, then $U \subseteq V(F)$.*

Proof. Suppose that $U \cap V(F) \neq \emptyset$ and that there is a vertex $u_r \in U$ which does not belong to $V(F)$. Consider the triple $\{x_r, y_r, z_r\}$. By Properties 1 and 3, $W \subseteq V(F)$. Hence, each of the vertices y_r and z_r is adjacent to exactly one vertex in $W \cap V(F)$. This means that exactly two vertices from A_r belong to the cubic subgraph F, which implies $d_F(x_r) = 2$, a contradiction. □

The remaining part of the proof will break into two cases.

Case 1. $U \cap V(F) \neq \emptyset$.
By Property 4, $U \subseteq V(F)$. Every vertex in $X = \{v_1, v_2, \ldots, v_{3k}\}$ is adjacent to two vertices in U. Therefore if H has a cubic subgraph F, then there must be a subset $Y' \subseteq Y$ of k vertices, such that the subgraph induced by the set $Y' \cup X$ is a $(1, 3)$-biregular subgraph in which every vertex in X has the degree 1. It is easy to see that $G - Y'$ is a cubic subgraph of G which covers all vertices of degree 4.

Case 2. $U \cap V(F) = \emptyset$.
Since the sets U and $V(F)$ are disjoint, all vertices of F which are adjacent to vertices in X must be in Y. This obviously implies that G has a cubic subgraph covering all vertices of degree 4.

We have polynomially reduced Problem 1 to Problem 2 by proving that the graph $H = H(G, k)$ has an interval 6-coloring if and only if the graph G has a cubic subgraph covering all vertices of degree 4. Since Problem 1 is $\mathcal{NP}$-complete, Problem 2 is also $\mathcal{NP}$-complete. □

Remark. The method suggested in the proof of Theorem 3.1 can be used for constructing $(6, 3)$-biregular bipartite graphs which do not admit interval 6-colorings. To see this, let G be a $(4, 3)$-biregular bipartite graph which does not have a cubic subgraph covering all vertices of degree 4 (take, for example, the graph G in Figure 2). Then the graph $H(G, k)$ constructed in the proof of Theorem 3.1 does not have an interval 6-coloring.

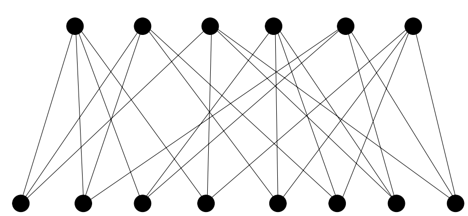

Figure 2: The graph G.

Now we will prove the $\mathcal{NP}$-completeness of the following problem:

Problem 3. *Interval t-coloring of an (α, β)-biregular bipartite graph, such that β is a divisor of α.*
Instance: An integer t and an (α, β)-biregular bipartite graph $H^{'}$ where $\alpha > \beta \geq 3$ and β is a divisor of α.
Question: Does there exist an interval t-coloring of $H^{'}$?

Theorem 3.2. *Problem 3 is $\mathcal{NP}$-complete.*

Proof. We can polynomially reduce Problem 2 to Problem 3 by setting $H' = H$, $t = 6$, $\alpha = 6$ and $\beta = 3$. Since Problem 2 is $\mathcal{NP}$-complete, Problem 3 is also $\mathcal{NP}$- complete. □

4 The number of colors in an interval coloring of an (α, β)-biregular bipartite graph

In [1, 2] Asratian and Kamalian showed that if a simple bipartite graph G has an interval t-coloring, then $t \leq |V(G)| - 1$. We improve that result for (α, β)-biregular bipartite graphs with more than $2(\alpha + \beta)$ vertices.

