INVERSION ARRANGEMENTS AND BRUHAT INTERVALS

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Abstract. Let W be a finite Coxeter group. For a given $w \in W$, the following assertion may or may not be satisfied:

($*$) The principal Bruhat order ideal of w contains as many elements as there are regions in the inversion hyperplane arrangement of w.

We present a type independent combinatorial criterion which characterises the elements $w \in W$ that satisfy ($*$). A couple of immediate consequences are derived:

(1) The criterion only involves the order ideal of w as an abstract poset. In this sense, ($*$) is a poset-theoretic property.
(2) For W of type A, another characterisation of ($*$), in terms of pattern avoidance, was previously given in collaboration with Linusson, Shareshian and Sjöstrand. We obtain a short and simple proof of that result.
(3) If W is a Weyl group and the Schubert variety indexed by $w \in W$ is rationally smooth, then w satisfies ($*$).

1. Introduction

Let n be a positive integer. Given indices $1 \le i < j \le n$, define a hyperplane

$$H_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}.$$

The arrangement of all such hyperplanes

$$\mathcal{A}_n = \{H_{i,j} \mid 1 \le i < j \le n\}$$

is known as the *braid arrangement*. The orthogonal reflections in the hyperplanes $H_{i,j}$ generate a finite reflection group isomorphic to the symmetric group $\mathfrak{S}_n$; a natural isomorphism is given by associating a reflection through $H_{i,j}$ with the transposition $(i,j) \in \mathfrak{S}_n$.

Given a permutation $w \in \mathfrak{S}_n$, we define its *inversion arrangement* as the following subarrangement of $\mathcal{A}_n$:

$$\mathcal{A}_w = \{H_{i,j} \mid 1 \le i < j \le n,\ w(i) > w(j)\}.$$

In particular, $\mathcal{A}_{w_0} = \mathcal{A}_n$, where $w_0 \in \mathfrak{S}_n$ is the reverse permutation $i \mapsto n+1-i$.

The inversion arrangement $\mathcal{A}_w$ cuts the ambient space into a set $\operatorname{reg}(w)$ of *regions*, a region being a connected component of the complement $\mathbb{R}^n \setminus \cup\mathcal{A}_w$.

Let $[\cdot,\cdot]$ denote closed intervals in the Bruhat order on $\mathfrak{S}_n$ (the definition of which is recalled in Section 2). Postnikov [13] discovered a numerical relationship between $\operatorname{reg}(w)$ and the Bruhat order ideal $[e,w]$, where $e \in \mathfrak{S}_n$ is the identity permutation. When w is a Grassmannian permutation, he proved that the sets are equinumerous; both are in 1-1 correspondence with certain cells in a CW decomposition of the totally nonnegative Grassmannian. For arbitrary w, he conjectured the following results that were subsequently proven in [8]:

(A) For all $w \in \mathfrak{S}_n$, $\#\mathrm{reg}(w) \leq \#[e, w]$.
(B) Equality holds in (A) if and only if w avoids the patterns 4231, 35142, 42513 and 351624.

The reader who is not familiar with the terminology employed in (B) may find an explanation in Section 4.

We have just defined $\mathcal{A}_w$ using $\mathfrak{S}_n$-specific language. It is, however, completely natural to replace $\mathfrak{S}_n$ by an arbitrary finite Coxeter group W and consider $\mathcal{A}_w$, $\mathrm{reg}(w)$ and $[e, w]$ for any $w \in W$; see Section 2 for details of the definitions. In fact, it was not (A) but the following result which was established in [8]:

(A′) Given a finite Coxeter group W and any $w \in W$, $\#\mathrm{reg}(w) \leq \#[e, w]$.

This generalises (A),[1] but notice that there is no statement (B′). Indeed, the problem of how to characterise those $w \in W$ for which equality holds in (A′) was posed as [8, Open problem 10.3]. Such a characterisation is the main result of the present paper. The precise assertion is stated in Theorem 3.2. It essentially says that equality holds in (A′) if and only if the following property is satisfied for every $u \leq w$: among all paths of shortest length from u to w in the Cayley graph of W (with edges generated by reflections), there is one which visits vertices in order of increasing Coxeter length.

A number of consequences are derived from the main result:

First, we conclude that the characterising property is poset-theoretic. That is, whether or not equality holds in (A′) can be determined by merely looking at $[e, w]$ as an abstract poset.

