

# THE FINITE ANTICHAIN PROPERTY IN COXETER GROUPS

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ABSTRACT. We prove that the weak order on an infinite Coxeter group contains infinite antichains if and only if the group is not affine.

## 1. INTRODUCTION

Given an infinite poset, a natural problem is to decide whether or not it contains infinite antichains (sets of pairwise incomparable elements). It is known [1, 9] that every antichain in the Bruhat order on any Coxeter group is finite. Here, we consider the other of the two most common ways to order a Coxeter group, namely the weak order. One observes that the answer must depend on the group; it is straightforward to check that the antichains in the infinite dihedral group are finite, whereas there are infinite ones in the universal Coxeter group of rank 3. The open problem of characterising the groups with infinite antichains is [2, Exercise 3.11]. The main result of this paper is that the answer is the following new characterisation of affine Weyl groups:

**Theorem 1.1.** *The weak order on an infinite Coxeter group contains an infinite antichain if and only if the group is not affine.*

After establishing notation in Section 2, we use the remaining two sections to prove our main result. In Section 3, we show that affine groups do not possess infinite antichains, whereas (irreducible) not locally finite ones do. The groups that remain are the compact hyperbolic Coxeter groups. In Section 4, it is shown that they all have infinite antichains, thereby finishing the proof of Theorem 1.1. While the proofs in Section 3 are uniform, we have been forced to resort to a case-by-case argument in Section 4.

**Remark 1.2.** The Poincaré series of an affine Weyl group is given by a simple formula first proved by Bott [3], see [10, Theorem 8.9]. From it, it immediately follows that every affine Weyl group except the infinite dihedral group  $I_2(\infty)$  has the following property: the number of elements of Coxeter length  $k$  grows arbitrarily large as  $k$  tends to infinity. Distinct elements of the same length are always incomparable under the weak order. Therefore, the only infinite Coxeter groups with bounded antichain size are  $W \times I_2(\infty)$ , for finite  $W$ . All other infinite groups have arbitrarily large finite antichains.

## 2. PRELIMINARIES

We assume the reader to be familiar with basic theory of Coxeter groups and root systems as can be found e.g. in [2] or [10]. Here, we review scattered pieces of the theory in order to agree on notation. For the most part, we borrow our terminology from [2].

Throughout the paper,  $(W, S)$  will denote a Coxeter system with  $|S| < \infty$ . Given  $J \subseteq S$ ,  $W_J = \langle J \rangle$  is the parabolic subgroup generated by  $J$ . Every coset in  $W/W_J$  has a unique representative of minimal length; the set of such representatives is denoted by  $W^J$ .

We use  $\ell(w)$  to denote the Coxeter length of  $w \in W$ . The *right descent set* of  $w$  is

$$D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}.$$

**Definition 2.1.** *The (right) weak order on  $W$  is defined by  $v \leq_R w$  if and only if there exists  $u \in W$  such that  $w = vu$  and  $\ell(w) = \ell(v) + \ell(u)$ .*

One can also define the left weak order  $\leq_L$  in the obvious way. In this paper, “weak order” always refers to the right weak order. The results are of course equally valid for the left version.

Every group element  $w \in W$  can be expressed as a word in the free monoid  $S^*$ . If such a word has length  $\ell(w)$  it is called a *reduced expression*. The combinatorics of reduced expressions is a key to many properties of Coxeter groups and plays a prominent role in our arguments. Abusing notation, we will sometimes blur the distinction between elements of  $W$  and their representatives in  $S^*$ . We trust the context to make the meaning clear.

For  $s, s' \in S$ , let  $m(s, s')$  denote the order of  $ss'$ . This information is collected in the *Coxeter diagram* which is a complete graph on the vertex set  $S$  in which the edge  $\{s, s'\}$  is labelled with  $m(s, s')$ . For convenience, we agree to suppress edges with label 2 and labels that are equal to 3.

Consider a word in  $S^*$  representing  $w \in W$ . Deleting a factor  $ss$ , we obtain another word representing  $w$ . Similarly, replacing a factor  $ss'ss's \dots$  of length  $m(s, s')$  with the factor  $s'ss'ss' \dots$  of the same length, we again obtain a representative of  $w$ . The former operation is called a *nil move*, the latter a *braid move*. Obviously, nil moves can never be performed on reduced expressions. The following important result is due to Tits [13].

