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New Phenomena in the World of Peaked Solitons

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Abstract

The aim of this work is to present new contributions to the theory of peaked solitons. The thesis consists of two papers, which are named “New solutions with peakon creation in the Camassa–Holm and Novikov equations” and “Peakon-antipeakon solutions of the Novikov equation” respectively.

In Paper I, a new kind of peakon-like solution to the Novikov equation is discovered, by transforming the one-peakon solution via a Lie symmetry transformation. This new kind of solution is unbounded as $x \to +\infty$ and/or $x \to -\infty$, and has a peak, though only for some interval of time. Thus, the solutions exhibit creation and/or destruction of peaks. We make sure that the peakon-like function is still a solution in the weak sense for those times where the function is non-differentiable. We find that similar solutions, with peaks living only for some interval in time, are valid weak solutions to the Camassa–Holm equation, though it appears that these can not be obtained via a symmetry transformation.

In Paper II we investigate multipeakon solutions of the Novikov equation, in particular interactions between peakons with positive amplitude and antipeakons with negative amplitude. The solutions are given by explicit formulas, which makes it possible to analyze them in great detail. As in the Camassa–Holm case, the slope of the wave develops a singularity when a peakon collides with an antipeakon, while the wave itself remains continuous and can be continued past the collision to provide a global weak solution. However, the Novikov equation differs in several interesting ways from other peakon equations, especially regarding asymptotics for large times. For example, peakons and antipeakons both travel to the right, making it possible for several peakons and antipeakons to travel together with the same speed and collide infinitely many times. Such clusters may exhibit very intricate periodic or quasi-periodic interactions. It is also possible for peakons to have the same asymptotic velocity but separate at a logarithmic rate; this phenomenon is associated with coinciding eigenvalues in the spectral problem coming from the Lax pair, and requires nontrivial modifications to the previously known solution formulas which assume that all eigenvalues are simple.

Populärvetenskaplig sammanfattning

Inom vågteori studeras så kallade solitoner, vilka kan beskrivas som vågpaket som rör sig med konstant form och hastighet. Typiska egenskaper är att utbredningen i rummet är begränsad, samt att två solitoner som kolliderar kan passera genom varandra utan att ändra form.

Fenomenet beskrevs redan 1834 av John Scott Russell, som ridande längs en kanal följde en ”rundad, slät, väldefinierad upphöjning av vatten, vilken fortsatte sin bana längs kanalen synbarligen utan att ändra form eller förlora fart”. Dåvarande våglära kunde inte förklara uppkomsten av sådana vågor, men moderna hydrodynamiska teorier innehåller ett antal modeller där solitoner är ett naturligt koncept.

I denna avhandling studeras vågekvationer som tillåter en särskild typ av spetsiga solitoner, så kallade peakoner (från engelskans ”peaked soliton”). Avhandlingen utgörs av två artiklar som på olika sätt utvidgar förståelsen av detta fenomen.

I Artikel 1 beskrivs en ny typ av peakon-liknande våg där den spetsiga vågtoppen endast existerar under ett visst tidsintervall. Denna typ av våg kan visas förekomma i flera moderna vågekvationer, såsom Camassa–Holm-ekvationen och Novikovs ekvation.

I Artikel 2 studeras, i fallet med Novikovs ekvation, samspelet mellan peakoner och så kallade antipeakoner, vilket är vågor med spetsig vågdal istället för vågtopp. Artikeln beskriver vad som händer då peakoner kolliderar med antipeakoner, både i allmänhet och i några specialfall.

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Contents

Introduction

Introduction

This thesis presents several new contributions to the theory of peaked soliton solutions, so called peakons, in a number of different partial differential equations. The results are presented in two separate articles. Paper I is named “New solutions with peakon creation in the Camassa–Holm and Novikov equations”, and is published in Journal of Nonlinear Mathematical Physics, 2015, 22(1). Paper II is coauthored with Hans Lundmark, and is named “Peakon–antipeakon solutions of the Novikov equation”.

We begin the study of soliton equations by looking at the the KdV equation, which is the archetypical example of such equations. This gives us a chance to comment on some of the terminology used in the thesis. Then the relevant equations are introduced together with the concept of peakons, which are solutions to these equations in a certain weak sense.

Background

Before we get into the equations used in the papers, let us study the Korteweg–deVries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{1}$$

where $u = u(x,t)$ is a function of space and time, and subscripts denote partial derivatives. This equation was introduced in the 19th century as a mathematical model for shallow water waves, and provides a nice example of features that we will see later in other equations, such as soliton solutions, integrability and inverse scattering. The constant in front of the term uu_x has no particular relevance, as it can be changed to any other constant by a scaling transformation

$$u = \frac{k}{6}v,$$

so that v solves the equation

$$v_t + v_{xxx} + kvv_x = 0$$

if and only if u solves equation (1). In the literature, a number of different choices of k are used, though we will stick to $k = 6$, following Strauss's book [25].

Solitons

By soliton solutions, we mean localized traveling wave solutions, that interact with each other in a certain stable fashion. To find soliton solutions to the KdV equation, one makes the travelling wave ansatz

$$u(x,t) = f(x - ct).$$

and replaces the partial derivatives $u_x = f'$, $u_t = -cf'$ into equation (1) to get the ordinary differential equation

$$-cf' + f''' + 6ff' = 0.$$

This equation is easily integrated to

$$-cf + f'' + 3f^2 = C, \quad C \in \mathbb{R}.$$

Multiplying by $2f'$ and integrating again gives

$$-cf^2 + (f')^2 + 2f^3 = Cf + D, \quad C, D \in \mathbb{R}.$$

We require that $f(x)$ and its derivatives tend to zero as $x \to \pm\infty$, since we are searching for localized waves. We thus put $C = D = 0$, and solve the ODE

$$(f')^2 = cf^2 - 2f^3.$$

Besides the trivial solution $f(x) = 0$, solutions are found by separating the equation into

$$\int \frac{df}{f\sqrt{c-2f}} = \int dx,$$

which gives

$$\frac{1}{\sqrt{c}} \ln \left| \frac{\sqrt{c-2f} - \sqrt{c}}{\sqrt{c-2f} + \sqrt{c}} \right| = x - x_0.$$

Solving for f, we find that the travelling wave shape of the KdV equation is

$$f(x-ct) = \frac{c}{2}\left(1 - \left(\frac{1 + e^{\sqrt{c}(x-x_0-ct)}}{1 - e^{\sqrt{c}(x-x_0-ct)}}\right)^2\right)$$

which we write as

$$f(x-ct) = \frac{c}{2}\cosh^{-2}\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right).$$

Note that this is a whole family of solutions, parametrized by the constants c and x_0. For each choice of parameters one gets a soliton of fixed shape, where x_0 is the center of the wave profile at $t = 0$, the amplitude is $\frac{c}{2}$ and the speed is equal to c. We thus see that taller solitons move faster than shorter ones. See Figure 1 for an example of a KdV soliton.

