A Generalized Class-Teacher Model for Some Timetabling Problems *

A.S. Asratian
Department of Mathematics, Luleå University of Technology
S-971 87 Luleå, Sweden
D. de Werra
Department of Mathematics, Ecole Polytechnique Fédérale de Lausanne
CH 1015 Lausanne, Switzerland

Abstract. We consider a theoretical model which extends the basic "class-teacher model" of timetabling and which corresponds to some situations which occur frequently in the basic training programmes of universities and schools:

We are given m teachers $T_1, ..., T_m$ and n classes $C_1, ..., C_n$. The set of classes is partitioned into p disjoint subsets $G_1, ..., G_p$ in such a way that in addition to the lectures given by one teacher to one class, there are some lectures given by one teacher to the students of all classes in group G_l, $1 \leq l \leq p$. Such lectures will be called group-lectures. The number a_{lj} of one hour group-lectures which teacher T_j must deliver to group G_l and the number b_{ij} of one hour class-teaching which T_j must give to class C_i are given. Is there a timetable of t hours (or length t), so that each class C_i and each group G_l receives all their lectures, but no student is scheduled to be taught by more than one teacher in each hour, and no teacher must teach to more than one group or class in each hour? We show that this problem is NP-complete and find some sufficient conditions for the existence of a timetable of length t. We also describe an algorithm for constructing a timetable corresponding to the requirement matrices $A = (a_{lj})$ and $B = (b_{ij})$ and show that under a natural assumption on A and B this algorithm finds a timetable within $\frac{7}{6}$ of the optimum length.

Keywords: timetabling, university timetable, edge coloring, chromatic index

1. Introduction

Timetabling problems have very often been formulated in mathematical terms (see for instance, [12,16,19,21,23,25]). However these problems strongly depend on the types of schools, universities and educational systems and they can be very

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different. Therefore, there is no universal timetabling model which could be applied everywhere. A huge variety of timetabling models has been described in the OR literature [3-6,8,10,12,15-17,20-26]. The most popular model is the so-called *class-teacher model* which is defined as follows:

Class-teacher model. There are m teachers $T_1, ..., T_m$ and n classes $C_1, ..., C_n$. A class consists of a set of students who follow exactly the same program. We are given an $m \times n$ requirement matrix $B = (b_{ij})$ where b_{ij} is the number of lectures involving class C_i and teacher T_j. We shall assume that all lectures have the same duration (say one period or one hour) and the set $\{1, 2, ..., t\}$ of all periods is given.

In the simplest case, without including all constraints which are usually present in real situations, the timetabling problem associated to this model is formulated as follows:
Problem 1. Assign each lecture to some period in $\{1, 2, ..., t\}$ in such a way that no teacher (resp. no class) is involved in more than one lecture at a time.

Such a timetable when it exists is called a timetable of length t corresponding to the matrix B. It is known (see the next section) that a timetable of length t corresponding to B exists if and only if

$$t \geq \max(\max_i \sum_{j=1}^{m} b_{ij}, \max_j \sum_{i=1}^{n} b_{ij}). \tag{1.1}$$

Moreover if t satisfies this inequalitiy, a timetable can be constructed in polynomial time.

Class-teacher models with some additional requirements where the timetabling problem can still be solved polynomially are described in [22,24]. But in general, the various real-life restrictions are such that the corresponding timetabling problem becomes NP-complete (see [13] for concepts related to complexity). For example, the NP-completeness of the following problem was proved in [9].

Problem 2. There are n classes $C_1, ..., C_n$ and m teachers $T_1, ..., T_m$. The number of lectures which teacher T_j must give to class C_i, is b_{ij}. We are also given for each teacher T_j a set $L(T_j) \subseteq \{1, ..., t\}$ of periods during which T_j is available for teaching, and for each class C_i a set $L(C_i) \subseteq \{1, ..., t\}$ of periods during which C_i is available for teaching. The problem is to determine whether there exists a timetable of length t, so that each class receives all its teaching corresponding to the matrix

$B = (b_{ij})$, the availabilities of the teachers and classes are not violated and no class or teacher is involved in more than one lecture at a time?

All lectures given by one teacher to one class will be called *individual lectures* in the remainder of the paper.

In this paper we shall consider the following theoretical model which extends the class-teacher model and which corresponds to some situations which occur frequently in the basic training programmes of universities and schools:

A generalized class-teacher model: We are given m teachers $T_1, ..., T_m$ and n classes $C_1, ..., C_n$. The set of classes is partitioned into p disjoint subsets $G_1, ..., G_p$ in such a way that in addition to the individual lectures given by one teacher to one class, there are some lectures given by one teacher to the students of all classes in group G_l, $1 \leq l \leq p$. (Such lectures will be called *group-lectures*). The number a_{lj} of group-lectures which teacher T_j must deliver to group G_l and the number b_{ij} of individual lectures which T_j must give to class C_i are given. We shall assume that all lectures have the same duration (say one period) and the set $\{1, 2, ..., t\}$ of all periods is given.

Clearly if all elements of A are 0 or if all groups contain exactly one class, then we have the class-teacher model.

In the simplest case, without including some constraints which are usually present in real situations, the timetabling problem associated to the generalized class-teacher model can be formulated as follows:

Problem 3. Is there a timetable of t periods (or of length t), where each class C_i and each group G_l receive all their lectures, but no student is scheduled to be taught by more than one teacher in each period, and no teacher must teach to more than one group or class at a time?

