SMOOTHNESS OF SCHUBERT VARIETIES INDEXED BY INVOLUTIONS IN FINITE SIMPLY LACED TYPES

AXEL HULTMAN AND VINCENT UMUTABAZI

Abstract. We prove that in finite, simply laced types, every Schubert variety indexed by an involution which is not the longest element of some parabolic subgroup is singular.

1. Introduction

Let w be an involution in the symmetric group S_n. In [12] Hohlweg proved that the Schubert variety X_w is smooth if and only if w is the longest element of some parabolic subgroup of S_n. He arrived at this result by exploiting Lakshmibai and Sandhya's [14] classical pattern avoidance criterion for smoothness of type A Schubert varieties.

The main result of this paper extends Hohlweg's result to arbitrary finite, simply laced types. Namely, if W is a simply laced Weyl group, and $w \in W$ is an involution, X_w is smooth if and only if w is the longest element of a parabolic subgroup of W.

It seems likely that it would be possible to arrive at this result in a case by case fashion using the general root system pattern avoidance criteria for smoothness pioneered by Billey and Postnikov [2]. Instead of investigating this approach, we provide a uniform proof based on Carrell-Peterson type criteria in terms of the associated Bruhat graphs.

In Section 2, we recall properties of Coxeter systems and Bruhat graphs which can be used to study smoothness of Schubert varieties in a combinatorial way. In Section 3, we prove Theorem 3.1 which is the main result.

2. Preliminaries

In this section some properties of Bruhat graphs of Coxeter groups are recalled. For more on these concepts, see e.g. [3] and [9].

A *Coxeter group* is a group W generated by a set S of *simple reflections* s under relations of the form $s^2 = e$ and $(ss')^{m(s,s')} = e$ for all $s, s' \in S$ where e is the identity element and $m(s', s) = m(s, s')$ ≥ 2 is the order of ss' for $s \neq s'$. The pair (W, S) is called a *Coxeter system*, and each element $w \in W$ is a product of generators $s_i \in S$, i.e., $w = s_1 s_2 \cdots s_j$. If j is minimal among all such expressions for w, then j is called the *length* of w, denoted $\ell(w) = j$. The Coxeter system (W, S) is *simply laced* if $m(s, s') \leq 3$ for all $s, s' \in S$; otherwise it is *multiply laced*.

If W is finite, there exists a *longest element* $w_0 \in W$. It is an involution and satisfies $\ell(v) < \ell(w_0)$ for all other elements $v \in W$. In fact w_0 is the unique element in w such that $\ell(sw_0) < \ell(w_0)$ for all $s \in S$.

From now on let us fix a Coxeter system (W, S). Let $T = \{wsw^{-1} : w \in W, s \in S\}$ be the set of *reflections* in W. For $v, w \in W$ define:

(i) $v \to w$ if $w = vt$ for some $t \in T$ with $\ell(v) < \ell(w)$.
(ii) $v \leq w$ if $v = v_0 \to v_1 \to \cdots \to v_m = w$ for some $v_i \in W$.

The *Bruhat graph* $\mathrm{Bg}_S(W)$ of (W, S) is the directed graph whose vertex set is W and whose edge set is $E_{\mathrm{Bg}_S(W)} = \{(u, w) : u \to w\}$. The *Bruhat order* is the partial order relation on W given by (ii).

Example 2.1. *Denote by $W(A_2)$ the Coxeter group of type A_2 with set of simple reflections $S(A_2) = \{s_1, s_2\}$ satisfying $m(s_1, s_2) = 3$. Then $\mathrm{Bg}_{S(A_2)}(W(A_2))$ is as depicted in Figure 1.*

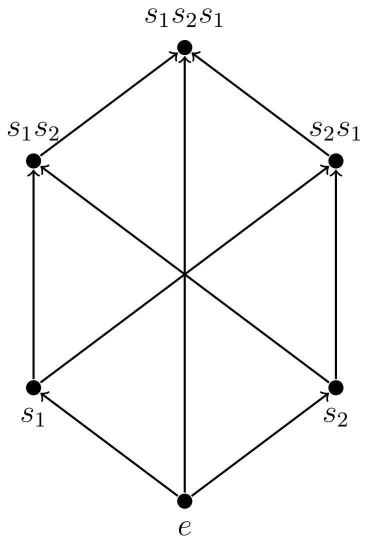

FIGURE 1. The Bruhat graph of $(W(A_2), S(A_2))$.

