

SUPERSOLVABILITY AND THE KOSZUL PROPERTY OF ROOT IDEAL ARRANGEMENTS

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ABSTRACT. A *root ideal arrangement* \mathcal{A}_I is the set of reflecting hyperplanes corresponding to the roots in an order ideal $I \subseteq \Phi^+$ of the root poset on the positive roots of a finite crystallographic root system Φ . A characterisation of supersolvable root ideal arrangements is obtained. Namely, \mathcal{A}_I is supersolvable if and only if I is *chain peelable*, meaning that it is possible to reach the empty poset from I by in each step removing a maximal chain which is also an order filter. In particular, supersolvability is preserved under taking subideals. We identify the minimal ideals that correspond to non-supersolvable arrangements. There are essentially two such ideals, one in type D_4 and one in type F_4 . By showing that \mathcal{A}_I is not line-closed if I contains one of these, we deduce that the Orlik-Solomon algebra $\mathcal{OS}(\mathcal{A}_I)$ has the Koszul property if and only if \mathcal{A}_I is supersolvable.

1. INTRODUCTION

Let Φ be a finite, crystallographic root system with set of positive roots Φ^+ . The orthogonal complements of the elements of Φ^+ form a hyperplane arrangement known as the *Weyl arrangement*, or *Coxeter arrangement*, of Φ ; this is the set of reflecting hyperplanes of the corresponding reflection group.

With any set of positive roots $R \subseteq \Phi^+$ is associated the corresponding subarrangement \mathcal{A}_R of the Weyl arrangement. That is,

$$\mathcal{A}_R = \{H_\gamma \mid \gamma \in R\},$$

where H_γ is the orthogonal complement of γ .

In [15], Sommers and Tymoczko initiated the study of an interesting family of subarrangements of the Weyl arrangement. Suppose $I \subseteq \Phi^+$ is an order ideal in the root poset on Φ^+ . That is, if $\gamma_1 \in I$ and $\gamma_2 < \gamma_1$, then $\gamma_2 \in I$. It was shown in [15] for Φ of classical type (and conjectured in general) that every *root ideal arrangement* \mathcal{A}_I is free in the sense of Terao [17]. This result was extended to arbitrary finite crystallographic type by Abe *et al.* [1].

A stronger property than freeness is supersolvability. Although being free, root ideal arrangements are not in general supersolvable. Indeed, for irreducible root systems, it is known that \mathcal{A}_{Φ^+} is supersolvable if and only if Φ is of type A_n , B_n , C_n or G_2 [2].

Any matroid \mathcal{A} has an associated Orlik-Solomon algebra $\mathcal{OS}(\mathcal{A})$, as introduced in [11]. We refer to [12] or [18] for accounts of the many attractive properties of $\mathcal{OS}(\mathcal{A})$. Let us here merely recall that if \mathcal{A} is a complex hyperplane arrangement (meaning that a flat of the matroid is the collection

of all hyperplanes containing a given subspace), $\mathcal{OS}(\mathcal{A})$ is isomorphic to the cohomology algebra of the complement of $\cup\mathcal{A}$.

Shelton and Yuzvinsky [14] proved that $\mathcal{OS}(\mathcal{A})$ is a Koszul algebra whenever \mathcal{A} is supersolvable. Whether the converse holds remains an open question which has been answered affirmatively for certain classes of arrangements. For example, it is known for hypersolvable arrangements [9], graphic arrangements [13] and arrangements whose minimal broken circuits are pairwise disjoint [10].

Let us call a poset chain peelable if it is possible to remove all its elements by using moves that remove, in each step, a set of elements that simultaneously form a maximal totally ordered subset and an order filter (i.e. the complement of an order ideal). The first main result of the present paper is the following characterisation of supersolvable root ideal arrangements:

Theorem 1.1. *Suppose Φ is a finite crystallographic root system and that $I \subseteq \Phi^+$ is an order ideal in the root poset on the positive roots. Then, the root ideal arrangement \mathcal{A}_I is supersolvable if and only if the poset I is chain peelable.*

It is clear that chain peelability of a poset is inherited by all its ideals. This means that there exist minimal “bad” root poset ideals in the sense that \mathcal{A}_I is not supersolvable if and only if I contains one of the bad ideals as a subset. We identify these bad ideals, finding that there are essentially only two of them. One is generated by the roots of height 3 in type D_4 , whereas the other is generated by the roots of height 4 in type F_4 .

Falk [6] introduced the concept of line-closedness and showed for a matroid \mathcal{A} that having this property is a necessary condition for $\mathcal{OS}(\mathcal{A})$ being quadratic. This, in turn, is a necessary condition for $\mathcal{OS}(\mathcal{A})$ being Koszul; see e.g. [3]. By showing that a root ideal arrangement \mathcal{A}_I cannot be line-closed if I contains a bad ideal, we deduce

Theorem 1.2. *A root ideal arrangement is supersolvable if and only if its Orlik-Solomon algebra is Koszul.*

Thus, root ideal arrangements form yet another class of arrangements for which the Koszul property is equivalent to supersolvability.

The remainder of this paper is organised as follows. In the next section, we review some properties of root posets and agree on root system notation. In Section 3, conventions on arrangements are established and a convenient characterisation of supersolvability is recalled. We then start pursuing Theorem 1.1. General restrictions on the structure of supersolvable root ideal arrangements are collected in Section 4. Chain peelable posets are formally defined in Section 5 where it is shown that \mathcal{A}_I is supersolvable if I is chain peelable. The converse is established for simply laced Φ in Section 6, whereas multiply laced root systems are handled in Section 7. In the final section, the equivalence between the Koszul property of $\mathcal{OS}(\mathcal{A}_I)$ and supersolvability of \mathcal{A}_I is deduced.

