Some Results on an Edge Coloring Problem of Folkman and Fulkerson*

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Abstract. In 1968, Folkman and Fulkerson posed the following problem: Let G be a graph and let $(n_1, \ldots, n_t)$ be a sequence of positive integers. Does there exist a proper edge coloring of G with colors $1, 2, ..., t$ such that precisely n_i edges receive color i, for each $i = 1, \ldots, t$? If such a coloring exists then the sequence $(n_1, \ldots, n_t)$ is called color-feasible for G.

Some sufficient conditions for a sequence to be color-feasible for a bipartite graph where found by Folkman and Fulkerson, and de Werra.

In this paper we give a generalization of their results for bipartite graphs. Furthermore, we find a set of color-feasible sequences for an arbitrary simple graph. In particular, we describe the set of all sequences which are color-feasible for a connected simple graph G with $\Delta(G) \geq 3$, where every pair of vertices of degree at least 3 are non-adjacent.

Keywords: Graphs; Edge colorings; Color-feasible sequences

1. Introduction

We use Bondy and Murty [7] for terminology and notation not defined here. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph G. For each vertex u of G let $N_G(u)$ denote the set of vertices adjacent to u and $d_G(u)$ denote the degree of u. The maximum vertex degree of G is denoted by $\Delta(G)$. An edge t-coloring or simply t-coloring of G is a mapping $f : E(G) \longrightarrow \{1, ..., t\}$. If $e \in E(G)$ and $f(e) = k$ then we say that the edge e is colored k. A t-coloring of G is called proper if no pair of adjacent edges receives the same color. The minimum number t for which there exists a proper t-coloring of G is called the chromatic index of G and is denoted by $\chi'(G)$. A graph is simple if it has no loops and no two of its edges join the same pair of vertices.

In 1968, Folkman and Fulkerson [10] posed and investigated the following problem:

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Problem 1. Let G be a graph with q edges, and let $(n_1, \ldots, n_t)$ be a sequence of non-increasing positive integers, $\sum_{i=1}^{t} n_i = q$. Does there exist a proper t-coloring of G in which precisely n_i edges receive color i, for each $i = 1, \ldots, t$?

If such a coloring exists then the sequence $(n_1, \ldots, n_t)$ is called color-feasible for G.

If G is a bipartite graph with bipartition (V_1, V_2) where $V_1 = \{x_1, ..., x_n\}$ and $V_2 = \{y_1, ..., y_m\}$, then G can be represented by an integral $n \times m$ matrix $B = (b_{ij})$ where b_{ij} is the number of edges in G with endvertices x_i and $y_j, 1 \leq i \leq n, 1 \leq j \leq m$. Then Problem 1 has the following two reformulations:

Problem 1a. When can a matrix B with nonnegative integer entries be written as a sum

$$B = P_1 + P_2 + ... + P_t,$$

where each P_i is a permutation matrix of size n_i, that is, P_i has at most one 1 in each row and column and contains n_i 1's?

Problem 1b. In a school, there are n classes $C_1, ..., C_n$ and m teachers $T_1, ..., T_m$. Given that the number of one hour lectures which teacher T_j must give to class C_i, is b_{ij}. Also the number n_h of classrooms available at period h is given, for $h = 1, ..., t$. The problem is to determine whether there exists a timetable of t periods, so that each class receives all its teaching corresponding to the matrix $B = (b_{ij})$, precisely n_h classrooms are used in each hour h and no class or teacher is involved in more than one lecture at a time?

A general way for investigation of Problem 1 was suggested by Folkman and Fulkerson.

Let D_q denote the set of non-increasing sequences of positive integers which sum to q. For two sequences $P = (p_1, \ldots, p_m)$ and $N = (n_1, \ldots, n_t)$ from D_q, the sequence P is said to majorise N, written $P \succeq N$, if $m \leq t$ and $\sum_{i=1}^{r} p_i \geq \sum_{i=1}^{r} n_i$, for each $r = 1, \ldots, m-1$. Clearly the majorisation relation can be viewed as a partial order on the set D_q.

Theorem 1.1([10]). If a sequence $P \in D_q$ is color-feasible for G then every sequence $N \in D_q$ for which $P \succ N$ is also color-feasible for G.

The proof of Theorem 1.1 in [10] provides a polynomial algorithm for constructing a coloring corresponding to N if a coloring corresponding to P is given.

The sequences in the set D_q can be classified in the following way [18]. With each non-increasing sequence $N = (n_1, \dots, n_t)$ we associate a new sequence $s(N) = (s(0), s(1), \dots, s(l))$, where $s(0) = 0$ and $s(i+1) = \max\{j : 1 + n_j \geq n_{s(i)+1}\}$, for $i = 0, 1, \dots, l-1$. For example, if $N = (9, 8, 8, 7, 4, 3)$ then $s(N) = (0, 3, 4, 6)$. In fact the integer l in the definition of $s(N)$ is the minimum number of disjoint subsequences, in each of which any two members differ by at most 1. If the sequence $s(N)$ consists of l positive members then we call N an l-step sequence.

The next result follows from Theorem 1.1.

Corollary 1.2 ([10,18]). A 1-step sequence $N \in D_q$ of length t is color-feasible for a graph G with q edges if and only if $\chi'(G) \leq t$.

