PERMUTATION STATISTICS OF PRODUCTS OF RANDOM PERMUTATIONS

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Abstract. Given a permutation statistic $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$, define the *mean statistic* $\bar{\mathfrak{s}}$ as the class function giving the mean of $\mathfrak{s}$ over conjugacy classes. We describe a way to calculate the expected value of $\mathfrak{s}$ on a product of t independently chosen elements from the uniform distribution on a union of conjugacy classes $\Gamma \subseteq \mathfrak{S}_n$. In order to apply the formula, one needs to express the class function $\bar{\mathfrak{s}}$ as a linear combination of irreducible $\mathfrak{S}_n$-characters. We provide such expressions for several commonly studied permutation statistics, including the excedance number, inversion number, descent number, major index and k-cycle number. In particular, this leads to formulae for the expected values of said statistics.

1. Introduction

Consider the symmetric group $\mathfrak{S}_n$ of permutations of $[n] = \{1, \ldots, n\}$. For $\Gamma \subseteq \mathfrak{S}_n$ one may study the behaviour of various permutation statistics $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$ on products $\gamma_1 \cdots \gamma_t \in \mathfrak{S}_n$ of random $\gamma_i \in \Gamma$.

Definition 1.1. *Choose a subset $\Gamma \subseteq \mathfrak{S}_n$, a function $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$ and a nonnegative integer t. We denote by $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ the expected value of $\mathfrak{s}$ on a product of t elements independently chosen from the uniform distribution on Γ.*

A product of t random elements of Γ corresponds to a t-step random walk on the Cayley graph of $\mathfrak{S}_n$ induced by Γ. Random walks on Cayley graphs form a classical and well studied subject in probability theory; a good general reference is [13]. In the present paper, we specifically address the problem of computing $\mathbb{E}_\Gamma(\mathfrak{s}, t)$. Recent work in this vein includes the following. The case of Γ being the set of adjacent transpositions and $\mathfrak{s}$ counting inversions was studied by Eriksson et al. [6], Eriksen [4] and Bousquet-Mélou [2]. Turning, instead, to Γ comprised of all transpositions, $\mathfrak{s}$ being the absolute length function (essentially counting disjoint cycles) was studied in [5], whereas Jönsson [10] considered the fixed point counting function $\mathfrak{s}$ and Sjöstrand [14] found the solution when $\mathfrak{s}$ counts inversions.

We shall describe a method to attack the general problem. Although it could potentially be of use for more general Γ (as indicated by the hypotheses of Theorem 3.1 below), we shall apply it to situations where Γ is a union of conjugacy classes. The technique makes use of representation theory of $\mathfrak{S}_n$. Similar ideas have been frequent in the study of random walks on Cayley graphs since the seminal paper by Diaconis and Shahshahani [3]. In particular, our method is heavily inspired by that described in [5]. We are concerned with more general Γ, but the principal novelty here is to dispose of the requisite of [5] that $\mathfrak{s}$ be a class function. As we shall see, removing this restriction significantly improves the versatility of the method. The aforementioned results from [5, 10, 14], as well as an abundance of new ones, are now accessible in a uniform way.

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Here is a brief sketch of the content of the paper. In the next section, we review some facts about symmetric group characters. In Section 3, we then describe how they connect with the problem of computing $\mathbb{E}_\Gamma(\mathfrak{s}, t)$. The main observations of that section, Theorem 3.1 and Theorem 3.2, provide a recipe for calculating $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ whenever Γ is a union of conjugacy classes. In order to express $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ explicitly for a given statistic $\mathfrak{s}$, the remaining task is to decompose the *mean statistic* $\overline{\mathfrak{s}}$ as a linear combination of irreducible $\mathfrak{S}_n$-characters. This turns out to be a quite manageable task for many standard permutation statistics. We provide explicit decompositions for the mean statistics corresponding to the k-cycle number, excedance number, weak excedance number, inversion number, major index and descent number statistics in Sections 4, 5 and 6. In particular, $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ is determined for $\mathfrak{s}$ being any of these statistics (and conjugation invariant Γ). We conclude with explicit examples in Section 7.