2. ROOT SYSTEMS AND ROOT POSETS

Some useful root system properties are collected in this section. A thorough account of the subject can be found in Humphreys’ book [8].

Consider a finite, crystallographic root system Φ with set of positive roots $\Phi^+ \subset \Phi$. Denote by $\Delta \subseteq \Phi^+$ the set of simple roots. Then, Δ is a basis for the ambient Euclidean space $V \cong \mathbb{R}^{\text{rank } \Phi}$. For $\gamma \in \Phi^+$ and $\alpha \in \Delta$, let γ_α be the corresponding coordinate. Thus,

$$\gamma = \sum_{\alpha \in \Delta} \gamma_\alpha \alpha,$$

where all γ_α are nonnegative integers. The *height* of γ is defined by

$$\text{ht}(\gamma) = \sum_{\alpha \in \Delta} \gamma_\alpha$$

and the *support* of γ is

$$\text{supp } \gamma = \{\alpha \in \Delta \mid \gamma_\alpha \neq 0\}.$$

This support induces a connected subgraph of the Dynkin diagram of Δ and, conversely, the sum of the nodes of any connected subgraph of the Dynkin diagram is a positive root.

The reflections through the orthogonal complements of the roots are the reflections of a finite reflection group. Let s_α denote the reflection corresponding to the simple root $\alpha \in \Delta$. Then, s_α permutes $\Phi^+ \setminus \{\alpha\}$ whereas $s_\alpha(\alpha) = -\alpha$.

The *root poset* on Φ^+ is the partial order defined by $\gamma \leq \gamma'$ if and only if $\gamma_\alpha \leq \gamma'_\alpha$ for all $\alpha \in \Delta$. The cover relation \lhd satisfies $\text{ht}(\gamma) = \text{ht}(\gamma') - 1$ whenever $\gamma \lhd \gamma'$. In other words, $\gamma \lhd \gamma'$ means $\gamma' - \gamma \in \Delta$. The root posets of type D_4 and F_4 are depicted in Figures 1 and 2, respectively.

For any $S \subseteq \Phi$, $\Phi \cap \text{span } S$ is a root subsystem of Φ . We collect here convenient properties of such subsystems that shall be used frequently, often without explicit mentioning. The first observation follows, for example, by considering root lengths and using the fact that every root is in the orbit of a simple root under the action of the associated reflection group.

Lemma 2.1. *If Φ has a doubly (respectively, triply) laced root subsystem, then Φ is doubly (respectively, triply) laced.*

In particular, every root subsystem of a simply laced system is itself simply laced.

Lemma 2.2. *Any rank two subsystem of Φ contains at most one pair of positive roots that are incomparable in the root poset on Φ^+ . If they exist, these two elements are minimal among all positive roots in the subsystem.*

Proof. Suppose $\Psi \subseteq \Phi$ is a rank two subsystem. Let γ_1, γ_2 denote the simple roots of Ψ that correspond to the choice of positive roots $\Psi^+ = \Phi^+ \cap \Psi$. The other elements of Ψ^+ are positive linear combinations of the γ_i , hence larger than γ_i . Inspection of all rank two root systems reveals that $\Psi^+ \setminus \{\gamma_1, \gamma_2\}$ is totally ordered. The assertion follows. \square

Note that γ_1 and γ_2 in the above proof may be comparable in the root poset on Φ^+ . Hence, it is not in general possible to replace “at most” by “exactly” in the statement of Lemma 2.2.

Let (\cdot, \cdot) denote the Euclidean inner product on V . By inspecting all simply and doubly laced rank two root systems (namely, $A_1 \times A_1$, A_2 and $B_2 = C_2$), one readily verifies the following lemma.

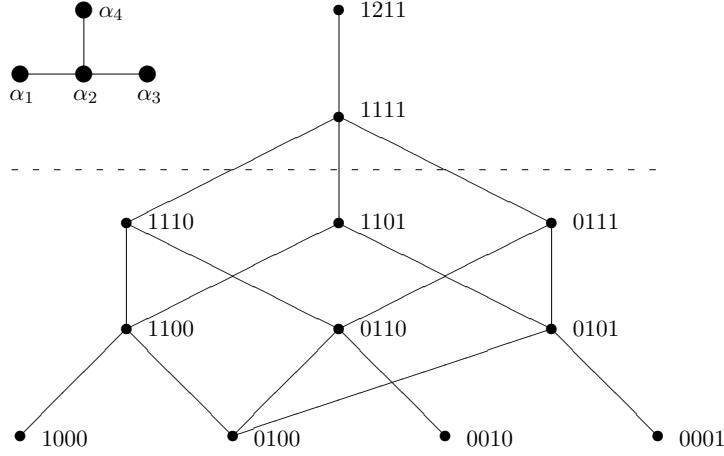


FIGURE 1. The type D_4 Dynkin diagram and root poset. Root poset elements are labelled by coordinate sequences. For example, the label “0101” indicates the root $\alpha_2 + \alpha_4$. The elements below the dotted line form the minimal star ideal.

Lemma 2.3. *Suppose Φ is a finite crystallographic root system without triple bonds in the Dynkin diagram. For any two distinct positive roots $\beta, \gamma \in \Phi^+$, the following assertions hold:*

- If $(\beta, \gamma) > 0$, then $\beta - \gamma \in \Phi$ and $\beta + \gamma \notin \Phi$.
- If $(\beta, \gamma) < 0$, then $\beta - \gamma \notin \Phi$ and $\beta + \gamma \in \Phi$.
- If $(\beta, \gamma) = 0$ and Φ is simply laced, then $\beta - \gamma \notin \Phi$ and $\beta + \gamma \notin \Phi$.

