

THE TOPOLOGY OF THE SPACE OF MATRICES OF BARVINOK RANK TWO

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ABSTRACT. The Barvinok rank of a $d \times n$ matrix is the minimum number of points in \mathbb{R}^d such that the tropical convex hull of the points contains all columns of the matrix. The concept originated in work by Barvinok and others on the travelling salesman problem. Our object of study is the space of real $d \times n$ matrices of Barvinok rank two. Let $B_{d,n}$ denote this space modulo rescaling and translation. We show that $B_{d,n}$ is a manifold, thereby settling a conjecture due to Develin. In fact, $B_{d,n}$ is homeomorphic to the quotient of the product of spheres $S^{d-2} \times S^{n-2}$ under the involution which sends each point to its antipode simultaneously in both components. In addition, using discrete Morse theory, we compute the integral homology of $B_{d,n}$. Assuming $d \geq n$, for odd d the homology turns out to be isomorphic to that of $S^{d-2} \times \mathbb{R}P^{n-2}$. This is true also for even d up to degree $d-3$, but the two cases differ from degree $d-2$ and up. The homology computation straightforwardly extends to more general complexes of the form $(S^{d-2} \times X)/\mathbb{Z}_2$, where X is a finite cell complex of dimension at most $d-2$ admitting a free \mathbb{Z}_2 -action.

1. INTRODUCTION

In the *tropical semiring* $(\mathbb{R}, \odot, \oplus)$ one defines “multiplication” and “addition” by $a \odot b = a + b$ and $a \oplus b = \min(a, b)$, respectively. Real n -space \mathbb{R}^n has the natural structure of a semimodule over the tropical semiring so that one obtains a theory of “tropical geometry”.¹

Given any classical geometric notion, one may try to construct reasonable analogues in tropical geometry and study their properties. This has been a very lively direction of research over recent years. A brief introduction is [13]. For more background the reader may e.g. consult the extensive list of references appearing in [11].

Consider the classical concept of n points on a line in \mathbb{R}^d . Such point sets are characterized by the property that their convex hull is at most one-dimensional. An equivalent characterization is that all points lie in the convex hull of at most two of the points. In tropical geometry, there is a natural notion of *tropical convex hull* which leads to obvious analogues of the above characterizations. However, the two analogues differ; the latter is more restrictive. Considering the points as columns of matrices, the former situation deals with matrices of *tropical rank* at most 2, whereas the latter leads one to consider matrices of *Barvinok rank* at most 2. Various tropically motivated definitions of rank are discussed in [5].

The concept of Barvinok rank has found applications in optimization theory. Motivating the nomenclature, Barvinok et al. [1] showed that the maximum version

¹Some authors require of a semiring the existence of an additively neutral element, which the tropical semiring lacks as we have defined it. The issue is of no importance to us, but could be rectified by incorporating an infinity element.

of the travelling salesman problem can be solved in polynomial time if the Barvinok rank of the distance matrix is fixed (with \oplus denoting max rather than min).

Let $\mathcal{M}(d, n)$ denote the set of $d \times n$ matrices with real entries. In this paper, we are interested in the space of matrices $M \in \mathcal{M}(d, n)$ with Barvinok rank 2, corresponding to sets of n marked points in \mathbb{R}^d whose tropical convex hull is generated by two of the points. The space contains some topologically less interesting features in that it is invariant under rescaling and translation. Taking the quotient by the equivalence relation generated by these operations, one obtains a space $B_{d,n}$ which is our object of study. Develin [4] conjectured that $B_{d,n}$ is a manifold and that its homology “does not increase in complexity as n gets large”. He confirmed the conjecture for $d = 3$ and computed the integral homology in a few more accessible cases.

Our main results are summarized in the following two theorems:

Theorem 1.1. *The space $B_{d,n}$ is homeomorphic to $(S^{d-2} \times S^{n-2})/\mathbb{Z}_2$, the quotient of the product of two spheres under the involution which sends each point to its antipode simultaneously in both components.*

Theorem 1.2. *The reduced integral homology groups of $B_{d,n}$ are given by*

$$\tilde{H}_i(B_{d,n}; \mathbb{Z}) \cong \mathbb{Z}^{f(i)} \oplus \mathbb{Z}_2^{t(i)},$$

where

$$f(i) = \begin{cases} 2 & \text{if } i + 2 = d = n \text{ and } i \text{ is odd,} \\ 1 & \text{if } i + 2 = n \neq d \text{ and } i \text{ is odd,} \\ 1 & \text{if } i + 2 = d \neq n \text{ and } i \text{ is odd,} \\ 1 & \text{if } i = d + n - 4 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$t(i) = \begin{cases} 1 & \text{if } 1 \leq i \leq \min\{d, n\} - 3 \text{ and } i \text{ is odd,} \\ 1 & \text{if } \max\{d, n\} - 2 \leq i \leq d + n - 5 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

When a finite group acts freely on a manifold, the quotient is again a manifold. In particular, therefore, the manifold part of Develin’s conjecture follows from Theorem 1.1, whereas the “complexity of homology” part is implied by Theorem 1.2.

For $d \geq n$, note that the homology of $B_{d,n}$ coincides with that of $S^{d-2} \times (S^{n-2}/\mathbb{Z}_2) \cong S^{d-2} \times \mathbb{R}P^{n-2}$ for odd d . For even d , the homology groups of the two manifolds differ in higher degrees for reasons to be explained in Section 4.

For the computations that lead to Theorem 1.2, we have opted to work with slightly more general chain complexes, since this requires no additional effort. An advantage of this approach is that it highlights the connection between $B_{d,n} \cong (S^{d-2} \times S^{n-2})/\mathbb{Z}_2$ and $S^{d-2} \times (S^{n-2}/\mathbb{Z}_2)$. Specifically, for any finite cell complex X with a free \mathbb{Z}_2 -action, we show that $(S^{d-2} \times X)/\mathbb{Z}_2$ and $S^{d-2} \times (X/\mathbb{Z}_2)$ have the same homology whenever $\dim X \leq d - 2$ and d is odd. For even d , there is still a connection, but the situation is slightly more complicated; see Theorem 4.12 for details.

The remainder of the paper is organized as follows. We review some concepts from tropical geometry in the next section. In Section 3, $B_{d,n}$ is defined and given

an explicit simplicial decomposition in terms of trees. From it, Theorem 1.1 is deduced. The last section is devoted to the proof of Theorem 1.2. Our approach is based on Forman’s discrete Morse theory [8].

