Pseudodifferential Operators: The Fourier Transform, Orthogonality and Cancellation

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July 2016

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Last updated: 26th July 2016.

1 Introduction and Background

These notes formed the basis of a mini-course I gave at the Basque Centre for Applied Mathematics in July 2016. The aim of the course was to give a faily self-contained introduction to the study of pseudodifferential operators, starting from an introductory doctoral student level and ending with the study of fairly recent results. Much of the material follows [5] quite closely (in particular Chapters VI and VII) and that text is an excellent first port of call for further reading. I would like to thank the Basque Centre for Applied Mathematics for inviting me to give the mini-course and for the accompanying financial support. I have done my best to eliminate errors, but certainly many remain, for which I apologise in advance. Any feedback is welcome and feel free to email me about mistakes you find, as I can then correct them.

1.1 Function Spaces and the Fourier transform

We begin by introducing various function spaces we will make use of. Although we will assume the reader has a working knowledge of measure theory, it will not be crucial to understanding the material. For a measurable subset Ω of $\mathbf{R}^n$ we denote by $L^1(\Omega)$ the space of measurable functions $f\colon \Omega \to \mathbf{C}$ which are absolutely integrable. A measurable function f is said to be *absolutely integrable* if

$$\|f\|_{L^1(\Omega)} := \int_\Omega |f(x)|dx$$

is finite. The mapping $f \mapsto \|f\|_{L^1(\Omega)}$ defines a norm on the space $L^1(\Omega)$ which makes it a Banach space. For $p > 1$ we define $L^p(\Omega)$ similarly by replacing the norm $\|f\|_{L^1(\Omega)}$ with

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p}.$$

These spaces are also Banach spaces and $L^2(\Omega)$ is even a Hilbert space with inner product

$$(f,g)_{L^2(\Omega)} := \int_\Omega f(x)\overline{g(x)}dx.$$

We can even extend our definition of the space $L^p(\Omega)$ to the case $p = \infty$. In this case we define

$$\|f\|_{L^\infty(\Omega)} := \inf\{M \in \mathbf{R} \,|\, |f(x)| < M \text{ for almost all } x \in \Omega\}.$$

It is often convenient to calculate with functions which are better behaved than simply p-integrable.[1] *Schwarz functions* are smooth functions $\varphi\colon \mathbf{R}^n \to \mathbf{C}$ such that

$$\sup_{x\in\mathbf{R}^n} |x^\beta \partial^\alpha \varphi(x)| \tag{1.1}$$

is finite for each pair of multi-indices α and β. We denote the space of Schwarz functions by $\mathcal{S}$. They are dense in $L^p(\mathbf{R}^n)$ when $p < \infty$ but not for $p = \infty$. The expressions in (1.1) define semi-norms which turn $\mathcal{S}$ into a locally convex topological vector space. Convergence in such a space is define as convergence in each of the semi-norms: that is we say that $\varphi_j \to \varphi$ in $\mathcal{S}$ as $j \to \infty$ if $\varphi_j, \varphi \in \mathcal{S}$ and

$$\sup_{x\in\mathbf{R}^n} |x^\beta \partial^\alpha (\varphi_j - \varphi)(x)| \to 0$$

as $j \to \infty$ for each α and β.

For $f \in L^1(\mathbf{R}^n)$ we define the *Fourier transform* of f to be the function

$$\xi \mapsto \mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx.$$

[1] That is to say functions in $L^p(\mathbf{R}^n)$.

It is easy to see that we have the estimate

$$\|\widehat{f}\|_{L^\infty(\mathbf{R}^n)} \le \|f\|_{L^1(\mathbf{R}^n)}$$

The Fourier transform is useful not least because it turns differentiation into multiplication and via verse. The following theorem states precisely what we mean by this.

Theorem 1.1. *Assume $\varphi \in \mathcal{S}$.*

1. *The Fourier transform of $x \mapsto \partial_j \varphi(x)$ is $\xi \mapsto 2\pi i \xi_j \widehat{\varphi}(\xi)$, and*
2. *the Fourier transform of $x \mapsto -2\pi i x_j \varphi(x)$ is $\xi \mapsto \partial_j \widehat{\varphi}(\xi)$.*

Proof. The Fourier transform of $x \mapsto \partial_j \varphi(x)$ is

$$\int_{\mathbf{R}^n} \partial_j \varphi(x) e^{-2\pi i x\cdot\xi} dx = \int_{\mathbf{R}^n} \varphi(x) 2\pi i \xi_j e^{-2\pi i x\cdot\xi} dx = 2\pi i \xi_j \int_{\mathbf{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi} dx,$$

as can be seen by integration by parts. Equally, the Fourier transform of $x \mapsto -2\pi i x_j \varphi(x)$ is

$$\int_{\mathbf{R}^n} -2\pi i x_j \varphi(x) e^{-2\pi i x\cdot\xi} dx = \int_{\mathbf{R}^n} \varphi(x) \partial_{\xi_j} \left(e^{-2\pi i x\cdot\xi} \right) dx = \partial_{\xi_j} \left(\int_{\mathbf{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi} dx \right)$$

□

Theorem 1.2. *The Fourier transform $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ is continuous and invertible with inverse*

$$\mathcal{F}^{-1}(f)(x) = \int_{\mathbf{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

For two functions $f, g \in L^1(\mathbf{R}^n)$ we define their *convolution* to be

$$(f * g)(x) = \int_{\mathbf{R}^n} f(x-y) g(y) dy.$$

The Fourier transform interacts nicely with convolutions.

Theorem 1.3. *Let $\varphi, \psi \in \mathcal{S}$. Then*

1. $\widehat{\varphi * \psi}(\xi) = \widehat{\varphi}(\xi)\widehat{\psi}(\xi)$
2. $\widehat{\varphi\psi}(\xi) = (\widehat{\varphi} * \widehat{\psi})(\xi)$
3. $(\varphi, \psi)_{L^2(\mathbf{R}^n)} = (\widehat{\varphi}, \widehat{\psi})_{L^2(\mathbf{R}^n)}$

Theorem 1.4 (Plancherel)**.** *Let $f \in L^2(\mathbf{R}^n)$. Then $\widehat{f} \in L^2(\mathbf{R}^n)$ and*

$$\|\widehat{f}\|_{L^2(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}.$$

Moreover for all $f, g \in L^2(\mathbf{R}^n)$

$$(f, g)_{L^2(\mathbf{R}^n)} = (\widehat{f}, \widehat{g})_{L^2(\mathbf{R}^n)}.$$

1.2 Pseudodifferential Operators: Motivation

In order to motivate the definition of pseudodifferential operators we begin with an informal discussion of the kinds of questions we might want to answer. Given a differential operator L and a function $f\colon \mathbf{R}^n \to \mathbf{R}$ we are very often interested in solving the equation

$$Lu = f$$

for a function u in an appropriate function space. In practice it is often impossible to find an explicit formula for $u = L^{-1}f$ and even when it is possible, the formula obtained may not be terribly useful. What can be useful, however, is to compare the smoothness of f with that of u. To what extend does the equation propagate or mask the singularities of f in the solution u? With such a question in mind, we no longer need to find an inverse to L. It would be sufficient to find an operator P which, for example, was such that

$$PL = I + E,$$

where E is an smoothing operator. That is, is suffices to invert the differential operator up to smooth functions.

Consider the example of a second-order elliptic equation

$$L(u)(x) = \sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = f(x) \tag{1.2}$$

where the matrix $\{a_i j(x)\}_{ij}$ is real, symmetric and positive definite. To simplify our calculations we consider the constant coefficient operator with x in the coefficient matrix frozen at $x = x_0$:

$$L_{x_0}(u)(x) = \sum_{ij} a_{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = f(x)$$

Assuming this equation is somewhat similar to (1.2) at least near x_0 perhaps, we can take the Fourier transform to obtain

$$-4\pi^2 \sum_{ij} a_{ij}(x_0)\xi_i\xi_j \widehat{u}(\xi) = \widehat{f}(\xi).$$

From here it would be easy to find a formula for u by dividing by $-4\pi^2 \sum_{ij} a_{ij}(x_0)\xi_i\xi_j$ and then taking the inverse Fourier transform. However, the singular behaviour in ξ near the origin leads our conscience to introduce a smooth cut-off function η which is zero in a neighbourhood of the origin and $\eta(\xi) = 1$ for large ξ. This yields

$$P_{x_0}(f)(x) = \int_{\mathbf{R}^n} \left(-4\pi^2 \sum_{ij} a_{ij}(x_0)\xi_i\xi_j \right)^{-1} \eta(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi$$

as a candidate for a near-inverse. Observe then that

$$P_{x_0} L_{x_0} = I + E_{x_0},$$

where E_{x_0} is convolution with $\mathcal{F}^{-1}(\eta - 1)$ and hence smoothing.

Our hope is then that the operator

$$P(f)(x) = \int_{\mathbf{R}^n} \left(-4\pi^2 \sum_{ij} a_{ij}(x)\xi_i\xi_j \right)^{-1} \eta(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi} d\xi$$

with unfrozen x will act as a reasonable inverse to L modulo a smoothing operator. We will see later that this is the case although the smoothing operator will only gain one derivative.

1.3 Pseudodifferential Operators: Definition

We consider smooth functions $a\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ which, for a given $m \in \mathbf{R}$, satitisfy the estimates

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{1.3}$$

for each pair of multi-indices α and β. We denote the set of such functions S^m, the set of *symbols* of order m. The *pseudodifferential operator* T_a associated to a symbol $a \in S^m$ is defined to be

$$T_a(f)(x) = \int_{\mathbf{R}^n} a(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

for $f \in \mathcal{S}$. It is clear that $T_a(f)$ is well-defined for $f \in \mathcal{S}$, but we can also show that $T_a\colon \mathcal{S} \to \mathcal{S}$. Indeed, first observe that

$$(I-\Delta_\xi)^N e^{2\pi i x\cdot\xi} = (1+4\pi^2|x|^2)^N e^{2\pi i x\cdot\xi}$$

for each $N \in \mathbf{N}$, so

$$(1+4\pi^2|x|^2)^N T_a(f)(x) = \int_{\mathbf{R}^n} a(x,\xi)\widehat{f}(\xi)(I-\Delta_\xi)^N e^{2\pi i x\cdot\xi}d\xi = \int_{\mathbf{R}^n} \left[(I-\Delta_\xi)^N a(x,\xi)\widehat{f}(\xi)\right] e^{2\pi i x\cdot\xi}d\xi.$$

Since $f \in \mathcal{S}$, the integrand on the right is bounded by $(1+|\xi|^2)^{-n-1}$, for example. A similar argument applies to derivatives of $T_a(f)$, proving $T_a(f) \in \mathcal{S}$ and the mapping T_a is continuous.

Observe that formally by writing out the Fourier transform of f we can rewrite the operator T_a as

$$T_a(f)(x) = \iint_{\mathbf{R}^{2n}} a(x,\xi)f(y)e^{2\pi i(x-y)\cdot\xi}dyd\xi.$$

This integral does not necessarily converge, even for $f \in \mathcal{S}$, but does if we also assume that $a(x,\xi)$ has compact ξ-support.

1.4 The Hardy-Littlewood Maximal Function

In the section we take what appears to be a detour and study the maximal operators. Loosely speaking a maximal operator of a function takes the "maximal average" of that function at each point. It's not immediately obvious that such operators would be of use to us, but in fact they are ubiquitous in the study of PDEs and Harmonic Analysis. Given that the Fourier transform of a function is a description of the oscillations of which the function is composed, it is not surprising averages also should appear, as the role of cancellation is fundamental albeit subtle.