Such a timetable when it exists will be called a *university timetable of length t* corresponding to the matrices $A = (a_{lj})$ and $B = (b_{ij})$. We denote by $t_{min}(A, B)$ the minimum t for which there exists a university timetable of length t corresponding to A and B.

The generalized class-teacher model describes the situation which occurs in many universities and schools where a collection of programmes are offered to the students; these programmes contain a few topics which are common to all programmes. It is in general the case (in a science faculty or an engineering institution) of basic courses in mathematics, physics, chemistry, biology, etc. For these lectures students of different programmes (corresponding to the classes in the above formulation) are grouped to attend group-lectures in these topics. Such a problem arise for instance at Luleå University of Technology in Sweden and Ecole Polytechnique Fédérale de Lausanne in Switzerland; at the EPFL for instance twelve programmes are offered (12 classes). Groups of three or four classes are formed for these courses of basic science which now correspond to group-lectures. The groups are also the same for all courses of mathematics, physics, etc. Besides these group-lectures there are in each programme individual lectures which correspond to courses given to one class (students of one programme).

A similar system was used and is still applied in many universities of the former republics of the Soviet Union.

The generalized class-teacher model was for the first time investigated by Asratian [1]. Some sufficient conditions for existence of university timetable were obtained in the case when the set of $\{T_1, ..., T_m\}$ consists of two different types of teachers: *professors* and *lecturers*. The professors must lecture to groups $G_1, ..., G_p$ only and the lecturers can teach to classes $C_1, ..., C_n$ only. Such model is called the *professor-lecturer model*. In particular, it was proved in [1] that a timetable of length t corresponding to the matrices A and B exists if each teacher has at most $\frac{t}{2}$ lectures, the number of all lectures of each class does not exceed t and the number of group-lectures in each class does not exceed $\frac{t}{2}$.

The following results are obtained in the present paper:

1. We prove that the problem of deciding whether $t_{min}(A, B) \leq t$, for a given t is NP-complete.

2. We give a sharp upper bound for $t_{min}(A, B)$ and find some sufficient conditions for the existence of a timetable of length t.

3. We suggest a polynomial algorithm for constructing a timetable corresponding to the requirement matrices A and B and show that under a natural assumption on A and B this algorithm finds a timetable within $\frac{7}{6}$ of the optimum length.

4. In section 4 we consider more extensively the professor-lecturer model.

2. Timetables and edge colorings of graphs

Here we consider some interconnections between timetables and proper edge colorings of bipartite graphs. We use [7] for terminology and notation of graph theory not defined here. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and the edge set of a graph G. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$. An edge coloring of a graph G with colors $\alpha_1, \ldots, \alpha_t$ is an assignment of colors to the edges of G, one color from $\{\alpha_1, \ldots, \alpha_t\}$ to each edge. Such a coloring is called *proper* if no pair of adjacent edges receives the same color. More formally, an edge coloring of G with colors from the set $\{\alpha_1, \ldots, \alpha_t\}$ is a mapping $f : E(G) \longrightarrow \{\alpha_1, \ldots, \alpha_t\}$. If $f(e) = \alpha_k$ then we say that the edge e is colored α_k. The minimum t for which there exists a proper edge coloring of G with t colors is called the *chromatic index* of G and denoted by $\chi'(G)$. According to the well-known Kőnig's Coloring Theorem [18], $\chi'(G) = \Delta(G)$ for any bipartite graph G where $\Delta(G)$ denotes the maximum degree of G. This result provides the answer to Problem 1. But first we need a formal definition of timetables.

We have n classes $C_1, \ldots, C_n$, m teachers $T_1, \ldots, T_m$ and an $n \times m$ requirement matrix $B = (b_{ij})$. A timetable of length t, corresponding to the matrix B is an $n \times t$ array $\mathbf{S} = (s_{ih})$ satisfying the following three conditions:

(i) each entry of $\mathbf{S}$ is either one of the members of the set $\{T_1, \ldots, T_m\}$ or is empty;

(ii) the symbol T_j occurs precisely b_{ij} times in the i^{th} row of $\mathbf{S}$, for $j = 1, \ldots, m$;

(iii) in each column of $\mathbf{S}$ all symbols are different.

Now consider a bipartite graph $H = H(B)$, with bipartition (V_1, V_2) where $V_1 = \{C_1 \ldots C_n\}$ and $V_2 = \{T_1 \ldots, T_m\}$, where vertices C_i and T_j are joined by b_{ij} edges. Then there is a one–to–one correspondence between proper edge colorings of H with colors $1, 2, ..., t$ and timetables of length t, which respect the requirement matrix B. This correspondence is that the column numbers $1, 2, ..., t$ of a timetable $\mathbf{S} = (s_{ih})$ form the color set and $s_{ih} = T_j$ if and only if one of the edges with ends C_i and T_j is colored with color h. Hence Kőnig's Coloring Theorem implies the following result: A timetable of length t corresponding to B exists if and only if the inequality (1.1) holds.

A V_1*-sequential coloring* of $H = H(B)$ is a proper edge coloring where the edges incident with each vertex $C_i \in V_1$ are colored precisely with the colors $1, 2, ..., d_H(C_i)$. A timetable induced by this coloring is called a *sequential timetable* corresponding to the matrix B. In fact, this is a timetable in which all the classes have lessons without interruptions, and all begin at the same time. The problem of deciding whether a bipartite graph has a V_1-sequential coloring, is NP-complete

[3]. However, in many situations which arise in practice, sequential timetables exist. Probably the main reason is (see [1]) that a usual property of requirement matrices B in practice is that the number of lectures for each class is not less than the number of lectures taught by each teacher teaching that class, that is, if $\sum_{j=1}^{m} b_{rj} \geq \sum_{i=1}^{n} b_{is}$, for each pair r, s with $b_{rs} \neq 0$. It was proved in [1] that this condition is sufficient for existence of a sequential timetable. In terms of graphs it can be formulated as follows.