Second, we give a new proof of the difficult direction of (B). In [8], (A′) was proven by exhibiting an injective map ϕ from (essentially) $\mathrm{reg}(w)$ to $[e, w]$. Thus, proving (B) amounts to characterising surjectivity of ϕ in terms of pattern avoidance when $W = \mathfrak{S}_n$. That surjectivity implies the appropriate pattern avoidance is a reasonably straightforward consequence of the construction of ϕ; see [8, Section 4]. Contrastingly, the proof of the converse statement given in [8, Section 5] is a direct, fairly involved, counting argument which does not use ϕ at all. In light of our Theorem 3.2, surjectivity of ϕ can now, however, be related to pattern avoidance in a rather straightforward way.

Third, when W is a Weyl group, each element $w \in W$ corresponds to a Schubert variety $X(w)$. We derive from Theorem 3.2 that equality holds in (A′) whenever $X(w)$ is rationally smooth. To this end, we prove a variation of the classical Carrell-Peterson criteria for rational smoothness which should be of independent interest. It is to be noted that Oh and Yoo [11] recently derived a stronger q-analogue of equality in (A′) for rationally smooth $X(w)$.

Here is an outline of the structure of the remainder of the paper. In the next section we agree on basic notation and concepts related to Coxeter groups. In particular, the definition of the map ϕ is recalled from [8]. In Section 3, we establish our main result. The new proof of (B) is described in Section 4 before we conclude in Section 5 with the connection to rationally smooth Schubert varieties.

[1] An explanation of the implication (A′) $\Rightarrow$ (A) can be found in [8].

2. Coxeter groups and inversion arrangements

In this section, we recall some properties of finite Coxeter groups. The reader looking for more information should consult [2] or [9]. We also review parts of [8] that are needed for subsequent sections.

A finite Coxeter group is generated by a set S of *simple reflections* subject to relations of the form $s^2 = e$ for all $s \in S$ and $(ss')^{m(s,s')} = e$ for suitable $m(s, s')$. Here, $e \in W$ is the identity element.

For $w \in W$, the *Coxeter length* $\ell(w)$ is the smallest k such that $w = s_1 \cdots s_k$ for some $s_i \in S$. The expression $s_1 \cdots s_k$ is then called *reduced*.

The set T of *reflections* consists of all conjugates of simple reflections, i.e. $T = \{wsw^{-1} \mid w \in W\}$. The *absolute length* $\ell'(w)$ is the smallest k such that $t_1 \cdots t_k = w$ for some $t_i \in T$.

Choose a root system $\Phi \subset \mathbb{R}^n$ for W with set of positive roots Φ^+. In an incarnation of W as a group generated by orthogonal reflections in Euclidean space, the positive roots are in one-to-one correspondence with the reflections of W; the reflecting hyperplane fixed by a reflection is the orthogonal complement of the corresponding root.

When W is a symmetric group $\mathfrak{S}_n$, so that T is the set of transpositions, it is well known that $\ell'(w) = n - c(w)$, where $c(w)$ is the number of cycles in the disjoint cycle decomposition of w. This fact is generalised by the following fundamental result of Carter which connects the absolute length function with the underlying geometry.

Theorem 2.1 (Carter [4])**.** *Let W be a finite reflection group. Given $w \in W$, the following assertions hold.*

(a) *The codimension of the fixed point space of w equals $\ell'(w)$.*
(b) *Given reflections $t_1, \ldots, t_m \in T$, we have $\ell'(t_1 \cdots t_m) = m$ if and only if the corresponding roots $\alpha_{t_1}, \ldots, \alpha_{t_m} \in \Phi^+$ are linearly independent.*

Remark 2.2. A useful consequence is that if there are two minimal factorisations into reflections $t_1 \cdots t_m = r_1 \cdots r_m = w$, $\ell'(w) = m$, then we must have $\operatorname{span}\{\alpha_{t_1}, \ldots, \alpha_{t_m}\} = \operatorname{span}\{\alpha_{r_1}, \ldots, \alpha_{r_m}\}$ since both sides of the equality sign coincide with the orthogonal complement of the fixed point space of w.

The *Bruhat graph* $\operatorname{bg}(W)$ is the Cayley graph of W with edges directed towards greater Coxeter length. That is, the vertex set is W and we have directed edges $x \to tx$ for $x \in W$, $t \in T$, whenever $\ell(x) < \ell(tx)$.

Taking transitive closure of $\operatorname{bg}(W)$ yields the *Bruhat order* on W. In other words, $u \leq w$ if and only if $u \to \cdots \to w$. The subgraph of $\operatorname{bg}(W)$ which is induced by the principal order ideal $[e, w] = \{u \in W \mid u \leq w\}$ is denoted by $\operatorname{bg}(w)$. We refer to $\operatorname{bg}(w)$, too, as a Bruhat graph. An example can be found in Figure 1.