Now, consider a setup with a linear combination of two such KdV solitons, separated by a large distance at time $t = 0$, and let their amplitudes be chosen such that the left one is taller. Note that the KdV equation can be interpreted as an evolution equation

$$u_t = -u_{xxx} - 6uu_x.$$

Thus, given the initial wave profile $u(x,0)$, there should exist a well defined wave function $u(x,t)$, at least up to some time $t > 0$. Initially, the separation of the solitons and their exponential decay means that they will not interact much. In fact, if one writes the linear combination as $u = v + w$, it follows that the product

$$uu_x = vv_x + vw_x + wv_x + ww_x$$

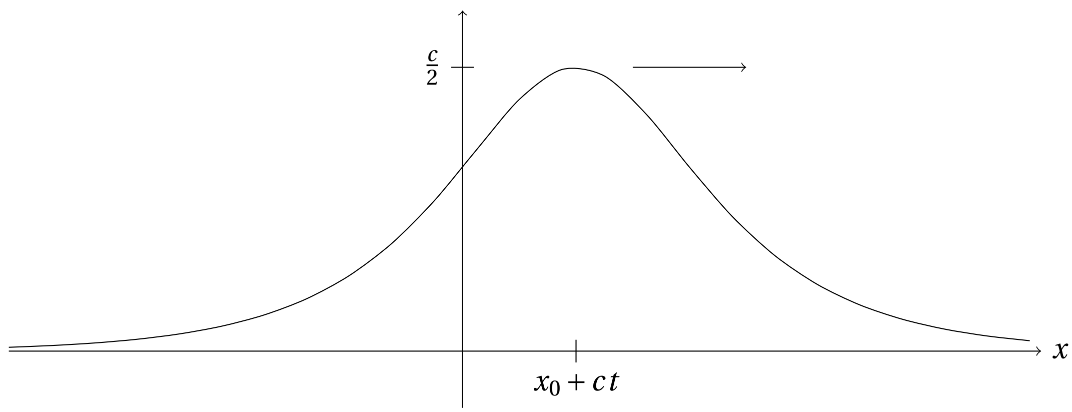

Figure 1: KdV soliton.

is initially approximately equal to $vv_x + ww_x$, since either v or w_x has to be close to zero for any given x, and the same holds for v_x and w. Thus, we expect the solitons to first move along relatively unperturbed, according to their respective speeds. But what happens when the left (faster) soliton catches up with the right one?

Running numerical simulations with such setups, Kruskal and Zabusky [17] found that, after a complicated nonlinear interaction, the solitons emerge unscathed, so that there is a faster wave to the right of a slower one as $t \to +\infty$, with initial velocities and shapes preserved. In fact, one can show that this kind of behaviour translates to any linear combination of solitons, so that the number of solitons is preserved after interactions, with faster solitons ending up to the right of slower ones. It is this property of stability under interactions that characterizes localized traveling waves as solitons.

Integrability

The KdV equation turns out to be somewhat special among PDEs. The fact that one can find explicit solutions is not at all obvious just from looking at the equation – a small change in the order of derivatives in one of the terms might destroy the possibility of finding soliton solutions. This indicates that there is some property of the KdV equation which makes it particularly easy to work with, and this property is what we will call integrability.

In what follows we will simply think of a PDE as integrable if it has an infinite number of constants of motions and we can find explicit solutions to it. Note though, that there are a number of different ways of defining integrability, depending on the context. See [27] for various takes on the term.

We now want to study the constants of motion for the KdV equation. The first few can be found by trying to write the equation in the form

$$\frac{\partial}{\partial t}P(u) + \frac{\partial}{\partial x}Q(u) = 0.$$

In fact, if one is able to find such a relation, then (provided the integrals converge)

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(u)dx = -\frac{\partial}{\partial x}\int_{-\infty}^{\infty} Q(u)dx = 0.$$

It is easy to see that the KdV equation (1) can be written in this form, with $P(u) = u$, $Q(u) = u_{xx} + 3u^2$, which means that

$$\int_{-\infty}^{\infty} u\,dx$$

is a constant of motion. Furthermore, multiplying the KdV equation by $2u$ gives

$$2uu_t + 2uu_{xxx} + 12u^2 u_x = 0$$

which is equivalent to

$$\frac{\partial}{\partial t}\left(u^2\right) + \frac{\partial}{\partial x}\left(2uu_{xx} - u_x^2 + 4u^3\right) = 0,$$

so there is another constant of motion

$$\int_{-\infty}^{\infty} u^2\,dx.$$

The next expression,

$$\int_{-\infty}^{\infty}\left(\frac{u_x^2}{2} - u^3\right)dx,$$

is a bit more tricky to find, although one can easily check that its time derivative is equal to zero. The physical interpretations of these three constants are conservation of mass, momentum and energy, respectively.

The fact that KdV has infinitely many constants of motion is due to Miura, Gardner and Kruskal [23]. Define recursively a sequence $P_i(u)$ via

$$\begin{cases} P_1 &= u, \\ P_i &= -\frac{\partial}{\partial x}P_{i-1} + \sum_{k=1}^{i-1} P_k P_{i-k}, \quad i \geq 2. \end{cases} \tag{2}$$

Studying this sequence, one can show that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} P_i(u)\,dx = 0, \quad i \geq 1,$$

which gives an infinite number of constants of motion. Note that the even-numbered ones are trivial, in the sense that

$$P_{2j} = \frac{\partial}{\partial x}Q_j,$$

for some functions $Q_j(u)$ that tend to zero as $x \to \pm\infty$. Thus the even-numbered expressions only give the trivial constant of motion, equal to zero. The odd-numbered P_i will also contain some (but not all) terms that are integrated to zero. Removing those superfluous terms in P_1, P_3 and P_5 will lead to the three constants of motion found above.