Proposition 2.1 [1] The graph $H = H(B)$ has a V_1-sequential coloring if $d_H(C_r) \geq d_H(T_s)$ for each pair of adjacent vertices C_r and T_s, that is, for each pair of vertices C_r and T_s with $b_{rs} \neq 0$.

It is clear that in terms of edge colorings of the graph $H = H(B)$ Problem 2 can be formulated as follows. For each vertex $v \in V(H)$, let $L(v) \subseteq \{1, ..., t\}$ be a set of colors assigned to v. Can H be given a proper edge coloring in which each edge incident with a vertex $v \in V(H)$ receives a color from the set $L(v)$?

If $L(T_j) = \{1, ..., t\}$ for $j = 1, ..., m$ and $|L(C_r)| \geq \sum_{j=1}^{m} b_{rj}$ for $r = 1, ..., n$, then L is called a V_1*-scheme.* A coloring of G corresponding to a V_1–scheme L is called an L*-coloring* of G. The next result gives a connection between L-colorings and V_1-sequential colorings.

Theorem 2.2 [14]. If a bipartite graph H with bipartition (V_1, V_2) has a V_1–sequential coloring then it has an L–coloring for any V_1–scheme L.

Proposition 2.1 and Theorem 2.2 imply the following result which we will use later.

Theorem 2.3 [2]. Let L be a V_1-scheme. The graph $H = H(B)$ has an L-coloring if $|L(C_r)| \geq d_H(T_s)$ for every pair of vertices C_r and T_s with $b_{rs} \neq 0$.

More about interconnection between timetables and edge colorings of bipartite graphs can be found in [2,17,23,25].

3. A general model

We begin by a formal definition of a university timetable corresponding to the matrices A and B.

A university timetable of length t corresponding to the matrices A and B is an $n \times t$ array $\mathbf{S} = (s_{ih})$ satisfying the following conditions:

(i) each entry of $\mathbf{S}$ is either one of the members of the set $\{T_1, \ldots, T_m\}$ or is empty;

(ii) if $C_i \subseteq G_l$ then the symbol T_j occurs precisely $a_{lj} + b_{ij}$ times in the i^{th} row of $\mathbf{S}$, for $j = 1, \ldots, m$;

(iii) any two symbols s_{i_1h} and s_{i_2h} in the h^{th} column of $\mathbf{S}$,$1 \le h \le t$ are different if classes C_{i_1} and C_{i_2} are contained in different groups.

(iv) if classes C_{i_1} and C_{i_2} are contained in the same group and $s_{i_1h} = T_j$ then $s_{i_2h} = T_j$ if and only if $s_{ih} = T_j$ for any class C_i of this group.

In other words, conditions (iii) and (iv) mean that in the h^{th} hour,$1 \le h \le t$, each lecturer can have either a group-lecture or an individual lecture. For instance Fig.1 shows a university timetable $\mathbf{S}$ of length 4 corresponding to the matrices A and B with three teachers T_1, T_2, T_3, four classes C_1, C_2, C_3, C_4 and two groups $G_1 = \{C_1, C_2\}$ and $G_2 = \{C_3, C_4\}$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{S} =$$

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>C_1</td><td rowspan="2">T_1</td><td rowspan="2">T_2</td><td>T_3</td><td>T_1</td></tr>
<tr><td>C_2</td><td>T_2</td><td>T_3</td></tr>
<tr><td>C_3</td><td>T_2</td><td>T_1</td><td rowspan="2">T_1</td><td rowspan="2">T_2</td></tr>
<tr><td>C_4</td><td></td><td>T_3</td></tr>
</table>

Fig.1

Now we will show that Problem 3 is NP-complete. It is known that the following restriction of Problem 2 is NP-complete. (We change notations for technical reasons).

The Restricted Problem 2. There are n' classes $C_1, ..., C_{n'}$ and m' teachers $T_1, ..., T_{m'}$. The number of lectures which teacher T_j must give to class C_i, is b'_{ij}. Also for each teacher T_j a set $L(T_j) \subseteq \{1, 2, 3\}$ of periods during which T_j is available for teaching, is given such that $|L(T_j)| = \sum_{i=1}^{n} b'_{ij} \ge 2$ for each $j = 1, ..., m$. Furthermore, at each period each class is available for teaching. Is there a timetable of length 3, such that each class receives all its lectures corresponding to the matrix $B' = (b'_{ij})$, the availabilities of the teachers are not violated and no class or teacher is involved in more than one lecture at a time?

Theorem 3.1. The problem of determining, whether there is a university timetable of length t, corresponding to the matrices A and B, is NP-complete even in the case $t = 3$ and $p \le 4$.

Proof. To prove the theorem it is sufficient to reduce polynomially the Restricted Problem 2 to Problem 3.

Since $t = 3$, $L(T_j)$ could be one of the sets $\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$ for each $j = 1, ..., m$. Let $R_{a,b}$ denote the set of teachers T_j with $L(T_j) = \{a, b\}$ and $r_{a,b}$ denote the cardinality of $R_{a,b}$, $1 \leq a < b \leq 3$. Now we will describe a model of a university timetabling problem for $t = 3$.