2. Symmetric group characters

In this section, we review elements of the representation theory of $\mathfrak{S}_n$. From this vast and classical subject only a few bits and pieces that we need in the sequel are extracted in order to agree on notation. We refer to [12] for a thorough background and much more information.

Let P_n denote the set of integer partitions $\lambda \vdash n$. The irreducible representations of $\mathfrak{S}_n$ are in bijection with P_n in a standard way. We use the notation ρ^λ for the representation indexed by $\lambda \vdash n$ and denote the corresponding character by χ^λ. These irreducible characters form a basis for the $\mathbb{C}$-vector space $\mathrm{Cl}_n = \{f : P_n \to \mathbb{C}\}$ of class functions. Moreover, this basis is orthonormal with respect to the standard Hermitian inner product on Cl_n

$$\langle f, g\rangle = \frac{1}{n!}\sum_{\lambda \vdash n} |C_\lambda| f(\lambda) g(\lambda)^*,$$

where C_λ denotes the conjugacy class of permutations π with $\mathrm{type}(\pi) = \lambda$ as cycle type.

Abusing notation, we at times consider class functions as defined on $\mathfrak{S}_n$ rather than on P_n. In other words, for $\pi \in \mathfrak{S}_n$ and $f \in \mathrm{Cl}_n$, $f(\pi)$ should be interpreted as $f(\mathrm{type}(\pi))$. We trust the context to prevent confusion.

It is convenient to encode partitions as weakly decreasing sequences of positive integers, sometimes employing exponent notation to signal repeated parts. For example, $(7, 3^4, 1)$ denotes the partition of 20 which consists of one part of size 7, four parts of size 3 and one part of size 1. In this notation, a *hook shape* is a partition of the form $(a, 1^b)$ for integers $a \geq 1$ and $b \geq 0$.

The trivial $\mathfrak{S}_n$-character is indexed by (n). Thus, $\chi^{(n)}(\mu) = 1$ for all $\mu \vdash n$. The next three lemmata collect a few more values of certain irreducible characters that we shall need in the sequel. All statements are readily verified using e.g. the Murnaghan-Nakayama rule.

Define $f^\lambda = \chi^\lambda((1^n))$. This is the dimension of the irreducible representation ρ^λ.

Lemma 2.1. *Hook shape characters satisfy*

$$f^{(n-k,1^k)} = \binom{n-1}{k}$$

and

$$\chi^{(n-k,1^k)}((n)) = (-1)^k,$$

whereas $\chi^\lambda((n)) = 0$ *if* λ *is not a hook shape.*

Lemma 2.2. *Let $\lambda \vdash n$. If λ has p parts of size 1 and q parts of size 2, then*

$$\chi^{(n-1,1)}(\lambda) = p - 1$$

and

$$\chi^{(n-2,1^2)}(\lambda) = \binom{p-1}{2} - q.$$

A frequently occurring quantity is the *content* of $\lambda \vdash n$. It is defined by

$$c_\lambda = \binom{n}{2} \frac{\chi^\lambda((2, 1^{n-2}))}{f^\lambda}.$$

Lemma 2.3. *Partitions of the form $\lambda = (a, b, 1^{n-a-b})$ have content given by*

$$c_\lambda = \binom{a}{2} + \binom{b-1}{2} - \binom{n-a-b+2}{2}.$$

In particular, taking $b = 1$ (or $b = 0$ if $a = n$), the content of a hook shape is

$$c_{(n-k,1^k)} = \frac{n(n-2k-1)}{2}.$$

3. Mean statistics and irreducible characters

Let $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$ be any real-valued function on the symmetric group. In all our subsequent applications, $\mathfrak{s}$ will be a permutation statistic associating a nonnegative integer with each permutation in $\mathfrak{S}_n$.