For a locally integrable function $f\colon \mathbf{R}^n \to \mathbf{C}$ we define the *centred Hardy-Littlewood maximal function* by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy$$

where $B_r(x) = \{y \in \mathbf{R}^n \,|\, |y-x| < r\}$ is a Euclidean ball of radius r centred at x. We also define the *(uncentred) Hardy-Littlewood maximal function* to be

$$M(f)(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|dy$$

where the supremum is taken over all Euclidean balls B which contain x. Clearly

$$M(f)(x) \leq \mathcal{M}(f)(x) \lesssim M(f)(x)$$

for all $x \in \mathbf{R}^n$, where the implicit constant only depends on n.

Theorem 1.5. *If $f \in L^p(\mathbf{R}^n)$ for $1 \leq p \leq \infty$ then $M(f)$ is finite almost everywhere. If $f \in L^1(\mathbf{R}^n)$, then for every $\alpha > 0$*

$$|\{x \in \mathbf{R}^n \,|\, M(f)(x) > \alpha\}| \leq \frac{3^n}{\alpha}\int_{\mathbf{R}^n} |f(y)|dy. \tag{1.4}$$

If $f \in L^p(\mathbf{R}^n)$ for $1 < p \leq \infty$ then $M(f) \in L^p(\mathbf{R}^n)$ and

$$\|M(f)\|_{L^p(\mathbf{R}^n)} \leq \frac{p3^n2^{p-1}}{p-1}\|f\|_{L^p(\mathbf{R}^n)}. \tag{1.5}$$

To prove the theorem we need a geometric lemma.

Lemma 1.6. *Given a finite collection of balls* $\{B_1, B_2, \ldots, B_k\}$ *in* $\mathbf{R}^n$*, there exists a finite subcollection* $\{B_{j_1}, B_{j_2}, \ldots, B_{j_m}\}$ *of pairwise disjoint balls such that*

$$\left|\bigcup_{i=1}^m B_{j_i}\right| \geq 3^{-n}\left|\bigcup_{i=1}^k B_i\right|.$$

Proof. We re-index the collection $\{B_1, B_2, \ldots, B_k\}$ so that

$$|B_1| \geq |B_2| \geq \cdots \geq |B_k|.$$

and describe how the subcollection is selected from this re-indexed collection: First select B_1 to be in the subcollection; Then for ℓ ranging from 2 to k select the ball B_ℓ to be in the subcollection precisely when B_ℓ is disjoint from $\bigcup_{i=1}^{\ell-1} B_{j_i}$.

As we started with a finite number of balls, will will have chosen a finite number for the subcollection, $\{B_{j_1}, B_{j_2}, \ldots, B_{j_m}\}$ for some $m \leq k$ say. If a ball B_ℓ was not selected it must intersect some previously selected ball B_{j_r} and, since we ordered the balls in descending radii, $B_\ell \subset 3B_{j_r}$. Therefore the union of the triples of selected balls contains all non-selected balls in addition, of course, to containing all selected balls. Therefore

$$\left|\bigcup_{i=1}^k B_i\right| \leq \left|\bigcup_{i=1}^m 3B_{j_i}\right| \leq \bigcup_{i=1}^m |3B_{j_i}| = 3^n \bigcup_{i=1}^m |B_{j_i}| = 3^n \left|\bigcup_{i=1}^m B_{j_i}\right|,$$

where the last equality follows as $\{B_{j_1}, B_{j_2}, \ldots, B_{j_m}\}$ were chosen to be disjoint. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We first prove (1.4). As a supremum of continuous functions $M(f)$ is lower semicontinuous and therefore $E_\alpha := \{x \in \mathbf{R}^n \,|\, M(f)(x) > \alpha\}$ is an open set. Let K be a compact subset of E_α. For each $x \in K$ there exists a ball B_x such that

$$\int_{B_x} |f(y)|dy > \alpha |B_x|.$$

Clearly $\{B_x\}_{x\in K}$ covers K and by compactness there exists a finite subcover $\{B_{x_1}, B_{x_2}, \ldots, B_{x_k}\}$. Applying Lemma 1.6 we can find a subcollection $\{B_{x_{j_1}}, B_{x_{j_2}}, \ldots, B_{x_{j_m}}\}$ such that

$$|K| \leq \left|\bigcup_{i=1}^k B_{x_i}\right| \leq 3^n \left|\bigcup_{i=1}^m B_{x_{j_i}}\right| \leq 3^n \sum_{i=1}^m |B_{x_{j_i}}| < \frac{3^n}{\alpha}\int_{B_{x_{j_i}}} |f(y)|dy \leq \frac{3^n}{\alpha}\int_{\mathbf{R}^n} |f(y)|dy$$

Taking the supremum over all compact sets K we obtain (1.4).

In order to prove (1.5) we assume the equality

$$\|g\|^p_{L^p(\mathbf{R}^n)} = p\int_0^\infty \alpha^{p-1}|\{x \in \mathbf{R}^n \,|\, |g(x)| > \alpha\}|d\alpha \tag{1.6}$$

whose proof we leave as an exercise. Define $f_1 \colon \mathbf{R}^n \to \mathbf{R}^n$ as equal to $f(x)$ if $f(x) > \alpha/2$ and 0 otherwise. Then $M(f) \leq M(f_1) + \alpha/2$ and so

$$\{x \in \mathbf{R}^n \,|\, M(f)(x) > \alpha\} \subseteq \{x \in \mathbf{R}^n \,|\, M(f_1)(x) > \alpha/2\}.$$

Therefore by (1.4)

$$|\{x \in \mathbf{R}^n \,|\, M(f)(x) > \alpha\}| \leq \frac{3^n}{\alpha}\int_{\{x\in\mathbf{R}^n \,|\, |f(x)|>\alpha/2\}} |f(y)|dy$$

and so by (1.6)

$$\begin{aligned}\|M(f)\|^p_{L^p(\mathbf{R}^n)} &= p\int_0^\infty \alpha^{p-1}|\{x\in\mathbf{R}^n \,|\, |M(f)(x)|>\alpha\}|d\alpha\\ &\leq p\int_0^\infty \frac{3^n}{\alpha}\alpha^{p-1}\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>\alpha/2\}}|f(y)|dyd\alpha\\ &\leq p\int_{\mathbf{R}^n}\int_0^{2|f(y)|}3^n\alpha^{p-2}d\alpha|f(y)|dy\\ &= \frac{3^n2^{p-1}p}{p-1}\int_{\mathbf{R}^n}|f(y)|^p dy\end{aligned}$$

which proves (1.5). That $M(f)$ is finite almost everywhere follows easily from (1.4) and (1.5). □

Theorem 1.7. *Suppose that $\varphi\colon \mathbf{R}^n\to\mathbf{R}$ is integrable non-increasing and radial. Then, for $f\in L^1$, we have*

$$\int \varphi(y)f(x-y)\,dy \leq \|\varphi\|_{L^1}M(f)(x)$$

for all $x\in\mathbf{R}^n$.

1.5 Exercises

1. Prove Theorem 1.3.
2. Prove that if $\{a_k\}_k$ is a pointwise convergent sequence of symbols (converging to some $a\in S^m$) that satisfy (1.3) uniformly in k, then $T_{a_k}(f)\to T_a(f)$ in $\mathcal{S}$ as $k\to\infty$.
3. Under the assumption that $a(x,\xi)\in S^m$ has compact ξ-support compute a formula for the adjoint T_a^* and show that $T_a^*\colon\mathcal{S}\to\mathcal{S}$.
4. Prove (1.6) for $0<p<\infty$.

2 Calculus and L^2-boundedness

2.1 Pseudo-local Behaviour of Pseudodifferential Operators

We begin this section with a theorem regarding symbols which have compact support in the x-variable. Although this assumption is very strong, the theorem will prove to be a useful tool in understanding general operators.

Theorem 2.1. *Suppose that $a(x,\xi)\in S^0$ has compact x-support uniformly in ξ. Then there exists a constant C such that*

$$\|T_a(f)\|_{L^2(\mathbf{R}^n)}\leq C\|f\|_{L^2(\mathbf{R}^n)}$$

for all $f\in\mathcal{S}$ and the operator T_a initially defined on $\mathcal{S}$ extends to a bounded operator from $L^2(\mathbf{R}^n)$ to itself.

Proof. The assumption of compact x-support allows us to take the Fourier transform in x, so we can write

$$a(x,\xi)=\int \widehat{a}(\lambda,\xi)e^{2\pi i\lambda\cdot x}d\lambda$$

where

$$\widehat{a}(\lambda,\xi)=\int a(x,\xi)e^{-2\pi ix\cdot\lambda}dx.$$

Since $x\mapsto a(x,\xi)$ is compactly supported, we can integrate by parts to show

$$(2\pi i\lambda)^\alpha\widehat{a}(\lambda,\xi)=\int(\partial_x^\alpha a(x,\xi))e^{-2\pi ix\cdot\lambda}d\lambda.$$

for each multi-index α and hence prove

$$\sup_{\xi} |\widehat{a}(\lambda,\xi)| \lesssim (1+|\lambda|^2)^{-N}. \tag{2.1}$$

Now define the symbol $b_\lambda(x,\xi) = \widehat{a}(\lambda,\xi)e^{2\pi i\lambda\cdot x}$ and compute

$$\begin{aligned} T_a(f)(x) &= \int a(x,\xi)\widehat{f}(\xi)e^{-2\pi i x\cdot\xi}d\xi \\ &= \iint \widehat{a}(\lambda,\xi)e^{2\pi i\lambda\cdot x}\widehat{f}(\xi)e^{-2\pi i x\cdot\xi}d\xi d\lambda \\ &= \int T_{b_\lambda}(f)(x)d\lambda \end{aligned}$$

But the operator T_{b_λ} is clearly bounded on $L^2(\mathbf{R}^n)$ uniformly in λ: This is because it is the composition of multiplier operator and a modulation. By Plancherel's Theorem (Theorem 1.4), the bound (2.1) shows the L^2 operator norm of the multiplier is bounded by $(1+|\lambda|^2)^{-N}$. The modulation, that is multiplication by the unitary complex number $e^{2\pi i\lambda\cdot x}$, is of course an L^2 operator bounded in uniformly in λ. Therefore

$$\begin{aligned} \|T_a(f)\|_{L^2(\mathbf{R}^n)} = \left\| \int T_{b_\lambda}(f)d\lambda \right\|_{L^2(\mathbf{R}^n)} &\leq \int \|T_{b_\lambda}(f)\|_{L^2(\mathbf{R}^n)}d\lambda \\ \lesssim \int \|f\|_{L^2(\mathbf{R}^n)}(1+|\lambda|^2)^{-N}d\lambda &\lesssim \|f\|_{L^2(\mathbf{R}^n)} \end{aligned}$$

completing the proof. □

The following theorem demostrates the pseudo-local nature of pseudodifferential operators.

Theorem 2.2. *Suppose that $a \in S^0$. For each $N \in \mathbf{N}$ there exists a constant $C_N > 0$ such that for all $x_0 \in \mathbf{R}^n$*

$$\int_{|x-x_0|\leq 1} |T_a(f)(x)|^2dx \leq C_N \int_{\mathbf{R}^n} \frac{|f(x)|^2}{(1+|x-x_0|)^N}dx.$$

Proof. We prove the theorem only in the special case $x_0 = 0$. Using a partition of unity we can split the function f into two parts: $f = f_1 + f_2$, such that f_1 is supported in the ball centred at the origin of radius 3 and f_2 is supported outside the ball of radius 2. Since $T_a(f) = T_a(f_1) + T_a(f_2)$ is sufficies to estimate the two terms of the right.