Theorem 4.1. *If a connected (α, β)-biregular bipartite graph G with $|V(G)| \geq 2(\alpha + \beta)$ has an interval t-coloring, then $t \leq |V(G)| - 3$.*

Proof. Suppose that the theorem is false. Let G be a connected (α, β)-biregular bipartite graph with bipartition (X, Y) and minimum number of vertices which has an interval t-coloring f, where $t > |V(G)| - 3$. By the proof of Asratian and Kamalian in [1] there is a path in G between an edge colored t and an edge colored 1 with labels decreasing along the path. We denote by θ the set of all such paths of minimum length. Let n be the length of paths in θ. Define the subsets $\theta_1, \ldots, \theta_n$ in the following way: $\theta_1 = \theta$ and θ_i is the subset of paths from θ_{i-1} with the greatest color of the ith edge, $i = 2, \ldots, n$. We choose some path $P = v_0 e_1 v_1 \ldots e_n v_n$ from θ_n.

Let us show that $n \geq 5$. Suppose that $n = 4$. According to the conditions $f(e_1) = t$. Suppose that $v_1 \in X$, then $f(e_2) \geq t - \alpha + 1$ and thus $f(e_3) \geq t - \alpha + 1 - \beta + 1$. By continuing in the same way we arrive at $f(e_4) \geq t - 2\alpha + 2 - \beta + 1$. According to our assumption $f(e_4) = 1$ and therefore $2\alpha + \beta - 2 \geq t$. This contradicts our assumption $t > |V(G)| - 3 \geq 2(\alpha + \beta) - 3$. Now suppose that $v_1 \in Y$. Proceeding in the same way we did before we have $\alpha + 2\beta - 2 \geq t$ and $t \geq 2(\alpha + \beta) - 3$, a contradiction in this case as well. With a similar argument, it is easy to prove that $n \neq 2, 3$. Hence, $n \geq 5$.

Let $A(i) = \{y \in N_G(v_i) : f(e_{i+1}) < f(v_i y) < f(e_i)\}$, for each $i = 1, \ldots, n-1$. We evidently have $|A(i)| = f(e_i) - f(e_{i+1}) - 1$, $i = 1, \ldots, n-1$.

Let us show that $A(i) \cap \{v_0, \ldots, v_n\} = \emptyset$, for each $i = 1, \ldots, n-1$. Assume that there exist i_0, j_0, for which $v_{i_0} \in A(j_0)$ or $v_{j_0} \in A(i_0)$. We will define a path P' in the following way: If $i_0 \neq 0$, $j_0 \neq n$, then $P' = v_0 e_1 v_1 \ldots v_{i_0} e' v_{j_0} \ldots v_n$, where $e' = v_{i_0} v_{j_0}$. If $i_0 = 0$, then $P' = v_1 e_1 v_0 e' v_{j_0} \ldots v_n$, where $e' = v_0 v_{j_0}$. If $j_0 = n$, then $P' = v_0 e_1 v_1 \ldots v_{i_0} e' v_n e_n v_{n-1}$, where $e' = v_{i_0} v_n$. In all cases the labels along P' are decreasing and P' is shorter than P which contradicts the choice of P. We now show that $A(i) \cap A(j) = \emptyset$, if $1 \leq i < j \leq n-1$. Suppose that there exist i_0, j_0, $1 \leq i_0 < j_0 \leq n-1$ for which $A(i_0) \cap A(j_0) \neq \emptyset$. Since G has no cycle of odd length, $j_0 - i_0 \geq 2$. Let $v \in A(i_0) \cap A(j_0)$. Consider a new path $P'' = v_0 e_1 v_1 \ldots v_{i_0} e' v e'' v_{j_0} \ldots v_{n-1} e_n v_n$, where e' is the edge joining v_{i_0} and v and e'' is the edge joining v and v_{j_0}. The colors along P'' are decreasing. If $j_0 - i_0 \geq 3$ then P'' is shorter than P and if $j_0 - i_0 = 2$, then $f(e') > f(e_{1+i_0})$, so in both cases we have a contradiction to the choice of P.

Let $A(0) = \{v \in N_G(v_0) : v \notin V(P)\}$. We will show that there is at least one vertex $u \in A(0)$, such that $u \notin A(i)$, $i = 1, \ldots, n-1$. Evidently, $A(0) \cap A(i) = \emptyset$ if $i \geq 4$, because otherwise we could form a shorter path between an edge of G colored t and an edge of G colored 1 with labels decreasing along the path, contradicting the choice of P. Furthermore, $A(0) \cap A(3) = A(0) \cap A(1) = \emptyset$, because G is bipartite. Since $|A(0)| \geq |A(2)| + 1$, we have $A(0) \setminus A(2) \neq \emptyset$. Thus, there is a vertex $u \in A(0) \setminus \bigcup_{i=1}^{n-1} A(i)$.