Let $\operatorname{al}(u, w)$ denote the distance from u to w in $\operatorname{bg}(w)$ (equivalently, in $\operatorname{bg}(W)$) in the directed, graph-theoretic sense. Thus, $\operatorname{al}(u, w)$ is finite precisely when $u \leq w$. Clearly, $\operatorname{al}(u, w) \geq \ell'(uw^{-1})$ in general, since the right hand side can be thought of as the distance from u to w in $\operatorname{bg}(W)$ if we disregard directions of edges.

A convenient characterisation of the Bruhat order can be given in terms of reduced expressions:

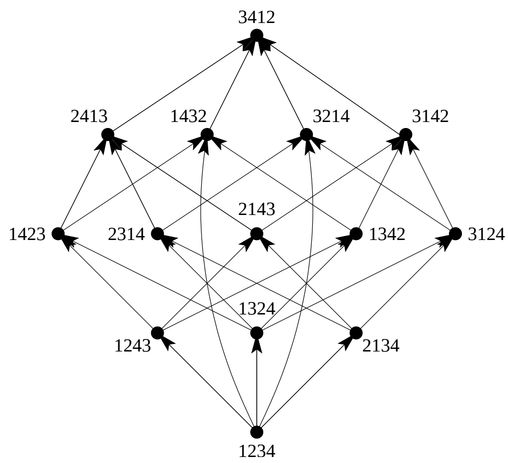

FIGURE 1. The Bruhat graph associated with the permutation $3412 \in \mathfrak{S}_4$ (one line notation). Disregarding the two curved edges yields the Hasse diagram of the Bruhat interval $[e = 1234, 3412]$.

Proposition 2.3. *Choose a reduced expression $s_1 \cdots s_k$ for $w \in W$. Then, $u \leq w$ if and only if $u = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_k$ for some $1 \leq i_1 < \cdots < i_m \leq k$, where a hat denotes omission of an element.*

The equivalence of these two appearances of the Bruhat order can be derived from the following fundamental fact.

Proposition 2.4 (Strong exchange property)**.** *If $u \to w$ and $s_1 \cdots s_k$ is any expression for $w \in W$, then $u = s_1 \cdots \widehat{s_i} \cdots s_k$ for some $i \in [k] = \{1, \dots, k\}$.*

For the remainder of this section, $s_1 \cdots s_k$ is a fixed reduced expression for some $w \in W$, where W is a finite Coxeter group. The *inversions* of w are the reflections of the form $t_i = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1$, $i \in [k]$. The set $\mathrm{inv}(w)$ of inversions of w is independent of the choice of reduced expression.

Let $\alpha_i \in \Phi^+$ be the root corresponding to t_i, and denote by $H_i = \alpha_i^\perp$ the associated hyperplane. The *inversion arrangement* of w is

$$\mathcal{A}_w = \{H_1, \dots, H_k\}.$$

The connected components of the complement of $\cup \mathcal{A}_w$ are called *regions* of $\mathcal{A}_w$. The set of such regions is denoted by $\mathrm{reg}(w)$.

At the heart of [8] one finds the construction of an injective map $\mathrm{reg}(w) \to [e, w]$. (More accurately, the domain of the map is not $\mathrm{reg}(w)$, but a set which is equinumerous with $\mathrm{reg}(w)$.) We shall study this map further in the present paper, so we review it here. For convenience, we deviate slightly from the presentation in [8], but the formulations are equivalent via standard facts from matroid theory.

It is convenient to order positive roots that correspond to inversions of w with respect to the indices. For example, $\{\alpha_{i_1} < \cdots < \alpha_{i_m}\}$ indicates the set $\{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ under the assumption $1 \leq i_1 < \cdots < i_m \leq k$.

A *circuit* is a minimal linearly dependent set $X = \{\alpha_{i_1} < \cdots < \alpha_{i_m}\} \subseteq \Phi^+$ of positive roots corresponding to inversions of w in the manner described above.

If X is a circuit, $\{\alpha_{i_1} < \cdots < \alpha_{i_{m-1}}\}$ is a *broken circuit*.[2] If $Y \subseteq \{\alpha_1, \ldots, \alpha_k\}$ does not have a subset which is a broken circuit, say that Y is an *NBC set*. We denote the family of NBC sets by $\mathrm{NBC}(w)$, although it of course depends not only on w but also on the choice of reduced expression $s_1 \cdots s_k$. The point is that $\#\mathrm{reg}(w) = \#\mathrm{NBC}(w)$. This well known fact follows for instance by combining two different interpretations of the characteristic polynomial of $\mathcal{A}_w$ evaluated at -1. The $\mathrm{reg}(w)$ part of the story is due to Zaslavsky [16] whereas the $\mathrm{NBC}(w)$ connection in this generality was presented by Rota [14]. A more thorough account of these matters can be found e.g. in [12].