Inverse scattering and Lax pairs

In their study of the KdV equation [10], Gardner, Greene, Kruskal and Miura found a method for transforming a PDE to the region of spectral (or scattering) data, where time evolution is trivial, which actually gives a method for solving the equation given initial conditions

$$u(x,0) = \Phi_0(x).$$

Lax [18] reformulated the problem as a condition of compatibility for two linear PDEs. Consider the system

$$\begin{cases} L\Phi = \lambda\Phi, \\ \Phi_t = B\Phi, \end{cases}$$

where L and B are some given differential operators, and λ is a fixed parameter. For this system to have solutions Φ, it is necessary that the time derivative of the first equation is consistent with the second equation. By that, we mean that the derivative

$$\frac{\partial}{\partial t}(L\Phi) = L_t\Phi + L\Phi_t = \lambda\Phi_t$$

is consistent with the second equation, which is the case only if

$$L_t\Phi + LB\Phi = BL\Phi.$$

This equation can be conveniently written as an operator equation

$$L_t = [B, L] \tag{3}$$

using the usual commutator notation. What Lax did was to find a pair of operators L and B, depending on a function u, such that the consistency condition (3) is equivalent to the KdV equation. In fact, with

$$\begin{aligned} L &= -\partial_x^2 - u, \\ B &= -4\partial_x^3 - 3u\partial_x - 3\partial_x u, \end{aligned}$$

one gets that L_t is just $-u_t$, while $[B, L]\Phi$ is equal to

$$(-4\partial_x^3 - 3u\partial_x - 3\partial_x u)(-\Phi_{xx} - u) - (-\partial_x^2 - u)(-4\Phi_{xxx} - 3u_x\Phi - 6u\Phi_x),$$

which after cancellation yields the expression

$$6uu_x\Phi + u_{xxx}\Phi.$$

Thus, the Lax equation with this choice for the operators B and L is equivalent to the KdV equation, as desired.

Let us now look at the problem from the other direction. Given any function u that solves the KdV equation, try to solve the eigenvalue problem

$$L\Phi = \lambda\Phi,$$

where one requires $\Phi(x)$ to be a nonzero function such that

$$\int |\Phi|\,dx < \infty.$$

With the operator $L = -\partial_x^2 - u$ taken from Lax's pair, this equation is known as the time-independent Schrödinger equation of quantum mechanics, and gives rise to a discrete spectrum of eigenvalues (see [25])

$$\lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_1 < 0,$$

with corresponding eigenfunctions $\Phi_n(x,t)$. The limiting behaviour of these eigenfunctions as $x \to \infty$ is

$$\Phi_n(x,t) \sim c_n(t) e^{-\sqrt{-\lambda_n} x},$$

where the time dependence (which can be found via the second equation of the Lax system) is simply given by

$$c_n(t) = c_n(0) e^{4(-\lambda_n)^{\frac{3}{2}} t}.$$

We can now describe (in somewhat vague terms) the steps that constitute the inverse scattering method for computing solutions to the KdV equation.

First, given $u(x,0) = \Phi_0(x)$, compute the scattering data at time 0. This data consists of the time independent spectrum $(\lambda_n)_{n=1}^N$, together with the parameters $(c_n(0))_{n=1}^N$. Since the time dependence is known, one immediately finds the scattering data (λ_n, c_n) at an arbitrary time $t > 0$. The third step, which is the hardest one, is known as the inverse scattering problem, where one must find $u(x,t)$ given scattering data at arbitrary time t. (It turns out that one more scattering variable is needed to uniquely reconstruct the function u, but the details are beyond the scope of this presentation.)

In general, the same kind of methods can be applied to any PDE which has an associated Lax pair. The difficult part is to find the inverse mapping from the spectral region back to the original function, and sometimes also finding enough spectral data to guarantee uniqueness.

Inverse scattering techniques are of particular importance in Paper II, where we consider solution formulas obtained through inverse scattering for the Novikov equation in [13].

Peakon equations

In Paper I and II, three more recently discovered partial differential equations are studied, all somewhat similar to the KdV equation in that they are integrable and admit soliton solutions. One important difference though, is that the soliton solutions to these equations turn out to have non-smooth peaks, which gives rise to the term 'peakons'. In this section we give a short historical overview describing the partial differential equations of interest to us, and what is known about peakon solutions to these equations.

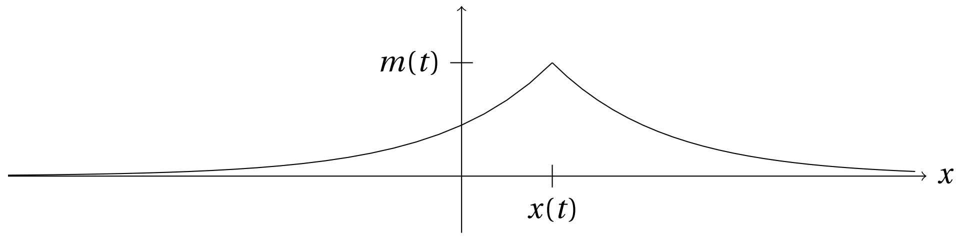

Figure 2: Peakon profile.

The Camassa–Holm b-family

The first object of interest is the following family of third order quadratically nonlinear PDEs,

$$u_t - u_{xxt} = -(b+1)uu_x + bu_x u_{xx} + uu_{xxx}, \quad (x,t) \in \mathbb{R}^2, \tag{4}$$

where b is a fixed parameter. Choosing $b = 2$ gives the dispersionless version of the much studied Camassa–Holm (CH) equation [4] which was first developed in 1993 as a new model of shallow water waves. If one instead chooses $b = 3$ in (4), it turns into the Degasperis–Procesi (DP) equation [9] from 1999.

Both equations are of interest in wave theory as they accomodate wave breaking, i.e., the slope of the wave profile may tend to infinity in finite time. It is interesting to note that the equations in the family are integrable only for $b = 2$ and $b = 3$, according to a number of integrability tests [9, 22, 14, 16].

Let us first study some properties of the Camassa–Holm equation, which appears as one of the main equations in Paper I. The original form of the equation includes a dispersion term,

$$u_t - u_{xxt} + \kappa u_x = -3uu_x + 2u_x u_{xx} + uu_{xxx},$$

with the constant κ related to the critical shallow water wave speed. Like for the KdV equation, soliton solutions can be shown to exist in this equation, but it is difficult to find explicit solution formulas. Camassa and Holm showed, that in the limit as $\kappa \to 0$, the solitons tend to to a very simple limiting shape, given by the function $e^{-|x|}$. Note though, that this shape is no longer smooth, as there is a peak where the left and right derivatives do not coincide. Camassa and Holm thus coined the term 'peakons', short for peaked solitons.