Now, consider the set $A(n) = \{v \in N_G(v_n) : v \notin V(P)\}$. We will show that there is at least one vertex $v \in A(n)$ such that $v \notin A(i)$, $i = 0, \ldots, n-1$. If $A(n) \cap A(i) \neq \emptyset$, when $i \leq n-4$, we would be able to form a shorter path between an edge of G colored t and an edge of G colored 1 with labels decreasing along the path, contradicting the choice of P. Hence, if $i \leq n-4$, then $A(n) \cap A(i) = \emptyset$. Furthermore, since G is bipartite, we have $A(n) \cap A(n-1) = A(n) \cap A(n-3) = \emptyset$. Because $|A(n)| \geq |A(n-2)| + 1$, we can conclude that $A(n) \setminus A(n-2) \neq \emptyset$. Therefore there is a vertex $v \in A(n) \setminus \bigcup_{i=0}^{n-1} A(i)$.

Since $n \geq 5$, P consists of $n+1$ vertices and there are at least two vertices which do not belong to P or $A(i)$, $i = 1, \ldots, n-1$, we can now conclude that

$$\begin{aligned} |V(G)| &\geq n + 1 + 2 + \sum_{i=1}^{n-1} |A(i)| = n + 3 + \sum_{i=1}^{n-1} (f(e_i) - f(e_{i+1}) - 1) \\ &= n + 3 + t - 1 - (n-1) = 3 + t. \end{aligned}$$

This clearly contradicts the choice of G. $\square$

Corollary 4.2. *Let G be a connected (α, β)-biregular bipartite graph such that* $\gcd(\alpha, \beta) = 1$ *and $G \neq K_{\alpha,\beta}$. If G has an interval t-coloring, then* $t \leq |V(G)| - 3$.

Proof. Let $X = \{x \in V(G) : d_G(x) = \alpha\}$ and $Y = \{y \in V(G) : d_G(y) = \beta\}$. By Lemma 2.1, $|X| = k\beta$ and $|Y| = k\alpha$ for some integer $k \geq 1$. Since $G \neq K_{\alpha,\beta}$, we have $k \geq 2$, that is, $|V(G)| \geq 2(\alpha + \beta)$. The result now follows from Theorem 4.1. $\square$

In Theorem 4.3 below we will show that the bound in Theorem 4.1 is tight. First we give some new notations.

In the following we will denote a graph G with a proper coloring f by (G, f). Let $H = K_{\alpha,\beta}$ be a complete graph with bipartition (X, Y) where $X = \{x_1, \dots, x_\alpha\}$ and $Y = \{y_1, \dots, y_\beta\}$, and let $H_1, \dots, H_p$ be p disjoint copies of H such that H_k has bipartition (X_k, Y_k) where $X_k = \{x_1^{(k)}, \dots, x_\alpha^{(k)}\}$ and $Y_k = \{y_1^{(k)}, \dots, y_\beta^{(k)}\}$, $k = 1, \dots, p$. Assume that f is a proper coloring of H using colors $1, \dots, t$. Without loss of generality we assume that $f(x_1 y_1) = 1$ and $f(x_\alpha y_\beta) = t$. For each $k = 1, \dots, p$ consider an edge coloring f_k of H_k, where $f_k(x_i^{(k)} x_j^{(k)}) = f(x_i x_j)$ for all i, j, $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$. We will say that the edge colorings f and f_k are *equivalent*. We define the *colored composition* $Q(H_1, \dots, H_p, f_1, \dots, f_p)$ of the colored graphs $(H_1, f_1), \dots, (H_p, f_p)$ in the following way:

If $p = 1$ then $Q(H_1, f_1)$ is the graph H_1 with coloring f_1.