Definition 2.5. *Construct a map* $\phi : \mathrm{NBC}(w) \to [e, w]$ *by* $\{\alpha_{i_1} < \cdots < \alpha_{i_m}\} \mapsto t_{i_1} \cdots t_{i_m} w$.

Proving statement (A′), it was shown in [8] that ϕ always is well defined and injective.

3. A surjectivity characterisation

Maintain the notation of the previous section. Thus, we keep fixed a finite Coxeter group W, an element $w \in W$ with a reduced expression $s_1 \cdots s_k$ and corresponding inversions t_i with their associated positive roots α_i, $i \in [k]$.

In this section, we determine when the map ϕ is surjective. The image of ϕ is dependent on the choice of reduced expression for w, but the cardinality of the image is not, since it coincides with $\#\mathrm{reg}(w)$. Thus, whether or not ϕ is surjective depends solely on the element w.

The next lemma is the main source from which this paper flows.

Lemma 3.1. *Assume* $\mathrm{al}(u, w) = \ell'(uw^{-1})$ *for all* $u \leq w$. *For fixed* $u \leq w$, *let* $m = \mathrm{al}(u, w)$ *and pick the lexicographically maximal sequence* $(i_m, \ldots, i_1)$ *such that* $u = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots s_k$.[3] *Then,* $\{\alpha_{i_1} < \cdots < \alpha_{i_m}\} \in \mathrm{NBC}(w)$.

Proof. Suppose u is such that the indices $1 \leq i_1 < \cdots < i_m \leq k$ yield a counterexample with m minimal. This minimality implies that if $(j_{m-1}, \ldots, j_1)$ is lexicographically maximal such that $s_1 \cdots \widehat{s_{j_1}} \cdots \widehat{s_{j_{m-1}}} \cdots s_k = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_{m-1}}} \cdots s_k$, then $\{\alpha_{j_1} < \cdots < \alpha_{j_{m-1}}\} \in \mathrm{NBC}(w)$.

If $j_{m-1} = i_m$, then $uw^{-1} = t_{j_1} \cdots t_{j_{m-2}}$ and, consequently, $\ell'(uw^{-1}) \leq m - 2$ which is a contradiction. Thus, $j_{m-1} \neq i_m$.

Define $V = \mathrm{span}\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$. By Carter's result (Theorem 2.1), $\dim V = m$. Let

$$n = \max\{i \in [k] \mid \alpha_i \in V\}.$$

We claim that $n > i_m$. If $j_{m-1} > i_m$, this is immediate since $\alpha_{j_{m-1}} \in V$ by Remark 2.2. If, on the other hand, $j_{m-1} < i_m$, we have $i_x = j_x$ for all $x \in [m-1]$ by maximality of $(i_m, \ldots, i_1)$. Any broken circuit which is a subset of $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ therefore contains α_{i_m}. By assumption, such a broken circuit exists, and the claim is established.

Having concluded $n > i_m$, observe $uw^{-1}t_n w = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_m}} \cdots \widehat{s_n} \cdots s_k \leq w$. Again by Carter's result, $\ell'(uw^{-1}t_n) \leq m$. Multiplication by a reflection changes the absolute length by exactly one, so we conclude $\ell'(uw^{-1}t_n) = m - 1$. Thus,

[2] Note that a broken circuit is a circuit missing its *largest* element. This convention is backwards compared to common matroid terminology but convenient for our purposes.

[3] By the strong exchange property, such a sequence exists.

$uw^{-1}t_n = t_{a_1} \cdots t_{a_{m-1}}$ for some NBC set $\{\alpha_{a_1} < \cdots < \alpha_{a_{m-1}}\} \subset V$. By Remark 2.2, $V = \mathrm{span}\{\alpha_{a_1}, \ldots, \alpha_{a_{m-1}}, \alpha_n\}$. Thus, $a_{m-1} < n$ and the fact that $uw^{-1} = t_{a_1} \cdots t_{a_{m-1}} t_n$ therefore contradicts maximality of the sequence $(i_m, \ldots, i_1)$. $\square$

The desired characterisation is now within reach. For symmetric groups, it was established in [8, Theorem 6.3]. The general case answers [8, Open problem 10.3].

Theorem 3.2. *The map* $\phi : \mathrm{NBC}(w) \to [e, w]$ *is surjective, hence bijective, if and only if* $\mathrm{al}(u, w) = \ell'(uw^{-1})$ *for all* $u \le w$.