First we define classes. Let $n = n' + r_{1,2} + r_{1,3} + r_{2,3}$. The set of n classes will consist of classes $C_1, ..., C_{n'}$ used in the Restricted Problem 2 and $r_{1,2}+r_{1,3}+r_{2,3}$ new classes which we define in the following way: For each teacher T_j with $|L(T_j)| = 2$, $1 \leq j \leq m'$, we introduce a new class consisting of one new student Q_j who is not in $\cup_{i=1}^{n'} C_i$ and who has not been considered yet. We will say that this class $\{Q_j\}$ corresponds to T_j. Thus, we obtain $n = n' + r_{1,2} + r_{1,3} + r_{2,3}$ disjoint classes $C_1, ..., C_{n'}, C_{1+n'}, ..., C_n$.

Put $p = 4$. We form four disjoint groups G_1, G_2, G_3, G_4 as follows:

$G_1 = \{C_1, ..., C_{n'}\}$,

G_2 consists of $r_{1,2}$ new classes corresponding to the teachers in $R_{1,2}$,

G_3 consists of $r_{1,3}$ new classes corresponding to the teachers in $R_{1,3}$, and

G_4 consists of $r_{2,3}$ new classes corresponding to the teachers in $R_{2,3}$.

Some of the groups G_2, G_3, G_4 may be empty but for convenience and without loss of generality we consider the case when $G_i \neq \emptyset$ for $i = 1, 2, 3, 4$.

Put $m = m' + 2$. The set of teachers will consist of the teachers $T_1, ..., T_{m'}$ who are mentioned in the Restricted Problem 2 and two new teachers, $T_{m'+1}$ and $T_{m'+2}$.

Now we have to define requirement matrices A and B. We define an $n \times (m'+2)$ matrix $B = (b_{ij})$ in the following way: $b_{ij} = b'_{ij}$ for $1 \leq i \leq n', 1 \leq j \leq m'$, $b_{ij} = 0$ for $1 \leq i \leq n', m' + 1 \leq j \leq m' + 2$; now we define b_{ij} for $1 + n' \leq i \leq n, 1 \leq j \leq m' + 2 : b_{ij} = 1$ if and only if the new class C_i corresponds to the teacher T_j and otherwise $b_{ij} = 0$. Futhermore, we define a $4 \times (m'+2)$ matrix $A = (a_{lj})$ as follows: $a_{2,m'+1} = 2, a_{3,m'+1} = 1, a_{3,m'+2} = 1, a_{4,m'+2} = 2$ and all other elements of A are 0.

Suppose that there exists a university timetable $\mathbf{S_1}$ of length 3 corresponding to the matrices A and B. It consists, in fact, of four timetables for groups G_1, G_2, G_3 and G_4, respectively. It is clear that the timetable for G_2 contains a column K_2 consisting of elements of the set $R_{1,2}$, the timetable for G_3 contains a column K_3 consisting of elements of the set $R_{1,3}$, and the timetable for G_4 contains a column K_4 consisting of elements of the set $R_{2,3}$. The definition of the matrix A implies that K_2, K_3, K_4 are parts of mutually different columns of $\mathbf{S_1}$. Therefore, possibly after permutation of columns of $\mathbf{S_1}$, we will transform $\mathbf{S_1}$ to the form in Fig.2. Then the timetable for G_1, which is denoted by $\mathbf{S}$ in Fig.2, will be a solution of the Restricted Problem 2.

$\mathbf{S_1} =$

	1	2	3
G_1		$\mathbf{S}$	
G_2	T_{m+1}	T_{m+1}	K_2
G_3	T_{m+2}	K_3	T_{m+1}
G_4	K_4	T_{m+2}	T_{m+2}

Fig.2

Conversely, if $\mathbf{S}$ is a solution of the Restricted Problem 2, then we can construct a university timetable $\mathbf{S_1}$ corresponding to the matrices A and B in the same way, as shown in Fig.2.

Now we reformulate Problem 3 in terms of graphs. We associate with a pair of matrices A and B a bipartite graph $H = H(A, B)$ with bipartition (X, Y) where

$$X = \{C_1, ..., C_n, G_1, ..., G_p\}, Y = \{T_1, ..., T_m\}$$

and the set $E(H)$ is defined as follows: the vertex T_j is joined by a_{lj} edges with the vertex G_l and by b_{ij} edges with the vertex C_i. Let $H(A)$ be the subgraph of $H(A, B)$ induced by the set $\{G_1, ..., G_p, T_1, ..., T_m\}$ and $H(B)$ be the subgraph induced by the set $\{C_1, ..., C_n, T_1, ..., T_m\}$. Clearly, $H(A, B) = H(A) \cup H(B)$.

Proposition 3.2. A university timetable of length t corresponding to the matrices A and B exists if and only if the graph $H(A, B)$ has a proper t-coloring such that all the edges incident with vertices G_l and C_i have different colors for each pair G_l and C_i with $C_i \in G_l$.

Proof. The proof follows easily from the following interpretation: An edge joining vertices T_j and G_l is colored h if and only if teacher T_j gives a group-lecture to the group G_l at period h; an edge joining vertices T_j and C_i is colored h if and only if the teacher T_j gives a individual lecture to the class C_i at period h.

Now we will give upper and lower bounds for $t_{min}(A, B)$. But first we introduce some parameters. The number

$$w(T_j) = \sum_{l=1}^{p} a_{lj} + \sum_{i=1}^{n} b_{ij}$$

is the working time of the tacher T_j, and the number

$$w(C_i) = \sum_{j=1}^{m} a_{lj} + \sum_{j=1}^{m} b_{ij}$$

where $C_i \in G_l$, is the working time of the class C_i.