3. ARRANGEMENTS AND SUPERSOLVABILITY

Like any vector collection, a subset $R \subseteq \Phi^+$ determines a matroid whose bases are the maximal linearly independent subsets of R . From a matroidal point of view, R is equivalent to the hyperplane arrangement \mathcal{A}_R consisting of the orthogonal complements of the roots in R . In what follows, we shall only consider properties of hyperplane arrangements that are completely determined by the matroidal structure. Therefore we can, and shall, identify any root ideal arrangement \mathcal{A}_I with its root ideal $I \subseteq \Phi^+$. In this spirit, an *arrangement* from now on is a collection \mathcal{A} of pairwise non-parallel vectors in some Euclidean space $V \cong \mathbb{R}^n$. For convenience, we shall assume that \mathcal{A} is *essential*, i.e. that it spans V .

Let \mathcal{A} be an arrangement. A *flat* of \mathcal{A} is a closed set with respect to the closure operator defined by $\text{cl}(S) = \mathcal{A} \cap \text{span } S$ for $S \subseteq \mathcal{A}$. The *rank* of a flat (or any other subset of \mathcal{A}) is the dimension of its span.

An arrangement is *supersolvable* if its lattice of flats is supersolvable in the sense of Stanley [16], i.e. if there is a maximal chain of modular flats. Björner, Edelman and Ziegler [4, Theorem 4.3] showed that the following alternative definition, which is better suited for our purposes, is equivalent. For brevity, we refer to a flat of rank k as a k -*flat*.

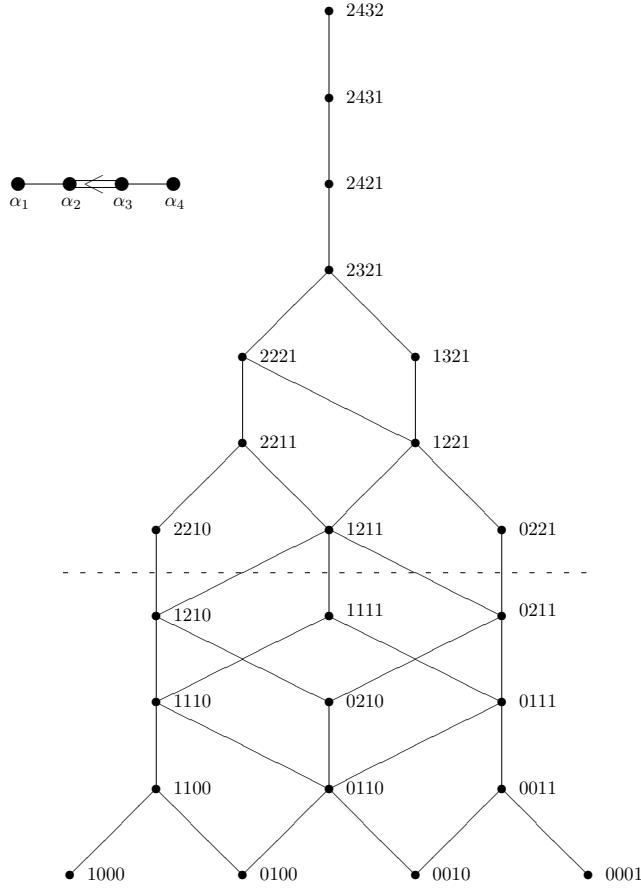


FIGURE 2. The type F_4 Dynkin diagram and root poset.
Roots are labelled by coordinate sequences, as in Figure 1.
The ideal \hat{I} consists of the elements below the dotted line.

Definition 3.1. An arrangement \mathcal{A} is supersolvable if it can be partitioned $\mathcal{A} = \Pi_1 \uplus \cdots \uplus \Pi_n$ in such a way that for all $i \in \{1, \dots, n\}$ the following conditions hold:

- The rank of the subarrangement $\mathcal{A}_i = \Pi_1 \uplus \cdots \uplus \Pi_i$ is i .
- No 2-flat of \mathcal{A}_i is a subset of Π_i .

Let us say that the expression $\Pi_1 \uplus \cdots \uplus \Pi_n$ is a *modular partition* of \mathcal{A} if it satisfies the conditions of Definition 3.1.

If \mathcal{A} is supersolvable, \mathcal{A} is *free* in the sense of Terao [17], meaning that the $\text{Sym } V^*$ -module $\mathcal{D}(\mathcal{A})$ of derivations is free. In this case, the multiset of cardinalities of the blocks of any modular partition coincides with the multiset of degrees of any homogeneous basis for $\mathcal{D}(\mathcal{A})$; these degrees are known as the *exponents* of \mathcal{A} . Recall from the introduction that all root ideal arrangements are free. The corresponding exponents seem to play a role for the Poincaré polynomials of Hessenberg varieties generalising that which is played by the ordinary exponents of Weyl groups for the corresponding flag varieties; see [15] for more details.