Since $\chi'(G) = \Delta(G)$ for a bipartite graph G, we obtain the following result.

Corollary 1.3 A 1-step sequence $N \in D_q$ of lenth t is color-feasible for a bipartite graph G with q edges if and only if $\Delta(G) \leq t$.

It is known that the problem of deciding whether $\chi'(G) \leq t$ is NP-complete even in the case when G is a simple t-regular graph [13,14]. Therefore, Corollary 1.2 implies that Problem 1 is NP-complete in general case.

However, given a little more notation, we can formulate a simple necessary condition for the color-feasibility of a sequence N.

Let k be a positive integer. An edge subset $F \subseteq E(G)$ is called a k-matching of G if each vertex of G is incident with at most k edges of F. A k-matching of maximum cardinality is called a maximum k-matching of G. We shall denote by $q_k(G)$ the number of edges in a maximum k-matching of G. Note that the number $q_k(G)$ can be found in polynomial time [5].

Let $D_q(G)$ denote the set of all sequences $(n_1, ..., n_t)$ in D_q which satisfy the following condition:

$$\left.\begin{array}{l} t \geq \Delta(G), \quad |E(G)| = \sum_{i=1}^{t} n_i \\ \text{and} \\ q_k(G) \geq \sum_{i=1}^{k} n_i, \quad \text{for } k = 1, \dots, \Delta(G) - 1. \end{array}\right\}$$

Clearly, this condition can be checked in polynomial time.

Property 1.4. If a sequence $N \in D_q$ is color-feasible for G, then $N \in D_q(G)$.

In some cases this necessary condition, $N \in D_q(G)$, is also sufficient for color-feasibility of N for G.

Theorem 1.5. ([18]) Let G be a bipartite graph with q edges. Then a 2-step sequence $N \in D_q$ is color-feasible for a bipartite graph G if and only if $N \in D_q(G)$.

Corollary 1.6. ([10]) Let G be a bipartite graph and $N = (n_1, \dots, n_t)$ be a sequence in D_q with $n_1 = \dots = n_k > n_{k+1} = \dots = n_t$. Then N is color-feasible for G if and only if $N \in D_q(G)$.

Some other properties of the set of color-feasible sequences for a bipartite graph G can be found in [1,2,8,10,15,18,19,20].

Corollary 1.3 and Theorem 1.5 imply that Problem 1 is solved polynomially if G is a bipartite graph and N is a 1- or 2-step sequence. However, even for a bipartite graph G Problem 1 is NP-complete if N is a 3-step sequence. (Indeed, Problem 1 is NP-complete even if G is bipartite, $\Delta(G) = 3$ and $N = (n_1, n_2, n_3)$ (see [4,12])).

Let G be a graph with $\Delta = \Delta(G)$ and, for each integer k, let G_k denote the subgraph induced by the set of vertices of degree at least k. Fournier [11] proved that if G is a simple graph and the subgraph G_Δ has no edges then $\chi'(G) = \Delta(G)$. Berge and Fournier [6] observed that the same proof actually gives the following broader result, which was stated earlier by Lovasz and Plummer ([16], 7.4.3).

Proposition 1.7 If G is a simple graph where the subgraph G_Δ is acyclic, then $\chi'(G) = \Delta(G)$.

In this paper we use similar ideas for investigation of Problem 1. Let $\Delta_1(G)$ denote the minimum integer k such that the subgraph G_{k+1} is acyclic. In particular, if the subgraph G_Δ contains a cycle, then $\Delta_1(G) = \Delta(G)$. Furthermore, let $\Delta_2(G)$ denote the minimum integer k such that the subgraph G_{k+1} contains no edges. It is clear that $\Delta_1(G) \leq \Delta_2(G) \leq \Delta(G)$. In this terminology Proposition 1.7 can be reformulated in the following way: if $\Delta_1(G) < \Delta(G)$ then $\chi'(G) = \Delta(G)$.

We say that the number $\Delta_i(G), i \in \{1, 2\}$, is a threshold for a sequence $N \in D_q$ with $s(N) = (s(0), s(1), ..., s(l))$ if $s(1) > \Delta_i(G)$.

The following results are obtained in this paper.

1.We give a sufficient condition for an l-step sequence $N \in D_q$ with $l \geq 2$ to be color-feasible for a bipartite graph. This result implies Theorem 1.5.

2. We investigate Problem 1 for an arbitrary simple graph G:

a) We prove that all sequences with threshold $\Delta_2(G)$ in the set $D_q(G)$ are color-feasible for G. We also prove that if G is connected and $\Delta_2(G) = 2 < \Delta(G)$, that is, every pair of vertices of degree at least 3 are non-adjacent, then all sequences in $D_q(G)$ are color-feasible for G.

b) We show that all 1- and 2-step sequences with threshold $\Delta_1(G)$ in the set $D_q(G)$ are color-feasible for G.

Note that we described in [3] a polynomial algorithm to solve the following problem: are all 3-step sequences with threshold $\Delta_1(G)$ in the set $D_q(G)$ color-feasible for G? By using this algorithm we can, in particular, determine for an arbitrary tree G: are all 3-step sequences in $D_q(G)$ color-feasible for G? For a tree with bounded degrees there exists a polynomial algorithm to determine all color-feasible sequences (see [21]).