To study time dependent peakon solutions, consider the expression

$$u(x,t) = m(t)e^{-|x-x(t)|} = \begin{cases} m(t)e^{-x+x(t)}, & x \geq x(t), \\ m(t)e^{x-x(t)}, & x \leq x(t). \end{cases} \tag{5}$$

See Figure 2 for a picture of the wave profile. If $m(t) < 0$, the wave instead has a trough (downward pointing peak), and we call this an *antipeakon*.

One reason for the interest in peakon solutions is that they behave nicely under taking linear combinations, like we saw for the KdV solitons. Let us use the term *multi-*

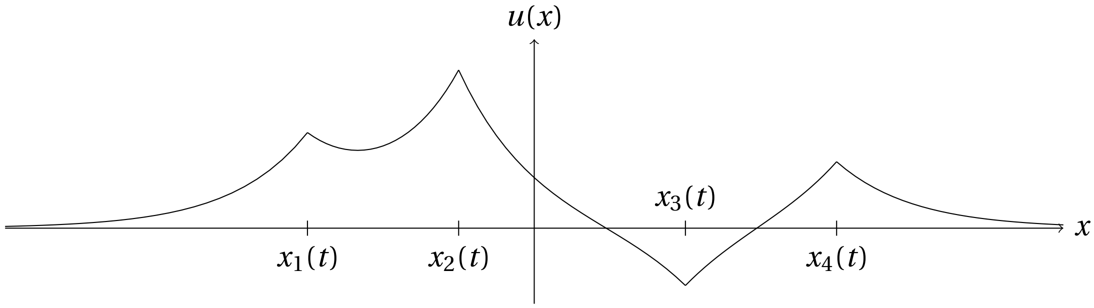

Figure 3: Multipeakon.

peakon to describe sums of peakons

$$u(x,t) = \sum_{k=1}^{n} m_k(t) e^{-|x-x_k(t)|}. \tag{6}$$

Sometimes one assumes $m_k > 0$, which is the so called pure peakon case, whereas all $m_k < 0$ corresponds to the pure antipeakon case. The remaining case, where not all m_k have the same sign, we call the mixed (peakon/antipeakon) case. See Figure 3 for an example of a multipeakon wave profile for a given time t.

Note that multipeakons cannot be solutions to the partial differential equations (4) in a strong sense, since they are non-differentiable. The problem lies in multiplying u_x with u_{xx}, since u_{xx} contains Dirac deltas exactly at the jump discontinuities of u_x. To get around this, one rewrites the b-family as

$$\left(1-\partial_x^2\right) u_t + \left(b+1-\partial_x^2\right) \partial_x \left(\frac{1}{2} u^2\right) + \partial_x \left(\frac{3-b}{2} u_x^2\right) = 0. \tag{7}$$

A function $u(x,t)$ is said to be a weak solution if $u(\cdot,t) \in W^{1,2}_{\text{loc}}(\mathbb{R})$ for each fixed t, which means that $u(\cdot,t)^2$ and $u_x(\cdot,t)^2$ are locally integrable functions, if $u_t(\cdot,t)$ defined as the limit of a difference quotient exists as a distribution, and if (7) is satisfied for all t in the distributional sense.

With the n-peakon ansatz (6), our PDEs are easily seen to be satisfied on the intervals where the multipeakon is differentiable, since each exponential function is a solution. Studying what goes on at the locations of each peak, one finds that the PDEs simplify into a system of $2n$ ODEs in the variables (x_k, m_k), which denote the position and height respectively of peakon k. For the b-family, this system is

$$\begin{cases} \dot{x}_k = \sum_{i=1}^{n} m_i e^{-|x_k - x_i|}, \\ \dot{m}_k = (b-1) m_k \sum_{i=1}^{n} m_i \operatorname{sgn}(x_k - x_i)\, e^{-|x_k - x_i|}, \end{cases} \tag{8}$$

where we use the convention that sgn 0 = 0.

For $n = 1$, the system reduces to

$$\begin{cases} \dot{x}_1 = m_1, \\ \dot{m}_1 = 0, \end{cases} \tag{9}$$

which means that the peakon $u(x,t) = m_1 e^{-|x-m_1 t|}$ clearly is a travelling wave solution, maintaining its shape and height, travelling with constant speed equal to its height. Note that in this case, antipeakons move to the left while peakons move to the right.

For $n > 1$, the interaction between peakons makes the ODE systems considerably more complicated. The system (8) was solved in the pure peakon sector using inverse scattering techniques for Camassa–Holm in [1], and the mixed case was solved in [2].

It turns out that in the pure peakon sector, there are no collisions amongst peakons, i.e., the coordinates $x_1(t) < x_2(t) < \cdots < x_n(t)$ remain separated for all times. This is not necessarily true in the mixed case, where collisions may occur, causing some m_k to tend to infinity. Thus one has to be careful with the meaning of continuing a solution beyond a collision. In [12] it was shown how to obtain global multipeakon solutions of the Camassa–Holm equation, by introducing a new system of ODEs which is well-posed even at collisions. See also [3] for how to resolve singularities for more general kinds of solutions of the Camassa–Holm equation.

Finally, a word on the Degasperis–Procesi equation. Most of what we wrote about the Camassa–Holm equation carries through to the DP case. It is worth noting though that the situation is slightly different when it comes to weak solutions. Functions in $W^{1,p}_{\text{loc}}(\mathbb{R})$ are continuous by the Sobolev embedding theorem, but if one puts $b = 3$ in (7), the term u_x^2 disappears, so one only has to require that $u(\cdot,t) \in L^2_{\text{loc}}(\mathbb{R})$. Thus the Degasperis–Procesi equation admits solutions that are not continuous, see for example [19, 5, 6], while CH does not.

The DP peakon ODEs where solved in [20] in the pure peakon case, again with inverse scattering, and [26] dealt with the mixed peakon-antipeakon case.

The Novikov equation

Most prevalent in the thesis is the Novikov equation [24, 15]

$$u_t - u_{xxt} = -4u^2 u_x + 3u u_x u_{xx} + u^2 u_{xxx}, \tag{10}$$

which appears in both papers. This equation has a similar structure to CH and DP, but instead has cubic nonlinearities in the right hand side.