The map $v \mapsto v^{-1}$ is an automorphism of the Bruhat order:

Lemma 2.2. *For all $v, w \in W$, $v < w$ if and only if $v^{-1} < w^{-1}$.*

Define the *left descent set* of w as $D_L(w) = \{s \in S : \ell(sw) < \ell(w)\}$. The following fundamental result about the Bruhat order is sometimes called the *Lifting property.*

Lemma 2.3 (Verma [16])**.** *Suppose $v < w$ and $s \in D_L(w) \setminus D_L(v)$. Then, $v \leq sw$ and $sv \leq w$.*

2.1. **Reflection subgroups.** Maintain the Coxeter system (W, S) and its set of reflections T as defined above. Then W' is a *reflection subgroup* of W if $W' = \langle W' \cap T \rangle$. A reflection subgroup W' is called *dihedral* if $W' = \langle t, t' \rangle$ for some $t, t' \in T$, with $t \neq t'$.

Lemma 2.4 (Dyer [9])**.** *Suppose $t_1, t_2, t_3, t_4 \in T$ and $t_1 t_2 = t_3 t_4 \neq e$. Then $W' = \langle t_1, t_2, t_3, t_4 \rangle$ is a dihedral reflection subgroup of W.*

It turns out that reflection subgroups of W are themselves Coxeter groups. For $w \in W$, define $N(w) := \{t \in T : \ell(tw) < \ell(w)\}$. This is the set of *inversions* of w.

Theorem 2.5 (Deodhar [7], Dyer [8])**.** *Let W' be a reflection subgroup of W and define $X = \{t \in T : N(t) \cap W' = \{t\}\}$. Then,*

(1) $W' \cap T = \{ut'u^{-1} : u \in W', t' \in X\}$.
(2) (W', X) *is a Coxeter system.*

Coxeter described all types of affine groups generated by reflections and their reflection subgroups [6]. The following lemma is a very special case. It can be seen directly e.g. by considering root lengths.

Lemma 2.6. *Every reflection subgroup of a finite simply laced group is itself simply laced.*

For any subset $Y \subseteq W$ define the Bruhat graph of Y, denoted $\mathrm{Bg}_S(Y)$, as the directed subgraph of $\mathrm{Bg}_S(W)$ induced by Y.

Theorem 2.7 (Dyer [9])**.** *Let W' be a reflection subgroup of W and let X be as in Theorem 2.5. Then $\mathrm{Bg}_S(W') = \mathrm{Bg}_X(W')$.*

2.2. **Schubert varieties.** Let G be an algebraic group over $\mathbb{C}$ and B a Borel subgroup containing a maximal torus T. Then G/B is called the *flag variety* and it is the disjoint union of *Schubert cells BwB/B* where $w \in W$ and $W = N(T)/T$ is the *Weyl group* (which is a finite Coxeter group). The closure $X_w := \overline{BwB/B}$ is called a *Schubert variety.* Note that $G/B = X_{w_0}$ for w_0 the longest element of W. More on Schubert varieties can be found e.g. in [1].

Next, we review ways to detect singularities of Schubert varieties by inspecting Bruhat graphs. For a lower interval $[e, w] = \{z \in W : e \leq z \leq w\}$ write $\mathrm{Bg}_S(w)$ for the Bruhat graph of $[e, w]$. Let z be a vertex in $\mathrm{Bg}_S(w)$. The *degree* of z, denoted $\deg_w(z)$, is the number of edges incident to z in $\mathrm{Bg}_S(w)$ (where directions of edges are ignored).

The following result holds in any Coxeter group. In that generality it is due to Dyer [10]. In our context, where W is a (finite) Weyl group, other proofs given by Carrell and Peterson [4] and Polo [15] also apply.