2. Color-feasibility for bipartite graphs

Let G be a graph with $|E(G)| = q$ and $\Delta(G) = \Delta$, and let $H(G) = (h_1, ..., h_\Delta)$ be a sequence where $h_1 = q_1(G)$ and $h_i = q_{i+1}(G) - q_i(G)$, for $i = 1, ..., \Delta - 1$. If $H(G)$ is non-increasing, that is, $H(G) \in D_q(G)$ then the condition $N \in D_q(G)$ is equivalent to the condition $H(G) \succeq N$.

Remark 2.1. It follows from Theorem 1.1 that if $H(G)$ is color-feasible for G then $D_q(G)$ is the set of all color-feasible sequences for G.

It is known [18] that if G is a bipartite graph then $H(G) \in D_q$. The following criterion for feasibility of $H(G)$ was found by de Werra [18]: Let G be a bipartite graph with $\Delta(G) = \Delta$ and let $s(H(G)) = (s(0), s(1), ..., s(l)), l \geq 2$. Then $H(G)$ is color-feasible for G if and only if there exist edge subsets $F_1, F_2, ..., F_l$ such that $F_1 \subset F_2 \subset ... \subset F_l$ and F_j is a maximum $s(j)$-matching of G, for each $j = 1, ..., l$.

An immediate corollary (not stated in an explicit form in [18]) is a criterion (Proposition 2.2) for an arbitrary sequence N to be color-feasible for a bipartite graph G. The interest of the proof given in the present paper is that it is constructive: if G and N satisfy this criterion then a corresponding coloring is constructed in polynomial time.

Proposition 2.2. Let G be a bipartite graph, and let $N = (n_1, \ldots, n_t)$ be a non-increasing sequence with $s(N) = (s(0), s(1), \ldots, s(l))$, $l \geq 2$. Then N is color-feasible for G if and only if there exist edge subsets $F_1, F_2, ..., F_l$ such that $F_1 \subset F_2 \subset ... \subset F_l$ and F_j is an $s(j)$-matching with $\sum_{i=1}^{s(j)} n_i$ edges, for each $i = 1, ..., l$.

Proof. Suppose that G has a proper t–coloring corresponding to N. Then the set F_j consisting of edges colored $1, 2, ..., s(j)$ is an $s(j)$–matching of G, $j = 1, ..., l$, and $F_1 \subset F_2 \subset ... \subset F_l$.

Conversely, suppose that there exist edge subsets $F_1, ..., F_l$ satisfying the condition of Proposition 2.2. We will prove that the edges in F_j can be properly colored with colors $1, \ldots, s(j)$ such that precisely n_i edges are colored i, for $i = 1, \ldots, s(j)$.

For $j = 1$ it follows from Corollary 1.3 because $(n_1, ..., n_{s(1)})$ is a 1-step sequence, $|F_1| = \sum_{i=1}^{s(1)} n_i$ and F_1 is an $s(1)$-matching. Suppose that the required coloring is already constructed for F_j, $1 \leq j < l$. We will color edges from $F_{j+1} \backslash F_j$ with colors from the set $C = \{1, 2, \ldots, s(j+1)\}$.

Let e be an edge which has so far not been colored, and u and v be the ends of e. If there is a color $\alpha \in C$ for which there is no edge colored α adjacent to e, then use color α to color e. Otherwise, since F_{j+1} is an $s(j+1)$-matching, and we have $s(j+1)$ colors, there are a color $t_v \in C$ which is not used to color an edge incident with u, and a color $t_u \in C$, $t_u \neq t_v$, which is not used to color an edge incident with v. Consider a path P of maximum length with initial vertex u whose edges are alternatively colored t_u and t_v. Clearly, P cannot pass through v, otherwise $E(P) \cup \{e\}$ forms an odd cycle in G, which contradicts G being bipartite. Thus if we interchange the two colors t_u and t_v along P, the color t_u will no longer be used on an edge adjacent to either vertex, and we can color e with t_u.

Suppose that the proper coloring of the edges of F_{j+1} produced by this procedure is such that n'_i edges are colored i, for each $i = 1, \ldots, s(j+1)$. We may assume (possibly after permuting the colors) that $n'_1 \geq n'_2 \geq \cdots \geq n'_{s(j+1)}$. It is not difficult to see that the above procedure of coloring certainly guarantees that $n'_i \geq n_i$, for each $i = 1, \ldots, s(j)$. Let $k(j+1)$ denote the maximal i with $n'_i > 0$. Then $s(j) < k(j+1) \leq s(j+1)$ and $(n'_1, \ldots, n'_{k(j+1)}) \succeq (n_1, \ldots, n_{s(j+1)})$ because $n_{1+s(j)} - n_{s(j+1)} \leq 1$.

If $(n'_1, \ldots, n'_{k(j+1)}) \neq (n_1, \ldots, n_{s(j+1)})$ then, by using the algorithm suggested in [10], we can polynomially transform the coloring of F_{j+1} corresponding to the sequence $(n'_1, ..., n'_{k(j+1)})$ to a proper $s(j+1)$-coloring of the edges of F_{j+1} corresponding to the sequence $(n_1, n_2, ..., n_{s(j+1)})$. ■

The next auxiliary lemma is a corollary of a result of Berge (see [5]).