4. THE STRUCTURE OF MODULAR PARTITIONS

Let I be a root poset ideal. The next lemma shows that any block of a modular partition of I must be of one of two specific kinds of sets that we now define. The first kind is indexed by a simple root $\alpha \in \Delta$. Let

$$F_I(\alpha) = \{\gamma \in I \mid \gamma \geq \alpha\}$$

be the order filter in I generated by α . For the second kind, choose simple roots $\alpha, \beta \in \Delta$ and positive integers a, b . Define

$$G_I(\alpha, \beta, a, b) = \{\gamma \in I \mid \gamma_\alpha \alpha + \gamma_\beta \beta \neq k(a\alpha + b\beta) \text{ for all } k \in \mathbb{N}\}.$$

Lemma 4.1. *Suppose that $I = \Pi_1 \dot{\cup} \dots \dot{\cup} \Pi_n$ is a modular partition. Then either (a) there is some $\alpha \in \Delta$ such that $\Pi_n = F_I(\alpha)$, or (b) there are $\alpha, \beta \in \Delta$ and positive integers a, b such that $\Pi_n = G_I(\alpha, \beta, a, b)$.*

Moreover, if (a) holds, $F_I(\alpha)$ is a chain. If (b) holds, $a\alpha + b\beta \in \Phi^+$.

Proof. Define $d = |\Pi_n \cap \Delta|$ and $I_{n-1} = I \setminus \Pi_n$. Since I_{n-1} has rank $n-1$, we have $d \geq 1$. It is not possible to have $d \geq 3$, because in that case Π_n would contain two orthogonal simple roots and, hence, a 2-flat. Thus, $d = 1$ or $d = 2$.

First assume $d = 1$; say $\Pi_n \cap \Delta = \{\alpha\}$. Thus, $I_{n-1} \subseteq \text{span}(\Delta \setminus \{\alpha\})$, because the dimension of this span is $n-1$. Therefore, $F_I(\alpha) \subseteq \Pi_n$. If equality does not hold, we find $\gamma \in \Pi_n \setminus F_I(\alpha)$. Since γ and α are incomparable, all roots in $I \cap \text{span}\{\alpha, \gamma\}$ except γ belong to $F_I(\alpha)$, but then Π_n contains an entire 2-flat, a contradiction. We conclude that $\Pi_n = F_I(\alpha)$ so that situation (a) is at hand. Being an order filter, $F_I(\alpha)$ cannot contain incomparable elements since, by Lemma 2.2, two such elements would generate a 2-flat entirely contained in $F_I(\alpha)$. Hence, $F_I(\alpha)$ is in fact a chain in this case.

Now consider the case $d = 2$ with $\Pi_n \cap \Delta = \{\alpha, \beta\}$. No 2-flat is contained in Π_n , so $a\alpha + b\beta \in I_{n-1}$ for some positive integers a and b . (In particular, $a\alpha + b\beta$ is a root.) The span of $\{a\alpha + b\beta\} \cup \Delta \setminus \{\alpha, \beta\}$ has dimension $n-1$, hence must be equal to $\text{span } I_{n-1}$. Thus, $\gamma_\alpha \alpha + \gamma_\beta \beta$ must be parallel to $a\alpha + b\beta$ for every $\gamma \in I_{n-1}$. This shows $\Pi_n \supseteq G_I(\alpha, \beta, a, b)$. Equality must in fact hold, for if $\gamma_\alpha \alpha + \gamma_\beta \beta$ is parallel to $a\alpha + b\beta$, then γ is the only positive root with this property in the 2-flat generated by α and γ , forcing $\gamma \in I_{n-1}$. Thus (b) is satisfied. \square

Remark 4.2. *There are just a few possible values that a and b can assume in case (b) of Lemma 4.1. Since $a\alpha + b\beta \in \Phi^+$, there is a bond between α and β in the Dynkin diagram. If the bond is simple, the only possibility is $(a, b) = (1, 1)$. If the bond is double, α and β have different lengths. If α is the long root, either $(a, b) = (1, 1)$ or $(a, b) = (1, 2)$.*

Since by construction $F_I(\alpha)$ is an order filter, $I \setminus F_I(\alpha)$ is an ideal in the root poset. Notice however, that $I \setminus G_I(\alpha, \beta, a, b)$ is not in general an ideal. Thus, Lemma 4.1 seems to suggest that in order to study supersolvable root ideals one is forced to leave the realm of ideals. Strictly speaking, this is true; there are root ideals with supersolvable partitions whose initial segments are

not all ideals.¹ The following lemma shows that these segments *are* however ideals in root subsystems.

Lemma 4.3. *If $\alpha, \beta \in \Delta$ and $a\alpha + b\beta \in \Phi^+$ for positive a and b , then $\Phi'^+ = \Phi^+ \setminus G_{\Phi^+}(\alpha, \beta, a, b)$ is the set of positive roots in a root subsystem Φ' of Φ with set of simple roots $\Delta' = \{a\alpha + b\beta\} \cup \Delta \setminus \{\alpha, \beta\}$. Moreover, the root poset on Φ'^+ coincides with the induced subposet of the root poset of Φ^+ . In particular, $I \setminus G_I(\alpha, \beta, a, b)$ is an ideal in the root poset on Φ'^+ .*

Proof. Define $\Phi' = \Phi \cap \text{span } \Delta'$. Being the intersection of Φ with a hyperplane spanned by roots, Φ' is a root subsystem of rank one less than Φ . The elements of $\Phi'^+ = \Phi^+ \cap \Phi'$ are positive linear combinations of the elements of Δ' . Hence, Δ' can be chosen as the simple root set of Φ' . With this choice, the order relation of the root poset on Φ'^+ is precisely that induced from Φ^+ . \square

5. CHAIN PEELINGS AND SUPERSOLVABILITY

In this section we define chain peelings of posets. First, we establish a key lemma about the structure of totally ordered intervals in root posets. From it, we deduce that chain peelings of I are modular partitions.