We will also use the following notations:

$$W = \max(\max_j w(T_j), \max_i w(C_i)),$$

$$c(A) = \max_j \sum_{l=1}^{p} a_{lj},$$

$$c(B) = \max_j \sum_{i=1}^{p} b_{ij},$$

$$r(A) = \max_l \sum_{j=1}^{m} a_{lj},$$

$$r(B) = \max_i \sum_{j=1}^{m} b_{ij},$$

Theorem 3.3.

$$W \leq t_{min}(A, B) \leq \max(r(A), c(A)) + \max(c(B), r(B)).$$

Proof. The lower bound is obvious. To prove the upper bound consider the subgraphs $H(A)$ and $H(B)$ defined above. Clearly, $\Delta(H(A)) = \max(r(A), c(A))$ and $\Delta(H(B)) = \max(c(B), r(B))$. First, we properly color edges of the graph $H(A)$ with $\Delta(H(A))$ colors $1, 2, ..., \Delta(H(A))$, and then we properly color the edges of $H(B)$ with $\Delta(H(B))$ colors $\Delta(H(A)) + 1, ..., \Delta(H(A)) + \Delta(H(B))$. We obtain a proper coloring of the graph $H(A, B)$ satisfying the condition of Proposition 3.2.

Corollary 3.4. If $\max(c(A), c(B)) \leq \min(r(A), r(B))$ then $t_{min}(A, B) \leq r(A) + r(B)$.

Theorem 3.5. There is a university timetable of length t,

$$t \le \max(r(A), r(B), \max_{1\le j\le m} w(T_j))) + \max(c(A), c(B), \lfloor W/2 \rfloor), \tag{3.1}$$

corresponding to A and B which can be constructed in polynomial time.

Proof. First we describe an algorithm which constructs a proper coloring of the edges of the graph $H(A, B)$ satisfying the condition of Proposition 3.2.
Step 1. Choose two integers, k and s, satisfying $s + k + 1 \ge W$ in the following way:

a) If $c(A) + c(B) + 1 \ge W$ then put $k = c(A)$ and $s = c(B)$.
b) Otherwise (if $c(A) + c(B) + 1 < W$) set $s = k = \lfloor W/2 \rfloor$.

Step 2. Let

$$X^* = \{C_i | d_H(C_i) > s\} \cup \{G_l | d_H(G_l) > k\}$$

and the set F of all edges in H having one end in the set X^*. Construct a proper coloring of the edges in F with colors $1, 2, .., t_0$ where $t_0 = \max(r(A), r(B), \max_j w(T_j))$.
Claim 1. We have $t_0 \ge \max(s, k)$.

Proof of the claim. In case a) we have

$$t_0 \ge \max(r(A), r(B), \max(c(A), c(B)) \ge \max(c(A), c(B)) = \max(s, k).$$

In case b) since $c(A) + c(B) + 1 < W$ we must have a class C_i and a group G_l containing C_i such that $W = \sum_{j=1}^m a_{lj} + \sum_{j=1}^m b_{ij}$, so

$$r(A) + r(B) = \max_l \sum_{j=1}^m a_{lj} + \max_i \sum_{j=1}^m b_{ij} \ge W$$

and hence $t_0 \ge \max(r(A), r(B)) \ge \lfloor W/2 \rfloor$.

Step 3. Let $l = \min(s, k)$ and let E_0 be the set of edges which are colored with the first $t_0 - l (\ge 0)$ colors. (If $t_0 = l$ then $E_0 = \emptyset$). We keep the coloring of edges in E_0 and we consider that the remaining edges in the graph $H - E_0$ are not colored (here $H - E_0$ is the graph of all remaining edges in H when E_0 has been removed). We construct a proper coloring of the edges of $H(A) - E_0$ with $\Delta(H(A) - E_0)$ new colors and a proper coloring of the edges of $H(B) - E_0$ with $\Delta(H(B) - E_0)$ new colors. This gives a proper coloring of $H = H(A, B)$ with

$$t_0 - l + \Delta(H(A) - E_0) + \Delta(H(B) - E_0)$$

colors.

Claim 2. The coloring constructed in steps 1 to 3 corresponds to a timetable (i.e. satisfies the condition of Proposition 3.2).

Proof of the claim. We only have to show that for each class C_i and each group G_l containing C_i, the edges incident with vertices G_l and C_i have all different colors. Clearly it is true for all such edges in $H - E_0$ from the construction. Let us now consider a group G_l and a class C_i in G_l. By definition

$$W \geq \sum_{j=1}^{m} a_{lj} + \sum_{j=1}^{m} b_{ij} = d_H(C_i) + d_H(G_l).$$

From the choice of s, k we have $s + k + 1 \geq W \geq d_H(G_l) + d_H(C_i)$. So at most one of the inequalities $d_H(C_i) > s$ and $d_H(G_l) > k$ can hold; therefore $C_i \in X^*$ only if $G_l \notin X^*$, and $G_l \in X^*$ only if $C_i \notin X^*$. This implies that the edges in F (and hence in E_0) incident with G_l and the edges in F (and hence in E_0) incident with C_i cannot have a color in common.

We observe that the algorithm is polynomial since there are polynomial algorithms to construct proper colorings in bipartite graphs (see, for example, [11]).