Lemma 2.3. Let s be a positive integer and F a subset of the set of edges in a graph G. Then F is a maximum s-matching of G if and only if there is no path P such that edges of P are alternatively in F and $E(G) \setminus F$, both the first and the last edge of P is in $E(G) \setminus F$ and the number of edges of F incident with the origin of P as well as the number of edges incident with the terminus of P is less than s.

Theorem 2.4. Let N be a sequence in D_q with $s(N) = (s(0), s(1), \ldots, s(l))$, $l \geq 2$. Then N is feasible for a bipartite graph G with q edges if every pair of vertices of degree more than $s(2)$ are non-adjacent in G and $H(G) \succeq N$.

Proof. Let X_1 be a maximum $s(1)$-matching of G. We shall construct edge subsets $X_2, \dots, X_l$ in the following way: suppose that $X_1, ..., X_{i-1}$ have already been constructed $(2 \leq i \leq l)$. If $s(i) \geq \Delta(G)$, put $X_i = E(G)$. Otherwise, at each vertex u with $d_G(u) > s(i)$ delete precisely $d_G(u) - s(i)$ edges from $E(G)\backslash X_{i-1}$. The remaining edges of $E(G)\backslash X_{i-1}$ together with X_{i-1} form the next edge subset X_i.

By our construction, the following property holds for each edge $uv \in E(G) \setminus X_i$:

if u is incident with less than $s(i)$ edges of X_i then $d_G(u) \leq s(i), d_G(v) > s(i)$ and v is incident with exactly $s(i)$ edges of X_i,

if u is incident with exactly $s(i)$ edges of X_i then $d_G(u) > s(i)$ and, therefore, $d_G(v) \leq s(i)$ since every pair of vertices of degree more than $s(2)$ are non-adjacent in G. It means that v is incident with less than $s(i)$ edges of X_i.

It is not difficult to check now that every alternating path $P = a_0 a_1 \dots a_{2r+1}$ relative to X_i (i.e., whose edges are alternatively in X_i and in $E(G) \setminus X_i$) with end edges in $E(G)\backslash X_i$ has the following property: if a_0 is incident with less than $s(i)$ edges of X_i then each of the vertices $a_1, a_3, \dots, a_{2r+1}$ is incident with exactly $s(i)$ edges of X_i, and each of the vertices $a_2, a_4, ..., a_{2r}$ is incident with less than $s(i)$ edges of X_i. We begin with the vertex a_1.

This property and Lemma 2.3 imply that X_i is a maximum $s(i)$-matching of G. Clearly $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_l$. From X_1 choose a subset of edges F_1 of size $\sum_{i=1}^{s(1)} n_i$, and from X_j choose a subset F_j of size $\sum_{i=1}^{s(j)} n_i$ containing F_{j-1}, for each $j = 2, \dots, l$. This is possible since $H(G) \succeq N$. Now the theorem follows from Proposition 2.2. ■

Corollary 2.5([18]). Let G be a bipartite graph with q edges. A sequence $N = (n_1, ..., n_t) \in D_q$ with $s(N) = (s(0), s(1), s(2))$ is color-feasible for G if and only if $H(G) \succeq N$.

Proof. Clearly, $s(2) = t$. If $H(G) \succeq N$ then $t \geq \Delta(G)$, and the conditions of Theorem 2.4 are trivially satisfied. Therefore N is color-feasible for G. ■

Theorem 2.6. Let G be a bipartite graph with bipartition (V_1, V_2) where $d_G(x) \geq d_G(y)$ for each edge (x, y) with $x \in V_1$ and $y \in V_2$. Then $D_q(G)$ is the set of all color-feasible sequences for G. Furthermore, the sequence $H(G) = (h_1, h_2, ..., h_\Delta)$ satisfies the condition: $h_i = |\{x \in V_1 / d_G(x) \geq i\}|$ for each $i = 1, ..., \Delta$.

Proof. We will show that the edges of G can be properly colored with colors $1, 2, ..., \Delta(G)$ such that edges incident to each vertex $x \in V_1$ are colored with colors

$1, 2, ..., d_G(x)$. This implies that $H(G)$ satisfies the above condition and is color-feasible for G. Then, by Remark 2.1, $D_q(G)$ is the set of all color-feasible sequences for G.

Let $V_1 = \{x_1, x_2, \ldots, x_n\}$ with $d_G(x_1) \geq d_G(x_2) \geq \cdots \geq d_G(x_n)$. Suppose that the edges incident with each vertex x_i, $i = 1, \ldots, k$, $k < n$, are already colored with the colors $1, 2, \ldots, d_G(x_i)$, and the edges incident with the vertices $x_{k+1}, ..., x_n$ are not colored yet. Let $e_1, e_2, \ldots, e_{d(x_{k+1})}$ be the edges incident with x_{k+1}. Then for each $j = 1, \ldots, d(x_{k+1})$ in turn we do the following:

Consider the vertex y_j which is the end in V_2 of the edge e_j. Since $d_G(x_{k+1}) \geq d(y_j)$ there is a color l such that $1 \leq l \leq d_G(x_{k+1})$ and there are no edges incident with y_j colored l. If $l = j$, then color e_j with color j.

Otherwise, consider a path P of maximum length with origin at y_j whose edges are alternatively colored j and l. Clearly, by construction, this path must end in V_2. Thus we may interchange the two colors along this path, to make color j available at y_j, and color e_j with color j. ■

The above conditions use the structure of a graph G. Now we will give some other type of conditions for color-feasibility of a sequence of length 3.