Proof. The only if part is a direct consequence of the following "going-down property" of ϕ ([8, Proposition 6.2]): if ϕ is surjective, then the NBC set $\phi^{-1}(u) = \{\alpha_{i_1} < \cdots < \alpha_{i_m}\}$, $m = \ell'(uw^{-1})$, corresponds to reflections $t_{i_1}, \ldots, t_{i_m} \in T$ such that $t_{i_{j-1}} \cdots t_{i_m} w \to t_{i_j} \cdots t_{i_m} w$ for all j. This immediately implies $\mathrm{al}(u, w) = m$.

Under the assumption $\mathrm{al}(u, w) = \ell'(u, w)$ for all $u \le w$, Lemma 3.1 provides a preimage $\phi^{-1}(v)$ for any $v \le w$, thereby establishing the if direction. $\square$

When looking for a shortest path, in the undirected sense, from u to w in the Bruhat graph, we *a priori* have to consider all of $\mathrm{bg}(W)$. Fortunately, the situation is simpler than that; the next lemma implies, in particular, that an undirected path from u to w of length $\ell'(uw^{-1})$ can be found inside $\mathrm{bg}(w)$ if $u \le w$.

Lemma 3.3. *Given any* $u, w \in W$*, there exists an element* $v \le u, w$ *such that* $\mathrm{al}(v, w) + \mathrm{al}(v, u) = \ell'(uw^{-1})$.

Proof. The Bruhat subgraph induced by a coset corresponding to a reflection subgroup $D = \langle t_1, t_2 \rangle \subseteq W$, where $t_1, t_2 \in T$, is isomorphic to the Bruhat graph of the dihedral Coxeter group which is isomorphic to D [5]. The simple structure of such Bruhat graphs shows that whenever $x \to y \leftarrow z$, there exists some y' with $x \leftarrow y' \to z$. This implies that, in the Bruhat graph $\mathrm{bg}(W)$, among all (not necessarily directed) paths from u to w of fixed length l, those that are minimal with respect to the sum of the Coxeter lengths of the vertices are of the form $u = x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_k \to x_{k+1} \to \cdots \to x_l = w$ for some $0 \le k \le l$. If we let $l = \ell'(uw^{-1})$, $v = x_k$ is an element with the prescribed properties. $\square$

As an example, one readily verifies that the directed distance from any vertex to the top element always coincides with the undirected distance in Figure 1. By Theorem 3.2 and Lemma 3.3, we may therefore conclude that ϕ is surjective when $w = 3412 \in \mathfrak{S}_4$. This, of course, is also immediate from the pattern avoidance condition in statement (B).

An interesting consequence is that $\#\mathrm{reg}(w) = \#[e, w]$ is a combinatorial property of the poset $[e, w]$. In the symmetric group setting, this was established in [8, Corollary 6.4].

Theorem 3.4. *If* $w, w' \in W$ *satisfy* $\#\mathrm{reg}(w) = \#[e, w]$ *and* $\#\mathrm{reg}(w') < \#[e, w']$*, then* $[e, w] \ncong [e, w']$ *as posets.*

Proof. Dyer [5] has shown that the Bruhat graph $\mathrm{bg}(w)$ is determined by the combinatorial structure of $[e, w]$. By Lemma 3.3, it is therefore possible to determine from the poset structure of $[e, w]$ whether or not $\mathrm{al}(u, w) = \ell'(uw^{-1})$ for all $u \le w$. Invoking Theorem 3.2, that is sufficient for deciding whether ϕ is surjective. $\square$

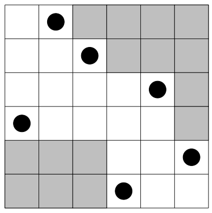

Figure 2. A set of dots forming the diagram of the permutation $235164 \in \mathfrak{S}_6$. The right hull rh(235164) consists of the non-shaded squares.

4. The symmetric group case revisited

We interpret composition of permutations from left to right. That is, $uw(i) = w(u(i))$ for $u, w \in \mathfrak{S}_n$, $i \in [n]$.[4]

For permutations $p \in \mathfrak{S}_m$ and $w \in \mathfrak{S}_n$, say that w *contains the pattern* p if there exist indices $1 \le i_1 < \cdots < i_m \le n$ such that for all $1 \le j < k \le m$, $p(j) < p(k)$ if and only if $w(i_j) < w(i_k)$. If w does not contain the pattern p, it *avoids* p.

If $w \in \mathfrak{S}_n$ avoids the patterns 4231, 35142, 42513 and 351624, then $\#[e, w] = \#\mathrm{reg}(w)$. This is the difficult direction of statement (B); the fairly involved proof given in [8] is based on deriving a common recurrence relation for $\#[e, w]$ and $\#\mathrm{reg}(w)$ and does not use any properties of the map ϕ. Finding a direct proof of surjectivity of ϕ was formulated as [8, Open problem 10.1]. The purpose of this section is to derive such a proof from the results of the previous section.