Claim 3. The number t of colors used satisfies

$$t \leq t_0 + \max(c(A), c(B), \lfloor W/2 \rfloor).$$

Proof of the claim. In case a) we have $s + k \geq c(A) + c(B) + 1 \geq W$ so that $\max(c(A), c(B)) \geq \lfloor W/2 \rfloor$. Thus we have to show that $t \leq t_0 + \max(c(A), c(B))$. In $H(A) - E_0$ vertices G_l have degrees at most l if $G_l \in X^*$ and at most k by definition of X^* if $G_l \notin X^*$, so vertices G_l have degree at most $c(A)$ because $l = \min(s, k) = \min(c(A), c(B))$.

Similarly one verifies that in $H(B) - E_0$ all vertices have degree at most $c(B)$. Hence the number of colors used satisfies

$$t \leq t_0 - l + \Delta(H(A) - E_0) + \Delta(H(B) - E_0)$$

$$\leq t_0 - \min(c(A), c(B)) + c(A) + c(B) = t_0 + \max(c(A), c(B)).$$

Let us now examine case b) where $l = s = k = \lfloor W/2 \rfloor$ and $c(A) + c(B) < W$. If $\max(c(A), c(B)) \leq \lfloor W/2 \rfloor$ we have to show that $t \leq t_0 + \lfloor W/2 \rfloor$. Since

$$t = t_0 - l + \Delta(H(A) - E_0) + \Delta(H(B) - E_0)$$

we have

$$t \leq t_0 - \lfloor W/2 \rfloor + \lfloor W/2 \rfloor + \lfloor W/2 \rfloor \leq t_0 + \lfloor W/2 \rfloor.$$

Suppose now that $\max(c(A), c(B)) > \lfloor W/2 \rfloor$. We have to show that $t \leq t_0 + \max(c(A), c(B))$. Since $c(A) + c(B) < W$, we have $\min(c(A), c(B)) < \lfloor W/2 \rfloor$. One of $(H(A) - E_0), (H(B) - E_0)$ has maximum degree at most $\lfloor W/2 \rfloor$. Therefore

$$t = t_0 - l + \Delta(H(A) - E_0) + \Delta(H(B) - E_0)$$

$$\leq t_0 - \lfloor W/2 \rfloor + \lfloor W/2 \rfloor + \max(c(A), c(B)).$$

To end the proof of Theorem 3.5 we observe that we have constructed with the algorithm a proper edge coloring (according to Claim 2) which defines a university timetable (see the proof of Proposition 3.2). From Claim 3, the number t of periods satisfies (3.1). The proof of Theorem 3.5 is complete.

The above proof gives in fact a polynomial algorithm for constructing a university timetable which has a number of periods satisfying (3.1).

Corollary 3.6.

$$W \leq t_{min}(A, B) \leq \max(r(A), r(B), \max_j w(T_j)) + \max(c(A), c(B), \lfloor \frac{W}{2} \rfloor)$$

Remark 3.7. It is not difficult to see that in the case when

$$\max(r(A), r(B), \max_j w(T_j)) \leq \frac{W}{2},$$

$$\max(c(A), c(B)) \leq \frac{W}{2}$$

both algorithms described in the proofs of Theorem 3.3 and Theorem 3.5 give university timetables of minimum length corresponding to the matrices A and B.

Now we deduce some corollaries from Theorem 3.5 and Corollary 3.6.

Corollary 3.8. If $\max(c(A), c(B)) \leq \lfloor \frac{W}{2} \rfloor$ and

$$\max(r(A), r(B), \max_j w(T_j)) \leq \lfloor \frac{W}{\lambda} \rfloor$$

for some $\lambda, 0 < \lambda \leq 2$, then there is a university timetable corresponding to A and B within $\frac{\lambda+2}{2\lambda}$ of the optimum length which can be constructed in polynomial time.

Proof. Let t denote the length of a timetable constructed by the algorithm described in the proof of Theorem 3.5. Then

$$W \leq t_{min}(A, B) \leq t \leq \frac{\lambda + 2}{2\lambda} W$$

and, therefore, $t \leq \frac{\lambda+2}{2\lambda} t_{min}(A, B)$.

By taking, for example, $\lambda = \frac{3}{2}$ we obtain the following result.

Corollary 3.9. If $\max(c(A), c(B)) \leq \lfloor W/2 \rfloor$ and

$$\max(r(A), r(B), \max_j w(T_j)) \leq \lfloor \frac{2}{3} W \rfloor$$

then there is a university timetable corresponding to A and B within $\frac{7}{6}$ of the optimum length which can be constructed in polynomial time.

Corollary 3.10. Let A and B be the requirement matrices and t be a positive integer such that

a) every lecturer has at most $\frac{1}{3}t$ group-lectures and at most $\frac{1}{3}t$ individual lectures,

b) the working time of every class and every lecturer does not exceed $\frac{2}{3}t$.

Then there is a university timetable of length t corresponding to the matrices A and B.

Proof. Clearly, $W \leq \frac{2}{3}t, c(A) \leq \frac{1}{3}t, c(B) \leq \frac{1}{3}t, r(A) \leq \frac{2}{3}t$ and $r(B) \leq \frac{2}{3}t$. Then the result follows from Theorem 3.5.

Let us compare the upper bounds in Theorem 3.3 and Corollary 3.6. We show that in some cases the bound in Theorem 3.3 is better and in other cases the bound of Corollary 3.6 is better. Let, for example,

$$\max(c(A), c(B)) \leq \min(r(A), r(B)) < \lfloor W/2 \rfloor.$$

Then the bound of Theorem 3.3, $r(A) + r(B)$, is better than the bound of Corollary 3.6, $\max(r(A), r(B)) + \lfloor W/2 \rfloor$. But if $\max(c(A), c(B)) \leq \min(r(A), r(B))$, both $r(A)$ and $r(B)$ are approximately $\lfloor \frac{2}{3} W \rfloor$, $\max(c(A), c(B)) \leq \lfloor W/2 \rfloor$ and

$$\max(r(A), r(B), \max_j w(T_j)) \leq \lfloor \frac{2}{3} W \rfloor,$$

then the bound in Theorem 3.3 is about $\lfloor \frac{4}{3}W \rfloor$ and the bound in Corollary 3.5 is at most $\lfloor \frac{7}{6}W \rfloor$, that is, better.