We mention here also the Geng–Xue (GX) system [11]

$$\begin{cases} u_{xxt} - u_t = (u_x - u_{xxx})uv + 3(u - u_{xx})v u_x, \\ v_{xxt} - v_t = (v_x - v_{xxx})uv + 3(v - v_{xx})u v_x, \end{cases} \tag{11}$$

which is studied briefly in the Appendix of Paper I. This system can be thought of as a two-component generalization of the Novikov equation, as the system reduces to two copies of (10) when replacing $u = v$.

To define weak solutions of the Novikov equation we write (10) as

$$\left(1-\partial_x^2\right)u_t + \left(4-\partial_x^2\right)\partial_x\left(\frac{1}{3}u^3\right) + \partial_x\left(\frac{3}{2}u u_x^2\right) + \frac{1}{2}u_x^3 = 0. \tag{12}$$

We require that $u(\cdot,t) \in W^{1,3}_{\text{loc}}(\mathbb{R})$ for all t, so that u^3 and u_x^3 are locally integrable. It then follows from Hölder's inequality with conjugate indices 3 and $\frac{3}{2}$ that the term uu_x^2 is locally integrable as well. We also require that the time derivative u_t exists as a distribution defined as the limit of a difference quotient for almost every t. Then it makes sense to call u a weak solution to the Novikov equation if the left hand side of (12) gives zero when acting on a test function, for all t where u_t exists.

The Novikov equation is of interest in this thesis, since it admits peakon solutions too. By using the multipeakon ansatz as before, a similar system to (8) can be constructed for peakon dynamics in the Novikov equation,

$$\begin{cases} \dot{x}_k = \left(\sum_{i=1}^n m_i e^{-|x_k-x_i|}\right)^2, \\ \dot{m}_k = m_k \left(\sum_{i=1}^n m_i e^{-|x_k-x_i|}\right)\left(\sum_{i=1}^n m_i \operatorname{sgn}(x_k-x_i)\, e^{-|x_k-x_i|}\right). \end{cases} \tag{13}$$

Note that here, a single peakon travels with constant speed equal to the *square* of its height, which is different from the previous equations, in that both peakons and antipeakons move to the right in the Novikov equation. Pure multipeakon solutions to Novikov and GX were studied in [13] and [21] respectively, while the peakon-antipeakon interactions are the topic of Paper II.

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Paper I

New solutions with peakon creation in the Camassa–Holm and Novikov equations

M. KARDELL

ABSTRACT. In this article we study a new kind of unbounded solutions to the Novikov equation, found via a Lie symmetry analysis. These solutions exhibit peakon creation, i.e., these solutions are smooth up until a certain finite time, at which a peak is created. We show that the functions are still weak solutions for those times where the peak lives. We also find similar unbounded solutions with peakon creation in the related Camassa–Holm equation, by making an ansatz inspired by the Novikov solutions. Finally, we see that the same ansatz for the Degasperis–Procesi equation yields unbounded solutions where a peakon is present for all times.

1 Introduction

In 1993, Camassa and Holm [3] discovered an integrable partial differential equation within the context of shallow water theory, an equation which has since been studied quite extensively. One reason for the interest in this equation is that it allows (weak) explicit solutions in the form of so called multipeakons. More recent equations with similar properties include the Degasperis–Procesi [7] and the Novikov [13] equations. See [6] for a discussion of the role of the Camassa–Holm and Degasperis–Procesi equations in hydrodynamics.

The results of this article originated from a Lie symmetry analysis of the Novikov equation. This framework gives a complete list of transformations such that each solution of the equation is mapped to another solution. In the resulting list of transformations, there are two nontrivial transformations which we use to produce new solutions to the Novikov equation.

In fact, applying the new transformations found in this article to the Novikov one-peakon solution gives an unbounded solution displaying quite interesting behaviour. (Though the peakon is a weak solution, it is piecewise smooth, so the transformation makes sense locally away from the peak.) We find that this new solution depends smoothly on x for some interval in time, and has peakon creation (or destruction, depending on the transformation) at some finite time t. We also show that these functions are still weak solutions for those times for which the peak lives.

By making an ansatz inspired by the Novikov solutions with peakon creation, we also find such solutions to the Camassa–Holm equation. It is interesting to note that, apparently, these solutions cannot be found using Camassa–Holm symmetries. Another thing to note is that the same ansatz does not give peakon creation in the closely related Degasperis–Procesi equation, instead we find a kind of unbounded peakon solution where the peak lives for all times.

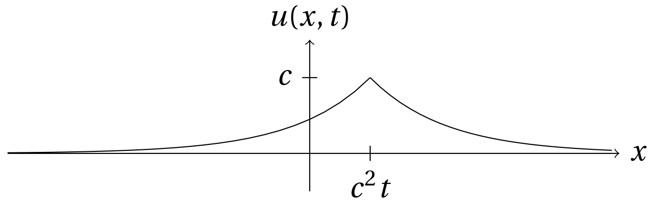

Figure 4: One-peakon solution

2 Novikov Solutions with Peakon Creation

The Novikov equation, given by

$$u_t - u_{xxt} = -4u^2 u_x + 3u u_x u_{xx} + u^2 u_{xxx}, \tag{2.1}$$

admits multi-peakon solutions

$$u(x,t) = \sum_{i=1}^{n} m_i(t) e^{-|x - x_i(t)|} \tag{2.2}$$

in a weak sense. The word peakon is short for 'peaked soliton', where peaked means that there is some point where the left and right derivatives do not coincide. The peakons interact in quite a complicated way; see [9] for explicit time dependence of the functions $\{x_i(t), m_i(t)\}$ and a weak formulation of the problem.

Consider the one-peakon solution $u(x,t) = ce^{-|x-c^2t|}$. This is a peakon traveling to the right, with constant speed equal to the square of the height of the peakon (which differs from Camassa–Holm and Degasperis–Procesi peakons, where the speed is just equal to the height). For fixed t, the peakon looks as in Figure 4. Peakons are important due to the fact that the travelling waves of greatest height of the governing equations for water waves (incompressible homogeneous Euler equations with a free boundary) present a peak at their crest, cf. [4, 5].

In the Appendix, Theorem A.3, we compute the Lie symmetries of the Novikov equation. These correspond to transformations that take known (strong) solutions of the equation to other solutions. We repeat here the result for convenience.