Definition. Let $N = (n_1, n_2, n_3)$ be a non-increasing sequence. We define the weight of N, denoted $w(N)$, by $w(N) = 3n_1 + 2n_2 + n_3$.

Theorem 2.7. Let G be a bipartite graph with $\Delta(G) = 3$, $|E(G)| = q$ and $H(G) = (h_1, h_2, h_3)$. Then every sequence $N = (n_1, n_2, n_3) \in D_q$, satisfying $H(G) \succeq N$ and

$$w(N) \leq w(H(G)) - \lfloor\frac{h_1 - h_2}{2}\rfloor - \lfloor\frac{h_2 - h_3}{2}\rfloor + 1,$$

is color-feasible for G.

Proof. The proposition is evident if $H(G)$ is color-feasible. Suppose that $H(G)$ is not color-feasible for G. It follows from Corollary 1.3 and Theorem 1.5 that $H(G)$ is a 3-step sequence. Therefore, $h_1 - h_2 \geq 2$ and $h_2 - h_3 \geq 2$. Define three sequences:

$N_0 = (h_1 - \lfloor\frac{h_1-h_2}{2}\rfloor, h_2 + \lfloor\frac{h_1-h_2}{2}\rfloor, h_3)$,

$N_1 = (h_1, h_2 - \lfloor\frac{h_2-h_3}{2}\rfloor, h_3 + \lfloor\frac{h_2-h_3}{2}\rfloor)$,

$N_2 = (h_1 - \lfloor\frac{h_1-h_2}{2}\rfloor, h_2 + \lfloor\frac{h_1-h_2}{2}\rfloor - \lfloor\frac{h_2-h_3}{2}\rfloor, h_3 + \lfloor\frac{h_2-h_3}{2}\rfloor)$.

Clearly, N_0 and N_1 are 2-step sequences and $H(G) \succ N_0, H(G) \succ N_1$. Then, by Theorem 1.5, N_0 and N_1 are color-feasible for G. Furthermore, it is clear that $w(N_2) = (3h_1 + 2h_2 + h_3) - \lfloor\frac{h_1-h_2}{2}\rfloor - \lfloor\frac{h_2-h_3}{2}\rfloor = w(H(G)) - \lfloor\frac{h_1-h_2}{2}\rfloor - \lfloor\frac{h_2-h_3}{2}\rfloor$.

Let $N = (n_1, n_2, n_3)$ be a sequence in D_q satisfying $H(G) \succeq N$ and $w(N) \leq w(N_2) + 1$. We will show that either $N_0 \succeq N$ or $N_1 \succeq N$. Suppose that this is not true. It means that $n_1 = h_1 - \lfloor\frac{h_1-h_2}{2}\rfloor + b_1$, where $0 < b_1 \leq \lfloor\frac{h_1-h_2}{2}\rfloor$, and $n_1 + n_2 = h_1 + h_2 - \lfloor\frac{h_2-h_3}{2}\rfloor + b_2$, where $0 < b_2 \leq \lfloor\frac{h_2-h_3}{2}\rfloor$. This implies that

$$n_2 = h_2 + \lfloor\frac{h_1 - h_2}{2}\rfloor - \lfloor\frac{h_2 - h_3}{2}\rfloor + b_2 - b_1, n_3 = h_3 + \lfloor\frac{h_2 - h_3}{2}\rfloor - b_2.$$

But then $w(N) = 3n_1 + 2n_2 + n_3 = w(N_2) + 3b_1 + 2(b_2 - b_1) - b_2 = w(N_2) + b_2 + b_1 > w(N_2) + 1$, which contradicts the condition $w(N) \leq w(N_2) + 1$. Therefore, either $N_0 \succeq N$ or $N_1 \succeq N$. Then, by Theorem 1.1, N is color-feasible for G. ■

The bound in Theorem 2.7 is sharp in the sense that for every $r \geq 1$ there exists a bipartite graph G with $q = 3r + 6$ edges and maximum degree 3 such that every sequence $N = (n_1, n_2, n_3) \in D_q$, satisfying $H(G) \succeq N$ and

$$w(N) \geq w(H(G)) - \lfloor\frac{h_1 - h_2}{2}\rfloor - \lfloor\frac{h_2 - h_3}{2}\rfloor + 2,$$

is not color-feasible for G.

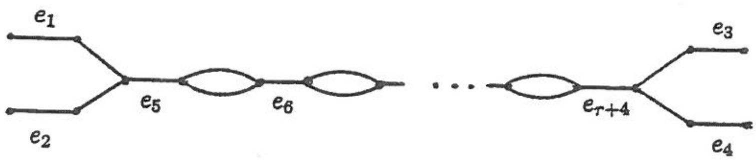

Fig. 1

Consider, for example, the graph G in Fig.1. Clearly, $H(G) = (r + 4, r + 2, r)$. Furthermore, $H(G)$ is not color-feasible for G, because G has the unique maximum matching $M_1 = \{e_1, e_2, ..., e_{r+4}\}$ and the graph $G - M_1$ has no matching of cardinality $r + 2$. Finally, by Theorem 2.7, every sequence of weight at least $w(H(G)) - 1$ is color-feasible for G.