Theorem 2.8. *Let $w \in W$. Then the degree of any vertex in $\mathrm{Bg}_S(w)$ is at least $\ell(w)$.*

In any Bruhat graph $\mathrm{Bg}_S(w)$, it is known that $\ell(w) = |N(w)| = \deg_w(w)$. In particular, if $\mathrm{Bg}_S(w)$ is regular (i.e., every vertex of $\mathrm{Bg}_S(w)$ has the same number of edges), then $\deg_w(e) = \ell(w)$.

Theorem 2.9 (Carrell-Peterson [4])**.** *The Schubert variety X_w is rationally smooth if and only if* $\mathrm{Bg}_S(w)$ *is regular.*

Smoothness and rational smoothness are equivalent for simply laced Weyl groups:

Theorem 2.10 (Carrell-Kuttler [5])**.** *Suppose W is simply laced. Then for any $w \in W$, X_w is smooth if and only if it is rationally smooth.*

Corollary 2.11. *If W is simply laced then X_w is smooth if and only if* $\mathrm{Bg}_S(w)$ *is regular.*

In general smoothness is stronger than rational smoothness when W is not simply laced. For example $X_{s_1 s_2 s_1}$ is rationally smooth but not smooth if W is of type C_2 generated by the simple reflections s_1 and s_2 with s_1 corresponding to the short root.

The next definition provides another characterization which can be used to prove that a given Schubert variety is not rationally smooth (see Theorem 2.14 below).

Definition 2.12. [13] *Let $x, u, v \leq w$. The Bruhat interval $[e, w]$ contains the* broken rhombus (x, u, v) *if the conditions below are satisfied:*

(1) *$x \leftarrow u \rightarrow v$;*
(2) *There is some $y \in W$ with $x \rightarrow y \leftarrow v$;*
(3) *If $x \rightarrow y \leftarrow v$, then $y \not\leq w$.*

Example 2.13. Consider the group $W(D_4)$ of type D_4 with set of simple reflections $S(D_4) = \{s_1, s_2, s_3, s_4\}$ where $m(s_i, s_2) = 3$ for $i = 1, 3, 4$ and $m(s_i, s_j) = 2$ for $i, j \neq 2$. In Figure 2 is depicted $\mathrm{Bg}_{S(D_4)}(w)$ for $w = s_2 s_1 s_3 s_4 s_2$. We use $1, 2, 3, 4$ for s_1, s_2, s_3, s_4 respectively for brevity. Thus, for example, w is represented by 21342. The interval $[e, w]$ contains the broken rhombus $(s_2 s_3, s_2, s_1 s_2)$ since there is no $y \leq w$ such that $s_2 s_3 \rightarrow y \leftarrow s_1 s_2$ although $s_2 s_3 \rightarrow s_1 s_2 s_3 \leftarrow s_1 s_2$. Moreover note that $\mathrm{Bg}_{S(D_4)}(w)$ is not regular. Hence, X_w is not smooth for $w = s_2 s_1 s_3 s_4 s_2 \in W(D_4)$.

The following is [13, Theorem 5.3]. It can also be obtained from the main result of Dyer [11].

Theorem 2.14. *The Schubert variety X_w is rationally smooth if and only if $[e, w]$ contains no broken rhombus.*

Corollary 2.15. *Suppose W is simply laced and let $w \in W$. Then, the Schubert variety X_w is smooth if and only if $[e, w]$ contains no broken rhombus.*

3. Schubert varieties indexed by involutions

In this section, which contains the main result, we will consider Schubert varieties indexed by involutions of finite simply laced groups.