Theorem 2.1. *If $u = f(x,t)$ solves the Novikov equation (2.1), then so do*

$$\begin{aligned} u_1 &= f(x-\varepsilon, t), \\ u_2 &= f(x, t-\varepsilon), \\ u_3 &= e^{\varepsilon/2} f(x, te^{\varepsilon}), \\ u_4 &= \sqrt{1+2\varepsilon e^{2x}}\, f\left(-\frac{1}{2}\ln\left(e^{-2x}+2\varepsilon\right), t\right), \\ u_5 &= \sqrt{1+2\varepsilon e^{-2x}}\, f\left(\frac{1}{2}\ln\left(e^{2x}+2\varepsilon\right), t\right). \end{aligned}$$

In this section we study the functions that one gets by transforming the one-peakon solution. Note though, that the one-peakon is not a smooth solution, so we can not say *a priori* whether this approach gives valid weak solutions of the Novikov equation, this has to be checked. Applying the first three transformations gives us translations and scaling of a peakon, hence no essentially new solutions come up. The fourth and fifth tranformations are more interesting. They give the functions

$$u_4(x,t) = c\sqrt{1+2\varepsilon e^{2x}}\,e^{-\left|\frac{1}{2}\ln(e^{-2x}+2\varepsilon)+c^2t\right|}, \tag{2.3a}$$

$$u_5(x,t) = c\sqrt{1+2\varepsilon e^{-2x}}\,e^{-\left|\frac{1}{2}\ln(e^{2x}+2\varepsilon)-c^2t\right|}. \tag{2.3b}$$

Note that these solutions do not tend to zero as $|x| \to \infty$. Let us first study the function $u_5(x,t)$.

Theorem 2.2. *The transformed Novikov peakon* $u_5(x,t) = c\sqrt{1+2\varepsilon e^{-2x}}\,e^{-\left|\frac{1}{2}\ln(e^{2x}+2\varepsilon)-c^2t\right|}$ *is a smooth solution to the Novikov equation up until* $t_0 = \frac{1}{2c^2}\ln(2\varepsilon)$*, when a peak is created at* $x = -\infty$*. After time* t_0*, the function is still a weak solution.*

Proof. Let us examine the expression inside the modulus signs in u_5. This expression is increasing in x, and has the only root $x = \frac{1}{2}\ln(e^{2c^2t} - 2\varepsilon)$. Thus, there can exist a value of x for which the expression changes sign, but only when $t > t_0 := \frac{1}{2c^2}\ln(2\varepsilon)$. Before time t_0, the function u_5 is smooth, and is thus a solution of the Novikov equation in the usual sense. At the time t_0 a peak is created at $x = -\infty$, so that for each time $t > t_0$ there exists a point where the left and right derivatives are unequal. After the creation, the peak moves in rapidly from the left.

More concretely, for $t \le t_0$, the expression (2.3b) simplifies significantly, since

$$u_5(x,t) = c\sqrt{1+2\varepsilon e^{-2x}}\,e^{-\frac{1}{2}\ln(e^{2x}+2\varepsilon)+c^2t} = c\frac{\sqrt{1+2\varepsilon e^{-2x}}}{\sqrt{e^{2x}+2\varepsilon}}e^{c^2t} = ce^{-x+c^2t}.$$

For $t > t_0$, one can simplify in a similar manner, depending on whether one is to the left or to the right of the peak at $B(t) := \frac{1}{2}\ln(e^{2c^2t} - 2\varepsilon)$, yielding

$$u_5(x,t) = \begin{cases} ce^{-x+c^2t}, & x \ge B(t) \\ c(e^x + 2\varepsilon e^{-x})e^{-c^2t}. & x \le B(t) \end{cases} \tag{2.4}$$

To check that a function is still a weak solution after time t_0, in the sense of [9], one needs to show that

$$\langle \left(1-\partial_x^2\right)u_t + \left(4-\partial_x^2\right)\partial_x\left(\frac{1}{3}u^3\right) + \partial_x\left(\frac{3}{2}uu_x^2\right) + \frac{1}{2}u_x^3, \varphi\rangle = 0, \quad \forall \varphi(x) \in C_0^\infty,$$

where $\langle\cdot,\cdot\rangle$ means action on test functions in the usual sense. Using the definition of distributional derivatives, one gets

$$\langle u_t, \left(1-\partial_x^2\right)\varphi\rangle + \langle \frac{1}{3}u^3, \partial_x\left(\partial_x^2-4\right)\varphi\rangle + \langle \frac{3}{2}uu_x^2, -\partial_x\varphi\rangle + \langle \frac{1}{2}u_x^3, \varphi\rangle = 0. \tag{2.5}$$

Let u^+ and u^- be the expressions of (2.4) to the right and left of the peak, respectively. Note that $u_5(x,t)$ is continuous at all points, with u_x and u_t piecewise continuous functions, so the lefthand side in (2.5) equals

$$\int_B^\infty u_t^+ \left(\varphi - \varphi_{xx}\right) dx + \int_{-\infty}^B u_t^- \left(\varphi - \varphi_{xx}\right) dx + \int_B^\infty \frac{1}{3} \left(u^+\right)^3 \left(\varphi_{xxx} - 4\varphi_x\right) dx +$$
$$+ \int_{-\infty}^B \frac{1}{3} \left(u^-\right)^3 \left(\varphi_{xxx} - 4\varphi_x\right) dx + \int_B^\infty \frac{3}{2} u^+ \left(u_x^+\right)^2 (-\varphi_x)\, dx + \int_{-\infty}^B \frac{3}{2} u^- \left(u_x^-\right)^2 (-\varphi_x)\, dx +$$
$$+ \int_B^\infty \frac{1}{2} \left(u_x^+\right)^3 \varphi\, dx + \int_{-\infty}^B \frac{1}{2} \left(u_x^-\right)^3 \varphi\, dx.$$

Using integration by parts to move the derivatives back to u, we get two kinds of terms. First we again get integrals, which combine to zero since u is a strong solution of the Novikov equation on each interval. The boundary values at infinity are all zero, since we integrate against a test function with compact support, but we also get boundary values at B:

$$U_1(B)\varphi(B) + U_2(B)\varphi_x(B) + U_3(B)\varphi_{xx}(B), \tag{2.6}$$

where we use the shorthand notation $f(B) = f(B(t), t)$, and

$$\begin{aligned}
U_1(B) :=& \left(u_t^-\right)_x (B) - \left(u_t^+\right)_x (B) + \frac{1}{3} \left((u^-)^3\right)_{xx} (B) - \frac{1}{3} \left(\left(u^+\right)^3\right)_{xx} (B) \\
&+ \frac{3}{2} u^+(B)(u_x^+(B))^2 - \frac{3}{2} u^-(B)(u_x^-(B))^2 + \frac{4}{3}(u^+)^3(B) - \frac{4}{3}(u^-)^3(B), \\
U_2(B) :=& u_t^+(B) - u_t^-(B) + \frac{1}{3} \left(\left(u^+\right)^3\right)_x (B) - \frac{1}{3} \left((u^-)^3\right)_x (B), \\
U_3(B) :=& \frac{1}{3} \left((u^-)^3\right)(B) - \frac{1}{3} \left(\left(u^+\right)^3\right)(B).
\end{aligned}$$