To estimate the first term we introduce a compactly supported smooth cut-off function η which is equal to 1 on the ball of radius 1. Then, using Theorem 2.1,

$$\begin{aligned} \int_{|x|\leq 1} |T_a(f_1)(x)|^2dx \leq \int |T_{\eta a}(f_1)(x)|^2dx &\leq C\int |f_1(x)|^2dx \\ \lesssim \int_{|x|\leq 3} |f(x)|^2dx &\leq C_N \int_{\mathbf{R}^n} \frac{|f(x)|^2}{(1+|x|)^N}dx. \end{aligned}$$

To estimate the second term $T_a(f_2)$ we must first consider the kernel of the operator T_a. Formally we can write the operator as

$$T_a(f)(x) = \iint a(x,\xi)e^{2\pi i(x-y)\cdot\xi}f(y)dyd\xi = \int k(x,x-y)f(y)dy \tag{2.2}$$

so $k(x,\cdot)$ is the distribution whose Fourier transform is $a(x,\cdot)$. Since $\xi \mapsto \partial_\xi^\alpha a(x,\xi)$ is integrable when $|\alpha| > n$, we can identify its inverse Fourier transform as the function $z \mapsto (-2\pi iz)^\alpha k(x,z)$. This integrability also provides the estimate

$$|z|^N|k(x,z)| \leq A_N \tag{2.3}$$

for $|z| \neq 0$ and each $N > n$. Furthermore it ensures we can indeed write

$$T_a(f)(x) = \int k(x, x-y) f(y) dy$$

for x not contained in the support of f. Since f_2 is supported away from the unit ball, we can use this representation together with (2.3) to estimate

$$\begin{aligned} |T_a(f_2)(x)| &= \left| \int_{|y|>2} k(x, x-y) f_2(y) dy \right| \leq \int_{|y|>2} |k(x, x-y)| \, |f_2(y)| \, dy \\ &\leq A_N \int_{|y|>2} |x-y|^{-N} \, |f(y)| \, dy \leq C_N \int_{\mathbf{R}^n} \frac{|f(y)|}{(1+|y|)^N} dy. \end{aligned}$$

Using Schwarz's inequality then completes the estimate for $T_a(f_2)$ and with it the proof of the theorem. $\square$

Corollary 2.3. *Suppose $a \in S^0$, then the operator T_a initially defined on $\mathcal{S}$ extends to a bounded operator from L^2 to L^2 and satisfies the estimate*

$$\|T_a(f)\|_{L^2(\mathbf{R}^n)} \leq C \|f\|_{L^2(\mathbf{R}^n)}$$

for some $C > 0$ independent of f.

Proof. Integrate the inequality in Theorem 2.2 with respect to x_0. $\square$

2.2 Symbolic Calculus

Theorem 2.4. *Suppose $a \in S^{m_1}$ and $b \in S^{m_2}$. Then there exists a symbol $c \in S^{m_1+m_2}$ such that*

$$T_c = T_a \circ T_b$$

and

$$c(x,\xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))$$

in the sense that

$$c(x,\xi) - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi)) \in S^{m_1+m_2-N}$$

for each $N \in \mathbf{N}$.

Proof. To simplify the proof we assume that both a and b are compactly supported in the ξ-variable and b is also compactly supported in the x-variable and leave the general case as an exercise for the interested reader. Writing

$$T_b(f)(y) = \iint b(y,\xi) e^{2\pi i \xi \cdot (y-z)} f(z) dz d\xi \quad \text{and} \quad T_a(f)(x) = \iint a(x,\eta) e^{2\pi i \eta \cdot (x-y)} f(y) dy d\eta$$

we can compute

$$\begin{aligned} T_a(T_b(f))(x) &= \iint a(x,\eta) e^{2\pi i \eta\cdot(x-y)} \iint b(y,\xi) e^{2\pi i \xi \cdot (y-z)} f(z) dz d\xi dy d\eta \\ &= \iint \left(\iint a(x,\eta) b(y,\xi) e^{2\pi i \eta \cdot (x-y)} e^{2\pi i \xi \cdot (y-z)} e^{-2\pi i \xi \cdot (x-z)} dy d\eta \right) e^{2\pi i \xi \cdot (x-z)} f(z) dz d\xi \\ &= \iint \left(\iint a(x,\eta) b(y,\xi) e^{2\pi i (\eta - \xi)\cdot(x-y)} dy d\eta \right) e^{2\pi i \xi \cdot (x-z)} f(z) dz d\xi \end{aligned}$$

so we hope to prove that

$$c(x,\xi) := \iint a(x,\eta)b(y,\xi)e^{2\pi i(\eta-\xi)\cdot(x-y)}dyd\eta$$

is a symbol satisfying the properties claimed in the statement of the theorem. Carrying out the integration in the y-variable we obtain

$$c(x,\xi) = \int a(x,\eta)\widehat{b}(\eta-\xi,\xi)e^{2\pi i(\eta-\xi)\cdot x}d\eta = \int a(x,\eta+\xi)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta \tag{2.4}$$

where $\eta \mapsto \widehat{b}(\eta,\xi)$ is the Fourier transform of $y \mapsto b(y,\xi)$.

Now we wish to replace $a(x,\eta+\xi)$ with its Taylor expansion about ξ:

$$a(x,\eta+\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\eta^\alpha + R_N(x,\xi,\eta).$$

Substituting this into (2.4) and using the Fourier inversion formula, it is easy to see that the α-th term is

$$\int \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\eta^\alpha\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta = \frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi)),$$

so we want to show the remainder contributes a term in $S^{m_1+m_2-N}$. We use the well-known estimate that bounds R_N by $|\eta|^N$ times the maximum of ξ-derivatives of order N along the line segment from ξ to $\xi+\eta$. This gives

$$|R_N(x,\xi,\eta)| \lesssim |\eta|^N(1+|\eta|)^{m_1-N} \quad \text{for } |\xi| \geq 2|\eta|$$

and

$$|R_N(x,\xi,\eta)| \lesssim |\eta|^N \quad \text{for all } \xi \text{ and } \eta.$$

Thus

$$\int R_N(x,\xi,\eta)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta = \int_{|\xi|\geq 2|\eta|} R_N(x,\xi,\eta)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta + \int_{|\xi|\leq 2|\eta|} R_N(x,\xi,\eta)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta.$$

Because $b \in S^{m_2}$ is assumed to have compact support in the x-variable, we have the estimate

$$|\widehat{b}(\eta,\xi)| \lesssim (1+|\eta|)^{-M}(1+|\xi|)^{m_2}$$

for each $M \in \mathbf{N}$ and so

$$\left|\int_{|\xi|\geq 2|\eta|} R_N(x,\xi,\eta)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta\right| \lesssim \int_{|\xi|\geq 2|\eta|} |\eta|^N(1+|\eta|)^{m_1-N-M}(1+|\xi|)^{m_2}d\eta \lesssim (1+|\xi|)^{m_1+m_2-N}$$

if M is chosen sufficiently large. Similarly

$$\left|\int_{|\xi|\leq 2|\eta|} R_N(x,\xi,\eta)\widehat{b}(\eta,\xi)e^{2\pi i\eta\cdot x}d\eta\right| \lesssim \int_{|\xi|\leq 2|\eta|} |\eta|^N(1+|\eta|)^{-M}(1+|\xi|)^{m_2}d\eta \lesssim (1+|\xi|)^{m_1+m_2-N}$$

again for sufficiently large M. □

2.3 Compound Symbols

A better understanding of pseudodifferential operators and, for example, their adjoints can be obtained by studying compond symbols. They are symbols similar to the class S^m but also depend on the y-variable as in (2.2). More precisely, we consider symbols $c(x,y,\xi)$ which satisfy the analogue of (1.3):

$$|\partial_y^\gamma\partial_x^\beta\partial_\xi^\alpha c(x,y,\xi)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{m-|\alpha|} \tag{2.5}$$

for each triple of multi-indices α, β and γ. To this symbol we can associate (at least formally) the operator

$$T_{[c]}(f)(x) = \iint c(x,y,\xi)e^{2\pi i(x-y)\cdot\xi}f(y)dyd\xi.$$

As we have seen earlier, to make sense of the operator in a rigorous way, we can assume compact support in the symbol and carefully take limits in $\mathcal{S}$, for example. A repeat of the proof of Theorem 2.4 shows us that the class of operators arising from compound symbols is in fact no bigger than the class arising from S^m.

Theorem 2.5. *Suppose that c is a compound symbol of order m (that is, satisfies (2.5)). Then there exists a symbol $a \in S^m$ such that*

$$T_{[c]} = T_a$$

and

$$a(x,\xi) - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha \partial_y^\alpha c(x,y,\xi)\big|_{y=x}) \in S^{m-N}$$

for each $N \in \mathbf{N}$.

What at first sight might appear disappointing, has an immediate and useful corollary: The class of pseudodifferential operators arising from symbols in S^m is closed under taking adjoints. This is easy to see, at least formally, by calculating

$$\begin{aligned}\langle f, T_a^*(g)\rangle = \langle T_a(f), g\rangle &= \int \left(\iint a(x,\xi)e^{2\pi i(x-y)\cdot\xi}f(y)dyd\xi\right)\overline{g(x)}dx \\ &= \int f(y)\overline{\left(\iint \overline{a(x,\xi)}e^{2\pi i(y-x)\cdot\xi}g(x)dxd\xi\right)}dy\end{aligned}$$

so $T_a^* = T_{[c]}$ where $c(x,y,\xi) = \overline{a(y,\xi)}$. Thus Theorem 2.5 says that, given $a \in S^m$, there exists a symbol $a^* \in S^m$ such that $T_a^* = T_{a^*}$ which satisfies

$$a^*(x,\xi) - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)} \in S^{m-N}$$

for each $N \in \mathbf{N}$.

2.4 Exercises

1. Use the special case of Theorem 2.2 when $x_0 = 0$ to help you prove the case $x_0 \neq 0$.
2. Remove the additional assumption in the proof of Theorem 2.4 that b is compactly supported in the x-variable. Hint: Use a cut-off function to split $b = b_1 + b_2$ where b_1 is supported near a arbitrary x_0. Then show that the symbol of $T_a \circ T_{b_2}$ belongs to $S^{m_1+m_2-N}$ for all $N \in \mathbf{N}$.
3. Using the proof of Theorem 2.4 as a guide, proof Theorem 2.5.

3 Singular Integrals and L^p-Boundedness

3.1 General Singular Integral Theory

Pseudodifferential operators arising from symbols in S^0 turn out to be examples of *singular integrals*, which are important operators that naturally appear in many situations. As such it is worth studying them in their own right. Our brief study here will provide us with a generalisation of Corollary 2.3 to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

A singular integral operator is an operator given by integration against a kernel K such that K is not quite integrable, but does enjoy some smoothness properties. More precisely and specifically for our purposes we require that it is an operator T which is continuous from $\mathcal{S}$ to $\mathcal{S}'$ and which can be represented in the form

$$T(f)(x) = \int_{\mathbf{R}^n} K(x,y)f(y)dy, \tag{3.1}$$

for some measurable function K at least when x is not an element of the support of f, which is such that

$$|K(x,y)| \leq \frac{A}{|x-y|^n}$$

and

$$\int_{|x-y|\geq 2\delta} |K(x,y) - K(x,z)|dx \leq A \quad \text{for } |z-y| \leq \delta. \tag{3.2}$$

The following theorem shows that if such an operator is bounded on $L^q(\mathbf{R}^n)$ then it is automatically bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < q$. We will outline how the proof goes later.

Theorem 3.1. *Assume that the operator T is of the form* (3.1) *for some measurable function K which satisfies* (3.2)*. If there also exists a constant $C_q > 0$ such that*

$$\|T(f)\|_{L^q(\mathbf{R}^n)} \leq C_q \|f\|_{L^q(\mathbf{R}^n)} \tag{3.3}$$

for some $q \leq \infty$ then for each $1 < p < q$ there exists a constant $C_p > 0$ such that

$$\|T(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}. \tag{3.4}$$

If we can verify the hypotheses of the theorem for T_a where $a \in S^0$ then Corollary 2.3 will imply that T_a is a bounded operator from $L^p(\mathbf{R}^n)$ to itself for all $1 < p \leq 2$. As we saw in section 2.3 the adjoint of T_a is also a pseudodifferential operator of order zero. Consequently T_a^* will also be bounded from $L^p(\mathbf{R}^n)$ to itself for all $1 < p \leq 2$ and so T_a will be bounded from $L^p(\mathbf{R}^n)$ to itself for all $2 \leq p < \infty$. In conclusion T_a will be a bounded operator from $L^p(\mathbf{R}^n)$ to itself for all $1 < p < \infty$. The following lemma suffices in order to apply Theorem 3.1, as we will explain below.