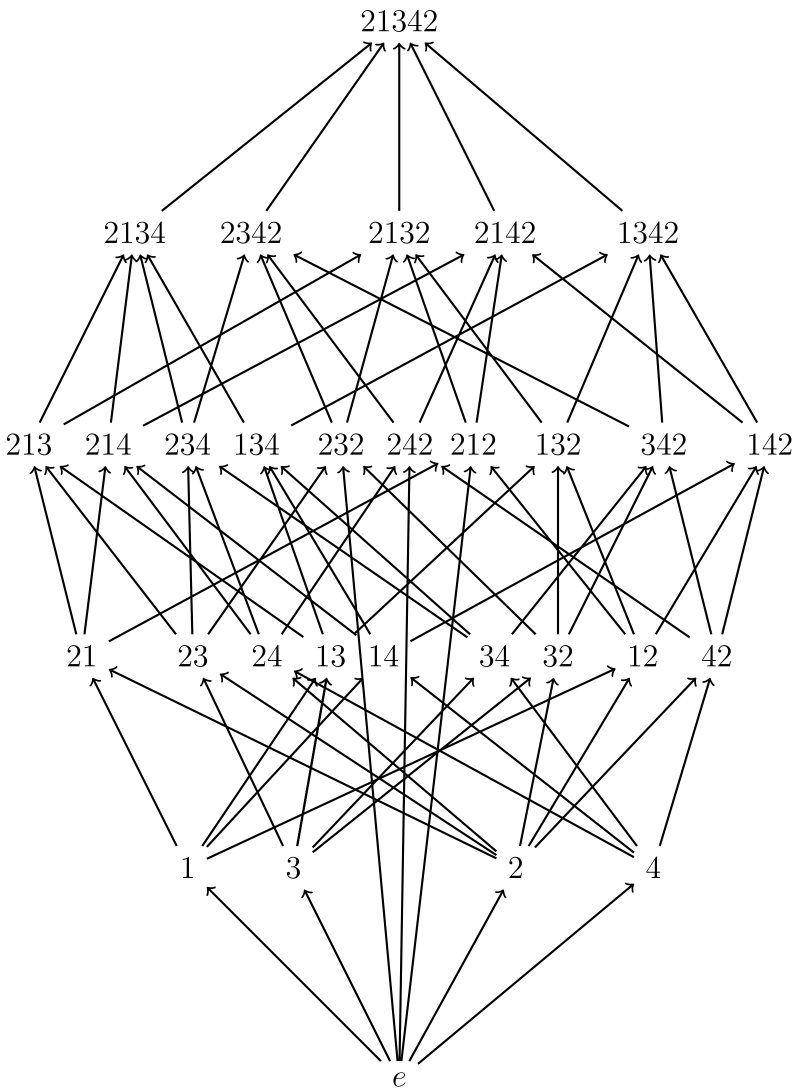

FIGURE 2. The Bruhat graph of $s_2s_1s_3s_4s_2 \in W(D_4)$.

Again let (W, S) be an arbitrary Coxeter system. A *parabolic subgroup* of W is a subgroup of the form $W_J = \langle J \rangle$ for $J \subseteq S$. If W_J is finite its longest element will be denoted by $w_0(J)$.

For $v \in W$ define $S(v) := \{s \in S : s \leq v\}$. Then $W_{S(v)}$ is the minimal parabolic subgroup of W which contains v.

Theorem 3.1. *Suppose (W, S) is finite and simply laced and let $v \in W$ be an involution. Then the Schubert variety X_v is smooth if and only if $v = w_0(J)$ for some $J \subseteq S$.*

Proof. The "if" assertion is obvious: $X_{w_0(J)}$ is a (smooth) flag variety. For the "only if" direction, let v be an involution which is not the longest element of any parabolic subgroup W_J of W. Since $v \neq w_0(S(v))$, there exists $s \in S(v)$ such that $v < sv$. If $\deg_v(e) \neq \ell(v)$,

$\mathrm{Bg}_S(v)$ is not regular and, by Corollary 2.11, X_v is not smooth. Thus, we may assume $\deg_v(e) = \ell(v)$. Since $\deg_{sv}(e)$ is the number of $t \in T$ such that $t \leq sv$ and that degree is at least $\ell(sv)$, there exists a reflection $t \leq sv$ such that $t \nleq v$. Since $t \leq sv$, by Lemma 2.2, $t^{-1} \leq v^{-1}s$ which implies that $t \leq vs$. Also we must have $st < t$. To see this we use Lemma 2.3 in the following way: Since $s \in D_L(sv)$, if we would have $t < st$ then by using the lifting property this would imply that $t \leqslant ssv = v$ which is a contradiction. Now since $st < t$, by Lemma 2.2 we see that $ts < t$. Because $ts < t$ and $t \leq vs$, we have $ts < vs$. Since $s \notin D_L(st)$, we have $s \in D_L(sv) \setminus D_L(st)$ and then by Lemma 2.3 we get $st \leq v$. Consider the dihedral subgroup $D = \langle s, sts \rangle$ of W. Since W is finite and simply laced, then by Lemma 2.6, D is also simply laced. Then (D, X) is either of type $A_1 \times A_1$ or of type A_2, where X is as in Theorem 2.5. By Theorem 2.7 $\mathrm{Bg}_S(D)$ equals the Bruhat graph of (D, X). Since, moreover, $st \to t$, D is not of type $A_1 \times A_1$ and hence D must be of type A_2. Therefore, $D = \{e, s, sts, ts, st, t\}$, $\mathrm{Bg}_S(D) \cong \mathrm{Bg}_{S(A_2)}(W(A_2))$, and $\mathrm{Bg}_S(D)$ is as shown in Figure 3.