We shall use a characterisation of the permutations that avoid the four patterns which is due to Sjöstrand [15]. To this end, define the *diagram* of a permutation $w \in \mathfrak{S}_n$ as the set $\mathrm{diag}(w) = \{(i, w(i)) \mid i \in [n]\} \subset [n]^2$. We think of it as a set of dots on an $n \times n$ chessboard with matrix conventions for row and column indices, so that, for instance, $(1, 1)$ is the upper left square.

Definition 4.1. *Given* $w \in \mathfrak{S}_n$*, the* right hull $\mathrm{rh}(w)$ *is the subset of* $[n]^2$ *which consists of those* (i, j) *such that each of the rectangles* $\{(x, y) \mid x \le i,\ y \ge j\}$ *and* $\{(x, y) \mid x \ge i,\ y \le j\}$ *has nonempty intersection with* $\mathrm{diag}(w)$.

These concepts are illustrated in Figure 2.

For $w \in \mathfrak{S}_n$ and $i, j \in [n]$, let

$$w[i, j] = \#\{x \in [n] \mid x \le i,\ w(x) \ge j\}.$$

The Bruhat order on a symmetric group has the following convenient characterisation, a proof of which can be found e.g. in [2]:

Proposition 4.2. *For* $u, w \in \mathfrak{S}_n$*,* $u \le w$ *if and only if* $u[i, j] \le w[i, j]$ *for all* $i, j \in [n]$*.*

Taking into account that 180° diagram rotation yields a Bruhat order automorphism, Proposition 4.2 makes it clear that $u \le w$ implies $\mathrm{diag}(u) \subseteq \mathrm{rh}(w)$. We are interested in the permutations w that satisfy the converse.

[4] When $W = \mathfrak{S}_n$, this makes our concept of inversions (defined in Section 2) coincide with that which is standard for permutations. Composing from right to left would require minor adjustments in the proofs, but not in the results.

Theorem 4.3 (Sjöstrand [15])**.** *For $w \in \mathfrak{S}_n$, the following are equivalent:*

- *w has the* right hull property*, meaning* $[e,w] = \{u \in \mathfrak{S}_n \mid \mathrm{diag}(u) \subseteq \mathrm{rh}(w)\}$*.*
- *w avoids* 4231*,* 35142*,* 42513 *and* 351624*.*

This section is motivated by the desire to find a simple new proof of (B), so since we are going to use Theorem 4.3 in that process, it is relevant to note that Sjöstrand's proof (in part based on ideas of Gasharov and Reiner [7]) is both elegant and conceptual.

In light of Theorem 4.3 and our main result, the if part of (B) now is equivalent to the following statement:

Lemma 4.4. *If $w \in \mathfrak{S}_n$ has the right hull property, then* $\mathrm{al}(u,w) = \ell'(uw^{-1})$ *for all $u \le w$.*

Proof. Assume w has the right hull property and pick $u < w$. To argue by induction, it suffices to find a transposition $t \in T$ such that $u \to tu \le w$ and $\ell'(uw^{-1}) = \ell'(tuw^{-1}) + 1$.

Choose a nontrivial cycle c in the disjoint cycle decomposition of uw^{-1}. Then, $cw < w$ because every dot in the diagram of cw also appears either in the diagram of w or in that of u, both of which are contained in $\mathrm{rh}(w)$.

Let $\mathrm{supp}(c) = \{i_1 < \cdots < i_m\} \subseteq [n]$ denote the set of non-fixed elements of c. Defining

$$\mathfrak{S}_c = \{x \in \mathfrak{S}_n \mid x(i) = w(i) \text{ for all } i \notin \mathrm{supp}(c)\},$$

we thus have $w, cw \in \mathfrak{S}_c$. A natural bijection $\mathfrak{S}_c \to \mathfrak{S}_m$, denoted $x \mapsto \widetilde{x}$, is constructed as follows. Starting with $\mathrm{diag}(x)$, obtain $\mathrm{diag}(\widetilde{x})$ by considering only rows indexed by $\mathrm{supp}(c)$ and columns indexed by $w(\mathrm{supp}(c))$. Proposition 4.2 shows that this correspondence is a Bruhat order isomorphism.