Choose $\Gamma \subseteq \mathfrak{S}_n$. We now describe a general procedure for computing the expected value of $\mathfrak{s}$ on a product of random elements of Γ taken independently from the uniform distribution.

The *mean statistic* $\bar{\mathfrak{s}}$ is the class function which computes the mean of $\mathfrak{s}$ over conjugacy classes. That is, $\bar{\mathfrak{s}} : P_n \to \mathbb{R}$ is defined by

$$\bar{\mathfrak{s}}(\lambda) = \frac{1}{|C_\lambda|} \sum_{\pi \in C_\lambda} \mathfrak{s}(\pi).$$

Hence, thinking of class functions as being defined on $\mathfrak{S}_n$, $\bar{\mathfrak{s}} = \mathfrak{s}$ if and only if $\mathfrak{s}$ is a class function. Considered in that particular setting, the remainder of this section resembles the procedure described in [5].[1]

Let $\pi \in \mathfrak{S}_n$. We need to keep track of the ways to express π as a product of elements from Γ. To this end, define a permutation statistic $\mathfrak{n}_t$ by

$$\mathfrak{n}_t(\pi) = \#\{(\gamma_1, \dots, \gamma_t) \in \Gamma^t \mid \gamma_1 \cdots \gamma_t = \pi\}.$$

Observe that all $\mathfrak{n}_t$ are class functions if and only if Γ is a union of conjugacy classes.

Theorem 3.1. *If at least one of the statistics $\mathfrak{n}_t$ and $\mathfrak{s}$ is a class function, then*

$$\mathbb{E}_\Gamma(\mathfrak{s}, t) = \frac{n!}{|\Gamma|^t} \langle \bar{\mathfrak{s}}, \overline{\mathfrak{n}_t} \rangle.$$

Proof. By definition,

$$\mathbb{E}_\Gamma(\mathfrak{s}, t) = \frac{1}{|\Gamma|^t} \sum_{\pi \in \mathfrak{S}_n} \mathfrak{n}_t(\pi)\mathfrak{s}(\pi).$$

[1] The context of [5] was that of $\mathfrak{s}(\pi)$ being the absolute length of π with Γ the set of transpositions. The technique, however, could clearly have been applied to any class function $\mathfrak{s}$.

Assume now that $\mathfrak{s}$ is a class function. Rewriting the right hand side by first summing over the conjugacy classes of $\mathfrak{S}_n$, we obtain

$$\begin{aligned}\mathbb{E}_\Gamma(\mathfrak{s},t) &= \frac{1}{|\Gamma|^t}\sum_{\lambda\vdash n}\bar{\mathfrak{s}}(\lambda)\sum_{\pi\in C_\lambda}\mathfrak{n}_t(\pi)\\ &= \frac{1}{|\Gamma|^t}\sum_{\lambda\vdash n}\bar{\mathfrak{s}}(\lambda)|C_\lambda|\overline{\mathfrak{n}_t}(\lambda)\\ &= \frac{n!}{|\Gamma|^t}\langle\bar{\mathfrak{s}},\overline{\mathfrak{n}_t}\rangle,\end{aligned}$$

as desired. If, instead, $\mathfrak{n}_t$ is a class function, the proof is completely analogous. □

Under the hypotheses of the preceding theorem, we are left with the task of evaluating the inner product of two mean statistics. This is easy if we are somehow able to express them in the orthonormal basis comprised of the irreducible $\mathfrak{S}_n$-characters. In other words, we want to find the coefficients a_λ and $b_\lambda^{(t)}$ defined by

$$\bar{\mathfrak{s}} = \sum_{\lambda\vdash n} a_\lambda\chi^\lambda$$

and

$$\overline{\mathfrak{n}_t} = \sum_{\lambda\vdash n} b_\lambda^{(t)}\chi^\lambda,$$

respectively.