2. TROPICAL CONVEXITY AND NOTIONS OF RANK

Recall that in the tropical semiring we define $a \odot b = a + b$ and $a \oplus b = \min(a, b)$ for $a, b \in \mathbb{R}$. For example,

$$0 \odot 3 \oplus (-2) \odot 3 = 1.$$

A natural semimodule structure on \mathbb{R}^n is provided by the “addition”

$$(x_1, \dots, x_n) \oplus (y_1, \dots, y_n) = (x_1 \oplus y_1, \dots, x_n \oplus y_n)$$

and the “multiplication by scalar”

$$\lambda \odot (x_1, \dots, x_n) = (\lambda \odot x_1, \dots, \lambda \odot x_n).$$

Following [6] we say that $S \subseteq \mathbb{R}^n$ is *tropically convex* if $\lambda \odot x \oplus \mu \odot y \in S$ for all $x, y \in S$ and $\lambda, \mu \in \mathbb{R}$. Note that if S is tropically convex, then $\lambda \odot x \in S$ for all $\lambda \in \mathbb{R}, x \in S$. Defining *tropical projective space*

$$\mathbb{TP}^{n-1} = \mathbb{R}^n / (1, \dots, 1)\mathbb{R},$$

any tropically convex set in \mathbb{R}^n is therefore uniquely determined by its image in \mathbb{TP}^{n-1} . We obtain a convenient set of representatives for the elements of tropical projective space by requiring the first coordinate to be zero.

The *tropical convex hull* $\text{tconv}(S)$ is the smallest tropically convex set which contains $S \subseteq \mathbb{R}^n$. It coincides with the set of finite tropical linear combinations:

$$\text{tconv}(S) = \left\{ \bigoplus_{x \in X} \lambda_x \odot x : \lambda_x \in \mathbb{R}, \emptyset \neq X \subseteq S, |X| < \infty \right\};$$

see [6]. Note that if $S \subseteq \mathbb{TP}^{n-1}$, i.e. if the points have vanishing first coordinate, we obtain (the image of) $\text{tconv}(S)$ inside \mathbb{TP}^{n-1} directly by only considering nonnegative scalars λ_x , at least one being equal to zero.

A *tropical line segment* is the tropical convex hull of two points in \mathbb{R}^n . It forms a piecewise linear curve in \mathbb{TP}^{n-1} .

In Figure 1, the tropical convex hulls of two three-point sets in \mathbb{TP}^2 are shown. According to the next definition, the 3×3 matrix with the three left-hand points as columns has tropical rank 3, whereas the points to the right form a matrix of tropical rank 2.

Definition 2.1 (cf. Theorem 4.2 in [5]). *The tropical rank of $M \in \mathcal{M}(d, n)$ equals one plus the dimension in \mathbb{TP}^{d-1} of the tropical convex hull of the columns of M .*

In particular, a matrix has tropical rank at most 2 if and only if the tropical convex hull of its columns is the union of the tropical line segments between all pairs of columns.

Definition 2.2. *The Barvinok rank of $M \in \mathcal{M}(d, n)$ equals the smallest number of points in \mathbb{R}^d whose tropical convex hull contains all columns of M .*

It is easy to see that the Barvinok rank cannot be smaller than the tropical rank. In general, the two notions are different. For example, the two matrices whose columns are the point sets in Figure 1 both have Barvinok rank 3.

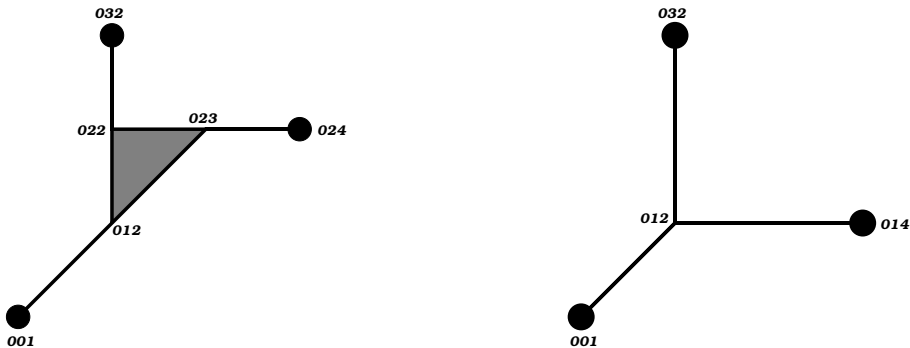


FIGURE 1. The tropical convex hulls of the point sets $\{(0, 0, 1), (0, 3, 2), (0, 2, 4)\}$ (left) and $\{(0, 0, 1), (0, 3, 2), (0, 1, 4)\}$ (right) in \mathbb{TP}^2 . The point $p = (0, \frac{3}{2}, \frac{5}{2})$, for instance, satisfies $p = \frac{3}{2} \odot (0, 0, 1) \oplus 1 \odot (0, 3, 2) \oplus 0 \odot (0, 2, 4)$. Therefore, it is part of the shaded region to the left.

Observe that a matrix has Barvinok rank at most 2 if and only if the tropical convex hull of its columns is a tropical line segment.

3. THE MANIFOLD OF BARVINOK RANK 2 MATRICES

In this section, we shall deduce Theorem 1.1. To begin with, we define the space $B_{d,n}$ which encodes the topologically interesting part of the space of matrices of Barvinok rank 2. We think of $B_{d,n}$ as sitting inside \mathbb{R}^{dn} with the subspace topology.

Fix positive integers d and n . Let $M \in \mathcal{M}(d, n)$. As usual, we consider the columns of M as a collection of n marked points in \mathbb{R}^d . Adding any real number to any row or any column of M preserves the Barvinok rank; adding to a row merely translates the point set, whereas adding to a column yields another representative for the same point in \mathbb{TP}^{d-1} . Similarly, multiplying M by any $\lambda \in \mathbb{R}$ does not increase the Barvinok rank (which is preserved if $\lambda \neq 0$).

In order to get unique representatives for matrices under the operations just described, the following definition is convenient.

Definition 3.1. Let $B_{d,n}$ be the set of matrices $M \in \mathcal{M}(d, n)$ satisfying

- (i) The first row of M is zero.
- (ii) The smallest entry in every row of M is zero.
- (iii) As a point in \mathbb{R}^{dn} , M is on the unit sphere.

3.1. A simplicial complex of trees. Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_n\}$ be disjoint sets of cardinality d and n , respectively.

We now describe an abstract simplicial complex with $B_{d,n}$ as geometric realization. The simplices are encoded by combinatorial trees whose leaves are marked using P and Q as label sets. This model is equivalent to that given by Develin in [4, § 3].² A completely analogous description in the context of matrices of tropical rank 2 was given by Markwig and Yu [12]; their complex is denoted $\mathcal{T}_{d,n}$ below.

²To translate from our trees to those of [4], simply replace the leaf labelled p_i and its incident edge by a leaf “heading off to infinity in the i -th coordinate direction”, and replace the leaf labelled q_i and its incident edge by the i -th marked point.