We have $\widetilde{cw} < \widetilde{w}$. There is some transposition $\widetilde{x} \in \mathfrak{S}_m$ such that $\widetilde{cw} \to \widetilde{x}\widetilde{cw} \le \widetilde{w}$. Observe that $\widetilde{x}\widetilde{cw} = \widetilde{tcw}$ for some transposition $t \in \mathfrak{S}_n$ with $\mathrm{supp}(t) \subseteq \mathrm{supp}(c)$. Thus, tuw^{-1} has one more cycle than uw^{-1} does (the cycle c of uw^{-1} is "split" upon multiplication by t). It follows that t has the desired properties. □

For convenience, let us record as a theorem the various equivalent conditions that have made appearances in this section.

Theorem 4.5. *Given a permutation $w \in \mathfrak{S}_n$, the following assertions are equivalent:*

(i) $\#\mathrm{reg}(w) = \#[e,w]$.
(ii) *w has the right hull property.*
(iii) *w avoids the patterns* 4231*,* 35142*,* 42513 *and* 351624*.*
(iv) $\mathrm{al}(u,w) = \ell'(uw^{-1})$ *for all $u \le w$.*

Proof. Theorem 3.2 shows (i) ⇔ (iv), the equivalence (ii) ⇔ (iii) is Sjöstrand's Theorem 4.3, (ii) ⇒ (iv) is Lemma 4.4 and, finally, (i) ⇒ (iii) is the less tricky direction of (B); see [8, Theorem 4.1]. □

Remark 4.6. A fifth equivalent assertion, which has not been used in this section, was given by Gasharov and Reiner in [7]. They showed that $w \in \mathfrak{S}_n$ satisfies condition (iii) of Theorem 4.5 if and only if the type A Schubert variety indexed by w is "defined by inclusions" (see [7] for the definition). Moreover, they discovered that these varieties admit a particularly nice cohomology presentation. It would be very interesting to understand more explicitly how the other equivalent conditions are connected to this picture. Regarding the type independent conditions (i) and (iv), this could perhaps lead to interesting cohomological information about Schubert varieties of other types.

5. Rational smoothness implies surjectivity

Suppose W is a Weyl group of a semisimple simply connected complex Lie group G. Then, W is a finite Coxeter group whose elements index the Schubert varieties in the (complete) flag variety of G. A lot of work has been devoted to understanding how singularities of Schubert varieties are reflected by combinatorial properties of W. A good general reference is [1].

Oh, Postnikov and Yoo established in [10] that when W is a symmetric group, a q-analogue of the equality $\#\mathrm{reg}(w) = \#[e, w]$ holds whenever the corresponding Schubert variety is rationally smooth. The same property was conjectured for all Weyl groups W. Recently, Oh and Yoo [11] presented a proof of this conjecture.

In this section, we shall see that the $q = 1$ case, i.e. the actual identity $\#\mathrm{reg}(w) = \#[e, w]$, of Oh and Yoo's result is a simple consequence of Theorem 3.2. In the process, we formulate a combinatorial criterion (Theorem 5.3 below) for detecting rational singularities of Schubert varieties.

Let $X(w)$ denote the Schubert variety indexed by $w \in W$. For the purposes of the present paper, the following classical criterion could be taken as the definition of $X(w)$ being rationally smooth.

Theorem 5.1 (Carrell-Peterson [3])**.** *The variety $X(w)$ is rationally smooth if and only if the Bruhat graph* $\mathrm{bg}(w)$ *is regular, i.e. has equally many edges (disregarding directions) incident with each vertex.*

For instance $\mathrm{bg}(3412)$, depicted in Figure 1, is not regular. Hence, $X(3412)$ is not rationally smooth.

If $w \in W$ is understood from the context and $u \leq w$, let

$$\mathcal{E}(u) = \{t \in T \mid tu \leq w\}.$$

Thus, $\mathcal{E}(u)$ can be thought of as the set of edges incident to u in $\mathrm{bg}(w)$. Define $\deg(u) = \#\mathcal{E}(u)$. Since $\mathcal{E}(w) = \mathrm{inv}(w)$, $\deg(w) = \ell(w)$. Hence, $\mathrm{bg}(w)$ is regular if and only if it is $\ell(w)$-regular.

Definition 5.2. *Suppose $x, y, z \leq w$. We say that $[e, w]$ contains the* broken rhombus *(x, y, z) if the following conditions are satisfied:*

(i) $x \leftarrow y \rightarrow z$.
(ii) *There is some $v \in W$ with $x \rightarrow v \leftarrow z$.*
(iii) *If $x \rightarrow v \leftarrow z$, then $v \not\leq w$.*

Returning to Figure 1, several broken rhombi can be found in $[e, 3412]$. One is given by $(2314, 1324, 1342)$, another is $(1432, 1234, 2134)$.