**Theorem 2.2** (Word Property).

- (a) *Any word in  $S^*$  can be brought to a reduced expression by a sequence of braid moves and nil moves.*
- (b) *Given a reduced expression for  $w$ , every other reduced expression for  $w$  can be obtained by a sequence of braid moves.*

One consequence of the Word Property is that every reduced expression for  $w \in W$  uses the same set of generators. We use  $S(w) \subseteq S$  to denote this set.

**2.1. A recognising automaton for reduced expressions.** It is a fundamental fact that the language of reduced expressions in  $W$  is regular, i.e. recognised by a finite state automaton. In other words, there is a directed graph on a finite vertex set whose edges are labelled with elements from  $S$ , such that the sequences of labels along directed paths beginning in some distinguished starting vertex are exactly the reduced expressions for elements in  $W$ . The existence of such an automaton is essentially due to Brink and Howlett [4]. They in fact showed that the language of so-called normal forms is regular, but this is enough due to a result of Davis and Shapiro [5].

At one point in Section 4, we will rely on explicit computations in a particular recognising automaton for reduced expressions. For this purpose, we briefly sketch

how the automaton works. For the sake of brevity, the reader will be kept on a need-to-know basis. A more thorough account of the construction can be found in [2, Section 4.8], which is largely based on material from the thesis of Eriksson [7]. See also Headley's thesis [8].

Suppose  $\Phi$  is a root system for  $W$  with simple roots  $\Delta = \{\alpha_s \mid s \in S\}$ . A symmetric bilinear form on  $V = \text{span } \Phi = \text{span } \Delta$  is defined by

$$(\alpha_s \mid \alpha_{s'}) = -\cos \frac{\pi}{m(s, s')}.$$

Given  $w \in W$ , we recursively define a corresponding set  $D_\Sigma(w) \subseteq \Phi^+$  of positive roots by

$$D_\Sigma(e) = \emptyset$$

(where  $e \in W$  is the identity element) and, if  $s \notin D_R(w)$ ,

$$D_\Sigma(ws) = \{\alpha_s\} \cup \{s(\beta) \mid \beta \in D_\Sigma(w) \text{ and } -1 < (\beta \mid \alpha_s) < 1\}.$$

The recognising automaton is constructed in the following way. Its vertex set (which turns out to be always finite) is

$$\{D_\Sigma(w) \mid w \in W\}.$$

The labelled edges are given by

$$D_\Sigma(w) \xrightarrow{s} D_\Sigma(ws),$$

whenever  $s \notin D_R(w)$ . Our distinguished starting vertex is  $D_\Sigma(e) = \emptyset$ .

### 3. AFFINE AND NOT LOCALLY FINITE GROUPS

A *well-partially-ordered* (wpo) set is a poset in which every non-empty subset has a minimal element and every antichain is finite. The origin of the following easy lemma is non-trivial to establish. See e.g. Kruskal's survey [11]. We include a proof for convenience and completeness.

**Lemma 3.1.** *Suppose  $P$  and  $Q$  are wpo posets. Then the product poset  $P \times Q$  is also wpo.*

*Proof.* It is easy to see that the non-empty subsets of  $P \times Q$  have minimal elements. Suppose, in order to deduce a contradiction, that  $A = \{(p_i, q_i)\}_{i \in \mathbb{N}}$  is an infinite antichain in  $P \times Q$ . We may assume that  $\{p_i\}_{i \in \mathbb{N}}$  and  $\{q_i\}_{i \in \mathbb{N}}$  are infinite; otherwise we could find an infinite subset  $B \subseteq A$  isomorphic to a subset of  $Q$  or  $P$ , respectively, giving a contradiction.

It follows from Ramsey's Theorem that every infinite poset either has an infinite chain or an infinite antichain (or both). The antichains in  $P$  are finite, so the set  $\{p_i\}_{i \in \mathbb{N}}$  contains an infinite chain. This chain has a smallest element since  $P$  is wpo. Without loss of generality we may therefore assume  $p_0 < p_1 < p_2 < \dots$ . Similarly, we may assume that  $\{q_i\}_{i \in \mathbb{N}}$  forms an infinite chain in  $Q$ . Since  $Q$  is wpo, we cannot have  $q_0 > q_1 > q_2 > \dots$ . Therefore, there exist indices  $i$  and  $j$  such that  $p_i < p_j$  and  $q_i \leq q_j$ , contradicting the fact that  $A$  is an antichain.  $\square$

Since non-empty subsets of Coxeter groups always contain minimal elements under weak order, Lemma 3.1 allows us to restrict attention to irreducible Coxeter groups — a group contains an infinite antichain if and only if one of its irreducible components does.