For positive roots β_1, β_2 , employ the following notation for intervals in the root poset:

$$[\beta_1, \beta_2] = \{\gamma \in \Phi^+ \mid \beta_1 \leq \gamma \leq \beta_2\}.$$

A *chain* in a poset is a totally ordered subposet.

Lemma 5.1. *Suppose $[\beta_1, \beta_2]$ is a nonempty chain in the root poset for distinct positive roots $\beta_1, \beta_2 \in \Phi^+$. Then, $\beta_2 - \beta_1 = k\beta$ for some $\beta \in \Phi^+$ and $k \in \{1, 2, 3\}$. If $k = 3$, the Dynkin diagram has a triple bond. If $k = 2$, the diagram has a double or a triple bond.*

Proof. By inspecting all rank two root systems, one readily checks that $\beta_2 - \beta_1 \in k\Phi^+$ can happen only if the Dynkin diagram of the rank two root system $\Phi^+ \cap \text{span}\{\beta_1, \beta_2\}$ has an ℓ -fold bond for some $\ell \geq k$. This observation verifies the concluding two sentences of the lemma.

Assume without loss of generality that Φ is irreducible. In type G_2 , the assertion is readily checked, so we may furthermore assume that there are no triple bonds. Thus, Lemma 2.3 is at our disposal. We shall use it frequently throughout the proof without explicit mentioning.

If β_2 covers β_1 , $\beta_2 - \beta_1$ is a simple root. If not, we have $\beta_1 \triangleleft \beta'_1 \leq \beta'_2 \triangleleft \beta_2$. Let $\alpha_1 = \beta'_1 - \beta_1$, $\alpha_2 = \beta_2 - \beta'_2$, $\gamma = \beta'_2 - \beta'_1$, $\gamma_1 = \gamma + \alpha_1$ and, finally, $\gamma_2 = \gamma + \alpha_2$. See Figure 3 for an illustration.

Proceed by induction on the length of the chain $[\beta_1, \beta_2]$. The induction hypothesis shows $\gamma_1, \gamma_2 \in \Phi^+ \cup 2\Phi^+$ and $\gamma \in \Phi^+ \cup 2\Phi^+ \cup \{0\}$. Moreover, $\alpha_1, \alpha_2 \in \Delta$.

There are two cases to consider, depending on whether α_1 and α_2 coincide. First, suppose $\alpha_1 \neq \alpha_2$. Since $[\beta_1, \beta_2]$ is a chain, $\beta_2 - \alpha_1$ is not a root. Thus, $(\beta_2, \alpha_1) \leq 0$. Similarly, $(\beta_1, \alpha_2) \geq 0$.

¹For example, if $I = \Phi^+$ in type A_2 with simple roots α_1 and α_2 , we may choose $\Pi_2 = \{\alpha_1, \alpha_2\} = G_I(\alpha_1, \alpha_2, 1, 1)$. Then, $\Pi_1 = \{\alpha_1 + \alpha_2\}$ is not an order ideal even though $\Pi_1 \uplus \Pi_2$ is modular.

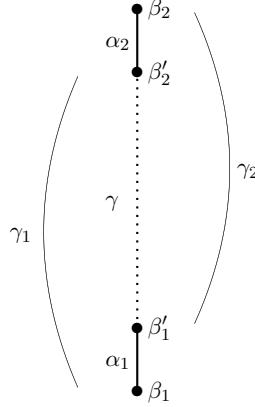


FIGURE 3. Illustration for Lemma 5.1.

Since $\gamma_2 = \gamma_1 + \alpha_2 - \alpha_1$ and the α_i are distinct, it is not possible to have $\gamma_1, \gamma_2 \in 2\Phi^+$. Let us assume $\gamma_1 \in \Phi^+$; the assumption $\gamma_2 \in \Phi^+$ admits a completely analogous proof.

If $(\gamma_1, \alpha_2) < 0$, $\alpha_2 + \gamma_1 \in \Phi^+$ and we are done. Otherwise, the fact that $\beta_2 - \gamma_1 = \beta_1 + \alpha_2 \notin \Phi^+$ yields

$$0 \geq (\gamma_1, \beta_2) = (\gamma_1, \beta_1) + |\gamma_1|^2 + (\gamma_1, \alpha_2) \geq (\gamma_1, \beta_1) + |\gamma_1|^2.$$

This is only possible if γ_1 is short and β_1 is long. Therefore,

$$(\beta_1, \beta_2) = |\beta_1|^2 + (\beta_1, \gamma_1) + (\beta_1, \alpha_2) \geq |\beta_1|^2 + (\beta_1, \gamma_1) > 0,$$

which leads to $\beta_2 - \beta_1 \in \Phi^+$, concluding the proof when $\alpha_1 \neq \alpha_2$.

Now suppose $\alpha_1 = \alpha_2 = \alpha$. If $\gamma = 0$, $\beta_2 - \beta_1 = 2\alpha$ and we are done. Otherwise, $\text{supp } \gamma \setminus \{\alpha\} \neq \emptyset$.