Although Theorem 3.1 applies if $\mathfrak{n}_t$ *or* $\mathfrak{s}$ is a class function, all our subsequent applications come from the former setting. The next theorem is the reason; it shows how to compute the $b_\lambda^{(t)}$ if Γ consists of conjugacy classes. Variations of the formula (and its proof) are numerous in the literature. With a bit of labour, it can be extracted e.g. from [3] or [8]. When Γ is a conjugacy class, the statement follows immediately from [11, Theorem A.1.9] which is attributed to Frobenius. We provide a self-contained proof for convenience.

Theorem 3.2. *Suppose Γ is a disjoint union of conjugacy classes $\Gamma_1,\ldots,\Gamma_k\subseteq\mathfrak{S}_n$. Let μ_i denote the cycle type of the permutations in Γ_i. Then,*

$$b_\lambda^{(t)} = \frac{1}{n!(f^\lambda)^{t-1}}\left(\sum_{i=1}^k|\Gamma_i|\chi^\lambda(\mu_i)\right)^t.$$

Proof. Let $\delta_{\cdot,\cdot}$ denote the Kronecker delta. By the Schur orthogonality relations,

$$\frac{1}{n!}\sum_{\lambda\vdash n} f^\lambda\chi^\lambda(\mu) = \delta_{(1^n),\mu} = \overline{\mathfrak{n}_0}(\mu).$$

This proves the $t=0$ case of the asserted statement.

Suppose f is any class function and define a linear map P_Γ on Cl_n by declaring

$$P_\Gamma(f)(\pi) = \sum_{\gamma\in\Gamma} f(\pi\gamma^{-1})$$

for $\pi\in\mathfrak{S}_n$. By definition, $\overline{\mathfrak{n}_t} = P_\Gamma^t(\overline{\mathfrak{n}_0})$. Thus, it suffices to show that χ^λ is an eigenfunction of P_Γ with eigenvalue $\frac{1}{f^\lambda}\sum_{i=1}^k|\Gamma_i|\chi^\lambda(\mu_i)$. To this end, we may by linearity assume without loss of generality that Γ is a single conjugacy class. Denote its cycle type simply by μ.

Define $M_\lambda = \sum_{\gamma\in\Gamma}\rho^\lambda(\gamma)$. Since Γ is a conjugacy class, M_λ and $\rho^\lambda(\pi)$ commute for all $\pi\in\mathfrak{S}_n$. By Schur's Lemma, $M_\lambda = h(\lambda)I$, where I is the identity map and $h(\lambda)\in\mathbb{C}$. Now observe that

$$P_\Gamma(\chi^\lambda)(\pi) = \mathrm{trace}\left(\rho^\lambda(\pi)M_\lambda\right) = h(\lambda)\chi^\lambda(\pi)$$

for $\pi \in \mathfrak{S}_n$. Plugging in $\pi = \mathrm{id}$ yields $|\Gamma|\chi^\lambda(\mu) = h(\lambda) f^\lambda$, proving the claim. □

For Γ satisfying the hypothesis of Theorem 3.2, we conclude that the remaining challenge is to decompose the mean statistic $\bar{\mathfrak{s}}$ as a linear combination of irreducible $\mathfrak{S}_n$-characters. The upcoming three sections are essentially devoted to such computations.

4. Cycle numbers

For $\pi \in \mathfrak{S}_n$ and a positive integer k, let $\mathrm{cyc}_k(\pi)$ denote the number of elements that are contained in the k-cycles of the disjoint cycle decomposition of π. Thus, π contains $\mathrm{cyc}_k(\pi)/k$ k-cycles. Clearly, cyc_k is a class function so that $\overline{\mathrm{cyc}_k} = \mathrm{cyc}_k$. The following result, providing a decomposition of this statistic, is due to Alon and Kozma [1]. We take this opportunity to state a shorter, independent proof.[2]

Theorem 4.1 (Theorem 3 in [1])**.** *Let $k \in [n]$. Regarded as a class function on $\mathfrak{S}_n$, cyc_k decomposes as*

$$\begin{aligned}\mathrm{cyc}_k = \chi^{(n)} &+ \sum_{i=1}^{\min(k,n-k)} (-1)^{k-i} \chi^{(n-k,i,1^{k-i})} \\ &+ \sum_{j=n-k+1}^{k-1} (-1)^{k-j} \chi^{(j,n-k+1,1^{k-j-1})}.\end{aligned}$$