If $p = 2$ then we obtain $Q(H_1, H_2, f_1, f_2)$ if we do the following: we delete the edge $x_1^{(1)} y_1^{(1)}$ of color 1 from (H_1, f_1) and the edge $x_1^{(2)} y_1^{(2)}$ of color 1 from (H_2, f_2), and then add the edges $x_1^{(1)} y_1^{(2)}$ and $y_1^{(1)} x_1^{(2)}$ colored 1.

If $p > 2$ consider the graphs $H_1', \dots, H_p'$ with colorings $f_1', \dots, f_p'$, respectively, which are defined as follows:

> (H_1', f_1') is obtained from (H_1, f_1) by deleting the edge $x_1^{(1)} y_1^{(1)}$,
>
> for each $k = 2, \dots, p-1$, (H_k', f_k') is obtained from (H_k, f_k) by deleting the edges $x_1^{(k)} y_1^{(k)}$ and $x_\alpha^{(k)} y_\beta^{(k)}$,
>
> (H_p', f_p') is obtained from (H_p, f_p) by deleting the edge $x_\alpha^{(p)} y_\beta^{(p)}$, if p is odd, or, by deleting the edge $x_1^{(p)} y_1^{(p)}$, if p is even.

We will obtain the colored graph $Q(H_1,\dots,H_p,f_1,\dots,f_p)$ from the colored graphs $(H_1',f_1'),\dots,(H_p',f_p')$ if for each $k<p$ we do the following: if k is odd we add the edges $x_1^{(k)}y_1^{(k+1)}$ and $y_1^{(k)}x_1^{(k+1)}$ of color 1 between H_k' and H_{k+1}', and if k is even we add the edges $x_\alpha^{(k)}y_\beta^{(k+1)}$ and $y_\beta^{(k)}x_\alpha^{(k+1)}$ of color t between H_k' and H_{k+1}'. Clearly, the graph $Q(H_1,\dots,H_p,f_1,\dots,f_p)$ is 2-connected.

Theorem 4.3. *For all integers n,α,β, where $n\geq 1$ and $\alpha>\beta\geq 3$, there is a connected (α,β)-biregular bipartite graph G with $m=n(\alpha+\beta)$ vertices, which for every integer t, such that $\alpha+\beta-\gcd(\alpha,\beta)\leq t\leq m-3$, has an interval t-coloring.*

Proof. Let $H=K_{\alpha,\beta}$ be a complete graph with bipartition (X,Y) where $X=\{x_1,\dots,x_\alpha\}$ and $Y=\{y_1,\dots,y_\beta\}$, and let t be an integer satisfying $\alpha+\beta-\gcd(\alpha,\beta)\leq t\leq n(\alpha+\beta)-3$.

Case 1. $\alpha+\beta-\gcd(\alpha,\beta)\leq t\leq \alpha+\beta-1$.
By Lemma 2.3, there is an interval t-coloring f of $K_{\alpha,\beta}$. Without loss of generality we assume that $f(x_1y_1)=1$ and $f(x_\alpha y_\beta)=t$. Let $H_1,\dots,H_n$ be n disjoint copies of H such that H_k has bipartition (X_k,Y_k) where $X_k=\{x_1^{(k)},\dots,x_\alpha^{(k)}\}$ and $Y_k=\{y_1^{(k)},\dots,y_\beta^{(k)}\}$, $k=1,\dots,n$. For each $k=1,\dots,n$ consider an edge coloring f_k of H_k which is equivalent to f. Then the colored composition $Q(H_1,\dots,H_n,f_1,\dots,f_n)$ of $(H_1,f_1),\dots,(H_n,f_n)$ is the required graph with an interval t-coloring.

Case 2. $\alpha+\beta\leq t\leq n(\alpha+\beta)-3$
Let $t=r(\alpha+\beta)-1+l$, where $1\leq r\leq n-1$, $0\leq l\leq \alpha+\beta-1$, and let $F_1,\dots,F_r,H_1,\dots,H_{n-r}$ be n disjoint copies of $H=K_{\alpha,\beta}$ such that

(a) H_k has bipartition (X_k,Y_k) where $X_k=\{x_1^{(k)},\dots,x_\alpha^{(k)}\}$ and $Y_k=\{y_1^{(k)},\dots,y_\beta^{(k)}\}$, $k=1,\dots,n-r$.