Consider the set of trees \mathcal{T} with leaf set $P \cup Q$ such that every internal vertex (i.e. non-leaf) has degree at least three. We may think of \mathcal{T} as a simplicial complex in the following way. The vertex set of the complex consists of all bipartitions of $P \cup Q$, and we identify a tree $\tau \in \mathcal{T}$ with the simplex comprised of the bipartitions induced by the connected components that arise when an internal edge (one not incident to a leaf) of τ is removed. (It is well-known, and easy to see, that there is at most one tree giving rise to any given set of bipartitions.) Clearly, the simplices in the boundary of τ are those obtained by contracting internal edges.

Let $\mathcal{T}_{d,n}$ be the subcomplex of \mathcal{T} which is induced by the bipartitions where both parts have nonempty intersection with both P and Q . Our main object of study is the subcomplex of $\mathcal{T}_{d,n}$ which consists of the trees whose internal vertices form a path as induced subgraph. Let us denote this complex by $\mathcal{B}_{d,n}$.

Proposition 3.2 (§ 3 in [4]). *A geometric realization of $\mathcal{B}_{d,n}$ is given by $B_{d,n}$.*

Specifically, a matrix $M \in B_{d,n}$ is associated with a tree in $\mathcal{B}_{d,n}$ as follows. Let c_1, \dots, c_n denote the columns of M . Then, $T = \text{tconv}(c_1, \dots, c_n)$ is a tropical line segment in \mathbb{TP}^{d-1} . Now construct a tree $\tau = \tau(M)$ whose internal vertex set is the union of $\{c_1, \dots, c_n\}$ and the set of points where the curve T is not smooth. Two internal vertices $v_1 \neq v_2$ are adjacent if and only if $\text{tconv}(v_1, v_2)$ (which is a subset of T) contains no other internal vertex. The leaf set of τ is $P \cup Q$. The leaf p_i is adjacent to the internal vertex which is closest to the origin (which, by Condition (ii) of Definition 3.1, is necessarily an internal vertex) among those where the i -th coordinate is maximized. Finally, the leaf q_i is adjacent to the vertex c_i . The resulting tree (seen as an abstract, leaf-labelled tree) is τ .

Example 3.3. *Consider the 6×5 matrix*

$$M = \begin{pmatrix} 6 & 1 & 4 & 6 & 3 \\ 2 & -3 & -1 & 2 & -1 \\ 5 & -2 & 0 & 4 & 2 \\ 5 & -2 & 0 & 4 & 2 \\ 0 & -5 & -1 & 0 & -3 \\ 7 & -2 & 0 & 4 & 4 \end{pmatrix}.$$

We shall see shortly that M has Barvinok rank 2. However, M does not satisfy the conditions of Definition 3.1, hence does not belong to $B_{6,5}$. Subtracting the first row of M from each row, adding an appropriate amount to each row and rescaling, we obtain the following matrix which represents the equivalence class of M in $B_{6,5}$:

$$M' = \frac{1}{\sqrt{97}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 2 & 5 \end{pmatrix}.$$

The tropical convex hull of the columns of M' is shown in Figure 2. It is generated by the third and fifth columns. Thus, M' (and M) has Barvinok rank 2. The associated tree $\tau(M')$ encoding the simplex in $B_{6,5}$ which contains M' is also displayed.

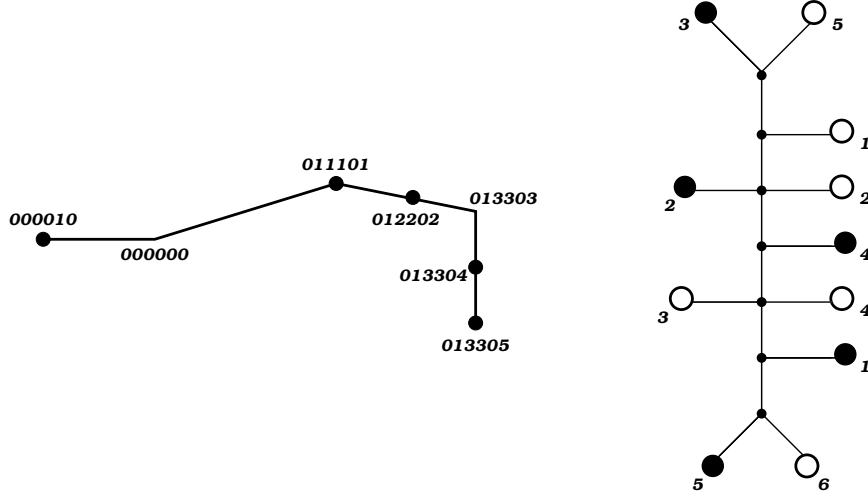


FIGURE 2. (Left) The tropical convex hull of five points on a tropical line segment in \mathbb{TP}^5 . Coordinates of the five points and the singular points on the curve are indicated; the point configuration from Example 3.3 has been inflated by a factor $\sqrt{97}$ in order to obtain integer coordinates. (Right) The tree in $\mathcal{B}_{6,5}$ associated with the five points. Leaves represented by white dots are elements of P , whereas black leaves belong to Q .

3.2. Proof of Theorem 1.1. Our next goal is to prove that $\mathcal{B}_{d,n}$ is homeomorphic to $(S^{d-2} \times S^{n-2})/\mathbb{Z}_2$, where the generator of \mathbb{Z}_2 acts by the antipodal map on both components.

For a set S , let $\mathfrak{b}(S)$ denote the proper part of the Boolean lattice on S . Thus, $\mathfrak{b}(S)$ is the poset of all proper, nonempty subsets of S ordered by inclusion.

Recall our sets P and Q . The chains in $\mathfrak{b}(P) \times \mathfrak{b}(Q)$ can be thought of as compositions (i.e. ordered set partitions) of $P \cup Q$ such that the first and the last block both have nonempty intersection with both P and Q ; let us for brevity call such compositions *balanced*. Namely, a chain $(S_1, T_1) < \dots < (S_k, T_k)$ corresponds to the balanced composition (C_1, \dots, C_{k+1}) , where

$$C_i = \left(\bigcup_{j=1}^i (S_j \cup T_j) \right) \setminus \left(\bigcup_{j=1}^{i-1} (S_j \cup T_j) \right),$$

identifying S_{k+1} and T_{k+1} with P and Q , respectively. Under this bijection, inclusion among chains corresponds to refinement among compositions.