Proof. If $\mu \vdash n-k$ is obtained from $\lambda \vdash n$ by removing a part of size k, we of course have $\mathrm{cyc}_k(\lambda) = \mathrm{cyc}_k(\mu) + k$. Thus, the symmetric function image of cyc_k under the characteristic map is

$$\sum_{\substack{\lambda \vdash n \\ \lambda \text{ has } k\text{-parts}}} \frac{p_\lambda \mathrm{cyc}_k(\lambda)}{z_\lambda} = \sum_{\mu \vdash n-k} \frac{p_\mu p_k (\mathrm{cyc}_k(\mu) + k)}{z_\mu k(\mathrm{cyc}_k(\mu)/k + 1)} = p_k \sum_{\mu \vdash n-k} \frac{p_\mu}{z_\mu} = p_k s_{n-k}.$$

Using [15, 7.72], we may write

$$p_k s_{n-k} = \sum (-1)^{\mathrm{ht}(\lambda/(n-k))} s_\lambda,$$

where the sum is over all partitions $\lambda \vdash n$ such that $\lambda/(n-k)$ is a border strip, and $\mathrm{ht}(\lambda/(n-k))$ is one less than the number of rows in the strip. Applying the inverse of the characteristic map, this is precisely the desired result. □

Theorem 4.1 refines results from [5] and [10]. The former work essentially revolved around the total number of cycles, i.e. the statistic $\sum_k \mathrm{cyc}_k/k$, whereas the fixed point number cyc_1 was studied in the latter.

5. Excedances

Recall that an *excedance* of $\pi \in \mathfrak{S}_n$ is an index $i \in [n]$ such that $\pi(i) > i$. Similarly, i is a *weak excedance* if $\pi(i) \geq i$. Let $\mathrm{exc}(\pi)$ and $\mathrm{wexc}(\pi)$ denote the number of excedances and weak excedances, respectively, of π. Clearly, neither exc nor wexc is a class function.

Theorem 5.1. *The mean statistics $\overline{\mathrm{exc}}$ and $\overline{\mathrm{wexc}}$ decompose as*

$$\overline{\mathrm{exc}} = \frac{n-1}{2}\chi^{(n)} - \frac{1}{2}\chi^{(n-1,1)}$$

[2]The proof employs standard terminology from the theory of symmetric functions. We refrain from reproducing the definitions since they are not used elsewhere in the paper. Everything can be found e.g. in [15, Chapter 7].

and

$$\overline{\text{wexc}} = \frac{n+1}{2}\chi^{(n)} + \frac{1}{2}\chi^{(n-1,1)},$$

respectively.

Proof. If i is not a fixed point of π, then i is an excedance of π if and only if $\pi(i)$ is not an excedance of π^{-1}. A fixed point is a weak excedance but not an excedance. Hence,

$$\overline{\text{exc}}(\lambda) = \frac{1}{2\#C_\lambda}\sum_{\pi\in C_\lambda}(\text{exc}(\pi) + \text{exc}(\pi^{-1})) = \frac{n-p}{2},$$

where p is the number of fixed points of any $\pi \in C_\lambda$, i.e. the number of parts that equal one in λ. Similarly,

$$\overline{\text{wexc}}(\lambda) = \frac{n+p}{2}.$$

The result now follows from Lemma 2.2. □

6. Inversions, descents and the major index

This section treats the mean statistics associated with three commonly occurring permutation statistics. First, we recall their definitions.

Let $\pi \in \mathfrak{S}_n$. A *descent* of π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. The number of descents of π is denoted by $\text{des}(\pi)$, whereas the *major index* $\text{maj}(\pi)$ is the sum of all descents of π.

An index pair $1 \le i < j \le n$ forms an *inversion* of π if $\pi(i) > \pi(j)$. Let $\text{inv}(\pi)$ be the number of inversions of π.