The continuity of u_5 gives $u^+(B) = u^-(B)$ which means that $U_3(B)$ is zero. It is not obvious, but easy to check with computer, that $U_1(B)$ and $U_2(B)$ are also zero. For example,

$$\left(u_t + \frac{1}{3}\left(u^3\right)_x\right)(B) = \frac{-2\varepsilon c^3 e^{c^2 t}}{\left(e^{2c^2 t} - 2\varepsilon\right)^{\frac{3}{2}}} \tag{2.7}$$

for both u^+ and u^-, showing that $U_2(B) = 0$. □

Note that as the peak moves in from the left, it is not actually a local maximum from the start (so it might be more accurate to call it a corner), as we can see from Figure 5. As time increases the corner really turns into a peak, indicated in Figure 6. The peak becomes increasingly separated from the large wave to the left, and one can see from the expression for $B(t)$ that, asymptotically, the peak moves to the right with constant speed $c^2 t$ like a one-peakon solution, unaffected by the wavefront. Figure 7 shows how the peak moves in space-time.

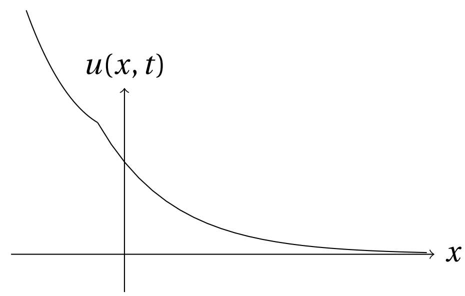

Figure 5: Wave profile of u_5, shortly after the time of creation

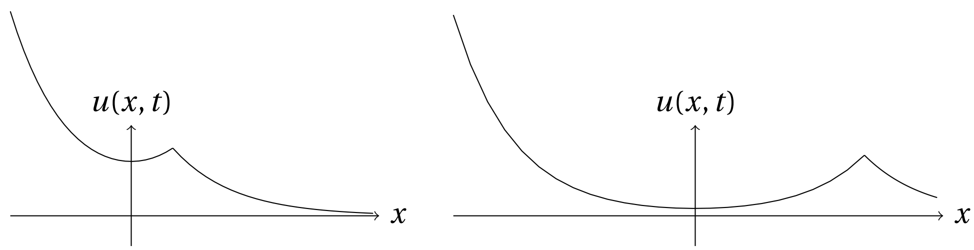

Figure 6: Wave profile of u_5, snapshots at two different later times

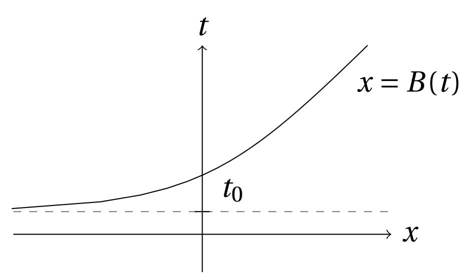

Figure 7: Movement of the peak in space-time

Let us also briefly consider the function $u_4(x,t)$. By modifying the argument above, one gets that this function also has a peak, but *before* a certain (finite) time, at which the position of the peak goes to $+\infty$. One can also check that u_4 is a weak solution until the peak is destroyed, after which it is a regular solution to the Novikov equation.

Finally, let us mention what happens if one combines the transformations above. Applying transformation 5 with parameter ε, followed by transformation 4 with parameter δ, gives the following function:

$$\tilde{u} = c\sqrt{1+2\delta e^{2x}}\sqrt{1+2\varepsilon(e^{-2x}+2\delta)}\,e^{-\left|\frac{1}{2}\ln\left(\frac{1}{e^{-2x}+2\delta}+2\varepsilon\right)-c^2 t\right|}.$$

It turns out that this function has a peak that is both created and destroyed in finite time. The precise interval for which the peak lives is

$$t \in \left(\frac{1}{2c^2}\ln(2\varepsilon), \frac{1}{2c^2}\ln\left(2\varepsilon+\frac{1}{2\delta}\right)\right).$$

Outside this interval, $\tilde{u}$ is a smooth function of x, and thus a regular solution as before. To find a function for which the peak lives between given times t_1 and t_2, choose

$$\begin{cases} \varepsilon = \frac{1}{2}e^{2c^2 t_1}, \\ \delta = \frac{1}{2\left(e^{2c^2 t_2} - e^{2c^2 t_1}\right)}, \end{cases} \qquad t_1 < t_2.$$

3 Peakon Creation in Related Equations

Finding unbounded solutions with peakon creation in the Novikov equation inspires us to look for solutions with similar behaviour in the related Camassa–Holm and Degasperis–Procesi equations.

3.1 Camassa–Holm solutions with peakon creation

The Camassa–Holm equation (CH), from [3], is given by

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{3.1}$$

It is known from [2] that the CH symmetry group only consists of translations and scalings. This means that we cannot find solutions with peakon creation just by transforming the one-peakon solution. Still, it turns out that there are solutions with peakon creation, that one can find via an ansatz inspired by the Novikov solutions found in the previous section.