Let $\Delta(\cdot)$ denote order complex³. We have a map of simplicial complexes $\varphi : \Delta(\mathfrak{b}(P) \times \mathfrak{b}(Q)) \rightarrow \mathcal{B}_{d,n}$ by sending a balanced composition $C = (C_1, \dots, C_k)$ to the unique tree $\varphi(C) \in \mathcal{B}_{d,n}$ in which C_i is the set of leaves adjacent to the i th internal vertex, counting along the internal path from one of the endpoints. As an example, there are two balanced compositions that are mapped to the tree

³The order complex of a finite poset is the abstract simplicial complex whose simplices are the totally ordered subsets.

in Figure 2, namely $(p_5q_3, p_1, p_2q_2, q_4, p_3p_4, q_1, p_6q_5)$ and its reverse composition $(p_6q_5, q_1, p_3p_4, q_4, p_2q_2, p_1, p_5q_3)$.

Conversely, for $\tau \in \mathcal{B}_{d,n}$, suppose the path from one endpoint of the internal path to the other traverses the internal vertices in the order v_1, \dots, v_k . Let $C_i \subset P \cup Q$ be the set of leaves adjacent to v_i . Clearly,

$$\varphi^{-1}(\tau) = \{(C_1, \dots, C_k), (C_k, \dots, C_1)\}.$$

This shows that φ induces an isomorphism of simplicial complexes:

$$\Delta(\mathfrak{b}(P) \times \mathfrak{b}(Q)) / \mathbb{Z}_2 \cong \mathcal{B}_{d,n},$$

where the generator of \mathbb{Z}_2 acts by taking complement inside $\mathfrak{b}(P)$ and $\mathfrak{b}(Q)$ simultaneously.

Given finite posets Π and Σ , the product space $\Delta(\Pi) \times \Delta(\Sigma)$ has a natural cell complex structure, where the cells are of the form $\Delta(C^\Pi) \times \Delta(C^\Sigma)$ for chains $C^\Pi \subseteq \Pi$, $C^\Sigma \subseteq \Sigma$. It is well known [7, Lemma 8.9] that $\Delta(\Pi \times \Sigma)$ is a simplicial subdivision of $\Delta(\Pi) \times \Delta(\Sigma)$, a cell $\Delta(C^\Pi) \times \Delta(C^\Sigma)$ being subdivided by $\Delta(C^\Pi \times C^\Sigma)$.

In particular, $\Delta(\mathfrak{b}(P) \times \mathfrak{b}(Q)) \cong \Delta(\mathfrak{b}(P)) \times \Delta(\mathfrak{b}(Q))$. Moreover, the \mathbb{Z}_2 -action clearly respects the subdivision so that we obtain

$$\mathcal{B}_{d,n} \cong (\Delta(\mathfrak{b}(P)) \times \Delta(\mathfrak{b}(Q))) / \mathbb{Z}_2,$$

where the generator of \mathbb{Z}_2 acts by taking complementary chains in both $\mathfrak{b}(P)$ and $\mathfrak{b}(Q)$.

Finally, it is well known that $\Delta(\mathfrak{b}(S))$ is homeomorphic to the $(|S| - 2)$ -sphere, and that the complement map on $\mathfrak{b}(S)$ corresponds to the antipodal map on the sphere. This concludes the proof of Theorem 1.1.

4. COMPUTING THE HOMOLOGY

The main aim of the remainder of the paper is to compute the integral homology groups of $B_{d,n}$, thereby proving Theorem 1.2. By Theorem 1.1, $B_{d,n}$ is homeomorphic to $(S^{d-2} \times S^{n-2}) / \mathbb{Z}_2$. To compute the homology of this manifold, we consider the standard cell decomposition into hemispheres of each of S^{d-2} and S^{n-2} ; see Section 4.1 for a description.

It is useful, however, to work with slightly more general chain complexes. Thus, we shift gears and temporarily forget about the context of the previous sections.

Let R be a principal ideal domain of odd or zero characteristic. Let

$$\begin{aligned} \mathbb{V} : \dots &\xrightarrow{\partial} V_{d+1} \xrightarrow{\partial} V_d \xrightarrow{\partial} V_{d-1} \xrightarrow{\partial} \dots \\ \mathbb{W} : \dots &\xrightarrow{\partial} W_{d+1} \xrightarrow{\partial} W_d \xrightarrow{\partial} W_{d-1} \xrightarrow{\partial} \dots \end{aligned}$$

be chain complexes of R -modules equipped with a degree-preserving \mathbb{Z}_2 -action. This means that for each of the two chain complexes there is a degree-preserving involutive automorphism ι commuting with ∂ .

Consider the tensor product $\mathbb{V} \otimes \mathbb{W}$ over R ; the k th chain group is equal to

$$\bigoplus_{i+j=k} V_i \otimes W_j,$$

and the boundary map is given by

$$\partial(v \otimes w) = \partial(v) \otimes w + (-1)^i v \otimes \partial(w)$$

for $v \in V_i$ and $w \in W_j$. We obtain a \mathbb{Z}_2 -action on $V \otimes W$ by

$$\iota(v \otimes w) = \iota(v) \otimes \iota(w).$$

For any chain complex C equipped with a \mathbb{Z}_2 -action induced by the involution ι , let C^+ be the chain complex obtained by identifying an element c with zero whenever $c + \iota(c) = 0$. Moreover, define C^- to be the chain complex obtained in the similar manner by identifying an element c with zero whenever $c - \iota(c) = 0$. Our goal is to examine $(V \otimes W)^+$.

4.1. The hemispherical chain complex. Let us consider the special case of interest in our calculation of the homology of $B_{d,n}$. Write $D = d - 2$ and $N = n - 2$. For $0 \leq i \leq D$, let V_i be a free R -module generated by two elements $\sigma_i^+ = \sigma_i^{+1}$ and $\sigma_i^- = \sigma_i^{-1}$; set $V_i = 0$ for $i < 0$ and $i > D$. We define

$$(1) \quad \partial(\sigma_i^\epsilon) = \sigma_{i-1}^\epsilon + (-1)^i \sigma_{i-1}^{-\epsilon}$$

for $\epsilon = \pm 1$. This means that V is the unreduced chain complex corresponding to the standard hemispherical cell decomposition of the D -sphere. A \mathbb{Z}_2 -action is given by mapping σ_i^+ to σ_i^- and vice versa. This corresponds to the antipodal action on the sphere, and V^+ consequently corresponds to the minimal cell decomposition of real projective D -space [9, Ex. 2.42]. We refer to V as the standard hemispherical chain complex over R of degree D . Using Theorem 1.1, we deduce the following.