In order to study these statistics all at once, it is convenient to define the quantity

$$I_\lambda(i,j) = \#\{\pi \in C_\lambda \mid \pi(i) > \pi(j)\}$$

for $\lambda \vdash n$ and $1 \le i < j \le n$.

Lemma 6.1. *Suppose $\lambda \vdash n$. Let p and q denote the number of 1-parts and the number of 2-parts, respectively, in λ. Then,*

$$I_\lambda(i,j) = \frac{\#C_\lambda}{2}\left(1 + \frac{2q - p(p-1)}{n(n-1)} + \frac{2(j-i-1)((n-p)(1-p)-2q)}{n(n-1)(n-2)}\right).$$

Proof. Fix $\lambda \vdash n$ and indices $1 \le i < j \le n$. Consider the following subsets of C_λ:

$$\begin{aligned}
T_1 &= \{\pi \in C_\lambda \mid \pi(i) = i \text{ and } \pi(j) = j\},\\
T_2 &= \{\pi \in C_\lambda \mid \pi(i) = j \text{ and } \pi(j) = i\},\\
T_3 &= \{\pi \in C_\lambda \mid \pi(i) = i \text{ and } \pi(j) = k \text{ for some } i < k < j\},\\
T_4 &= \{\pi \in C_\lambda \mid \pi(j) = j \text{ and } \pi(i) = k \text{ for some } i < k < j\},\\
T_5 &= \{\pi \in C_\lambda \mid \pi(i) = j \text{ and } \pi(j) = k \text{ for some } i < k < j\},\\
T_6 &= \{\pi \in C_\lambda \mid \pi(j) = i \text{ and } \pi(i) = k \text{ for some } i < k < j\},\\
T_7 &= C_\lambda \setminus (T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6).
\end{aligned}$$

Thus, C_λ is the disjoint union $C_\lambda = T_1 \uplus \cdots \uplus T_7$.

Let $f_{i,j} : C_\lambda \to C_\lambda$ be the involution $\pi \mapsto (i\,j)\pi(i\,j)$, where $(i\,j)$ denotes the transposition which interchanges i and j. Then, $f_{i,j}$ restricts to an involution $T_7 \to T_7$. This restriction has no fixed points since $T_1 \cup T_2$ is the fixed point set of $f_{i,j}$. Moreover, for a permutation $\pi \in T_7$, (i,j) is an inversion if and only if it is not an inversion of $f_{i,j}(\pi)$.

Observing that (i,j) is an inversion for all $\pi \in T_2 \cup T_5 \cup T_6$, whereas it is a non-inversion for all $\pi \in T_1 \cup T_3 \cup T_4$, we thus obtain

$$\begin{aligned} I_\lambda(i,j) &= \#T_2 + \#T_5 + \#T_6 + \frac{\#T_7}{2} \\ &= \frac{\#C_\lambda - \#T_1 + \#T_2 - \#T_3 - \#T_4 + \#T_5 + \#T_6}{2}. \end{aligned}$$

Computing

$$\begin{aligned} \#T_1 &= \frac{p(p-1)\#C_\lambda}{n(n-1)}, \\ \#T_2 &= \frac{2q\#C_\lambda}{n(n-1)}, \\ \#T_3 = \#T_4 &= \frac{p(n-p)(j-i-1)\#C_\lambda}{n(n-1)(n-2)}, \\ \#T_5 = \#T_6 &= \frac{(n-p-2q)(j-i-1)\#C_\lambda}{n(n-1)(n-2)} \end{aligned}$$

yields the desired result. □

Next, we exploit the fact that several familiar permutation statistics are obtained by taking appropriate sums of $I_\lambda(i,j)$.