We have $(\beta_1, \alpha) \leq 0$ and $(\beta_2, \alpha) \geq 0$ since $\beta_1 + \alpha \in \Phi^+$ and $\beta_2 - \alpha \in \Phi^+$. Thus, $s_\alpha(\beta_1) = \beta_1 + k_1\alpha$ and $s_\alpha(\beta_2) = \beta_2 - k_2\alpha$ for some $k_1, k_2 \in \{0, 1, 2\}$. It follows that

$$\beta_1 \leq s_\alpha(\beta_1) < s_\alpha(\beta_2) \leq \beta_2.$$

If one of the weak inequalities is strict, $s_\alpha(\beta_2 - \beta_1) \in (\Phi^+ \setminus \{\alpha\}) \cup 2(\Phi^+ \setminus \{\alpha\})$ by the induction hypothesis. Then, $\beta_2 - \beta_1 \in (\Phi^+ \setminus \{\alpha\}) \cup 2(\Phi^+ \setminus \{\alpha\})$ and we are done. If, on the other hand, neither of the weak inequalities is strict, we have

$$s_\alpha(\gamma) = s_\alpha(\beta_2 - \beta_1 - 2\alpha) = \beta_2 - \beta_1 + 2\alpha = \gamma + 4\alpha.$$

Hence $\gamma \notin \Phi^+$. By the induction hypothesis, this implies $\gamma = 2\gamma'$, $\gamma' \in \Phi^+$, and $s_\alpha(\gamma') = \gamma' + 2\alpha \in \Phi^+$. Thus, $\gamma' + \alpha \in \Phi^+$. Observing $\beta_2 - \beta_1 = 2(\gamma' + \alpha)$ concludes the proof. \square

Definition 5.2. Suppose P is a finite poset. If P is a chain, P is a chain peeling of itself. Otherwise a chain peeling of P is a partition of its elements $P = P_1 \uplus \cdots \uplus P_n$, $n \geq 2$, such that

- The set P_n is an order filter containing a minimal element of P .
- The set P_n is a chain.
- The partition $P_1 \uplus \cdots \uplus P_{n-1}$ is a chain peeling of $P \setminus P_n$.

In other words, if the order filter generated by a minimal element of P is a chain, we may “peel off” this chain and obtain a smaller poset with one minimal element less. If this process can go on until the empty poset is all that remains, P has a chain peeling.

A poset that has a chain peeling is *chain peelable*. Observe that if P is chain peelable, then so is every ideal of P .

For root poset ideals, as we shall see, chain peelability turns out to be equivalent to supersolvability. One direction of this correspondence is fairly straightforward to establish.

Lemma 5.3. *If I is a root poset ideal, any chain peeling of I is a modular partition of I .*

Proof. By induction on the rank, it is sufficient to show that if $F_I(\alpha)$ is a chain for some $\alpha \in \Delta$, then this chain does not contain an entire 2-flat.

Suppose $F_I(\alpha)$ is a chain. Assume it contains two distinct roots, say $\gamma, \gamma' \in F_I(\alpha)$. Let $\delta > \delta'$ be the two smallest elements in $F_I(\alpha) \cap \text{span}\{\gamma, \gamma'\}$. Lemma 5.1 shows that $\delta - \delta'$ is a positive multiple of some $\beta \in \Phi^+$. Since $\beta < \delta$, $\beta \in I \cap \text{span}\{\gamma, \gamma'\}$. By minimality of δ and δ' , $\beta \notin F_I(\alpha)$. Thus, $F_I(\alpha)$ does not contain the 2-flat $I \cap \text{span}\{\gamma, \gamma'\}$. \square

6. SIMPLY LACED ROOT SYSTEMS

For the moment, let us focus on simply laced root systems, leaving multiple laces to the next section. The arguments in this section are roughly classification independent. More precisely, we rely on the classification of finite root systems for two facts about the Dynkin diagram, namely that it contains no circuit and that if there is a node with degree at least three, then the degree is precisely three and this node is unique in its connected component. Both facts are easily shown from first principles without consulting the full classification machinery.

When Φ is of type D_4 , the root poset ideal which consists of all elements strictly below the sum of all simple roots has special properties. In Figure 1, this ideal is highlighted by the dotted line. We shall see that it is in some sense the unique minimal non-supersolvable ideal in the simply laced setting. Let us introduce a name for ideals that contain (an isomorphic copy of) it.

Definition 6.1. *An ideal $I \subseteq \Phi^+$ is a star ideal if there are distinct simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Delta$ such that $\gamma_4 = \alpha_1 + \alpha_2 + \alpha_3$, $\gamma_3 = \alpha_1 + \alpha_2 + \alpha_4$ and $\gamma_1 = \alpha_2 + \alpha_3 + \alpha_4$ all are roots that belong to I .*

Suppose α_i are as in Definition 6.1. Using that all roots have connected support and that there are no circuits in the Dynkin diagram, it is easy to see that $\alpha_1, \alpha_3, \alpha_4$ must all be neighbours of α_2 . In particular, $\alpha_1, \alpha_3, \alpha_4$ are pairwise orthogonal and α_2 has no other neighbours.

Lemma 6.2. *If I is a star ideal, then I is not supersolvable.*

Proof. Assume in order to obtain a contradiction that I is a star ideal with modular partition $I = \Pi_1 \uplus \cdots \uplus \Pi_n$ and that n is minimal among all modular partitions of star ideals. Let γ_i and α_j be as in Definition 6.1. By Lemma

4.1 and Lemma 4.3, $I_n \setminus \Pi_n$ is an ideal either in Φ^+ or in the subsystem Φ'^+ defined in Lemma 4.3. By minimality of n , this ideal is not a star ideal.

Suppose first that we are in case (a) of Lemma 4.1. Then, $\Pi_n = F_I(\alpha_i)$ for some $i \in \{1, 2, 3, 4\}$. Thus, at least two of the incomparable γ_i belong to Π_n , contradicting that $F_I(\alpha_i)$ is a chain.