Lemma 4.1. *Let V and W be the standard hemispherical chain complexes over R of degree $D = d - 2$ and $N = n - 2$, respectively. Then, the homology of $(V \otimes W)^+$ is isomorphic to the unreduced homology over R of $B_{d,n}$.*

Lemma 4.2. *Suppose that V is the standard hemispherical chain complex over R of degree D . Then,*

$$H_i(V^+) \cong \begin{cases} R & \text{if } i = 0, \\ R/(2R) & \text{if } 1 \leq i < D \text{ and } i \text{ is odd,} \\ R & \text{if } i = D \text{ and } D \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H_i(V^-) \cong \begin{cases} R/(2R) & \text{if } 0 \leq i < D \text{ and } i \text{ is even,} \\ R & \text{if } i = D \text{ and } D \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For each k , we have that $\sigma_k^+ = \sigma_k^-$ in the group V_k^+ . Hence,

$$\partial(\sigma_i^+) = \sigma_{i-1}^+ + (-1)^i \sigma_{i-1}^- = (1 + (-1)^i) \sigma_{i-1}^+$$

in V_{i-1}^+ , which is $2\sigma_{i-1}^+$ if i is even and 0 if i is odd. Since the characteristic of R is odd or zero, the first claim follows.

For the second claim, note that $\sigma_k^- = -\sigma_k^+$ in the group V_k^- . Therefore,

$$\partial(\sigma_i^+) = \sigma_{i-1}^+ + (-1)^i \sigma_{i-1}^- = (1 - (-1)^i) \sigma_{i-1}^+$$

in V_{i-1}^- , which is $2\sigma_{i-1}^+$ if i is odd and 0 if i is even. This proves the claim. \square

4.2. **2 is a unit in R .** First we consider the case that 2 is a unit in R .

Let \mathbf{C} be a chain complex of R -modules with an involution ι . Then, we may write each element c uniquely as a sum $c = a + b$ such that $a = \iota(a)$ and $b = -\iota(b)$. Namely, $a = \frac{1}{2}(c + \iota(c))$ and $b = \frac{1}{2}(c - \iota(c))$. This means that \mathbf{C}^+ can be identified with the subcomplex of elements a satisfying $a - \iota(a) = 0$ and \mathbf{C}^- with the subcomplex of elements b satisfying $b + \iota(b) = 0$. Moreover, we may identify \mathbf{C} with the direct sum $\mathbf{C}^+ \oplus \mathbf{C}^-$.

Applying the above to each of \mathbf{V} and \mathbf{W} , we obtain that

$$\mathbf{V} \otimes \mathbf{W} = (\mathbf{V}^+ \otimes \mathbf{W}^+) \oplus (\mathbf{V}^- \otimes \mathbf{W}^-) \oplus (\mathbf{V}^+ \otimes \mathbf{W}^-) \oplus (\mathbf{V}^- \otimes \mathbf{W}^+).$$

We have that $\iota(x) = x$ if

$$x \in (\mathbf{V}^+ \otimes \mathbf{W}^+) \oplus (\mathbf{V}^- \otimes \mathbf{W}^-)$$

and $\iota(x) = -x$ if

$$x \in (\mathbf{V}^+ \otimes \mathbf{W}^-) \oplus (\mathbf{V}^- \otimes \mathbf{W}^+).$$

As a consequence,

$$(\mathbf{V} \otimes \mathbf{W})^+ = (\mathbf{V}^+ \otimes \mathbf{W}^+) \oplus (\mathbf{V}^- \otimes \mathbf{W}^-).$$

Applying Künneth's theorem [9, Th. 3B.5], we obtain the following result.

Proposition 4.3. *If $H_i(\mathbf{V}^+)$ and $H_i(\mathbf{V}^-)$ are finitely generated free R -modules for each i , then*

$$H_d((\mathbf{V} \otimes \mathbf{W})^+) \cong \sum_{i+j=d} (H_i(\mathbf{V}^+) \otimes H_j(\mathbf{W}^+)) \oplus (H_i(\mathbf{V}^-) \otimes H_j(\mathbf{W}^-)).$$

Suppose that \mathbf{V} is the standard hemispherical chain complex of degree D . By Lemma 4.2, we have that $H_i(\mathbf{V}^+) \cong H_i(\mathbf{V}^-) \cong 0$ unless $i = 0$ or $i = D$. Moreover, $H_0(\mathbf{V}^+) \cong R$ and $H_0(\mathbf{V}^-) \cong 0$. Finally, if D is odd, then $H_D(\mathbf{V}^+) \cong R$ and $H_D(\mathbf{V}^-) \cong 0$. If instead D is even, then $H_D(\mathbf{V}^+) \cong 0$ and $H_D(\mathbf{V}^-) \cong R$. The following assertion is an immediate consequence:

Proposition 4.4. *If \mathbf{V} is the standard hemispherical chain complex of degree D , then the homology of $(\mathbf{V} \otimes \mathbf{W})^+$ consists of one copy of $H_j(\mathbf{W}^+)$ in degree j for each j and one copy of either $H_j(\mathbf{W}^+)$ or $H_j(\mathbf{W}^-)$ in degree $D + j$ for each j . The former is the case if D is odd, the latter if D is even.*

Identifying \mathbf{V} and \mathbf{W} with the cellular chain complexes corresponding to the hemispherical cell decompositions of the $(d-2)$ -sphere and the $(n-2)$ -sphere, respectively, yields the following corollary, which could also be deduced using transfer methods; see e.g. Bredon [2, § III].

Corollary 4.5. *If 2 is invertible in the coefficient ring R , then the reduced homology of $B_{d,n}$ is given by*

$$\tilde{H}_i(B_{d,n}) \cong \begin{cases} R^2 & \text{if } i + 2 = d = n \text{ and } i \text{ is odd,} \\ R & \text{if } i + 2 = n \neq d \text{ and } i \text{ is odd,} \\ R & \text{if } i + 2 = d \neq n \text{ and } i \text{ is odd,} \\ R & \text{if } i = d + n - 4 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the free part of the integral homology of $B_{d,n}$ is as described in Theorem 1.2.

Proof. This is an immediate consequence of Lemma 4.2 and Proposition 4.4. For the free part of the integral homology, set $R = \mathbb{Q}$ and apply the universal coefficient theorem; see e.g. [9, Cor. 3A.6]. \square

4.3. Discrete Morse theory on chain complexes. To compute the homology of $(V \otimes W)^+$ in the case that 2 is not a unit in R , we will use an algebraic version [10, § 4.4] of discrete Morse theory [8].

The general situation is that we have a chain complex C of finitely generated R -modules C_i . Write $C = \bigoplus_i C_i$. Assume that C can be written as a direct sum of three R -modules A , B , and U such that $f = \alpha \circ \partial$ defines an isomorphism $f : B \rightarrow A$, where $\alpha(a + b + u) = a$ for $a \in A, b \in B, u \in U$. Let $h : A \rightarrow B$ be the inverse of $f|_B$. For any chain group element x , define $\beta(x) = h \circ f(x)$ and $\hat{U} = (\text{id} - \beta)(U)$. Let \hat{U}_k be the component of \hat{U} in degree k .