Theorem 6.2. *The mean statistics associated with* des*,* maj *and* inv *can be written as the following linear combinations of irreducible characters:*

$$\begin{aligned} \overline{\mathrm{des}} &= \frac{n-1}{2}\chi^{(n)} - \frac{1}{n}\chi^{(n-1,1)} - \frac{1}{n}\chi^{(n-2,1,1)}, \\ \overline{\mathrm{maj}} &= \frac{n}{2}\overline{\mathrm{des}} = \frac{n(n-1)}{4}\chi^{(n)} - \frac{1}{2}\chi^{(n-1,1)} - \frac{1}{2}\chi^{(n-2,1,1)}, \\ \overline{\mathrm{inv}} &= \frac{n(n-1)}{4}\chi^{(n)} - \frac{n+1}{6}\chi^{(n-1,1)} - \frac{1}{6}\chi^{(n-2,1,1)}. \end{aligned}$$

Proof. Let $\lambda \vdash n$. Applying Lemma 6.1, we obtain the identities

$$\begin{aligned} \overline{\mathrm{des}}(\lambda) &= \sum_{i=1}^{n-1} \frac{I_\lambda(i,i+1)}{\#C_\lambda} = \frac{n-1}{2} + \frac{q}{n} - \frac{1}{n}\binom{p}{2}, \\ \overline{\mathrm{maj}}(\lambda) &= \sum_{i=1}^{n-1} \frac{iI_\lambda(i,i+1)}{\#C_\lambda} = \frac{n(n-1)}{4} + \frac{q}{2} - \frac{1}{2}\binom{p}{2}, \\ \overline{\mathrm{inv}}(\lambda) &= \sum_{1\le i<j\le n} \frac{I_\lambda(i,j)}{\#C_\lambda} = \frac{n(n-1)}{4} - \frac{p(p-1)}{12} - \frac{n(p-1)}{6} + \frac{q}{6}, \end{aligned}$$

where p and q are as in Lemma 6.1. That these equations are equivalent to the asserted ones is readily shown using Lemma 2.2. □

We remark that the formula for $\overline{\mathrm{des}}(\lambda)$ which appears in the above proof was previously found by Fulman [7, Theorem 2] using different methods.

7. Examples

If Γ is a union of conjugacy classes, we may combine Theorem 3.1 with Theorem 3.2 in order to explicitly compute $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ for any of the permutation statistics $\mathfrak{s}$ which were studied in the previous sections. The paper is now concluded with some sample computations of this kind. The case of Γ being the set of transpositions is treated in some detail. Among the results thus obtained, formulae from [5, 9, 10, 14] are recovered. First, however, let us look at two short examples involving other generating sets Γ.

Proposition 7.1. *For $k \in [n-1]$, the expected number of k-cycles in a product of $t > 0$ random n-cycles in $\mathfrak{S}_n$ is*

$$\frac{1}{k}\mathbb{E}_{C_{(n)}}(\mathrm{cyc}_k, t) = \frac{1}{k} + \frac{(-1)^{k(t+1)-1}}{k\binom{n-1}{k}^{t-1}}.$$

Proof. Let $\Gamma = C_{(n)} \subseteq \mathfrak{S}_n$ be the set of n-cycles. Assuming now that $t > 0$, Theorem 3.2 and Lemma 2.1 show that $b_\lambda^{(t)} = 0$ unless we have a hook shape $\lambda = (n-j, 1^j)$. Moreover,

$$b^{(t)}_{(n-j,1^j)} = \frac{1}{n!}\left(\frac{(n-1)!(-1)^j}{\binom{n-1}{j}}\right)^t \binom{n-1}{j}.$$

Suppose $k < n$ and consider the statistic cyc_k which counts elements that belong to k-cycles. According to Theorem 4.1, exactly two terms (namely $\chi^{(n)}$ and $(-1)^{k-1}\chi^{(n-k,1^k)}$) which correspond to hook shapes appear in the expansion of cyc_k. Applying Theorem 3.1, the asserted result follows. □

Proposition 7.2. *Let Γ be any union of conjugacy classes in which every permutation has exactly one fixed point. Then the expected number of excedances of a product of $t > 0$ random elements from Γ is*

$$\mathbb{E}_\Gamma(\mathrm{exc}, t) = \frac{n-1}{2},$$

whereas for weak excedances one obtains

$$\mathbb{E}_\Gamma(\mathrm{wexc}, t) = \frac{n+1}{2},$$

independently of t.