3. Color-feasibility for arbitrary simple graphs

Let G be a simple graph with $\Delta(G) = m$ where the subgraph induced by the set of vertices of degree m is acyclic. Assume that a subset $E_0 \subset E(G)$ is properly colored with colors $1, 2, ..., m$ such that exactly n_i edges are colored i, for $i = 1, ..., m$. For each vertex y let $C(y)$ denote the set of colors of the edges incident with y and

$\overline{C(y)} = \{1, 2, ..., m\} \setminus C(y)$. A generalization of Vizing's theorem [17] was obtained in [6]. We use similar considerations for investigation of Problem 1.

Definition([6]). Let $e = (x_0, y_0)$ be an uncolored edge of G and α_1 be a color such that $\alpha_1 \notin C(y_0)$ and $\alpha_1 \in C(x_0)$. We define a sequence $S(x_0, \alpha_1)$ of distinct edges $e_0, e_1, e_2, ...$ all incident with x_0, together with a function f that associates to each edge e_i of the sequence a color $\alpha_{i+1} = f(e_i)$, according to the following iterative procedure.

(I) Put $e_0 = e, f(e_0) = \alpha_1$.

(II) Suppose that the edges $e_0 = (x_0, y_0), ..., e_{i-1} = (x_0, y_{i-1})$ are already included in $S(x_0, \alpha_1)$ and $f(e_0) = \alpha_1, ..., f(e_{i-1}) = \alpha_i$ are already defined, $i \geq 1$.

a) If $\alpha_i \in C(x_0)$ and $\alpha_i \neq f(e_j)$ for all $j < i - 1$, consider the edge $e_i = (x_0, y_i)$ incident with x_0 that is colored with α_i; let $\alpha_{i+1} = f(e_i)$ be a color satisfying the condition $\alpha_{i+1} \notin C(y_i)$.

b) If either $\overline{C(y_{i-1})} = \emptyset$, or $\alpha_i \notin C(x_0)$, or $\alpha_i = f(e_j)$ for an index $j < i - 1$, then we stop, and the sequence $S(x_0, \alpha_1)$ is achieved, $S(x_0, \alpha_1) = (e_0, e_1, ..., e_{i-1})$.

Proposition 3.1. Let G be a simple graph with $\Delta(G) = m$ where the subgraph induced by the set of vertices of degree m is acyclic. Assume that a subset $E_0 \subset E(G)$ is colored with colors $1, 2, ..., m$ such that precisely n_i edges are colored i, for $i = 1, ..., m$. Then for an uncolored edge e the set $E_0 \cup \{e\}$ can be colored with colors $1, ..., m$ such that at least n_i edges are colored i, for each $i = 1, ..., m$.

Proof. Without loss of generality we suppose that the only uncolored edge is e, that is, $E(G) = E_0 \cup \{e\}$. Let $e = (b_0, b_1)$. Consider the following algorithm. First we label vertices b_0 and b_1.

Step $r (r \geq 0)$. Suppose that the vertices $b_0, b_1, ..., b_{r+1}$ have been already labelled and (b_r, b_{r+1}) is the only uncolored edge of G. Choose a color $\alpha_1 \notin C(b_r)$. If $\alpha_1 \notin C(b_{r+1})$ then color the edge (b_r, b_{r+1}) with α_1. If $\alpha_1 \in C(b_{r+1})$ then construct a sequence $S(b_{r+1}, \alpha_1)$. Let (b_{r+1}, b_{r+2}) be the last edge in $S(b_{r+1}, \alpha_1)$.

a) If $\overline{C(b_{r+2})} \neq \emptyset$ then, by using the same considerations as in [6], the edges in $E_0 \cup \{e\}$ can be properly colored with colors $1, ..., m$. It is not difficult to check that the method of coloring described in [6] guarantees that at least n_i edges are colored i, for $i = 1, ..., m$.

b) Suppose that $\overline{C(b_{r+2})} = \emptyset$ and $S(b_{r+1}, \alpha_1) = (e_0, e_1, ..., e_t)$ where $e_0 = (b_r, b_{r+1}), e_t = (b_{r+1}, b_{r+2})$, e_j is colored α_j and $f(e_j) = \alpha_{j+1}$, $j = 1, ..., t$. For each

$j = 1, ..., t$ remove the color α_j from e_j and assign it instead to e_{j-1}. Now the only uncolored edge is (b_{r+1}, b_{r+2}). Label the vertex b_{r+2} and go to Step $(r + 1)$.

It is not difficult to see that if the required coloring is not constructed on Step r then $\overline{C(b_{r+2})} = \emptyset$, that is, the new labelled vertex b_{r+2} has degree $\Delta(G)$. Since the subgraph of G induced by the vertices of degree $\Delta(G)$ is acyclic, the vertices $b_2, b_3, ...$ constructed by the algorithm, are different. Therefore, on some step of the algorithm the required coloring of $E_0 \cup \{e\}$ will be constructed. ■

Proposition 3.2. Let G be a simple graph with q edges, and let $N = (n_1, ..., n_t)$ be a sequence in the set D_q with $s(N) = (s(0), s(1), ..., s(l))$ such that $l \geq 2$ and $s(2) > \Delta_1(G)$. Then N is color-feasible for G if and only if there exist subsets $F_1, ..., F_l$ such that $F_1 \subset F_2 \subset ... \subset F_l$, the set F_j is an $s(j)$-matching with $\sum_{i=1}^{s(j)} n_i$ edges, for $j = 1, ..., l$, and edges of F_1 can be properly colored with $s(1)$ colors.