**Theorem 3.2.** *Affine Weyl groups have no infinite antichains.*

*Proof.* Let  $W$  be a finite Weyl group with root system  $\Phi$  and associated affine group  $\widetilde{W}$ . Consider the realisation of  $\widetilde{W}$  as a group generated by affine reflections in  $V = \text{span } \Phi$  (see e.g. [10, Section 4]). Identifying  $V$  with its dual, the reflecting (affine) hyperplanes are given by  $H_{\alpha,k} = \{\lambda \in V \mid \langle \lambda, \alpha \rangle = k\}$  for  $\alpha \in \Phi^+$ ,  $k \in \mathbb{Z}$ . The complement  $V \setminus \cup_{\alpha,k} H_{\alpha,k}$  is a disjoint union of connected open *alcoves*. The set of alcoves is in bijection with  $\widetilde{W}$ . The alcove corresponding to  $w \in \widetilde{W}$  is defined by a (possibly redundant) set of inequalities of the form  $n_w^\alpha < \langle \lambda, \alpha \rangle < n_w^\alpha + 1$ , where  $n_w^\alpha \in \mathbb{Z}$  for all  $\alpha \in \Phi^+$ . Corresponding to the identity  $e \in \widetilde{W}$ , the *fundamental alcove* is obtained by putting all  $n_e^\alpha = 0$ .

Define a partial order  $\preceq$  on  $\mathbb{Z}$  by letting  $i \preceq j$  iff  $|i| \leq |j|$  and either  $i = 0$  or  $\text{sgn}(i) = \text{sgn}(j)$ . Thus,  $\dots \succ 2 \succ 1 \succ 0 \prec -1 \prec -2 \prec \dots$ . Let  $Z$  denote this poset. It is known that the weak order on  $\widetilde{W}$  corresponds to inclusion on the sets of hyperplanes that separate the various alcoves from the fundamental one. This amounts to saying that, choosing some total ordering of  $\Phi^+$ , the map  $\varphi : \widetilde{W} \rightarrow Z^{|\Phi^+|}$  given by  $w \mapsto (n_w^\alpha)_{\alpha \in \Phi^+}$  is a poset automorphism from the weak order on  $\widetilde{W}$  to the image of  $\varphi$ .

By Lemma 3.1, the antichains in  $Z^{|\Phi^+|}$  are finite, and the theorem follows.  $\square$

A Coxeter group  $W$  is called *locally finite* if  $|W_J| < \infty$  for all  $J \subsetneq S$ .

**Theorem 3.3.** *If  $W$  is irreducible and not locally finite, then it has an infinite antichain.*

*Proof.* Suppose  $W_J$  is infinite and irreducible for  $J \subsetneq S$ . Choose  $s \in J$  and  $s' \in S \setminus J$  which are neighbours in the Coxeter diagram of  $W$ , i.e.  $s$  and  $s'$  do not commute. It follows from [6, Proposition 4.2] that  $W_J^{\setminus \{s\}}$  is infinite. Observe that  $D_R(w) = \{s\}$  for all  $w \in W_J^{\setminus \{s\}}$ . By the Word Property, since  $s$  and  $s'$  do not commute, every reduced expression for  $ws'$ ,  $w \in W_J^{\setminus \{s\}}$ , contains exactly one  $s'$ , and this is necessarily the last letter. This implies that the infinite set  $\{ws' \mid w \in W_J^{\setminus \{s\}}\}$  is an antichain under weak order.  $\square$

#### 4. COMPACT HYPERBOLIC GROUPS

Lannér [12] showed that the locally finite Coxeter groups that are neither finite nor affine are precisely the compact hyperbolic ones. In rank 3, every infinite, non-affine group is compact hyperbolic. The diagrams of the remaining irreducible compact hyperbolic Coxeter groups are shown in Figure 1.

In light of Theorems 3.2 and 3.3, the next result concludes the proof of Theorem 1.1.

**Theorem 4.1.** *Every compact hyperbolic Coxeter group has an infinite antichain.*

Proving this theorem is the topic of the remainder of the paper. Proceeding in a case-by-case fashion, the proof is somewhat unsatisfactory. In particular, our argument that the group at the bottom of Figure 1 has an infinite antichain relies on computer aided calculations and the structure of the automaton discussed in Section 2. It would be very interesting to have a type-independent proof of Theorem 4.1, perhaps in terms of general properties of the symmetric bilinear form  $(\cdot \mid \cdot)$ ; see [10, Section 6.8] for details.