4. Professor–lecturer model

In this section we consider a special case of Problem 3 where some type of teachers has only group-lectures and the other has only individual lectures. We call the first type of teachers *professors* and the second type of teachers *lecturers*. Our model in this special case is called the *professor-lecturer model*. Clearly, every column in A corresponding to a lecturer consists of 0's only, and every column in B corresponding to a professor also consists of 0's only. This model was considered for the first time in [1].

Note that Problem 3 remains NP-complete even in this case because in the proof of Theorem 3.1 we have constructed, in fact, a professor-lecturer model.

Now we give a sharp bound of $t_{min}(A, B)$ in the case of the professor-lecturer model. Let for each $T_s, 1 \le s \le m$, and each $G_l, 1 \le l \le p$,

$$w(G_l, T_s) = \sum_{j=1}^{m} a_{lj} + \sum_{i=1}^{n} b_{is}$$

and $W_1(A, B) = \max w(G_l, T_s)$, where the maximum is taken over all pairs l, s with $\sum_{i:C_i \in G_l} b_{is} \ne 0$.

Furthermore, let $V_1 = \{C_1, ..., C_n\}, X_1 = \{G_1, ..., G_p\}$, and let V_2 denote the set of lecturers and X_2 the set of professors. Then the graph $H(B)$ has bipartition (V_1, V_2) and the subgraph $H(A)$ has bipartition (X_1, X_2).

Theorem 4.1 In the case of the professor-lecturer model

$$W \le t_{min}(A, B) \le \max(W, W_1(A, B)).$$

Proof We will prove the upper bound. Let $W_0 = \max(W, W_1(A, B))$. Clearly, we can properly color edges of the subgraph $H(A)$ with colors $1, 2, ..., W_0$. Now consider the subgraph $H = H(B)$. To each vertex $C_r \in V_1$ we assign the subset $L(C_r) \subseteq \{1, 2, ..., W_0\}$ of colors which are not used for coloring the edges incident with the vertex G_l where G_l is the group which contains the class C_r. Then, $|L(C_r)| = W_0 - \sum_{j=1}^{m} a_{lj}$, if $C_r \in G_l$. Clearly, $|L(C_r)| \ge \sum_{j=1}^{m} b_{rj}$ for each $r = 1, ..., n$ because $W_0 \ge W \ge \max_r w(C_r)$. Furthermore, $|L(C_r)| \ge \sum_{i=1}^{n} b_{is}$ for each pair C_r and T_s with $b_{rs} \ne 0$. (Otherwise

$$W_1(A, B) - \sum_{j=1}^{m} a_{lj} < \sum_{i=1}^{n} b_{is}$$

which contradicts the definition of $W_1(A,B)$). This means that the graph $H = H(B)$ satisfies the condition of Theorem 2.3. Therefore there exists an L-coloring of H, that is, a proper edge coloring of H in which each edge incident with a vertex C_r receives a color from the set $L(C_r)$. Together with the proper edge coloring of $H(A)$ this L-coloring gives a proper W_0-coloring of $H(A,B)$ which satisfies the condition of Proposition 3.2. Therefore there exists a university timetable of length W_0 which corresponds to the matrices A and B.

Now we will compare the upper bound in Theorem 4.1 and the upper bound for the general case in Corollary 3.6.

If $\max(W, W_1(A,B)) = W$ then, by Theorem 4.1, $t_{min}(A,B) = W$. Thus, in this case the bound in Theorem 4.1 is the best.

Now let $\max(W, W_1(A,B)) = W_1(A,B)$. We have

$$W_1(A,B) \leq \max_l \sum_{s=1}^{m} a_{ls} + \max_j \sum_{i=1}^{n} b_{ij} = r(A) + c(B)$$

$$\leq \max(r(A), r(B), \max_j w(T_j)) + \max(c(A), c(B), \lfloor W/2 \rfloor).$$

Thus, in this case also the bound in Theorem 4.1 is not larger than the bound in Corollary 3.6.

Remark 4.2. The following example shows that the upper bound in Theorem 4.1 is sharp. Moreover, in this case the upper bound in Theorem 4.1 is less than the upper bound in Corollary 3.6. Let $p = 1, n = m$ and the elements of A and B are defined as follows: $a_{1m} = 0$ and $a_{1i} = 1$ for $i = 1, ..., m-1$, and $b_{im} = 1$ for $i = 1, ..., m$ and all other elements of B are 0. Clearly, $W = m, c(A) = 1 = r(B)$, $c(B) = m$ and $r(A) = m-1$. Then the upper bound in Corollary 3.6 is $2m$ and the upper bound in Theorem 4.1 is $2m-1$. On the other hand, consider a timetable of length $t_{min}(A,B)$ corresponding to the matrices A and B. Clearly, $m-1$ columns are occupied by group-lectures. Thus, we have only $t_{min}(A,B) - m + 1$ columns for scheduling m individual lectures of the teacher T_m . This implies that $m \leq t_{min}(A,B) - m + 1$. Therefore, $t_{min}(A,B) = 2m - 1$ and the upper bound in Theorem 4.1 is sharp.