Proof. The necessity is evident: if G has a proper t–coloring corresponding to N then the set of edges F_i consisting of edges colored $1, 2, ..., s(i)$ is a $s(i)$–matching for each $i = 1, ..., l$, and $F_1 \subset F_2 \subset ... \subset F_l$.

Conversely, suppose that there exist subsets $F_1, ..., F_l$ satisfying the condition of the proposition. We will prove that the edges in F_j can be properly colored with colors $1, \ldots, s(j)$ such that precisely n_i edges are colored i, for $i = 1, \ldots, s(j)$. By the assumption, the edges in F_1 can properly colored with colors $1, ..., s(1)$. Therefore, by Corollary 1.2, there is a proper $s(1)$-coloring of F_1 corresponding to the sequence $(n_1, ..., n_{s(1)})$.

Suppose that the required coloring is already constructed for F_j, $1 \leq j < l$. Let H_{j+1} denote the subgraph induced by the set F_{j+1}. Since $s(j + 1) > \Delta_1(G)$, the subgraph of H_{j+1} induced by the set of vertices of degree $s(j+1)$ in H_{j+1}, is acyclic. Then, by Proposition 3.1, the edges in F_{j+1} can be colored with colors $1, 2, ..., s(j+1)$ such that at least n_i edges in F_{j+1} are colored i, for each $i = 1, 2, ..., s(j)$.

Suppose that precisely n'_i edges are colored i, for $i = 1, \ldots, s(j + 1)$. We may assume (possibly after permuting the colors) that $n'_1 \geq n'_2 \geq \cdots \geq n'_{s(j+1)}$ and $n'_i \geq n_i$, for each $i = 1, \ldots, s(j)$. Let $k(j + 1)$ denote the maximal i with $n'_i > 0$. Then $s(j) < k(j + 1) \leq s(j + 1)$ and $(n'_1, \ldots, n'_{k(j+1)}) \succeq (n_1, \ldots, n_{s(j+1)})$ because $n_{1+s(j)} - n_{s(j+1)} \leq 1$.

If $(n'_1, \ldots, n'_{k(j+1)}) \neq (n_1, \ldots, n_{s(j+1)})$ then, by using the algorithm, suggested in [10] we can polynomially transform the coloring of F_{j+1} corresponding to the sequence $(n'_1, ..., n'_{k(j+1)})$, to a proper $s(j+1)$-coloring of the edges of F_{j+1} corresponding to the sequence $(n_1, n_2, ..., n_{s(j+1)})$. ■

Note that the proof of Proposition 3.2 provides a polynomial algorithm for constructing a coloring corresponding to the sequence N, if the required sets $F_1, ..., F_l$ are given.

Proposition 3.3. Let G be a simple graph. Then all 1- and 2-step sequences with threshold $\Delta_1(G)$ in the set $D_q(G)$ are color-feasible for G.

Proof. Proposition 1.7 and Corollary 1.2 imply that all 1-step sequences with threshold $\Delta_1(G)$ in $D_q(G)$ are color-feasible for G. Now consider a 2-step sequence $N = (n_1, ..., n_t)$ in $D_q(G)$ with $s(N) = (s(0), s(1), s(2))$ and $s(1) > \Delta_1(G)$. Construct a maximum $s(1)$-matching F of G. Clearly, $\sum_{i=1}^{s(1)} n_i \le |F|$ since $N \in D_q(G)$. Choose in F a subset F_1 of $\sum_{i=1}^{s(1)} n_i$ edges. Let H denote the subgraph induced by F_1. Since $s(1) > \Delta_1(G)$, the subgraph of H induced by vertices of degree $s(1)$ is acyclic. By Proposition 1.7, edges of F_1 can be properly colored by $s(1)$ colors. Put $F_2 = E(G)$. Then F_1 and F_2 satisfy the condition of Proposition 3.2. Therefore, N is color-feasible for G. ■

Theorem 3.4. Let G be a simple graph. Then all sequences with threshold $\Delta_2(G)$ in the set $D_q(G)$ are color-feasible for G.

Proof. Let $N = (n_1, ..., n_t) \in D_q(G), s(N) = (s(0), s(1), ..., s(l))$ and $s(1) > \Delta_2(G)$. Then $s(1) > \Delta_1(G)$ since $\Delta_2(G) \ge \Delta_1(G)$. If $l \le 2$ then, by Proposition 3.3, N is color-feasible for G. Now suppose that $l \ge 3$.