Proposition 4.6 ([10, Th. 4.16, Cor. 4.17]). *With notation as above, we have that*

$$\hat{U} : \dots \xrightarrow{\partial} \hat{U}_{k+1} \xrightarrow{\partial} \hat{U}_k \xrightarrow{\partial} \hat{U}_{k-1} \xrightarrow{\partial} \dots$$

forms a chain complex with the same homology as the original chain complex C . Moreover, for each $u \in U$, the element $\beta(u)$ is the unique element $b \in B$ with the property that $\partial(u - b) \in B + U$.

Proof. For the reader's convenience, we give a proof outline. Let $u \in U$. Note that

$$f(u - \beta(u)) = f(u) - f \circ h \circ f(u) = f(u) - f(u) = 0;$$

hence $\partial(u - \beta(u))$ is of the form $u_0 - b_0$, where $u_0 \in U$ and $b_0 \in B$. Since $u_0 - b_0$ is a cycle, we have that $f(b_0) = f(u_0)$ and hence

$$u_0 - b_0 = u_0 - h \circ f(b_0) = u_0 - h \circ f(u_0) = u_0 - \beta(u_0),$$

which implies that $\partial(\hat{U}) \subseteq \hat{U}$.

We claim that we may write C as a direct sum $\partial(B) \oplus B \oplus \hat{U}$. Namely, let $x \in C$, and define $\hat{a} = \partial \circ h \circ \alpha(x)$. We have that $\alpha(\hat{a}) = \alpha(x)$, which implies that $x - \hat{a} = b + u$ for some $b \in B$ and $u \in U$. Defining $\hat{b} = b + \beta(u)$ and $\hat{u} = u - \beta(u)$, we obtain that we may write $x = \hat{a} + \hat{b} + \hat{u}$, where $\hat{a} \in \partial(B)$, $\hat{b} \in B$, and $\hat{u} \in \hat{U}$. It is easy to show that this decomposition of x is unique; hence we obtain the claim.

Write $M = \partial(B) \oplus B$. As $\partial(M) \subseteq M$, we deduce that C splits into the direct sum of \hat{U} and

$$M : \dots \xrightarrow{\partial} M_{k+1} \xrightarrow{\partial} M_k \xrightarrow{\partial} M_{k-1} \xrightarrow{\partial} \dots$$

The homology of the latter complex is zero, because $\partial : B \rightarrow \partial(B)$ is an isomorphism. As a consequence, we are done. The very last statement in the proposition is immediate from the fact that $f : B \rightarrow A$ is an isomorphism. \square

For the connection to discrete Morse theory [8], consider a matching on the set of cells in a cell complex such that each pair in the matching is of the form (σ, τ) , where σ is a regular codimension one face of τ . Let A be the free R -module generated by cells matched with larger cells, let B be generated by cells matched with smaller cells, and let U be generated by unmatched cells. Then, the map $f : B \rightarrow A$ is an isomorphism if the matching is a Morse matching [3], and \hat{U} is the Morse complex associated to the matching.

4.4. 2 is not a unit in R . In the case that 2 is not a unit, the discussion in Section 4.2 does not apply, as the chain complex no longer splits in the manner described. In fact, the situation is considerably more complicated. For this reason, we only examine the special case that \mathbf{V} is the standard hemispherical chain complex of degree D .

We also need some assumptions on \mathbf{W} . Specifically, we assume that we may write W_j as a direct product $L_j \times L_j$, where L_j is a finitely generated R -module for each $j \in \mathbb{Z}$. Moreover, we assume that $\iota(w_0, w_1) = (w_1, w_0)$ for each element $(w_0, w_1) \in W_j$. For our main result to hold, we must assume that $L_j = 0$ unless $0 \leq j \leq D$. However, we will not actually use this assumption until Lemma 4.10.

We make no specific assumptions on the boundary operator on \mathbf{W} , which hence is of the general form

$$\partial(w_0, w_1) = (p(w_0) + r(w_1), q(w_0) + s(w_1)),$$

where p, q, r, s are maps $L_j \rightarrow L_{j-1}$ such that $\partial^2 = 0$ and $\iota\partial = \partial\iota$. Since $\iota(w, 0) = (0, w)$, we have that

$$(q(w), p(w)) = \iota \circ \partial(w, 0) = \partial \circ \iota(w, 0) = (r(w), s(w))$$

and hence that

$$\partial(w_0, w_1) = (p(w_0) + q(w_1), p(w_1) + q(w_0)).$$

For ∂^2 to be zero, it is necessary and sufficient that $p^2 + q^2 = pq + qp = 0$.

Recall that we want to examine $(\mathbf{V} \otimes \mathbf{W})^+$. In this chain complex, we have for each i and $w_0, w_1 \in L_j$ the identity

$$\sigma_i^- \otimes (w_0, w_1) = \sigma_i^+ \otimes (w_1, w_0).$$

In particular, we may identify $(V_i \otimes W_j)^+$ with $\langle \sigma_i^+ \rangle \otimes W_j \cong W_j \mathbf{e}_{i,j}$, where $\mathbf{e}_{i,j}$ is a formal variable. Writing $\epsilon = -1$ for compactness, also note that

$$\begin{aligned} \partial(\sigma_i^+ \otimes (w_0, w_1)) &= (\sigma_{i-1}^+ + \epsilon^i \sigma_{i-1}^-) \otimes (w_0, w_1) + \epsilon^i \sigma_i^+ \otimes \partial(w_0, w_1) \\ &= \sigma_{i-1}^+ \otimes (w_0 + \epsilon^i w_1, w_1 + \epsilon^i w_0) + \epsilon^i \sigma_i^+ \otimes \partial(w_0, w_1). \end{aligned}$$

Identifying $(V_i \otimes W_j)^+$ with $W_j \mathbf{e}_{i,j}$ as described above, we may express this as

$$\partial((w_0, w_1) \mathbf{e}_{i,j}) = (w_0 + \epsilon^i w_1, w_1 + \epsilon^i w_0) \mathbf{e}_{i-1,j} + \epsilon^i \partial(w_0, w_1) \mathbf{e}_{i,j-1}.$$

We want to use Proposition 4.6 to simplify $(\mathbf{V} \otimes \mathbf{W})^+$. For $0 \leq i \leq D$ and $j \in \mathbb{Z}$, define

$$\begin{aligned} \mathbf{a}_{i,j} &= (1, 0) \mathbf{e}_{i,j}, \\ \mathbf{b}_{i,j} &= (0, 1) \mathbf{e}_{i,j}. \end{aligned}$$