The following rational smoothness criterion can be easily derived from the main result of Dyer's manuscript [6]; thanks are due to an anonymous referee for directing us to that reference. We state here a direct proof based on Theorem 5.1.

Theorem 5.3. *The Schubert variety $X(w)$ is rationally smooth if and only if $[e, w]$ contains no broken rhombi.*

Proof. For a fixed reflection $t \in T$, we partition $T \setminus \{t\}$ in the following way. For $r \in T \setminus \{t\}$, let

$$C_t(r) = f^{-1}\left(\mathrm{span}(\{\alpha_r, \alpha_t\}) \cap \Phi^+\right),$$

where $f : T \to \Phi^+$ is the natural 1–1 correspondence $r \mapsto \alpha_r$ between reflections and positive roots. In other words, $C_t(r)$ consists of all reflections that correspond to roots in the plane spanned by α_t and α_r, and $\langle C_t(r)\rangle$ is a dihedral reflection subgroup of W. Now, $\{C_t(r) \setminus \{t\} \mid r \in T \setminus \{t\}\}$ is a partition of $T \setminus \{t\}$.

Any subgroup of W generated by reflections is a Coxeter group in its own right with a canonically defined set of Coxeter generators [5]. As was mentioned in the proof of Lemma 3.3, there is an isomorphism of directed graphs from the subgraph of $\mathrm{bg}(W)$ induced by a coset $\langle C_t(r)\rangle u$ to the Bruhat graph of the dihedral Coxeter group $D \cong \langle C_t(r)\rangle$. The image of $[e, w] \cap \langle C_t(r)\rangle u$ is a Bruhat order ideal I in D. The special structure of dihedral Bruhat intervals shows that either the number of elements of odd respectively of even lengths in I are equal, or they differ by one. Assuming I contains at least two elements, in the former case I has a unique maximum and in the latter it has two maximal elements $m_1 \neq m_2$ of the same Coxeter length. In this case, let x and z be the preimages of m_1 and m_2, respectively, and choose $y \in \langle C_t(r)\rangle u$ such that $x \leftarrow y \to z$. Then, Dyer's [5, Lemma 3.1] shows that $x \to v \leftarrow z$ implies $v \in \langle C_t(r)\rangle u$. Thus, (x, y, z) forms a broken rhombus in $[e, w]$.

Observe that in the Bruhat graph of a dihedral group, u and v are adjacent if and only if $\ell(u)$ and $\ell(v)$ have different parity.

Suppose $tu \to u \leq w$, $t \in T$. If $[e, w]$ contains no broken rhombi, the above considerations show that $|\mathcal{E}(u) \cap C_t(r)| = |\mathcal{E}(tu) \cap C_t(r)|$ for all $r \in T \setminus \{t\}$. Thus, $\deg(tu) = \deg(u)$ so that in fact all vertices in $[e, w]$ have degree $\deg(w)$, and $X(w)$ is rationally smooth by the Carrell-Peterson criterion.

For the converse statement, assume (x, y, z) is a broken rhombus in $[e, w]$ with $\ell(y)$ maximal. Let $t = xy^{-1}$ and $r = zy^{-1}$. Then, y has one more neighbour in $\langle C_t(r)\rangle y$ than x does. That is, $|\mathcal{E}(y) \cap C_t(r)| = |\mathcal{E}(x) \cap C_t(r)| + 1$. Moreover, by maximality of y, there is no $r' \in T$ with $|\mathcal{E}(y) \cap C_t(r')| = |\mathcal{E}(x) \cap C_t(r')| - 1$. Therefore $\deg(y) > \deg(x)$, implying that $X(w)$ is rationally singular. □

With this criterion and Theorem 3.2 at our disposal, the $q = 1$ case of Oh and Yoo's result is little more than an observation:

Corollary 5.4. *The map ϕ is surjective, hence bijective, if $X(w)$ is rationally smooth.*

Proof. Suppose ϕ is not surjective. Assume $z \leq w$ is such that $\mathrm{al}(z, w) > \ell'(zw^{-1})$ and $\ell'(zw^{-1})$ is minimal among all z with this property. By Lemma 3.3, there exist $x, y \leq w$ such that $x \leftarrow y \to z$ and $\ell'(xw^{-1}) = \ell'(yw^{-1}) - 1 = \ell'(zw^{-1}) - 2$. Now, $x \to v \leftarrow z$ implies $v \not\leq w$; otherwise a directed path of length $\mathrm{al}(v, w) + 1 = \ell'(vw^{-1}) + 1 \leq \ell'(xw^{-1}) + 2$ would exist from z to w, contradicting the assumptions. Hence, (x, y, z) is a broken rhombus. Theorem 5.3 concludes the proof. □

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