(b) F_s has bipartition (U_s,Z_s) where $U_s=\{u_1^{(s)},\dots,u_\alpha^{(s)}\}$ and $Z_s=\{z_1^{(s)},\dots,z_\beta^{(s)}\}$, $s=1,\dots,r$.

Consider an interval $(\alpha+\beta-1)$-coloring f of H where $f(x_iy_j)=i+j-1$ for $i=1,\dots,\alpha$, $j=1,\dots,\beta$. Let f_k be an edge coloring of H_k which is equivalent to f, $k=1,\dots,n-r$.

We define a proper coloring g_1 of F_1 by setting $g_1(u_i^{(s)}z_j^{(s)})=f(x_iy_j)$, $1\leq i\leq\alpha$, $1\leq j\leq\beta$, and for each graph F_s, $1<s\leq r$, we define a

proper coloring g_s as follows: $g_s(u_i^{(s)}z_j^{(s)}) = f(x_iy_j) + (s-1)(\alpha+\beta) - 1$, $1 \le i \le \alpha,\ 1 \le j \le \beta$. Now, using the colored graphs $(F_1, g_1), \ldots, (F_r, g_r)$ we define the colored graph $T(F_1, \ldots, F_r, g_1, \ldots, g_r)$ in the following way: If $r = 1$ then $T(F_1, g_1)$ is obtained from (F_1, g_1) by deleting the edge $u_\alpha^{(1)}z_\beta^{(1)}$ of color $\alpha + \beta - 1$.

If $r \ge 2$ then $T(F_1, \ldots, F_r, g_1, \ldots, g_r)$ is obtained if we

(i) delete the edge $u_\alpha^{(1)}z_\beta^{(1)}$ of color $\alpha + \beta - 1$ from (F_1, g_1),

(ii) for each $s = 2, \ldots, r$, delete the edge $u_1^{(s)}z_\beta^{(s)}$ of color $\beta - 1 + (s-1)(\alpha+\beta)$ and the edge $z_1^{(s)}u_\alpha^{(s)}$ of color $\alpha - 1 + (s-1)(\alpha+\beta)$ from (F_s, g_s),

(iii) for each $s = 1, \ldots, r-1$ add the edges $u_\alpha^{(s)}z_1^{(s+1)}$ and $z_\beta^{(s)}u_1^{(s+1)}$ of color $s(\alpha+\beta) - 1$.

Remark. The coloring of $T(F_1, \ldots, F_r, g_1, \ldots, g_r)$ is an interval coloring with colors $1, 2, \ldots, r(\alpha+\beta) - 2$. The colors of the edges incident with the vertex $u_\alpha^{(s)}$, $z_\beta^{(s)}$, $u_\alpha^{(r)}$, $z_\alpha^{(r)}$ form the intervals $[s(\alpha+\beta)-\beta, s(\alpha+\beta)-1]$, $[s(\alpha+\beta)-\alpha, s(\alpha+\beta)-1]$, $[r(\alpha+\beta)-\beta, r(\alpha+\beta)-2]$, $[r(\alpha+\beta)-\alpha, r(\alpha+\beta)-2]$, respectively.

Subcase 2a. $l \le \alpha + \beta - 2$, that is, $t \le (r+1)(\alpha+\beta) - 3$.
We construct the colored composition $Q(H_1, \ldots, H_p, f_1, \ldots, f_p)$ of the graphs $(H_1, f_1), \ldots, (H_p, f_p)$, where $p = n - r$. It follows from the definition of $Q(H_1, \ldots, H_p, f_1, \ldots, f_p)$ that the edge $x_\alpha^{(1)}y_\beta^{(1)}$ has the color $\alpha + \beta - 1$. The coloring of the graph $Q(H_1, \ldots, H_p, f_1, \ldots, f_p)$ is an interval coloring with colors $1, 2, \ldots, \alpha + \beta - 1$, that is, each edge received a color (number) from the set $\{1, 2, \ldots, \alpha + \beta - 1\}$. Now we add to the color of each edge of $Q(H_1, \ldots, H_p, f_1, \ldots, f_p)$ the number $t - (\alpha + \beta - 1)$. The obtained graph, colored with colors $t - (\alpha+\beta) + 2, t - (\alpha+\beta) + 1, \ldots, t$, we denote by $Q_t(H_1, \ldots, H_p, f_1, \ldots, f_p)$. Clearly, for every number (color) c from the set $\{t - (\alpha+\beta) + 2, t - (\alpha+\beta) + 1, \ldots, t\}$ there is at least one edge in $Q_t(H_1, \ldots, H_p, f_1, \ldots, f_p)$ colored c. Since $t \le (r+1)(\alpha+\beta) - 3$, we have $t - (\alpha+\beta) + 2 \le r(\alpha+\beta) - 1$. Therefore there is an edge e in $Q_t(H_1, \ldots, H_p, f_1, \ldots, f_p)$ colored $r(\alpha+\beta) - 1$. Clearly, $e = x_i^{(k)}y_j^{(s)}$ for some i, j, k, s, where $1 \le i \le \alpha$, $1 \le j \le \beta$, $1 \le k, s \le p$.