The next two corollaries follow easily from Theorem 4.1.

Corollary 4.3. Consider the professor-lecturer model and let t and k be positive integers, $k \leq t$, such that

a) the working time of each class and each professor does not exceed t,

b) each group has at most k group-lectures, and

c) the working time of each lecturer does not exceed $t-k$.

Then there exists a university timetable of length t corresponding to the matrices A and B.

Corollary 4.4 [1]. Consider the professor-lecturer model and let t be a positive integer such that

a) the working time of each class does not exceed t,

b) the working time of each professor and each lecturer does not exceed $\lfloor \frac{t}{2} \rfloor$,

c) each group has at most $\lfloor \frac{t}{2} \rfloor$ group-lectures.

Then there exists a university timetable of length t corresponding to the matrices A and B.

Theorem 4.5. Consider the professor-lecturer model and let t be a positive integer such that working time of each class and each professor does not exceed t. If there exists a sequential timetable corresponding to the matrix B then there exists a university timetable of length t corresponding to the matrices A and B.

Proof. First we properly color the edges of the graph $H(A)$ with colors $1, 2, ..., t$. Now to each vertex $C_r \in V_1$ of the graph $H(B)$ we assign a set of colors $L(C_r) \subseteq \{1, 2, ..., t\}$ which are not used for coloring of the edges incident with the vertex G_l where G_l is the group which contains the class C_r. Clearly, $|L(C_r)| = t - \sum_{j=1}^{m} a_{lj} \geq \sum_{j=1}^{m} b_{rj}$ because the working time of C_r is at most t. The existence of a sequential timetable corresponding to the matrix B is equivalent to the existence of V_1-sequential coloring of the graph $H(B)$. Then, by Theorem 2.2, there exists an L-coloring of $H(B)$. Together with the proper edge coloring of $H(A)$ this L-coloring gives a proper edge coloring of $H(A, B)$ which satisfies the conditions of Proposition 3.2. Therefore there exists a university timetable of length t, corresponding to the matrices A and B.

The next result follows from Theorem 4.5 and Proposition 2.1.

Corollary 4.6. Consider the professor-lecturer model and let t be a positive integer such that the working time of each class and each professor does not exceed t. If for each class C_i the number of individual lectures in C_i is not less than the working time of any lecturer teaching this class, then there exists a university timetable of length t corresponding to the matrices A and B.

A university timetable corresponding to the matrices A and B is called sequential if all classes have lessons without interruptions and all begin at the same time.

Proposition 4.7. Consider the professor-lecturer model. If there exists a sequential timetable corresponding to the matrix A, and a sequential timetable corresponding to the matrix B, then there exists a sequential university timetable corresponding to the matrices A and B.

Proof. Consider the graph $H(A)$ with bipartition (X_1, X_2) where $X_1 = \{G_1, ..., G_p\}$ and X_2 is the set of professors. First construct an X_1-sequential coloring of the graph $H(A)$. Now for each vertex $C_i \in V_1$ in $H(B)$ we define the set of colors

$$L(C_i) = \{1 + \sum_{j=1}^{m} a_{lj}, ..., w(C_i)\}.$$

Clearly, $H(B)$ has a V_1-sequential coloring. Then, by Theorem 2.2, there exists an L–coloring of $H(B)$. Together with the X_1-sequential coloring of $H(A)$ this L-coloring defines a sequentional university timetable corresponding to the matrices A and B.

Theorem 4.8. Consider the professor-lecturer model where the working time of each professor does not exceed the working time of each class, and the number of individual lectures for each class C_i is not less than the working time of each lecturer teaching that class. Then there exists a sequential university timetable corresponding to the matrices A and B.

Proof. First, consider the graph $H(A)$ with bipartition (X_1, X_2). To each $G_l \in X_1$ we assign a set of colors $L'(G_l) = \{1, 2, ..., w_l\}$, where $w_l = \min_{i:C_i \subseteq G_l} w(C_i)$. For each $l = 1, ..., p$ the number $|L'(G_l)|$ is not less than the working time of each professor teaching G_l, that is, $|L'(G_l)| \geq d_{H(A)}(T_j)$ for every pair G_l, T_j with $a_{lj} \neq 0$. This condition for $H(A)$ is similar to the condition for $H(B)$ in Theorem 2.3. Therefore we can state,in a similar way as for $H(B)$, that there exists an L'–coloring of $H(A)$.

Now to each vertex $C_i \in V_1$ we assign the subset $L(C_i) \subseteq \{1, 2, ..., w(C_i)\}$ of colors which are not used for coloring the edges incident with the vertex G_l where G_l is the group which contains the class C_i. We have $|L(C_i)| = \sum_{j=1}^{m} b_{ij}$ and $|L(C_i)|$ is not less than the working time of each lecturer teaching C_i. This means that the graph $H(B)$ satisfies the condition of Theorem 2.3. Therefore there exists an L–coloring of $H(B)$. Together with the L'-coloring of $H(A)$ this L-coloring of $H(B)$ defines a sequential university timetable corresponding to the matrices A and B.

5. Conclusion

We have examined a generalized class-teacher model which is able to handle situations where there are several disjoint groups of classes which have to take some group-lectures. This model present some interest in itself since the presence of groups is a characteristic of some specific timetabling problems. In addition to the results presented here, further research will be needed to deal with practical problems where many additional types of requirements are to be taken into account. Our purpose was simply to give a formulation of the group constraints and to examine some basic properties of this model.

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