Note that $(V_i \otimes W_j)^+ \cong (L_j \times L_j) \mathbf{e}_{i,j}$ is isomorphic to the direct sum of $L_j \mathbf{a}_{i,j}$ and $L_j \mathbf{b}_{i,j}$. Define

TABLE 1. The table indicates for different i the groups in which $L_j \mathbf{a}_{i,j}$ and $L_j \mathbf{b}_{i,j}$ are contained.

| i | $L_j \mathbf{a}_{i,j}$ | $L_j \mathbf{b}_{i,j}$ |
|---------|------------------------|------------------------|
| D | $U^{(D)}$ | B |
| $D-1$ | A | B |
| \dots | \dots | \dots |
| 1 | A | B |
| 0 | A | $U^{(0)}$ |

$$\begin{aligned}
A &= \bigoplus_{i=0}^{D-1} \bigoplus_j L_j \mathbf{a}_{i,j}, \\
B &= \bigoplus_{i=1}^D \bigoplus_j L_j \mathbf{b}_{i,j}, \\
U^{(D)} &= \bigoplus_j U_{D+j}^{(D)}, \text{ where } U_{D+j}^{(D)} = L_j \mathbf{a}_{D,j}, \\
U^{(0)} &= \bigoplus_j U_j^{(0)}, \text{ where } U_j^{(0)} = L_j \mathbf{b}_{0,j}.
\end{aligned}$$

See Table 1 for a schematic description. Write $U = U^{(0)} \oplus U^{(D)}$. Note that the direct sum of A , B , and U constitutes the full chain complex $(\mathbf{V} \otimes \mathbf{W})^+$.

It is clear that we obtain an isomorphism $g : B \rightarrow A$ by assigning $g(w \mathbf{b}_{i,j}) = w \mathbf{a}_{i-1,j}$ for each i and j and each $w \in L_j$. We now show that discrete Morse theory indeed yields an isomorphism between B and A , though the assignment is slightly more complicated. Let α be the projection map from $A + B + U$ to A as defined in Section 4.3, and let $f = \alpha \circ \partial$.

Lemma 4.7. *We have that $f|_B : B \rightarrow A$ defines an isomorphism.*

Proof. For $w \in L_j$, note that

$$\begin{aligned}
\partial(w \mathbf{b}_{i,j}) &= (\epsilon^i w, w) \mathbf{e}_{i-1,j} + \epsilon^i (q(w), p(w)) \mathbf{e}_{i,j-1} \\
&= \epsilon^i w \mathbf{a}_{i-1,j} + w \mathbf{b}_{i-1,j} + \epsilon^i q(w) \mathbf{a}_{i,j-1} + \epsilon^i p(w) \mathbf{b}_{i,j-1}.
\end{aligned}$$

In particular,

$$f(w \mathbf{b}_{i,j}) = \begin{cases} \epsilon^i w \mathbf{a}_{i-1,j} & \text{if } i = D, \\ \epsilon^i (w \mathbf{a}_{i-1,j} + q(w) \mathbf{a}_{i,j-1}) & \text{if } 1 \leq i \leq D-1. \end{cases}$$

For $i = D$, the term $q(w) \mathbf{a}_{D,j-1}$ is not present, as it belongs to U rather than A .

Each element x of degree k in B is of the form $x = \sum_{i=1}^D w_{k-i} \mathbf{b}_{i,k-i}$, where $w_{k-i} \in L_{k-i}$. We may express $f(x)$ in operator matrix form as

$$f(x) = \begin{pmatrix} \epsilon^1 I & 0 & 0 & \dots & 0 & 0 \\ \epsilon^1 q & \epsilon^2 I & 0 & \dots & 0 & 0 \\ 0 & \epsilon^2 q & \epsilon^3 I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \epsilon^{D-1} I & 0 \\ 0 & 0 & 0 & \dots & \epsilon^{D-1} q & \epsilon^D I \end{pmatrix} \begin{pmatrix} w_{k-1} \\ w_{k-2} \\ w_{k-3} \\ \dots \\ w_{k-D+1} \\ w_{k-D} \end{pmatrix}$$

in the basis $(\mathbf{a}_{0,k-1}, \mathbf{a}_{1,k-2}, \dots, \mathbf{a}_{D-3,k-D+2}, \mathbf{a}_{D-2,k-D+1}, \mathbf{a}_{D-1,k-D})$. Now, the operator matrix is invertible; its inverse is

$$\begin{pmatrix} \epsilon^1 I & 0 & 0 & \cdots & 0 & 0 \\ \epsilon^1 q & \epsilon^2 I & 0 & \cdots & 0 & 0 \\ \epsilon^1 q^2 & \epsilon^2 q & \epsilon^3 I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon^1 q^{D-2} & \epsilon^2 q^{D-3} & \epsilon^3 q^{D-4} & \cdots & \epsilon^{D-1} I & 0 \\ \epsilon^1 q^{D-1} & \epsilon^2 q^{D-2} & \epsilon^3 q^{D-3} & \cdots & \epsilon^{D-1} q & \epsilon^D I \end{pmatrix}.$$

Since this is true in each degree k , we deduce that $f|_B$ is an isomorphism. \square

Let β and \hat{U} be defined as in Proposition 4.6.

Corollary 4.8. *We have that*

$$\hat{U} : \cdots \xrightarrow{\partial} \hat{U}_{k+1} \xrightarrow{\partial} \hat{U}_k \xrightarrow{\partial} \hat{U}_{k-1} \xrightarrow{\partial} \cdots$$

forms a chain complex with the same homology as $(\mathbf{V} \otimes \mathbf{W})^+$.

Proof. Apply Proposition 4.6 and Lemma 4.7. \square

Write $\hat{u} = u - \beta(u)$, $\hat{U}_k^{(D)} = (\text{id} - \beta)(U_k^{(D)})$, and $\hat{U}_k^{(0)} = (\text{id} - \beta)(U_k^{(0)})$. Moreover, define $\hat{\mathbf{a}}_{D,j} = \mathbf{a}_{D,j} + \epsilon^{D+1} \mathbf{b}_{D,j} = (1, \epsilon^{D+1}) \mathbf{e}_{D,j}$.

Lemma 4.9. *Consider an element in $U^{(D)}$ of the form $u = w \mathbf{a}_{D,j}$, where $w \in L_j$. Then, we have that $\hat{u} = w \hat{\mathbf{a}}_{D,j}$ and*

$$\partial(w \hat{\mathbf{a}}_{D,j}) = \epsilon^D (p(w) + \epsilon^{D+1} q(w)) \hat{\mathbf{a}}_{D,j-1}.$$

In particular, the groups $\hat{U}_j^{(D)}$ constitute a subcomplex $\hat{U}^{(D)}$ of \hat{U} , and

$$H_*(\hat{U}^{(D)}) \cong \begin{cases} H_{*-D}(\mathbf{W}^+) & \text{if } D \text{ is odd,} \\ H_{*-D}(\mathbf{W}^-) & \text{if } D \text{ is even.} \end{cases}$$

Proof. The formula for $\partial(w \hat{\mathbf{a}}_{D,j})$ is just a straightforward computation. By Proposition 4.6, it follows that $\hat{u} = w \hat{\mathbf{a}}_{D,j}$.