Proof. In this case, Theorem 3.2 in conjunction with Lemma 2.2 shows $b^{(t)}_{(n-1,1)} = 0$ for all $t > 0$. Combining Theorem 3.1 and Theorem 5.1 concludes the proof. □

With these sample computations under the belt, let us now give the transposition setting a fairly thorough treatment. We start with the statistics studied in Sections 5 and 6.

Proposition 7.3. *A product of t random transpositions has the following expected values of the (weak) excedance number, descent number, major index and inversion number, respectively:*

$$\mathbb{E}_T(\mathrm{exc}, t) = \frac{n-1}{2}\left(1 - \left(1 - \frac{2}{n-1}\right)^t\right),$$

$$\mathbb{E}_T(\mathrm{wexc}, t) = \frac{n+1}{2}\left(1 + \frac{n-1}{n+1}\left(1 - \frac{2}{n-1}\right)^t\right),$$

$$\mathbb{E}_T(\mathrm{des}, t) = \frac{n-1}{2}\left(1 - \frac{2}{n}\left(1 - \frac{2}{n-1}\right)^t - \frac{n-2}{n}\left(1 - \frac{4}{n-1}\right)^t\right),$$

$$\mathbb{E}_T(\mathrm{maj}, t) = \frac{n(n-1)}{4}\left(1 - \frac{2}{n}\left(1 - \frac{2}{n-1}\right)^t - \frac{n-2}{n}\left(1 - \frac{4}{n-1}\right)^t\right),$$

$$\mathbb{E}_T(\mathrm{inv}, t) = \frac{n(n-1)}{4}\left(1 - \frac{2(n+1)}{3n}\left(1 - \frac{2}{n-1}\right)^t - \frac{n-2}{3n}\left(1 - \frac{4}{n-1}\right)^t\right).$$

Proof. Suppose $\Gamma = T \subseteq \mathfrak{S}_n$ is the set of transpositions. Recall the contents c_λ considered in Lemma 2.3. Observe that Theorem 3.2 specializes to

$$b_\lambda^{(t)} = \frac{1}{n!}\left(\frac{\binom{n}{2}\chi^\lambda((2,1^{n-2}))}{f^\lambda}\right)^t f^\lambda = \frac{c_\lambda^t f^\lambda}{n!}.$$

Combining this with the decompositions found in Theorem 5.1 and Theorem 6.2, we may invoke Theorem 3.1 to obtain the claimed statements. □

The formula for $\mathbb{E}_T(\mathrm{inv}, t)$ obtained in Proposition 7.3 was previously found by Sjöstrand [14, Theorem 5.1].

Finally, let us state an explicit formula for $\mathbb{E}_T(\mathrm{cyc}_k, t)$.

Proposition 7.4. *Suppose $k \in [n]$. The expected number of elements contained in the k-cycles of a product of t random transpositions is*

$$1 + \frac{(-1)^k}{n^t(n-1)^t}\binom{n}{k}\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}\frac{r+k-n}{n-r}\left(n^2-(2k+1)n+2rk\right)^t.$$

Proof. The computation is completely analogous to that of Proposition 7.3. This time, the relevant coefficients a_λ are those given by Theorem 4.1. After a few elementary manipulations, the asserted expression is obtained. □

With $k = 1$, Proposition 7.4 recovers results on fixed points from [10], whereas the sum over all k leads to the expected cycle numbers that were computed in [5]. Also, note that $\mathbb{E}_T(\mathrm{cyc}_n, t)/n$ is nothing but the probability that a product of t random transpositions forms an n-cycle. From that probability, one easily derives the number of factorisations of an n-cycle into t transpositions. Working out the details, one arrives at the formula counting such factorisations which appears in Jackson [9].

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