Let X_1 be a maximum $s(1)$-matching of G. We shall construct edge subsets $X_2, \ldots, X_l$ in the following way: suppose that $X_1, \ldots, X_{i-1}$ are already constructed $(i \le l)$. If $s(i) \ge \Delta(G)$, put $X_i = E(G)$. Otherwise, at each vertex u with $d_G(u) > s(i)$ delete precisely $d_G(u) - s(i)$ edges from $E(G) \backslash X_{i-1}$. The remaining edges of $E(G) \backslash X_{i-1}$ together with X_{i-1} form the next edge subset X_i. It is not difficult to check that every alternating path $P = a_0 a_1 \ldots a_{2r+1}$ relative to X_i with end edges in $E(G) \backslash X_i$ has the property that if a_0 is incident with less than $s(i)$ edges of X_i then each of the vertices $a_1, a_3, \ldots, a_{2r+1}$ is incident with exactly $s(i)$ edges of X_i, and each of the vertices $a_2, a_4, \ldots a_{2r}$ is incident with less than $s(i)$ edges of X_i. This property and Lemma 2.3 imply that X_i is a maximum $s(i)$-matching of G. Clearly $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_l$. From X_1 choose a subset of edges F_1 of size $\sum_{i=1}^{s(1)} n_i$, and from X_j choose a subset F_j of size $\sum_{i=1}^{s(j)} n_i$ containing F_{j-1}, for each

$j = 2, \ldots, l$. This is possible since $N \in D_q(G)$. By Proposition 1.7, edges of F_1 are colorable with $s(1)$ colors because $s(1) > \Delta_1(G)$. Now the theorem follows from Proposition 3.2. ■

Remark 3.5. It is known [8] that almost all simple graphs have only one vertex of maximum degree. Therefore, $\Delta_2(G) < \Delta(G)$ for almost all simple graphs.

Let G be a simple graph with $|E(G)| = q$ and $\Delta(G) = \Delta$, and let $H(G) = (h_1, ..., h_\Delta)$ be a sequence which was defined in Section 2. It is known that $H(G) \in D_q$ if G is bipartite and it may not be true if G is non-bipartite [18]. The next result describes a class of graphs where $H(G) \in D_q$ and, moreover, $H(G)$ is color-feasible for G.

Theorem 3.6. Let G be a connected simple graph with q edges where $\Delta(G) \geq 3$ and $\Delta_2(G) = 2$, that is, every pair of vertices of degree at least 3 are non-adjacent. Then $H(G)$ is color-feasible for G and $D_q(G)$ is the set of all color-feasible sequences for G.

Proof. Let $H(G) = (h_1, ..., h_\Delta)$ where $\Delta = \Delta(G)$. We will show that $H(G)$ is color-feasible for G. Let F_1 be a maximum matching of G. We will sequentially construct edge subsets $F_2, ..., F_\Delta$.

At each vertex x with $d_G(x) > 2$ delete precisely $d_G(x) - 2$ edges from $E(G) \backslash F_1$. The remaining set of edges we denote by F_2. It is clear that F_2 is a maximum 2-matching of G.

Suppose that the set F_2 induces a non-bipartite graph. Consider in this graph a cycle C of odd length. Since G is connected and $\Delta \geq 3$, there is an edge (x, y) in C and a vertex $z \notin C$ such that $d_G(x) \geq 3, d_G(y) = 2$ and $(x, z) \in E(G)$. Clearly, $d_G(z) \leq 2$. Now we delete the edge (x, y) from F_2 and introduce (x, z), that is, $F_2 := (F_2 \setminus \{(x, y)\}) \cup \{(x, z)\}$.

Then the number of odd cycles in the subgraph induced by F_2 decreases by 1. We repeat this procedure until F_2 induces a bipartite graph.

Suppose that we have already constructed subsets $F_1, ..., F_{i-1}$ where $2 < i \leq \Delta(G)$ and $F_1 \subset ... \subset F_{i-1}$. At each vertex x with $d_G(x) > i$ delete precisely $d_G(x) - i$ edges from $E(G) \setminus F_{i-1}$. The remaining set of edges we denote by F_i. It is not difficult to check that every alternating path $P = a_0 a_1 \ldots a_{2r+1}$ relative to F_i with end edges in $E(G) \backslash F_i$ has the property that if a_0 is incident with less than i edges of F_i then each of the vertices $a_1, a_3, \ldots, a_{2r+1}$ is incident with exactly i

edges of F_i, and each of the vertices $a_2, a_4, \dots a_{2r}$ is incident with less than i edges of F_i. This property and Lemma 2.3 imply that F_i is a maximum i-matching of G.

By repeating this process we obtain the sets $F_1, ..., F_\Delta$ such that F_i is a maximum i-matching of G, for $i = 1, ..., \Delta$, and $F_1 \subset F_2 \subset ... \subset F_\Delta$.

Let H_i be the subgraph induced by the set $F_i, i = 1, ..., \Delta$. Since H_2 is a bipartite graph with $\Delta(H_2) = 2$ and F_1 is a maximum matching, $q_1(G) \geq q_2(G) - q_1(G)$, that is , $h_1 \geq h_2$. Moreover, it is not difficult to see that the edges of H_2 can be colored with colors 1 and 2 such that h_1 edges colored 1 and h_2 edges colored 2.

Suppose that we have already properly colored edges in F_i with $i \geq 2$ colors $1, ..., i$ such that precisely h_j edges colored j, for $j = 1, ..., i$. If $i < \Delta$ then, by Proposition 3.1, edges in F_{i+1} can properly colored with $i+1$ colors such that at least h_j edges are colored j, for $j = 1, 2, ..., i$. The condition $\sum_{r=1}^{j} h_r = q_j(G)$ implies that under this coloring precisely h_j edges receive color j, for each $j = 1, 2, ..., i+1$.

By repeating this process we obtain a proper Δ-coloring corresponding to the sequence $H(G)$. Therefore, by Remark 2.1, $D_q(G)$ is the set of all color-feasible sequences for G. ■

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