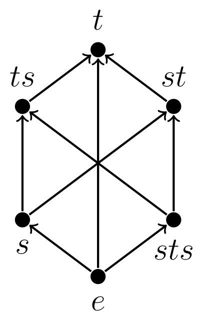

FIGURE 3. The Bruhat graph of D.

Now $ts, st \leq v$ but $t \nleq v$. From Figure 3, (st, sts, ts) is a broken rhombus of $[e, v]$ because there is no $x \leq v$ such that there are directed edges from st and ts to x. To see this, suppose that $st \to x \leftarrow ts$. So there exist $t', t'' \in T$ with $t' \neq t''$ such that $stt' = tst''$. Then $t't'' = tsts \neq e$. By Lemma 2.4 we therefore have a dihedral subgroup $W' = \langle sts, t, t', t'' \rangle$ of W, and W' is simply laced since W is (by Lemma 2.6). Clearly $D \subseteq W'$. Since W' has no more than six elements, $W' = D$. So $stt' = x \in D$. Since there is a directed edge from st to x, $x = t \nleq v$. Since (st, sts, ts) is a broken rhombus of $[e, v]$, by Corollary 2.15, X_v is not smooth. $\square$

When W is not simply laced, there may exist an involution $w \in W$ which is not the longest element of any parabolic subgroup of W but for which X_w is smooth. For example, in type C_2 there are two involutions of length three. One of them indexes a smooth Schubert variety (and, as was mentioned above, the other one indexes a rationally smooth but not smooth Schubert variety). This example shows that Theorem 3.1 cannot be extended to multiply laced types. We do, however, not know what happens in infinite simply laced types.

References

1. S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000.
2. S. Billey and A. Postnikov, *Smoothness of Schubert varieties via patterns in root subsystems*, Adv. in Appl. Math. **34** (2005), 447–466.
3. A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, vol. 231, Springer, New York, 2006.
4. J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., (1994), pp. 53–61.
5. J. B. Carrell and J. Kuttler, *Smooth points of T-stable varieties in G/B and the Peterson map*, Invent. Math. **151** (2003), 353–379.
6. H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. **35** (1934), 588–621.
7. Vinay V. Deodhar, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. (Basel) **53** (1989), 543–546.
8. M. Dyer, *Reflection subgroups of Coxeter systems*, J. Algebra **135** (1990), 57–73.
9. ———, *On the "Bruhat graph" of a Coxeter system*, Compositio Math. **78** (1991), 185–191.
10. ———, *The nil Hecke ring and Deodhar's conjecture on Bruhat intervals*, Invent. Math. **111** (1993), 571–574.
11. ———, *Rank two detection of singularities of Schubert varieties*, preprint, 2001.
12. C. Hohlweg, *Minimal and maximal elements in two-sided cells of S_n and Robinson-Schensted correspondence*, Discrete Math. **304** (2005), 79–87.
13. A. Hultman, *Inversion arrangements and Bruhat intervals*, J. Combin. Theory Ser. A **118** (2011), 1897–1906.
14. V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in* $\mathrm{Sl}(n)/B$, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), 45–52.
15. P. Polo, *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math. (N.S.) **5** (1994), 483–493.
16. D. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. École Norm. Sup. **4** (1971), 393–398.

Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

E-mail address: axel.hultman@liu.se

E-mail address: vincent.umutabazi@liu.se