The following simple lemma turns out to produce infinite antichains in most compact hyperbolic groups.

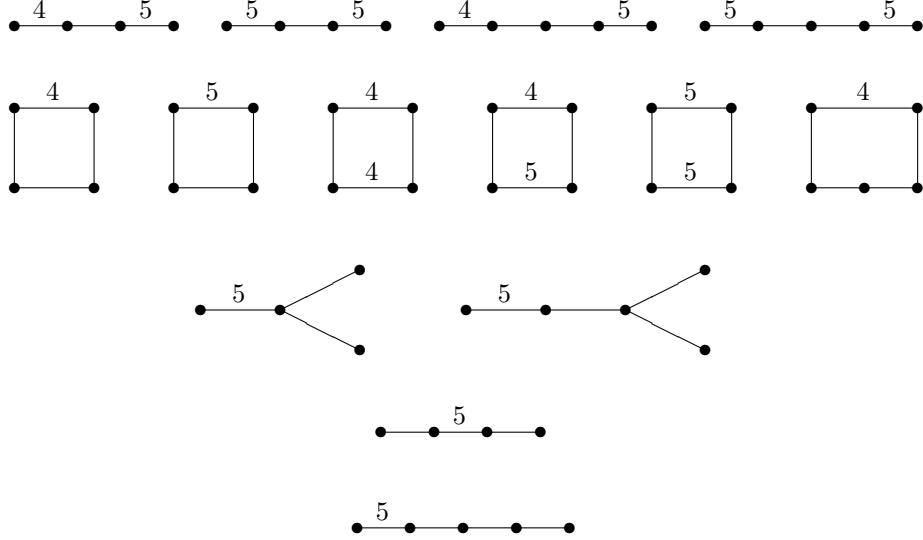


FIGURE 1. All irreducible compact hyperbolic Coxeter groups of rank at least 4.

**Lemma 4.2.** *Suppose that  $u, w \in W$  fulfil the following requirements:*

- (i)  $\ell(u) \leq \ell(w)$ .
- (ii)  $u \not\leq_R w$ .
- (iii)  $|S(w)| \geq 3$ .
- (iv) *Every reduced expression for  $wu$  is a concatenation of a reduced expression for  $w$  and a reduced expression for  $u$ .*
- (v) *Every reduced expression for  $w^2$  is a concatenation of two reduced expressions for  $w$ .*

*Then,  $\{w^k u \mid k \in \mathbb{N}\}$  is an infinite antichain in  $W$ .*

*Proof.* Suppose  $u$  and  $w$  satisfy the hypotheses. We claim that every reduced expression for  $w^k u$ ,  $k \in \mathbb{N}$ , is a concatenation of  $k$  reduced expressions for  $w$  and one for  $u$ . To see this, take an expression for  $w^k u$  of the form just described. By (iii), it allows no braid move which involves an entire copy of  $w$ . Thus, (iv) and (v) imply that every braid move simply replaces one copy of  $w$  (or  $u$ ) with another. Moreover, nil moves cannot be possible since it would mean either that  $\ell(wu) < \ell(w) + \ell(u)$  (contradicting (iv)) or that  $\ell(w^2) < 2\ell(w)$  (contradicting (v)). The claim is proved.

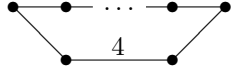
Now assume  $w^k u <_R w^l u$  for some  $k < l$ . By (i) and the above claim, this means that some reduced expression for  $w$  has a reduced expression for  $u$  as a prefix, contradicting (ii). We conclude that  $\{w^k u\}$  is indeed an antichain.  $\square$

We say that  $u$  and  $w$  form a *good pair* if they satisfy the hypotheses of Lemma 4.2.

**Lemma 4.3.** *Suppose  $W'$  is a Coxeter group obtained from  $W$  by increasing some edge labels in the Coxeter diagram. If  $W$  has infinite antichains, then so does  $W'$ .*

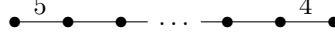
*Proof.* Take an infinite antichain  $\{w_1, w_2, \dots\} \subset W$ . Pick reduced expressions for the  $w_i$ . These expressions are reduced in  $W'$ , too. This is because any sequence of braid moves applicable in the context of  $W'$  is also applicable in  $W$ ; otherwise the expression would not be reduced in  $W$ . Thus, the sequence never leads to a nil move. The corresponding elements therefore form an antichain in  $W'$ , too.  $\square$

*Proof of Theorem 4.1.* The proof is divided into six different cases that combine to exhaust all irreducible compact hyperbolic groups (after allowing edge labels to increase, using Lemma 4.3). The rank three groups are covered by Cases I–III. The remaining groups are those in Figure 1. The first row is covered by Case II, the second by Case I and the third by Case IV. Finally, the singleton fourth and fifth rows are covered by Cases V and VI, respectively. In each case except the last one, we apply Lemma 4.2 by producing a good pair of elements in the corresponding group.