3.2 Pseudodifferential Operators as Singular Integrals

Lemma 3.2. *Suppose $a \in S^m$. Then T_a can be written as in* (3.1) *with*

$$K(x,y) = k(x, y-x)$$

where $k(x,z)$ is smooth away from $z = 0$ and satisfies

$$|\partial_x^\beta \partial_z^\alpha k(x,z)| \leq C_{\alpha,\beta,N} |z|^{-n-m-|\alpha|-N}$$

for all α, β and $N \geq 0$ so that $n + m + |\alpha| + N > 0$.

With this lemma at hand we can easily show that (3.2) is satified: If we set $\zeta = (1-t)y + tz$ for $t \in [0,1]$ we have

$$|\zeta - x| = |((1-t)y + tz) - x| = |(y-x) + t(z-y)| \leq |y-x| + t|z-y| \leq |y-x| + \delta \leq (3/2)|y-x|$$

if $|x-y| \geq 2\delta$. so applying Lemma 3.2 with $|\alpha| = 1$, $|\beta| = 0$ and $m = N = 0$ we have that

$$\sup_{\zeta\in[y,z]} |\nabla_\zeta k(x, \zeta - x)| \lesssim \sup_{\zeta\in[y,z]} |\zeta - x|^{n-1} \lesssim |y-x|^{-n-1},$$

where $[y,z]$ denotes the line segment from y to z. Thus

$$\begin{aligned}\int_{|x-y|\geq 2\delta} |k(x,y-x) - k(x,z-x)|dx &\leq \int_{|x-y|\geq 2\delta} \sup_{\zeta\in[y,z]} |\nabla_\zeta k(x,\zeta - x)||y-z|dx \\ &\leq \int_{|x-y|\geq 2\delta} |y-x|^{-n-1}|y-z|dx \lesssim 1\end{aligned}$$

for $|z-y| \leq \delta$, which is exactly (3.2) for T_a.

Before we return to the proof of Theorem 3.1 we first prove Lemma 4.2.

Proof of Lemma 3.2. This proof makes use of a very useful technique called a Littlewood-Paley decomposition. In involves spliting up the symbol a onto parts a_j supported where the frequency variable ξ is of size 2^j, a so-called *dyadic* decomposition. Such a splitting is advantageous here, as each a_j is also a symbol in S^m.

Consider a smooth function $\phi\colon \mathbf{R}^n \to \mathbf{R}$ which is compactly supported in the ball of radius 2 centred at the origin and equal to 1 on the unit ball centred at the origin. Set $\phi_j(\xi) = \phi(2^{-j}\xi) - \phi(2^{1-j}\xi)$. Then

$$\phi(\xi) + \sum_{j=1}^{\infty} \phi_j(\xi) = \lim_{j\to\infty} \phi(2^{-j}\xi) = 1$$

for all $\xi \in \mathbf{R}^n$. Moreover, $|\partial_\xi^\alpha \phi_j(\xi)| \lesssim 2^{-j|\alpha|} \leq (1+|\xi|)^{-|\alpha|}$ so $a_j(x,\xi) := a(x,\xi)\phi_j(\xi) \in S^m$ for $j \geq 1$ with constants in (1.3) comparable to those of a, uniformly in j. We also set $a_0(x,\xi) := a(x,\xi)\phi(\xi)$. We then obtain the operator identity

$$T_a = \sum_{j=0}^{\infty} T_{a_j}$$

and that each operator T_{a_j} has the kernel

$$k_j(x,z) = \int a_j(x,\xi) e^{2\pi i \xi\cdot z} d\xi$$

Just as in the proof of Theorem 2.2 we have that

$$(-2\pi i z)^\gamma \partial_x^\beta \partial_z^\alpha k_j(x,z) = \int \partial_\xi^\gamma((2\pi i \xi)^\alpha \partial_x^\beta a_j(x,\xi)) e^{2\pi i \xi\cdot z} d\xi.$$

The integrand here has support of size 2^{nj} and the integrand itself is bounded by $2^{j(m+|\alpha|-|\gamma|)}$, thus we can obtain the estimate

$$|\partial_x^\beta \partial_z^\alpha k_j(x,z)| \leq |z|^{-M} 2^{j(n+m-M+|\alpha|)} \tag{3.5}$$

for each $M \geq 0$.

We can now complete the proof of the lemma. First, if $|z| \geq 1$, then

$$|\partial_x^\beta \partial_z^\alpha k(x,z)| \leq \sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \lesssim \sum_{j=0}^{\infty} |z|^{-M} 2^{j(n+m-M+|\alpha|)} \lesssim |z|^{-M} \lesssim |z|^{-n-m-|\alpha|-N}$$

provided $M > n+m+|\alpha|+N$. Secondly, if $|z| < 1$ it again suffices to estimate

$$|\partial_x^\beta \partial_z^\alpha k(x,z)| \leq \sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x,z)|$$

but this time we split the sum

$$\sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| = \sum_{2^j \leq |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| + \sum_{2^j > |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x,z)|.$$

To the first sum we apply (3.5) with $M = 0$:

$$\sum_{2^j \leq |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \leq \sum_{2^j \leq |z|^{-1}} 2^{j(n+m+|\alpha|)} \lesssim \left\{ \begin{array}{ll} |z|^{-n-m-|\alpha|} & (\text{if } n+m+|\alpha| > 0) \\ \ln(|z|^{-1}) + 1 & (\text{if } n+m+|\alpha| \leq 0) \end{array} \right\} \lesssim |z|^{-n-m-|\alpha|-N}$$

for $N \geq 0$ since $|z| < 1$. For the second sum we apply (3.5) with $M > n+m+|\alpha|+N$:

$$\sum_{2^j > |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \leq \sum_{2^j > |z|^{-1}} |z|^{-M} 2^{j(n+m-M+|\alpha|)} \lesssim |z|^{-M} \lesssim |z|^{-n-m-|\alpha|-N}$$

since $|z| \geq 1$. □

3.3 The Calderón-Zygmund Decomposition

Before we begin the proof of Theorem 3.1 we first draw a couple of parallels with the proof of Theorem 1.5. The first is that we again proof a weak-type estimate like (1.4). From this we will draw the conclusion (3.4) by interpolating between the weak-type estimate and (3.3). As this argument is similar, we leave it to the interested reader. Recall that this interpolation argument required a decomposition of the function f into two pieces. Here we will also use a decomposition, although a more sophisticated one this time, called the *Calderón-Zygmund decomposition.* We state it here, but refer the reader to [5, p. 17] or [1, p. 284, p. 299], for example, for a proof.

Theorem 3.3 (Calderón-Zygmund decomposition)**.** *Given a function $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ there exists a decomposition of $f = g + b$ with $b = \sum_k b_k$ and a sequence of balls $\{B_k\}_k$ such that:*

1. *$g(x) = f(x)$ for $x \notin \cup_k B_k$ and $|g(x)| \leq c\alpha$ for almost every $x \in \mathbf{R}^n$;*
2. *Each function b_k is supported in B_k,*
$$\int |b_k(x)|dx \leq c\alpha|B_k| \quad \textit{and} \quad \int b_k(x)dx = 0;$$
and
3. *$\sum_k |B_k| \leq \frac{c}{\alpha}\|f\|_{L^1(\mathbf{R}^n)}$.*

Proof of Theorem 3.1. We only prove the theorem in the case $q < \infty$. As discussed above it suffices to prove
$$|\{x \,|\, |T(f)(x)| > \alpha\}| \leq \frac{C}{\alpha}\|f\|_{L^1(\mathbf{R}^n)}.$$
We apply Theorem 3.3 to f and then it suffices to prove
$$|\{x \,|\, |T(g)(x)| > \alpha/2\}| + |\{x \,|\, |T(b)(x)| > \alpha/2\}| \leq \frac{C}{\alpha}\|f\|_{L^1(\mathbf{R}^n)}. \tag{3.6}$$
Applying properties 1 and 3 we see that
$$\int |g|^q = \int_{\cup_k B_k} |g|^q + \int_{\mathbf{R}^n\setminus(\cup_k B_k)} |g|^q \leq c\sum_k |B_k|\alpha^q + c^{q-1}\int_{\mathbf{R}^n\setminus(\cup_k B_k)} \alpha^{q-1}|f| \lesssim \alpha^{q-1}\|f\|_{L^1(\mathbf{R}^n)}.$$
Chebycheff's inequality and (3.3) then yields
$$|\{x \,|\, |T(g)(x)| > \alpha/2\}| \leq (\alpha/2)^{-q}\|T(g)\|^q_{L^q(\mathbf{R}^n)} \lesssim (\alpha/2)^{-q}\|g\|^q_{L^q(\mathbf{R}^n)} \lesssim \alpha^{-1}\|f\|^q_{L^1(\mathbf{R}^n)} \tag{3.7}$$

Turning to $b = \sum_k b_k$, let B_k^* denote the concentric double of the ball B_k. By (3.2) we have that
$$\int_{\mathbf{R}^n\setminus(\cup_k B_k^*)} |K(x,y) - K(x,z_k)|dx \leq A \quad \text{for } y \in B_k,$$
where z_k is the centre of B_k. Using 2 we can write
$$T(b_k)(x) = \int (K(x,y) - K(x,z_k))b_k(y)dy$$
for $x \in \mathbf{R}^n \setminus (\cup_k B_k^*)$ and so
$$\int_{\mathbf{R}^n\setminus(\cup_k B_k^*)} |T(b_k)(x)|dx \leq \int_{\mathbf{R}^n\setminus(\cup_k B_k^*)} |K(x,y) - K(x,z_k)|dx \int |b_k(y)|dy \leq Ac\alpha|B_k|.$$
By 3 we have
$$\int_{\mathbf{R}^n\setminus(\cup_k B_k^*)} |T(b)(x)|dx \leq \sum_k \int_{\mathbf{R}^n\setminus(\cup_k B_k^*)} |T(b_k)(x)|dx \leq Ac\alpha\sum_k |B_k| \leq Ac^2\|f\|_{L^1(\mathbf{R}^n)}. \tag{3.8}$$

Finally

$$\begin{aligned}|\{x\,|\,|T(b)(x)|>\alpha/2\}| &\le |\{x\notin\cup_k B_k^*\,|\,|T(b)(x)|>\alpha/2\}|+\sum_k |B_k^*|\\ &\le \frac{1}{\alpha}\int_{\mathbf{R}^n\setminus(\cup_k B_k^*)}|T(b)(x)|dx+\sum_k|B_k^*|\\ &\lesssim \frac{1}{\alpha}\|f\|_{L^1(\mathbf{R}^n)}\end{aligned}$$

by (3.8) and again 3. This together with (3.7) proves (3.6) and we leave the details of the interpolation argument to obtain (3.4) to the reader. □

Corollary 3.4. *If $a\in S^0$ then T_a extends to a bounded operator from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1<p<\infty$.*

3.4 Exercises

1. Use (3.6) and (3.3) to prove (3.4).
2. Go through the proof of Theorem 3.1 and modify the argument to prove (3.6) when $q=\infty$.
3. Prove Corollary 3.4.

4 Cancellation and $T(1)$-type Theorems

4.1 Good Kernel, Bad Operator

We now broaden our horizons a little by considering a wider range of symbols. We say that $a\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{C}$ belongs to the class $S^m_{\rho,\delta}$ when

$$|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

Therefore clearly $S^m=S^m_{1,0}$. Just as in Section 1.3 we can make sense of T_a for $a\in S^m_{\rho,\delta}$ and show that it maps $\mathcal{S}$ continuously to $\mathcal{S}$.

Our next result shows that the kernel representation from Lemma 3.2 can be repeated under the considerably weaker assumption that $a\in S^0_{1,0}$. However, this is not sufficient to obtain the L^p-boundedness of such operators, which can in general fail.