If instead situation (b) of Lemma 4.1 holds, we may without loss of generality assume that either $\Pi_n = G_I(\alpha_1, \beta, a, b)$ for some $\beta \neq \alpha_i$, or else $\Pi_n = G_I(\alpha_1, \alpha_2, a, b)$. In the former case, the flat generated by γ_3 and γ_4 is contained in Π_n . In the latter, Π_n instead contains the flat generated by α_2 and γ_1 . Either way, Π_n contains an entire 2-flat, providing the contradiction. \square

If the subgraph of the Dynkin diagram induced by some simple roots $\alpha_1, \dots, \alpha_k \in \Delta$ is a path, then $\alpha_1 + \dots + \alpha_k \in \Phi^+$ since, in particular, this subgraph is connected. It is worthwhile to introduce some notation pertaining to roots of this form.

Definition 6.3. Let path Φ denote the set of positive roots $\gamma \in \Phi^+$ satisfying that $\text{supp } \gamma$ is a path in the Dynkin diagram of Δ and $\gamma_\alpha = 1$ for every $\alpha \in \text{supp } \gamma$.

Lemma 6.4. If Φ is simply laced and I is not a star ideal, then $I \subseteq \text{path } \Phi$.

Proof. If $I \not\subseteq \text{path } \Phi$, there is some $\gamma \in I \setminus \text{path } \Phi$ satisfying $\gamma' \in \text{path } \Phi$ for all $\gamma' < \gamma$. For some $\alpha \in \Delta$ and $\gamma' < \gamma$, we have $\gamma = \alpha + \gamma'$. By Lemma 2.3, $\alpha \notin \text{supp } \gamma'$ since Φ is simply laced and $\gamma' \in \text{path } \Phi$. Therefore, the support of γ does not form a path in the Dynkin diagram of Φ , so this support must contain the diagram of D_4 as a subgraph. Thus, I is a star ideal. \square

Since Φ is finite, the Dynkin diagram contains no circuits. If $\alpha_1, \alpha_2 \in \Delta$ belong to the same irreducible component, the diagram thus contains exactly one path with α_1, α_2 as endpoints. Let $(\alpha_1 \frown \alpha_2) \in \text{path } \Phi$ denote the sum of the simple roots that comprise this path.

Lemma 6.5. If Φ is simply laced and I is not a star ideal, then I is chain peelable.

Proof. Without loss of generality, assume $\Delta \subseteq I$ and that Φ is irreducible.

Suppose I is not a star ideal. By Lemma 6.4, $I \subseteq \text{path } \Phi$. Consider any leaf $\alpha \in \Delta$ in the Dynkin diagram. If $F_I(\alpha)$ is not a chain, there are $\alpha', \alpha'' \in \Delta$ such that $p' = (\alpha \frown \alpha') \in I$ and $p'' = (\alpha \frown \alpha'') \in I$ are incomparable. In particular, $p', p'' \geq \alpha_2$ for some $\alpha_2 \in \Delta$ which has degree at least 3 in the Dynkin diagram. By the classification of simply laced diagrams, α_2 must be the unique node with degree 3. Since I is not a star ideal, we may denote the neighbours of α_2 by $\alpha_1, \alpha_3, \alpha_4 \in \Delta$, in an appropriate order, so that $(\alpha_1 \frown \alpha_3) \notin I$.

If we now let α be the leaf satisfying $\alpha_1 \in \text{supp}(\alpha \frown \alpha_2)$, we cannot find $\alpha', \alpha'' \in I$ as above, since p' or p'' would have to be larger than $(\alpha_1 \frown \alpha_3)$. Thus, $F_I(\alpha)$ is a chain. The lemma follows by induction on the rank. \square

We are now in a position to characterise supersolvable root ideal arrangements in simply laced types. In particular, Theorem 1.1 is established for such root systems.

Theorem 6.6. *If Φ is simply laced, the following assertions are equivalent for an ideal $I \subseteq \Phi^+$ in the root poset:*

- (a) I is supersolvable.
- (b) I is chain peelable.
- (c) I is not a star ideal.

Proof. Lemma 6.2 shows (a) \Rightarrow (c), whereas (c) \Rightarrow (b) is Lemma 6.5. Finally, the implication (b) \Rightarrow (a) is established by Lemma 5.3. \square

In particular, all ideals in type A root posets are supersolvable whereas every root poset of type D or E contains a unique minimal star ideal, containment of which characterises non-supersolvability.

7. MULTIPLY LACED ROOT SYSTEMS

In order to complete the proof of Theorem 1.1, we now turn to root systems that are not simply laced. Here we make use of the classification; the irreducible types that we need to consider are B_n , C_n , F_4 and G_2 . The only case which requires a little care is type F_4 .

Theorem 7.1. *If Φ is of type B_n , C_n or G_2 , every root poset ideal $I \subseteq \Phi^+$ is both chain peelable and supersolvable.*

Proof. Every chain peelable ideal is supersolvable by Lemma 5.3. Since chain peelability is inherited by subideals, it suffices to show that the entire root poset Φ^+ is chain peelable. In types B_n and C_n , the statement follows by induction on n . Namely, if $n \geq 3$, it is enough to note that $F_{\Phi^+}(\alpha)$ is a chain if α is the simple root corresponding to the leaf which is not incident to the double bond in the Dynkin diagram. In the base case $B_2 = C_2$, as well as in type G_2 , chain peelability is readily verified. \square

Now consider the type F_4 root poset with simple roots labelled as in Figure 2. Let \hat{I} be the ideal generated by the three roots $\eta_1 = \alpha_1 + 2\alpha_2 + \alpha_3$, $\eta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\eta_3 = 2\alpha_2 + \alpha_3 + \alpha_4$. Thus, \hat{I} consists of the positive roots of height at most 4, as is illustrated in Figure 2.

Concluding the proof of Theorem 1.1, we now verify that containment of \hat{I} characterises both supersolvability and chain peelability in type F_4 .