On the other hand there is no edge colored $r(\alpha+\beta)-1$ at the vertices $u_\alpha^{(r)}$ and $z_\beta^{(r)}$. We obtain the required graph $G = G(n)$ with an interval t-coloring if we delete the edge $e = x_i^{(k)}y_j^{(s)}$ from the graph $Q_t(H_1, \ldots, H_p, f_1, \ldots, f_p)$

and add the edges $u_\alpha^{(r)} y_j^{(s)}$ and $z_\beta^{(r)} x_i^{(k)}$ of color $r(\alpha+\beta)-1$ between the graphs $Q_t(H_1,\dots,H_p,f_1,\dots,f_p)$ and $T(F_1,\dots,F_r,g_1,\dots,g_r)$.

Subcase 2b. $l=\alpha+\beta-1$, that is, $t=(r+1)(\alpha+\beta)-2$.
The condition $(r+1)(\alpha+\beta)-2=t\le n(\alpha+\beta)-3$ implies that $n-r\ge 2$. Let $p=n-r$. We construct the colored composition $Q(H_1,\dots,H_{p-1},f_1,\dots,f_{p-1})$ of the graphs $(H_1,f_1),\dots,(H_{p-1},f_{p-1})$. Now we define a coloring g of the graph H_p as follows: $g(x_i^{(r)} y_j^{(r)})=i+j$, for all $1\le i\le\alpha, 1\le j\le\beta$. Then we delete the edge $x_1^{(1)} y_2^{(1)}$ of color 2 from $Q(H_1,\dots,H_{p-1},f_1,\dots,f_{p-1})$ and the edge $x_1^{(r)} y_1^{(r)}$ of color 2 from (H_p,g) and add the edges $x_1^{(1)} y_1^{(r)}$ and $y_2^{(1)} x_1^{(r)}$ colored 2. We obtain a 2-connected graph with an interval coloring using colors $1,2,\dots,\alpha+\beta$. Now we add to the color of each edge of this graph the number $t-(\alpha+\beta)$. The new graph, colored with colors $t-(\alpha+\beta)+1, t-(\alpha+\beta)+2,\dots,t$, we denote by $Q_t(H_1,\dots,H_p,f_1,\dots,f_p)$. Note that for every number (color) c from the set $\{t-(\alpha+\beta)+1, t-(\alpha+\beta)+2,\dots,t\}$ there is at least one edge in $Q_t(H_1,\dots,H_p,f_1,\dots,f_p)$ colored c. In particular, the edge $x_1^{(1)} y_1^{(1)}$ receives the color $t-(\alpha+\beta)+1$. We have that $t-(\alpha+\beta)+1=r(\alpha+\beta)-1$. We obtain the required graph $G=G(n)$ with an interval t-coloring if we delete the edge $x_1^{(1)} y_1^{(1)}$ of color $r(\alpha+\beta)-1$ from $Q_t(H_1,\dots,H_p,f_1,\dots,f_p)$ and add the edges $u_\alpha^{(r)} y_1^{(r)}$ and $z_\beta^{(r)} x_1^{(r)}$ colored $r(\alpha+\beta)-1$. $\square$

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