Lemma 4.1. *Suppose $a\in S^0_{1,1}$. Then T_a can be written as in (3.1) with*

$$K(x,y)=k(x,y-x)$$

where $k(x,z)$ is smooth away from $z=0$ and satisfies

$$|\partial_x^\beta\partial_z^\alpha k(x,z)|\le C_{\alpha,\beta,N}|z|^{-n-|\alpha|-|\beta|} \tag{4.1}$$

for all α, β and $N\ge 0$ so that $n+m+|\alpha|+N>0$.

Proof. The proof is a reprise of that of Lemma 3.2, the estimate (3.5) is replaced by

$$|\partial_x^\beta\partial_z^\alpha k_j(x,z)|\le |z|^{-M}2^{j(n+m-M+|\alpha|)}$$

since each differentiation with respect to x now worsens the estimate by a factor of 2^j. □

Despite this hopeful beginning, the following counter-example shows us that $S^0_{1,1}$ is in fact worse than $S^0_{1,0}$.

Theorem 4.2. *There exists a symbol $a\in S^0_{1,1}$ which is not bounded on L^2 and hence not bounded on L^p for any $1<p\le 2$.*

Proof. Here we take $n = 1$. Consider the same Littlewood-Paley decomposition as we did in Section 3.2. Observe that each $\phi_j(\xi)$ is supported in the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$, so by considering just even j (that is, $j = 2k$ for $k \in \mathbf{N}$), the supports of ϕ_j are mutually disjoint. We are also free to choose $\phi_j(\xi)$ equal to 1 for $2^{j-1/2} \leq |\xi| \leq 2^{j+1/2}$. Define the symbol

$$a(x,\xi) = \sum_{k=1}^{\infty} e^{-2\pi i 2^{2k} x} \phi_{2k}(\xi).$$

It is easy to check that $a \in S^0_{1,1}$. Next choose f_0 so that its Fourier transform is supported in $|\xi| \leq 1/2$ and define f_N via the formula

$$\widehat{f_N}(\xi) = \sum_{k=1}^{N} \frac{1}{2k} \widehat{f}_0(\xi - 2^{2k}),$$

so the terms have mutually disjoint support. Thus,

$$\|f_N\|^2_{L^2(\mathbf{R}^n)} = \|\widehat{f_N}\|^2_{L^2(\mathbf{R}^n)} = \sum_{k=1}^{N} \frac{1}{4k^2} \|\widehat{f}_0(\cdot - 2^{2k})\|^2_{L^2(\mathbf{R}^n)} = \sum_{k=1}^{N} \frac{1}{4k^2} \|f_0\|^2_{L^2(\mathbf{R}^n)} \leq C \|f_0\|^2_{L^2(\mathbf{R}^n)}.$$

Now we can compute

$$a(x,\xi)\widehat{f_N}(\xi) = \sum_{k=1}^{N} e^{-2\pi i 2^{2k} x} \phi_{2k}(\xi) \frac{1}{2k} \widehat{f}_0(\xi - 2^{2k}) = \sum_{k=1}^{N} e^{-2\pi i 2^{2k} x} \frac{1}{2k} \widehat{f}_0(\xi - 2^{2k})$$

and therefore

$$T_a(f_N)(x) = \int_{\mathbf{R}^n} a(x,\xi)\widehat{f_N}(\xi) e^{2\pi i x\cdot\xi} d\xi = \sum_{k=1}^{N} e^{-2\pi i 2^{2k} x} \frac{e^{2\pi i 2^{2k} x}}{2k} f_0(x) = \sum_{k=1}^{N} \frac{1}{2k} f_0(x).$$

Clearly then

$$\|T_a(f_N)\|_{L^2(\mathbf{R}^n)} = \sum_{k=1}^{N} \frac{1}{2k} \|f_0\|_{L^2(\mathbf{R}^n)} \geq c \|f_0\|_{L^2(\mathbf{R}^n)} \ln N,$$

which proves the theorem. □

The following theorem bears even more bad news.

Theorem 4.3. *Suppose that $K(x,y)$ is a given function defined for $x \neq y$ which satisfies the inequality $|K(x,y)| \geq c|x-y|^{-n}$ for some $c > 0$. Then there does not exist an operator T that is bounded on $L^2(\mathbf{R}^n)$ for which K is the kernel in the sense that*

$$T(f)(x) = \int_{\mathbf{R}^n} K(x,y) f(y) dy$$

for x outside the support of f.

Proof. Assume that such an operator T is in fact a bounded operator — we wish to obtain a contradiction. Consider balls $B_{\mathbf{k}}$ of radius 1/4 centred at $\mathbf{k} \in \mathbf{Z}^n$ and denote its concentric double by $B'_{\mathbf{k}}$. Then define

$$S_R = \bigcup_{|\mathbf{k}| \leq 2R} B_{\mathbf{k}} \quad \text{and} \quad S'_R = \bigcup_{|\mathbf{k}| \leq 2R} B'_{\mathbf{k}}.$$

We have the estimates

$$R^n \lesssim |S'_R \cap \{x : |x| \leq R\}| \lesssim R^n.$$

for $R > 1$.

Take $f_R = \chi_{S_R}$. Then, if $x \notin \bigcup_{\mathbf{k}\in\mathbf{Z}^n} B'_{\mathbf{k}}$,

$$T(f_R)(x) = \int_{\mathbf{R}^n} K(x,y) f_R(y)dy \geq c \sum_{|\mathbf{k}|\leq 2R} \int_{B_{\mathbf{k}}} \frac{dy}{|x-y|^n}$$

and thus, if we also have $|x| \leq R$, then

$$T(f_R)(x) \gtrsim \sum_{0<|\mathbf{k}|\leq R} |\mathbf{k}|^{-n} \simeq \ln R$$

and so

$$\|T(f_R)\|^2_{L^2(\mathbf{R}^n)} \geq \int_{S'^c_R \cap \{x\,:\,|x|\leq R\}} |T(f_R)(x)|^2 dx \gtrsim (\ln R)^2 R^n.$$

However, $\|f_R\|^2_{L^2(\mathbf{R}^n)} \lesssim R^n$, which contradicts the boundedness of T. □

4.2 Redeaming Features

In this section we investigate what possible additional properties could be sufficient to conclude $L^2(\mathbf{R}^n)$-boundedness of operators with kernels which satisfy conditions such as (4.1). We will not prove the main result in this section (Theorem 4.5) but instead just concentrate on motivating why the result is reasonable.

Theorem 4.4. *Suppose the distribution k agrees with a function away from the origin and satisfies the estimate*

$$|k(x)| \leq A|x|^{-n} \tag{4.2}$$

for $x \neq 0$. If the operator $T(f) = k * f$ initially defined on $\mathcal{S}$ extends to a bounded operator on $L^2(\mathbf{R}^n)$, then there exists a constant C so that

$$\left| \int_{\varepsilon<|x|<N} k(y)dy \right| \leq C \tag{4.3}$$

for all $0 < \varepsilon < N < \infty$.

We might interpret Theorem 4.4 as follows. If we avoid technical details, we can formally take the limits $\varepsilon \to 0$ and $N \to \infty$ in (4.3) to obtain

$$|T(1)(x)| = \left| \int_{\mathbf{R}^n} k(y)dy \right| \leq C.$$

Thus it appears that $L^2(\mathbf{R}^n)$-boundedness implies that when the operator T maps constants to bounded functions. Intresting this implication can in fact be reversed. That is, checking T behaves well on a few specific functions can imply the $L^2(\mathbf{R}^n)$-boundedness of T. Such results are powerful tools for proving the $L^2(\mathbf{R}^n)$-boundedness of operators and there are now many different forms of such theorems. Typically such theorems go under the name of $T(1)$ or $T(b)$ theorems, because the main condition to check is that $T(1)$, or more generally $T(b)$, is well-behaved for an appropriate b.[2] Here, the correct notion of well-behaved is not really $L^\infty(\mathbf{R}^n)$-boundedness, as we might guess from the above discussion, but that is beyond the scope of this course.

Proof. The proof relies on the fact that $L^2(\mathbf{R}^n)$-boundedness implies the Fourier transform $\widehat{k}$ of k is bounded. First choose a smooth function ϕ which is supported in the unit ball, equal to 1 for $|x| \leq 1/2$ and $0 \leq \phi(x) \leq 1$. Then we can approximate

$$\int_{\varepsilon<|x|<N} k(x)dx$$

[2] For example, a b which is uniformly bounded both from above and below.

by

$$k * (\phi^N - \phi^\varepsilon)(0) = \int k(x)(\phi(x/N) - \phi(x/\varepsilon))dx$$

where $\phi^R(x) = \phi(x/R)$. Indeed, the difference

$$\int_{\varepsilon<|x|<N} k(x)dx - k * (\phi^N - \phi^\varepsilon)(0)$$

is dominated by

$$\int_{\varepsilon/2<|x|<\varepsilon} |k(x)|dx + \int_{N/2<|x|<N} |k(x)|dx \lesssim 1$$

via the kernel estimate (4.2).

However,

$$|k * \phi^N(0)| = \left|\int k(x)\phi(x/N)dx\right| = \left|\int \widehat{k}(x)\widehat{\phi}(Nx)N^n dx\right| \lesssim 1$$

since $\widehat{\phi}$ is bounded and similarly $|k * \phi^\varepsilon(0)| \lesssim 1$ □

We now set the context for our main theorem, which, as we said above, will roughly be a converse to Theorem 4.4. Assume now that T is a continuous operator from $\mathcal{S}$ to $\mathcal{S}'$ and associated to it is a kernel $K(x, y)$ defined for $x \neq y$ in the sense that, for compactly supported $f \in \mathcal{S}$,

$$T(f)(x) = \int_{\mathbf{R}^n} K(x, y)f(y)dy \tag{4.4}$$

when x is not contained in the support of f. The kernel is also assumed to satisfy the following estimates: For some $A > 0$ and $0 < \gamma \leq 1$ we have

$$\begin{aligned} |K(x, y)| &\leq A|x - y|^{-n}; \\ |K(x, y) - K(x', y)| &\leq A\frac{|x - x'|^\gamma}{|x - y|^{n+\gamma}}, \quad \text{for } |x - x'| \leq |x - y|/2; \text{ and} \\ |K(x, y) - K(x, y')| &\leq A\frac{|y - y'|^\gamma}{|x - y|^{n+\gamma}}, \quad \text{for } |y - y'| \leq |x - y|/2. \end{aligned} \tag{4.5}$$

The proof of Theorem 4.4 suggests that it may be possible to replace a '$T(1)$ condition' with a condition regarding how the operator acts on certain smooth bump functions. This is in fact possible and allows us to state the theorem in a rather slick manner that avoids other technicalities. We define *normalised bump functions* to be smooth functions supported in the unit ball which satisfy

$$|\partial_x^\alpha \phi(x)| \leq 1, \quad \text{for } 0 \leq |\alpha| \leq N,$$

for some specific N — the precise value is not of any interest to us here. We then define the translate and dilate of such a ϕ as

$$\phi^{R,x_0}(x) = \phi\left(\frac{x - x_0}{R}\right).$$

The condition we will impose on normalised bump functions is that

$$\|T(\phi^{R,x_0})\|_{L^2(\mathbf{R}^n)} + \|T^*(\phi^{R,x_0})\|_{L^2(\mathbf{R}^n)} \leq AR^{n/2}, \tag{4.6}$$

where A can only depend on N and T^* is the adjoint of T. Observe that if T is bounded on $L^2(\mathbf{R}^n)$, then (4.6) follows immediately. When (4.6) holds for all ϕ^{R,x_0}, we say T and T^* are restrictedly bounded.