Theorem 7.2. *Suppose Φ is of type F_4 . For an ideal $I \subseteq \Phi^+$ in the root poset, the following are equivalent:*

- (a) I is supersolvable.
- (b) I is chain peelable.
- (c) $I \not\supseteq \hat{I}$.

Proof. Since all roots of height 3 are covered by two roots each, it is clear that $F_I(\alpha)$ is not a chain for any simple root α if $I \supseteq \hat{I}$. Hence such an ideal cannot be chain peelable. Conversely, it is easy to see that the ideals $\Phi^+ \setminus F_{\Phi^+}(\eta_i)$ are chain peelable. If $I \not\supseteq \hat{I}$, then I is contained in at least one of these three ideals. Thus, I is chain peelable if and only if $I \not\supseteq \hat{I}$.

Now suppose $I \supseteq \hat{I}$. It remains to show that I is not supersolvable. Case (a) of Lemma 4.1 was ruled out above. Hence, if a modular partition $I = \Pi_1 \uplus \Pi_2 \uplus \Pi_3 \uplus \Pi_4$ were to exist, the only possibility would be to

have $\Pi_4 = G_I(\alpha, \beta, a, b)$ for suitably chosen α, β, a, b . Labelling the simple roots as in the Dynkin diagram in Figure 2, there are four possible choices to consider. For each case we now provide a 2-flat entirely contained in $G_I(\alpha, \beta, a, b)$, thereby ruling out the possibility of a modular partition:

- The first possibility is $G_I(\alpha_1, \alpha_2, 1, 1)$ which contains the 2-flat consisting of $2\alpha_2 + \alpha_3$ and $\alpha_2 + \alpha_3 + \alpha_4$.
- Second, $G_I(\alpha_2, \alpha_3, 1, 1)$ contains the 2-flat comprised of $\alpha_1 + \alpha_2$ and $2\alpha_2 + \alpha_3$.
- Third, $G_I(\alpha_3, \alpha_4, 1, 1)$ contains the 2-flat consisting of $2\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$.
- Finally, the set $G_I(\alpha_2, \alpha_3, 2, 1)$ contains the 2-flat which consists of $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_2 + \alpha_3 + \alpha_4$ and (if I contains it) $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$.

□

8. LINE-CLOSEDNESS, QUADRATICITY AND THE KOSZUL PROPERTY

In this final section, we shall apply the classification of supersolvable root ideal arrangements that was obtained in the previous sections in order to establish that these are the only root ideal arrangements whose Orlik-Solomon algebras are Koszul. For an account of how the Koszul property of Orlik-Solomon algebras is relevant to the study of arrangements, we refer to [18]. A general introduction to Koszul algebras can be found in [7].

In an effort to better understand what makes Orlik-Solomon algebras quadratic, Falk [6] used the concept of line-closedness:

Definition 8.1. *Let \mathcal{A} be an arrangement.*

- *A subset $S \subseteq \mathcal{A}$ is 2-closed if $\text{cl}(\{a, b\}) \subseteq S$ for all $a, b \in S$.*
- *The arrangement \mathcal{A} is line-closed if every 2-closed subset of \mathcal{A} is a flat of \mathcal{A} .*

Falk showed that every arrangement with a quadratic Orlik-Solomon algebra is line-closed.² In particular, being line-closed is a necessary condition for having a Koszul Orlik-Solomon algebra. We now observe that line-closed ideals cannot contain the minimal non-supersolvable ideals that were identified in Sections 6 and 7. Recall from those sections the definition of star ideals and the ideal \hat{I} in the type F_4 root poset, respectively.

Proposition 8.2. *Suppose Φ is a finite root system and $I \subseteq \Phi^+$ is an ideal in the root poset.*

- (a) *If Φ is simply laced and I is a star ideal, then I is not line-closed.*
- (b) *If Φ is of type F_4 and $I \supseteq \hat{I}$, then I is not line-closed.*

Proof. For (a), suppose I is a star ideal. Let γ_i and α_j be as in Definition 6.1. Observe that the four roots $\alpha_2, \gamma_1, \gamma_3$ and γ_4 form a 2-closed subset of I . Since (for example) $\alpha_1 = \frac{1}{2}(\gamma_3 + \gamma_4 - \gamma_1 - \alpha_2)$, this subset is not a flat. Hence, I is not line-closed.

Turning to (b), let α_i and η_j be as defined prior to Theorem 7.2. Suppose $\hat{I} \subseteq I$ and define

$$S = \{\eta_1, \eta_2, \eta_3, \alpha_3\} \cup (I \cap \{\alpha_3 + \eta_3, \eta_1 + \eta_2\}).$$

²A counterexample to the conjectured converse was found by Yuzvinsky; cf. [5].

Then, S is a 2-closed subset of I . However, $\alpha_2 = \frac{1}{3}(\eta_1 - \eta_2 + \eta_3 - \alpha_3) \in I \setminus S$ so that S is not a flat and I is not line-closed. \square

Corollary 8.3. *For a root ideal arrangement \mathcal{A}_I , the following assertions are equivalent:*

- (a) \mathcal{A}_I is supersolvable.
- (b) \mathcal{A}_I is line-closed.
- (c) $\mathcal{OS}(\mathcal{A}_I)$ is Koszul.

Proof. Proposition 8.2 and Theorems 6.6, 7.1 and 7.2 show (b) \Rightarrow (a). As was previously mentioned, (a) \Rightarrow (c) and (c) \Rightarrow (b) follow from results of Shelton and Yuzvinsky [14] and Falk [6], respectively. \square

In particular, Theorem 1.2 is established.

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