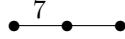
**Case I.**

Suppose the Coxeter generators form a cycle  $s_1, \dots, s_n, s_1$  in the Coxeter diagram, and assume  $m(s_1, s_n) = 4$ . Then,  $s_n$  and  $s_2 \dots s_n s_1$  form a good pair.



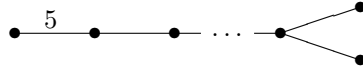
**Case II.**

Assume  $s_1, \dots, s_n$  is a path in the Coxeter diagram, with  $m(s_1, s_2) = 5$  and  $m(s_{n-1}, s_n) = 4$ . Now,  $s_1 s_2 s_1$  and  $s_1 \dots s_n s_{n-1} \dots s_2$  yield a good pair.



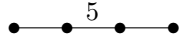
**Case III.**

Suppose we have three generators  $s, t, u \in S$  with  $m(s, t) = 7$  and  $m(t, u) \geq 3$ . A good pair is given by  $st$  and  $sutst$ .



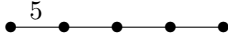
**Case IV.**

Let  $s_1, \dots, s_{n-1}$  be a path in the diagram with  $m(s_1, s_2) = 5$ , and add  $s_n$  with the relation  $m(s_{n-2}, s_n) = 3$ . We get a “fork-shaped” diagram like the one depicted. In this case, we may recycle the solution from Case II above, regarding the product of the “ends of the fork” as a single generator. We get a good pair consisting of the elements  $s_1 s_2 s_1$  and  $s_1 \dots s_n s_{n-2} s_{n-3} \dots s_2$ .



**Case V.**

Suppose  $S = \{s, t, u, v\}$  with  $m(s, t) = m(u, v) = 3$  and  $m(t, u) = 5$ . Consider the element  $\omega = utvsut = utvust = uwtust = utsvut = uwtvut$ , and note that these are all the reduced expressions for  $\omega$ . One easily checks that  $\omega$  satisfies condition (v) of Lemma 4.2 by inspecting the 25 concatenations of two such expressions, observing that none of them admits a braid move involving both copies of  $\omega$ . Similarly, we make sure that  $\omega$  together with  $\nu = utvt = utvut$  obey condition (iv). Conditions (i)–(iii) are immediate, implying that  $(\nu, \omega)$  is a good pair.

Case VI. 

Finally, we assume  $S = \{s, t, u, v, w\}$ ,  $m(s, t) = 5$  and  $m(t, u) = m(u, v) = m(v, w) = 3$ . Let  $\alpha = stuvwstuv$ . We claim that

$$\{\alpha^k w \mid k \in \mathbb{N} \text{ and } k \equiv 0 \pmod{6}\}$$

is an infinite antichain. Aided by the computer, we have established the following facts:

- (1)  $D_\Sigma(w\alpha^6) = D_\Sigma(w\alpha^7)$ .
- (2)  $\ell(w\alpha^7 w) = 2 + 7\ell(\alpha) = 65$ .

Now consider the recognising automaton for the language of reduced expressions described in Section 2. Combining (1) and (2), we see that there is a cycle of length  $\ell(\alpha) = 9$  corresponding to  $\alpha$  which begins and ends in  $D_\Sigma(w\alpha^6)$ . Repeatedly traversing this cycle produces reduced expressions. Furthermore, fact (2) shows that after walking this cycle a number of times, we may use an edge labelled  $w$ . Thus,  $\ell(w\alpha^k w) = 2 + k\ell(\alpha)$  for all  $k \geq 6$ . This proves  $w \not\prec_R \alpha^k w$  for such  $k$ . Since  $\alpha^k w \prec_R \alpha^l w \Leftrightarrow w \prec_R \alpha^{l-k} w$ , for  $k < l$ , we conclude that  $\{\alpha^k w \mid k \in 6\mathbb{N}\}$  is indeed an infinite antichain. □

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