Theorem 4.5. *Assume T is a continuous linear mapping from $\mathcal{S}$ to $\mathcal{S}'$ associated to a kernel K in the sense of* (4.4) *which satisfies* (4.5)*. Then T extends to a bounded linear operator on $L^2(\mathbf{R}^n)$ if and only if both T and T^* are restrictedly bounded in the sense of* (4.6)*.*

4.3 The Cauchy Integral

As an application of Theorem 4.5 we now give an outline of the proof of the boundedness of the Cauchy integral operator on $L^2(\mathbf{R}^n)$ on Lipschitz curves, at least when the Lipschitz constant is sufficiently small. You are probably familiar with the Cauchy integral from complex analysis, which, for $f\colon \mathbf{C}\to\mathbf{C}$, is given by

$$C_\gamma(f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

and defined for $z\notin\gamma$, where γ is a closed curve in the complex plane. If, instead of being closed, γ is assumed to be parametrised by a real variable, a closely related operator is the corresponding Hilbert transform

$$H_\gamma(f)(x) = \text{p.v.}\int_{-\infty}^{\infty}\frac{f(y)\gamma'(y)}{\gamma(x)-\gamma(y)}dy$$

and many questions regarding the Cauchy integral operator are closely related to the same questions regarding H_γ. Indeed, the question of $L^2(\mathbf{R})$-boundedness was the original motivation for the development of the singular integral theory discussed above.

If we assume γ is the graph of a Lipschitz function A, then γ is the image of $x\mapsto x+iA(x)$ in the complex plane and the kernel of H_γ may be written as

$$\frac{1+A'(y)}{x-y+i(A(x)-A(y))}.$$

Forgeting about the factor $1+iA'(y)$, which is irrelevant to the $L^2(\mathbf{R})$-boundedness of the operator, and expanding as a geometric series we obtain

$$\frac{1}{x-y+i(A(x)-A(y))} = \frac{1}{x-y}\sum_{k=0}^{\infty}(-i)^k\left(\frac{A(x)-A(y)}{x-y}\right)^k$$

where the sum will converge at least when the Lipschitz constant of A (that is, M such that $|A(x)-A(y)|\le M|x-y|$) is less than one. Thus is it natural to consider the presumably related operators

$$C_k(f)(x) = \text{p.v.}\int_{-\infty}^{\infty}\frac{(A(x)-A(y))^k}{(x-y)^{k+1}}f(y)dy$$

called *Calderón commutators*. Clearly C_0 is the usual Hilbert transform and therefore $L^2(\mathbf{R})$-boundedness is known. The boundedness of the other Calderón commutators can be obtained by induction as an application of Theorem 4.5. Indeed, the kernel estimates (4.5) are straightforward to prove and we can check (4.6) in the following way. Estimating the norm on the left-hand side of (4.6) outside the ball $B_{2R}(x_0)$ centred at x_0, with radius R, can be done easily using the kernel estimates. To estimate $\|C_k(\phi^{R,x_0})\|_{L^2(B_{2R}(x_0))}$ we compute

$$\begin{aligned}C_k(f)(x) &= \frac{1}{k}\int(A(x)-A(y))^k f(y)\frac{d[(x-y)^{-k}]}{dy}dy\\ &= -\frac{1}{k}\int(A(x)-A(y))^k\frac{df}{dy}(y)(x-y)^{-k}dy - \frac{1}{k}\int\frac{d[(A(x)-A(y))^k]}{dy}f(y)(x-y)^{-k}dy\end{aligned}$$

and can easily see the first term can be estimated appropriately, since ϕ is a normalised bump function. The second term is equal to

$$\int(A(x)-A(y))^{k-1}A'(y)f(y)(x-y)^{-k}dy = C_{k-1}(A'f)(x),$$

which is $L^2(\mathbf{R})$-bounded by the inductive hypothesis and the fact that A' is bounded. Therefore we have proved that each C_k is $L^2(\mathbf{R})$-bounded and consequently H_γ is $L^2(\mathbf{R})$-bounded provided the Lipschitz constant associated to γ is sufficiently small.

4.4 Exercises

1. Check carefully the proof of Lemma 4.1.
2. Estimate the norm $\|C_k(\phi^{R,x_0})\|_{L^2(\mathbf{R}\setminus B_{2R}(x_0)}$ in order to complete the proof of (4.6) as claimed in Section 4.3

5 Almost Orthogonality

5.1 Cotlar-Stein Lemma

Here we return to the subject of pseudodifferential operators. Our main result here, Theorem 5.1, provides an alternative condition for $L^2(\mathbf{R}^n)$-boundedness based breaking up the operator into 'almost orthogonal' pieces. Recall that the counter-example in Theorem 4.2 worked because the operator piled up orthogonal parts of function on top of each other, thus producing a function with a large $L^2(\mathbf{R}^n)$-norm. In this section we investigate the $L^2(\mathbf{R}^n)$-boundedness of operators that do not permit such behavior and will go on to apply Theorem 5.1 to symbols of type $S^0_{\rho,\rho}$ $(\rho < 1)$ in the next section.

Theorem 5.1 (The Cotlar-Stein Lemma). *Suppose that an operator T initially defined on $\mathcal{S}$ admits a decomposition*

$$T = \sum_{\mathbf{j}\in I} T_{\mathbf{j}}, \tag{5.1}$$

for some finite index set I. Furthermore, suppose there exists a non-negative sequence $\{\gamma(\mathbf{j})\}_{\mathbf{j}\in I}$ such that

$$A = \sum_{\mathbf{j}\in I} \gamma(\mathbf{j}) < \infty \tag{5.2}$$

$$\|T^*_{\mathbf{i}} T_{\mathbf{j}}\|_{L^2\to L^2} \le \gamma(\mathbf{i}-\mathbf{j})^2 \quad \textit{and} \tag{5.3}$$

$$\|T_{\mathbf{i}} T^*_{\mathbf{j}}\|_{L^2\to L^2} \le \gamma(\mathbf{i}-\mathbf{j})^2. \tag{5.4}$$

Then T extends to a bounded operator on $L^2(\mathbf{R}^n)$ with norm independent of the number of terms in the decomposition (5.1)*. More precisely, we conclude $\|T\|_{L^2\to L^2} \le A$.*

Proof. Here we write $\|\cdot\|_{L^2\to L^2} = \|\cdot\|$. Recall (or see exercise below) that $\|T^*T\| = \|T\|^2$, so for self-adjoint T, $\|T^2\| = \|T\|^2$. Therefore by induction $\|T^m\| = \|T\|^m$ at least when m is a power of 2. Applying this to the self-adjoint operator T^*T, we see that

$$\|(T^*T)^m\| = \|T\|^{2m}.$$

Making use of the decomposition (5.1), we can write

$$(T^*T)^m = \sum_{\mathbf{j}_1,\mathbf{j}_2,\dots,\mathbf{j}_{2m}} (T^*_{\mathbf{j}_1}T_{\mathbf{j}_2})(T^*_{\mathbf{j}_3}T_{\mathbf{j}_4})\dots(T^*_{\mathbf{j}_{2m-1}}T_{\mathbf{j}_{2m}})$$

and easily estimate the norm of each term in the sum using (5.3):

$$\begin{aligned}\|(T^*_{\mathbf{j}_1}T_{\mathbf{j}_2})(T^*_{\mathbf{j}_3}T_{\mathbf{j}_4})\dots(T^*_{\mathbf{j}_{2m-1}}T_{\mathbf{j}_{2m}})\| &\le \|T^*_{\mathbf{j}_1}T_{\mathbf{j}_2}\|\|T^*_{\mathbf{j}_3}T_{\mathbf{j}_4}\|\dots\|T^*_{\mathbf{j}_{2m-1}}T_{\mathbf{j}_{2m}}\| \\ &\le \gamma(\mathbf{j}_1-\mathbf{j}_2)^2\gamma(\mathbf{j}_3-\mathbf{j}_4)^2\dots\gamma(\mathbf{j}_{2m-1}-\mathbf{j}_m)^2.\end{aligned} \tag{5.5}$$

Alternatively we can write

$$(T^*_{\mathbf{j}_1}T_{\mathbf{j}_2})(T^*_{\mathbf{j}_3}T_{\mathbf{j}_4})\dots(T^*_{\mathbf{j}_{2m-1}}T_{\mathbf{j}_{2m}}) = T^*_{\mathbf{j}_1}(T_{\mathbf{j}_2}T^*_{\mathbf{j}_3})(T_{\mathbf{j}_4}T^*_{\mathbf{j}_5})\dots(T_{\mathbf{j}_{2m-2}}T^*_{\mathbf{j}_{2m-1}})T_{\mathbf{j}_m}$$

and then estimate using (5.4) and then (5.2):

$$\begin{aligned}&\|T^*_{\mathbf{j}_1}(T_{\mathbf{j}_2}T^*_{\mathbf{j}_3})(T_{\mathbf{j}_4}T^*_{\mathbf{j}_5})\dots(T_{\mathbf{j}_{2m-2}}T^*_{\mathbf{j}_{2m-1}})T_{\mathbf{j}_m}\| \\ &\le \|T^*_{\mathbf{j}_1}\|\|T_{\mathbf{j}_2}T^*_{\mathbf{j}_3}\|\|T_{\mathbf{j}_4}T^*_{\mathbf{j}_5}\|\dots\|T_{\mathbf{j}_{2m-2}}T^*_{\mathbf{j}_{2m-1}}\|\|T_{\mathbf{j}_m}\| \\ &\le \gamma(\mathbf{j}_1)\gamma(\mathbf{j}_2-\mathbf{j}_3)^2\gamma(\mathbf{j}_4-\mathbf{j}_5)^2\dots\gamma(\mathbf{j}_{2m-2}-\mathbf{j}_{2m-1})^2\gamma(\mathbf{j}_{2m}) \\ &\le A^2\gamma(\mathbf{j}_2-\mathbf{j}_3)^2\gamma(\mathbf{j}_4-\mathbf{j}_5)^2\dots\gamma(\mathbf{j}_{2m-2}-\mathbf{j}_{2m-1})^2.\end{aligned} \tag{5.6}$$

Taking the geometric mean of (5.5) and (5.6), we obtain

$$\|(T^*T)^m\| \leq A \sum_{\mathbf{j}_1,\mathbf{j}_2,\dots,\mathbf{j}_{2m}} \gamma(\mathbf{j}_1-\mathbf{j}_2)\gamma(\mathbf{j}_2-\mathbf{j}_3)\dots\gamma(\mathbf{i}_{2m-1}-\mathbf{j}_{2m}),$$

We can then estimate the sum by successively taking the sum in $\mathbf{j}_1$, $\mathbf{j}_2$, until $\mathbf{j}_{2m}$:

$$\begin{aligned} &A \sum_{\mathbf{j}_1,\mathbf{j}_2,\dots,\mathbf{j}_{2m}} \gamma(\mathbf{j}_1-\mathbf{j}_2)\gamma(\mathbf{j}_2-\mathbf{j}_3)\dots\gamma(\mathbf{j}_{2m-1}-\mathbf{j}_{2m}) \\ &\leq A^2 \sum_{\mathbf{j}_2,\mathbf{j}_3,\dots,\mathbf{j}_{2m}} \gamma(\mathbf{j}_2-\mathbf{j}_3)\dots\gamma(\mathbf{j}_{2m-1}-\mathbf{j}_{2m}) \\ &\leq \dots \leq A^{2m-1} \sum_{\mathbf{j}_{2m-1},\mathbf{j}_{2m}} \gamma(\mathbf{j}_{2m-1}-\mathbf{j}_{2m}) \leq A^{2m}\sum_{\mathbf{j}_{2m}} 1 \leq A^{2m}N, \end{aligned}$$

assuming the size of the index set I is N. Thus $\|T\|^{2m} \leq A^{2m}N$ and so

$$\|T\| \leq AN^{1/2m}.$$

Taking the limit $m \to \infty$ gives $\|T\| \leq A$ as claimed. □

5.2 Calderón-Vaillancourt Theorem

The following theorem is an application of Theorem 5.1.