For the last statement, we may identify \mathbf{W}^+ with the chain complex with chain groups L_j and with the boundary map given by

$$\partial(w) = p(w) + q(w).$$

Similarly, we may identify \mathbf{W}^- with the chain complex with chain groups L_j and with the boundary map given by

$$\partial(w) = p(w) - q(w).$$

Up to a shift in degree by D , the chain groups and the boundary map of $\hat{U}^{(D)}$ are isomorphic to those of either \mathbf{W}^+ or \mathbf{W}^- , depending on the parity of D . As a consequence, we obtain the statement. \square

For $a \in A$, $b \in B$, and $u \in U$, write $\gamma(a + b + u) = u$. Recall the assumption that $L_j = 0$ unless $0 \leq j \leq D$.

Lemma 4.10. *Let $0 \leq k \leq D$ and let $u = w \mathbf{b}_{0,k}$, where $w \in L_k$. Then,*

$$(2) \quad \hat{u} = \varphi(w \mathbf{b}_{0,k}) := \sum_{i=0}^D q^i(w) \mathbf{b}_{i,k-i}$$

and

$$(3) \quad \partial(\varphi(w\mathbf{b}_{0,k})) = \varphi((p(w) + q(w))\mathbf{b}_{0,k-1}).$$

In particular, the groups $\hat{U}_j^{(0)}$ constitute a subcomplex $\hat{U}^{(0)}$ of \hat{U} , and

$$H_*(\hat{U}^{(0)}) \cong H_*(W^+).$$

Proof. The boundary of the right-hand side of (2) equals

$$\begin{aligned} & \sum_{i=1}^D q^i(w)(\epsilon^i \mathbf{a}_{i-1,k-i} + \mathbf{b}_{i-1,k-i}) \\ & + \sum_{i=0}^D \epsilon^i q^{i+1}(w) \mathbf{a}_{i,k-i-1} + \sum_{i=0}^D \epsilon^i pq^i(w) \mathbf{b}_{i,k-i-1} \\ = & \sum_{i=0}^{D-1} \epsilon^i (pq^i(w) + \epsilon^i q^{i+1}(w)) \mathbf{b}_{i,k-i-1} \\ & + \epsilon^D q^{D+1}(w) \mathbf{a}_{D,k-D-1} + \epsilon^D pq^D(w) \mathbf{b}_{D,k-D-1} \\ = & \sum_{i=0}^{D-1} \epsilon^i (pq^i(w) + \epsilon^i q^{i+1}(w)) \mathbf{b}_{i,k-i-1} \\ = & \sum_{i=0}^{D-1} q^i(p(w) + q(w)) \mathbf{b}_{i,k-i-1} = \varphi((p(w) + q(w))\mathbf{b}_{0,k-1}). \end{aligned}$$

The second equality is because $k \leq D$ and hence $pq^D(w) = q^{D+1}(w) = 0$. The third equality follows from repeated application of the identity $pq = -qp$. This yields (3). Since the element in the right-hand side of (3) is an element in $B + U^{(0)}$, Proposition 4.6 yields (2).

For the final claim in the lemma, note that the chain groups and the boundary map of $\hat{U}^{(0)}$ are isomorphic to those of \hat{W}^+ . \square

Without the assumption that $L_j = 0$ unless $0 \leq j \leq D$, (3) would not necessarily be true, and the groups $\hat{U}_j^{(0)}$ might not constitute a subcomplex of \hat{U} . Namely, the term $q^{D+1}(w) \mathbf{a}_{D,k-D-1} \in U^{(D)}$ in the expansion of $\partial(\hat{u})$ is nonzero if $q^{D+1}(w)$ is nonzero.

Corollary 4.11. \hat{U} splits into two complexes

$$\begin{array}{ccccccc} \hat{U}^{(D)} : \hat{U}_{2D}^{(D)} & \xrightarrow{\partial} & \hat{U}_{2D-1}^{(D)} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \hat{U}_{D+1}^{(D)} & \xrightarrow{\partial} & \hat{U}_D^{(D)}, \\ \hat{U}^{(0)} : \hat{U}_D^{(0)} & \xrightarrow{\partial} & \hat{U}_{D-1}^{(0)} & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & \hat{U}_1^{(0)} & \xrightarrow{\partial} & \hat{U}_0^{(0)}. \end{array}$$

Proof. This is an immediate consequence of Lemmas 4.9 and 4.10. \square

Applying Lemmas 4.9 and 4.10 and using Corollaries 4.8 and 4.11, we obtain a description of the homology of $(V \otimes W)^+$ in terms of W^+ and W^- .

Theorem 4.12. *If D is odd, then*

$$H_i((V \otimes W)^+) \cong H_i(W^+) \oplus H_{i-D}(W^+) \\ \cong \begin{cases} H_i(W^+) & \text{if } 0 \leq i \leq D-1, \\ H_D(W^+) \oplus H_0(W^+) & \text{if } i = D, \\ H_{i-D}(W^+) & \text{if } D+1 \leq i \leq 2D, \\ 0 & \text{otherwise.} \end{cases}$$

If D is even, then

$$H_i((V \otimes W)^+) \cong H_i(W^+) \oplus H_{i-D}(W^-) \\ \cong \begin{cases} H_i(W^+) & \text{if } 0 \leq i \leq D-1, \\ H_D(W^+) \oplus H_0(W^-) & \text{if } i = D, \\ H_{i-D}(W^-) & \text{if } D+1 \leq i \leq 2D, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of $B_{d,n}$, we already know the free part of the homology by Corollary 4.5; hence we may focus on the torsion part.

Corollary 4.13. *Let $d \geq n$. Then, the torsion part $T_*(B_{d,n}; \mathbb{Z})$ of $H_*(B_{d,n}; \mathbb{Z})$ is an elementary 2-group satisfying*

$$T_i(B_{d,n}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 1 \leq i \leq n-3 \text{ and } i \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } d-2 \leq i \leq d+n-5 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is an immediate consequence of Lemma 4.2 and Theorem 4.12; let W be the standard hemispherical cell decomposition of the $(n-2)$ -sphere. \square

Combining Corollaries 4.5 and 4.13, we obtain Theorem 1.2.

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