Theorem 5.2 (Calderón-Vaillancourt Theorem)**.** *Suppose that $a \in S^0_{\rho,\rho}$ with $0 \leq \rho < 1$. Then the operator T_a initially defined on $\mathcal{S}$ has a bounded extension from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$.*

We give a proof only in the case $\rho = 0$.

Proof. By Plancherel's Theorem its enough to prove the $L^2(\mathbf{R}^n)$-boundedness of the operator

$$S(f)(x) = \int_{\mathbf{R}^n} a(x,\xi)e^{2\pi i x\cdot\xi} f(\xi)d\xi.$$

Given the now symmetric roles of x and ξ (in the context of $a \in S^0_{0,0}$) we decompose both the x and ξ-space in the same manner. To do this we choose a partition of unity $\{\phi_i\}_{i\in\mathbf{Z}^n}$ such that each ϕ_i is supported in the cube of unit radius centred at $i \in \mathbf{Z}^n$. Because it is a partition of unity $\sum_i \phi_i(x) = 1$ for all $x \in \mathbf{R}^n$ and, setting $\mathbf{i} = (i,i')$ $(i,i' \in \mathbf{Z}^n)$ we can decompose

$$a(x,\xi) = \sum_{\mathbf{i}} a_{\mathbf{i}}(x,\xi)$$

where $a_{\mathbf{i}}(x,\xi) = a(x,\xi)\phi_i(x)\phi_{i'}(\xi)$ and thus $T = \sum_{\mathbf{i}} T_{a_{\mathbf{i}}}$.

In order to apply Theorem 5.1 we must prove the almost-orthogonality estimates

$$\|T^*_{a_{\mathbf{i}}}T_{a_{\mathbf{j}}}\| \lesssim (1+|\mathbf{i}-\mathbf{j}|)^{-N} \tag{5.7}$$

and

$$\|T_{a_{\mathbf{j}}}T^*_{a_{\mathbf{i}}}\| \lesssim (1+|\mathbf{i}-\mathbf{j}|)^{-N} \tag{5.8}$$

for N sufficiently large.

It is straightforward to compute that the kernel of $T^*_{a_{\mathbf{i}}}T_{a_{\mathbf{j}}}$ is

$$a_{\mathbf{i},\mathbf{j}}(\xi,\eta) = \int_{\mathbf{R}^n} \overline{a_{\mathbf{i}}(x,\eta)}a_{\mathbf{j}}(x,\xi)e^{2\pi i x\cdot(\eta-\xi)}dx.$$

First observe that if $\mathbf{i} = (i, i')$ and $\mathbf{j} = (j, j')$ and $|i - j| \geq 2$, then the x-supports of $a_{\mathbf{i}}(x, \eta)$ and $a_{\mathbf{j}}(x, \xi)$ are disjoint, so $a_{\mathbf{i},\mathbf{j}}(\xi, \eta) = 0$. In the case $|i - j| < 2$ we can calculate

$$\begin{aligned}(1 + 4\pi^2|\eta - \xi|^2)^N a_{\mathbf{i},\mathbf{j}}(\xi, \eta) &= \int_{\mathbf{R}^n} \overline{a_{\mathbf{i}}(x, \eta)} a_{\mathbf{j}}(x, \xi)(1 - \Delta_x)^N e^{2\pi i x\cdot(\eta-\xi)} dx \\ &= \int_{\mathbf{R}^n} (1 - \Delta_x)^N \left[\overline{a_{\mathbf{i}}(x, \eta)} a_{\mathbf{j}}(x, \xi)\right] e^{2\pi i x\cdot(\eta-\xi)} dx\end{aligned}$$

which can be estimated uniformly in η and ξ. Bearing in mind the ξ and η-support properties of $a_{\mathbf{i}}(x, \eta)$ and $a_{\mathbf{j}}(x, \xi)$, we see this proves (5.7). The proof of (5.8) follows the same method, due to the symmetric roles of the variables. □

5.3 Exercises

1. Prove that for an bounded operator T on a Hilbert space $\|T^*T\| = \|T\|^2$.
2. Prove the following lemma.

   Lemma 5.3. *Suppose an operator S is given by*

   $$S(f)(x) = \int s(x, y)f(y)dy$$

   where the kernel s satisfies

   $$\sup_x \int |s(x, y)|dy \leq A \quad \text{and} \quad \sup_y \int |s(x, y)|dx \leq A.$$

   Then S is a $L^2(\mathbf{R}^n)$-bounded operator and $\|S\|_{L^2\to L^2} \leq A$.

6 Rough pseudodifferential operators

So far we have concerned ourselves with $L^2(\mathbf{R}^n)$-boundedness of operators associated to smooth symbols in the class $S^m_{\rho,\delta}$. It is also possible to prove the $L^p(\mathbf{R}^n)$-boundedness of such operators under appropriate assumptions on the parameters m, ρ and δ. We can explore these results and simultaneously greatly weaken the smoothness assumptions on our symbols. Although the results we can prove under these weaker assumptions on our symbols are, as one would expect, weaker than the sharp results for smooth symbols, they nevertheless are not so different. This approach also has the advantage of simultaneously weakening our hypothesis on our symbols and generalising our boundedness results to $L^p(\mathbf{R}^n)$ spaces.

6.1 Pseudo-pseudodifferential operators of Kenig-Staubach

Carlos Kenig and Wolfgang Staubach introduced the following symbol classes [2], which are sometimes called pseudo-pseudodifferential operators.

Definition 6.1. *Let $m \in \mathbf{R}$, $\rho \in [0, 1]$ and $\delta \in [0, 1]$. A function $a\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is said to belong to $L^\infty S^m_\rho$ when for each multi-index α there exists a constant C_α such that*

$$\|\partial^\alpha_\xi a(\cdot, \xi)\|_{L^\infty} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}.$$

Therefore, here only measurability in the x-variable needs to be assumed. Kenig-Staubach proved the following $L^p(\mathbf{R})$-boundedness result.

Theorem 6.2. *Fix $p \in [1, 2]$ and let $a \in L^\infty S^m_\rho$ with $0 \leq \rho \leq 1$ and $m < \frac{n}{p}(\rho - 1)$. Then T_a is a bounded operator on $L^q(\mathbf{R}^n)$ for each $q \geq p$.*

However, it is possible to obtain pointwise control of such operators by the Hardy-Littlewood maximal function [3]. This is what we shall investigate here.

Theorem 6.3. *Fix $p \in [1,2]$ and let $a \in L^\infty S^m_\rho$ with $0 \le \rho \le 1$ and $m < \frac{n}{p}(\rho - 1)$. Then there exists a constant C, depending only on n, p, m, ρ and a finite number of the constants C_α in Definition 6.1, such that*

$$|T_a(f)(x)| \le C\,(M(|f|^p)(x))^{1/p}\,,$$

for all $f \in \mathcal{S}$ and $x \in \mathbf{R}^n$.

Theorem 6.2 is an immediate corollary of the pointwise bound in Theorem 6.3.

Proof. To prove the theorem we use the Littlewood-Paley partition of unity introduced in the proof of Lemma 3.2, we decompose the symbol as

$$a(x,\xi) = a_0(x,\xi) + \sum_{k=1}^{\infty} a_k(x,\xi)$$

with $a_k(x,\xi) = a(x,\xi)\phi_k(\xi)$, $k \ge 0$.

First we consider the operator T_{a_0}. We have

$$T_{a_0}(f)(x) = \iint a_0(x,\xi)\, e^{2\pi i(x-y)\cdot\xi}\, f(y)\, dy\, d\xi = \int K_0(x,y) f(x-y)\, dy,$$

with

$$K_0(x,y) = \int a_0(x,\xi) e^{2\pi i(x-y)\cdot\xi} d\xi.$$

The same argument used to prove Lemma 3.2 gives us the estimate

$$|K_0(x,y)| \lesssim (1+|y|)^{-M},$$

for each $M > n$. Theorem 1.7 yields

$$|T_{a_0}(f)(x)| \lesssim \int (1+|y|)^{-M} |f(x-y)|\, dy \lesssim M(f)(x) \lesssim (M(|f|^p)(x))^{1/p}\,, \tag{6.1}$$

for all $1 \le p \le 2$.

Now let us analyse $T_{a_k}(f)(x) = \int a_k(x,\xi)\widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$ for $k \ge 1$. We note, just as before, that $T_{a_k}(f)(x)$ can be written as

$$T_{a_k}(f)(x) = \int K_k(x,y) f(x-y) dy$$

with

$$K_k(x,y) = \int a_k(x,\xi) e^{2\pi i(x-y)\cdot\xi} d\xi = \check{a}_k(x,y),$$

where $\check{a}_k$ here denotes the inverse Fourier transform of $a_k(x,\xi)$ with respect to ξ. One observes that

$$|T_{a_k}(f)(x)|^p = \Big|\int K_k(x,y) f(x-y)\, dy\Big|^p = \Big|\int K_k(x,y)\sigma_k(y)\frac{1}{\sigma_k(y)} f(x-y) dy\Big|^p,$$

with functions $\sigma_k(y)$ which will be chosen momentarily. Therefore, Hölder's inequality yields

$$|T_{a_k}(u)(x)|^p \le \Big\{\int |K_k(x,y)|^{p'} |\sigma_k(y)|^{p'} dy\Big\}^{\frac{p}{p'}} \Big\{\int \frac{|u(x-y)|^p}{|\sigma_k(y)|^p} dy\Big\}, \tag{6.2}$$

where $1/p + 1/p' = 1$. Now for an $l > n/p$, we define σ_k by

$$\sigma_k(y) = \begin{cases} 2^{\frac{-k\rho n}{p}}, & |y| \le 2^{-k\rho}; \\ 2^{-k\rho(\frac{n}{p} - l)} |y|^l, & |y| > 2^{-k\rho}. \end{cases}$$

By Hausdorff-Young's theorem and the symbol estimates in Definition 6.1, first for $\alpha = 0$ and then for $|\alpha| = l$, we have

$$\int 2^{\frac{-kp'\rho n}{p}} |K_k(x,y)|^{p'} dy \leq 2^{\frac{-kp'\rho n}{p}} \Big\{ \int |a_k(x,\xi)|^p d\xi \Big\}^{\frac{p'}{p}}$$
$$\lesssim 2^{\frac{-kp'\rho n}{p}} \Big\{ \int_{|\xi|\sim 2^k} 2^{pmk} d\xi \Big\}^{\frac{p'}{p}} \lesssim 2^{kp'(m-\frac{n}{p}(\rho-1))},$$

and

$$\int 2^{-k\rho p'(\frac{n}{p}-l)} |K_k(x,y)|^{p'} |y|^{p'l}\, dy \lesssim 2^{-k\rho p'(\frac{n}{p}-l)} \Big\{ \int |\nabla_\xi^l a_k(x,\xi)|^p \, d\xi \Big\}^{\frac{p'}{p}}$$
$$\lesssim 2^{-k\rho p'(\frac{n}{p}-l)} \Big\{ \int\limits_{|\xi|\sim 2^k} 2^{kp(m-\rho l)}\, d\xi \Big\}^{\frac{p'}{p}} \lesssim 2^{kp'(m-\frac{n}{p}(\rho-1))}.$$

Hence, splitting the integral into $|y| \leq 2^{-k\rho}$ and , $|y| > 2^{-k\rho}$ yields

$$\Big\{ \int |K_k(x,y)|^{p'} |\sigma_k(y)|^{p'} dy \Big\}^{\frac{p}{p'}} \lesssim \{2^{kp'(m-\frac{n}{p}(\rho-1))}\}^{\frac{p}{p'}} = 2^{kp(m-\frac{n}{p}(\rho-1))}.$$

Furthermore, once again using Theorem 1.7, we have

$$\int \frac{|f(x-y)|^p \, dy}{|\sigma_k(y)|^p} \lesssim M(|f|^p)(x)$$

with a constant that only depends on the dimension n. Thus (6.2) yields

$$|T_{a_k} f(x)|^p \lesssim 2^{k(m-\frac{n}{p}(\rho-1))} M(|f|^p)(x) \tag{6.3}$$

Summing in k using (6.1) and (6.3), we obtain

$$|T_a(f)(x)|^p \lesssim |T_{a_0}(f)(x)|^p + \sum_{k=1}^{\infty} |T_{a_k}(f)(x)|^p$$
$$\lesssim M(|f|^p)(x)\Big(1 + \sum_{k=1}^{\infty} 2^{k(m-\frac{n}{p}(\rho-1))}\Big)$$

Clearly the sum is convergent if $m < \frac{n}{p}(\rho - 1)$. ☐

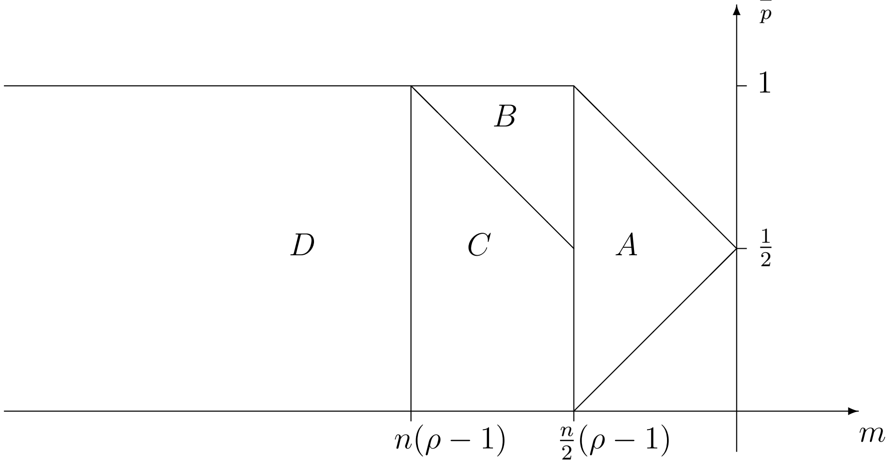

Figure 1: Regions of L^p-Boundedness

The interior of the region $\overline{C \cup D}$ in Figure 1 is the set of points $(m, 1/p)$ for which Theorem 6.3 ensures that T_a is $L^p(\mathbf{R}^n)$-bounded when $a \in L^\infty S^m_\rho$. The set $\overline{A \cup B \cup C \cup D}$ is the set for which smooth symbols in $S^m_{\rho,\delta}$ (for $\delta \leq \rho$, $\delta < 1$) produce $L^p(\mathbf{R}^n)$-bounded operators. This can be seen, except for the end-points, from the results we have proved here together with interpolation arguments. Indeed, Corollary 3.4 and Theorem 5.2 provide the required $L^2(\mathbf{R}^n)$-boundedness, remembering that $S^m_{\rho,\delta} \subset S^m_{\delta,\delta}$ when $\delta < \rho$. Theorem 6.3 provides the point $(m, 1/p) = (m, 0)$ for $m < n(\rho-1)/p$ (and thus just missing the end-point $(n(\rho-1)/p, 1/p)$) since $S^m_{\rho,\delta} \subset L^\infty S^m_\rho$. The symbolic calculus we proved for symbols in $S^m_{1,0}$ in Theorem 2.4 can be repeated for symbols in $S^m_{\rho,\delta}$ provided $\delta < \rho$ and so we have that the class of operators T_a with $a \in S^m_{\rho,\delta}$ is closed under taking adjoints. This gives us the boundedness for the point $(m, 1/p) = (m, 1)$ for $m < n(\rho-1)/p$. Interpolation then gives us boundedness on the interior of the pencil-shaped region $\overline{A \cup B \cup C \cup D}$.

6.2 Compound symbols revisited

In the context of pseudo-pseudodifferential operators, the symbolic calculus breaks down as we are not able to differential in the spatial variable x. This means if we consider symbols which also depend on a variable y, as we did earlier — so called compound symbols — they may no longer be included as alternative representations of operators which symbols in $L^\infty S^m_\rho$. Therefore we must study them separately. The following theorem gives the sharp value of m for which a function $a\colon \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ satisfying

$$|\partial^\alpha_\xi a(x,y,\xi)| \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|} \quad \text{for each multi-index } \alpha \tag{6.4}$$

gives rise to an $L^p(\mathbf{R}^n)$-bounded operator T_a.

Theorem 6.4. *Suppose $0 \leq \rho \leq 1$, $m < n(\rho-1)$ and the compound symbol a satisfies (6.4), then, for each $p > 1$, we have*

$$|T_a(f)(x)| \lesssim (M(|f|^p)(x))^{1/p},$$

and consequently

$$\|T_a(f)\|_{L^q(\mathbf{R}^n)} \lesssim \|f\|_{L^q(\mathbf{R}^n)}$$

for each $f \in \mathcal{S}$ and $1 < q \leq \infty$.

The points $(m, 1/p)$ for which we have boundedness are thus those in the interior of D shown in Figure 1. To fully understand the proof one needs to know a little bit about Muckenhoupt weights, which we will not have time to study here, although they are very interesting and beautiful. Nevertheless, the main ideas of the proof can still be understood if you are willing to take a few facts for granted. (See [3, Thm 3.7] for a full proof.)

Proof. Let $K(x,y,z) := \int a(x,y,\xi)\, e^{2\pi i z\cdot\xi}\, d\xi$, then we have

$$T_a f(x) = \int_{|x-y|\leq 1} K(x,y,x-y) f(y)\, dy + \int_{|x-y|>1} K(x,y,x-y) f(y)\, dy = \mathrm{I} + \mathrm{II}.$$

The fact that $m < n(\rho-1)$ allows us to use integration by parts (similarly to how we have done several times above) to show the kernel decays sufficiently quickly when $x-y$ is large. More precisely, it is possible to prove that $|K(x,y,x-y)| \leq C|x-y|^{-N}$ for sufficiently large N and $|x-y| \geq 1$. So II can be easily majorized by $M(f)(x)$.

We again use the Littlewood-Paley partition of unity $\{\phi_k\}_k$ just as we did in Theorem 6.3. Using that partition of unity and setting

$$K_k(x,y,z) := \int a_k(x,y,\xi)\, e^{2\pi i z\cdot\xi}\, d\xi$$

yields

$$\mathrm{I} = \sum_{k=0}^{\infty} \int_{|x-y|\leq 1} K_k(x,y,x-y) f(y)\, dy = \sum_{k=0}^{\infty} \mathrm{I}_k,$$

Now once again for $k = 0$ it is easy to check that $|K_0(x, y, x - y)| \lesssim \langle x - y \rangle^{-N}$ for all $N > 0$, hence

$$|I_0| \lesssim M(f)(x).$$

If we consider an individual term with $k \geq 1$, we have

$$\begin{aligned} |\mathrm{I}_k| &= \Big| \int_{|x-y|\leq 1} K_k(x, y, x-y) f(y)\, dy \Big| \\ &= \Big| \int_{|x-y|\leq 1} K_k(x, y, x-y) |b(x-y)|^r \frac{1}{|b(x-y)|^r} f(y) dy \Big|, \end{aligned}$$

where b and r are parameters to be chosen later. Therefore, Hölder's inequality yields

$$\begin{aligned} |\mathrm{I}_k| \leq & \Big\{ \int_{|x-y|\leq 1} |K_k(x, y, x-y)|^{p'} |b(x-y)|^{rp'} dy \Big\}^{\frac{1}{p'}} \\ & \times \Big\{ \int_{|x-y|\leq 1} \frac{|f(y)|^p}{|b(x-y)|^{rp}} dy \Big\}^{\frac{1}{p}}. \end{aligned}$$

By Theorem 1.7, for $r < \frac{n}{p}$, we have

$$\Big\{ \int_{|x-y|\leq 1} \frac{|f(y)|^p}{|b(x-y)|^{rp}} dy \Big\}^{\frac{1}{p}} \leq C b^{-r} \left(M(|f|^p)(x) \right)^{1/p},$$

therefore

$$|\mathrm{I}_k| \leq C \Big\{ \int |K_k(x, x-z, z)|^{p'} |bz|^{rp'} dz \Big\}^{\frac{1}{p'}} b^{-r} M_p\, u(x). \tag{6.5}$$

Considering the remaining integral, setting $\sigma_k^x(z, \xi) := a_k(x, x-z, \xi)$ we have

$$\begin{aligned} K_k(x, x-z, z) &= \int a_k(x, x-z, \xi) e^{iz\cdot\xi}\, d\xi \\ &= \int \sigma_k^x(z, \xi) e^{iz\cdot\xi}\, d\xi \\ &= \int \sigma_k^x(z, \xi)(\phi_{k-1}(\xi) + \phi_k(\xi) + \phi_{k+1}(\xi)) e^{iz\cdot\xi}\, d\xi \\ &= \int \sigma_k^x(z, \xi) \widehat{\psi}_k(\xi) e^{iz\cdot\xi}\, d\xi = T_{\sigma_k^x}(\psi_k)(z), \end{aligned}$$

where $\psi_k \in \mathcal{S}$ is defined via the identity

$$\widehat{\psi}_k(\xi) = \phi_{k-1}(\xi) + \phi_k(\xi) + \phi_{k+1}(\xi).$$

Observe that due to the scaling implicit in how we defined the Littlewood-Paley decomposition $\{\phi_k\}_k$, we must have $\psi_k(x) = 2^{nk}\psi(2^k x)$ for some $\psi \in \mathcal{S}$. Therefore, taking $b = 2^k$,

$$\Big\{ \int |K_k(x, x-z, z)|^{p'} |bz|^{rp'} dy \Big\}^{\frac{1}{p'}} = \Big\{ \int |T_{\sigma_k^x}(\psi_k)(z)|^{p'} |2^k z|^{rp'} dz \Big\}^{\frac{1}{p'}}$$

Now we observe that since x is fixed, σ_k^x belongs to the symbol class $L^\infty S^m_\rho$ with semi-norms that are uniform in x. Therefore to understand the integral above it suffices to understand how an operator with symbol in $L^\infty S^m_\rho$ acts on L^p spaces where we replace Lebesgue measure dx with the measure $|x|^{rp'} dx$. It turns out that such measures are examples of Muckenhoupt weights (rather, the weight $x \mapsto |x|^{rp'}$ in the measure $|x|^{rp'} dx$ is a Muckenhoupt weight). Moreover, such measures are precisely those for which the Hardy-Littlewood maximal function is bounded, so our pointwise estimate in Theorem 6.1 is perfectly suited to answer this question.

To simplify matters here, we will just assume that the Hardy-Littlewood maximal function M is bounded on the space $L^p(|x|^{rp'}dx)$. Now since $p' > 2$, we may apply (6.3) with the p in that estimate taken equal to 1, and obtain

$$\begin{aligned}
&\Big\{ \int |T_{\sigma_k^x}(\psi_k)(z)|^{p'} |2^k z|^{rp'} dz \Big\}^{\frac{1}{p'}} \\
&\le C 2^{k(m-n(\rho-1))} \Big\{ \int |M(\psi_k)(z)|^{p'} |2^k z|^{rp'} dz \Big\}^{\frac{1}{p'}} \\
&\le C 2^{k(m-n(\rho-1))} \Big\{ \int |\psi_k(z)|^{p'} |2^k z|^{rp'} dz \Big\}^{\frac{1}{p'}} \\
&= C 2^{k(m-n(\rho-1)+n/p)}.
\end{aligned} \tag{6.6}$$

Combining this with (6.5) we obtain

$$|\mathrm{I}_k| \le C 2^{k(m-n(\rho-1)-(r-n/p))} M_p(u)(x).$$

Therefore choosing r such that $r - n/p = (m - n(\rho - 1))/2$ and summing in k proves the theorem. □

6.3 Exercises

1. Prove Theorem 1.7.
2. Further reading: Read about Muckenhoupts weights in [5, Ch. V]. Then you should easily fully understand the proof of Theorem 6.4.

References

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