Theorem 3.1. *The function $u(x,t)$ defined by*

$$u(x,t) = \begin{cases} u^+ = a(t)e^{-x}, & x \ge B(t), \\ u^- = c(t)(e^x + e^{-x}), & x \le B(t), \end{cases} \quad \textit{for } t > t_0,$$

and $u(x,t) = a(t)e^{-x}$ *for* $t \le t_0$*, is a solution to the Camassa–Holm equation, with*

$$\begin{aligned} a(t) &= U\cosh[U(t-t_0)], \\ B(t) &= \ln(\sinh[U(t-t_0)]), \\ c(t) &= \frac{U}{\cosh[U(t-t_0)]}. \end{aligned}$$

Proof. We look for weak solutions of the kind

$$u(x,t) = \begin{cases} u^+ = a(t)e^{-x}, & x \ge B(t), \\ u^- = c(t)(e^x + e^{-x}), & x \le B(t), \end{cases} \tag{3.2}$$

where $a(t)$ and $c(t)$ are positive continuous functions, chosen in such a way that u is continuous at the peak $B(t)$ for all times. From the weak formulation of the Camassa–Holm equation found in [9], one has that u must satisfy

$$\langle \left(1-\partial_x^2\right) u_t + \left(3-\partial_x^2\right)\partial_x \left(\frac{1}{2}u^2\right) + \partial_x \left(\frac{1}{2}u_x^2\right), \varphi \rangle = 0 \tag{3.3}$$

for all test functions $\varphi(x) \in C_0^\infty$. Note that the function $u(x,t)$ is a strong solution of (3.1) on each interval. Thus integration by parts, as in the previous section, gives that

$$U_1(B)\varphi(B) + U_2(B)\varphi_x(B) + U_3(B)\varphi_{xx}(B) = 0$$

must be satisfied, where

$$\begin{aligned} U_1(B) &:= \left(u_t^-\right)_x(B) - \left(u_t^+\right)_x(B) + \frac{1}{2}\left((u^-)^2\right)_{xx}(B) - \frac{1}{2}\left(\left(u^+\right)^2\right)_{xx}(B) + \\ &\quad + \frac{1}{2}\left(u_x^+(B)\right)^2 - \frac{1}{2}\left(u_x^-(B)\right)^2 + \frac{3}{2}\left(\left(u^+\right)^2\right)(B) - \frac{3}{2}\left((u^-)^2\right)(B), \end{aligned} \tag{3.4a}$$

$$U_2(B) := u_t^+(B) - u_t^-(B) + \frac{1}{2}\left(\left(u^+\right)^2\right)_x(B) - \frac{1}{2}\left((u^-)^2\right)_x(B), \tag{3.4b}$$

$$U_3(B) := \frac{1}{2}\left((u^-)^2\right)(B) - \frac{1}{2}\left(\left(u^+\right)^2\right)(B). \tag{3.4c}$$

The condition (3.4c) = 0 is met because of continuity. Using continuity, we can also express $a(t)$ in terms of B and c, since

$$c\left(e^{-B} + e^B\right) = ae^{-B} \implies a = c\left(1+e^{2B}\right) \implies \frac{da}{dt} = \frac{dc}{dt}\left(1+e^{2B}\right) + 2\frac{dB}{dt}ce^{2B}.$$

Eliminating a and its time derivative in the conditions (3.4a) = (3.4b) = 0 gives the system

$$\frac{d}{dt}\left(ce^B\right) = c^2, \tag{3.5a}$$

$$\frac{dB}{dt} = c\left(e^B + e^{-B}\right). \tag{3.5b}$$

These conditions are simplified by a change of variables,

$$\begin{cases} G(t) = c(t)e^{B(t)}, \\ K(t) = \frac{1}{c^2(t)}, \end{cases} \implies \begin{cases} \frac{dG}{dt} = c^2 = \frac{1}{K}, \\ \frac{dK}{dt} = \frac{-2}{c^3}\frac{dc}{dt} = 2KG, \end{cases}$$

where the last line follows from the observation that

$$\frac{dc}{dt} = \frac{d}{dt}\left(\frac{G}{e^B}\right) = \frac{c^2}{e^B} - \frac{Gc\left(e^B + e^{-B}\right)e^B}{e^{2B}} = -c^2 e^B = -cG.$$

One can now get a separable differential equation and find a constant of motion:

$$\frac{dK}{dG} = \frac{\frac{dK}{dt}}{\frac{dG}{dt}} = 2K^2 G \implies \int \frac{dK}{K^2} = \int 2G\,dG \implies -\frac{1}{K} = G^2 + \text{constant}.$$

Apart from the trivial solution $a(t) = c(t) = 0$, G and $\frac{1}{K}$ are positive, so the constant has to be negative. Let the constant be named $-U^2$ for convenience. Then

$$\frac{dG}{dt} = \frac{1}{K} = U^2 - G^2 \implies \int \frac{dG}{U^2 - G^2} = \int dt$$

$$\implies \frac{1}{2U}\int\left(\frac{1}{U+G} + \frac{1}{U-G}\right)dU = \int dt \implies \frac{1}{2U}\ln\left(\frac{U+G}{U-G}\right) = t - t_0$$

$$\implies G = U\frac{e^{2U(t-t_0)} - 1}{e^{2U(t-t_0)} + 1} = U\tanh[U(t-t_0)].$$

From this one gets K as

$$K = \frac{1}{U^2 - G^2} = \frac{1}{U^2}\cdot\frac{1}{1 - \tanh^2[U(t-t_0)]} = \frac{\cosh^2[U(t-t_0)]}{U^2},$$

which gives expressions for $c(t)$, $B(t)$, and consequently $a(t)$:

$$c(t) = \frac{1}{\sqrt{K}} = \frac{U}{\cosh[U(t-t_0)]},$$

$$B(t) = \ln\left(G\sqrt{K}\right) = \ln(\sinh[U(t-t_0)]),$$

$$a(t) = c(t)\left(1 + e^{2B(t)}\right) = \frac{U}{\cosh[U(t-t_0)]}\left(1 + \sinh^2[U(t-t_0)]\right) = U\cosh[U(t-t_0)].$$

□

We note that our new solution behaves similarly to the Novikov solution with peakon creation in Theorem 2.2. Up to time t_0, the expression for $B(t)$ is undefined, so the function is a strong solution to the Camassa–Holm equation. At t_0 a peak is created at $x = -\infty$, which then moves rapidly in from the left. Note that the exact time dependencies are not the same as for the Novikov peakon-creation solution, even though the qualitative behaviour is the same.

3.2 Degasperis–Procesi solutions with peakon creation?

The Degasperis–Procesi (DP) equation [7] is given by

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{3.6}$$

Like Camassa–Holm, it only has scaling and translation symmetries [15], so we try to find peakon-creation solutions using the same method as in the last section.

Theorem 3.2. *For every $t \in \mathbb{R}$, the function*

$$u(x,t) = \begin{cases} u^+ = a(t)e^{-x}, & x \geq B(t), \\ u^- = c(t)(e^x + e^{-x}), & x \leq B(t), \end{cases}$$

where

$$a(t) = \sqrt{\frac{C_1}{C_0}} \left(\frac{1 + C_0 C_1 e^{2Ut}}{e^{Ut} + \frac{e^{-Ut}}{UC_0}} \right),$$
$$B(t) = \ln\sqrt{C_0 C_1} + Ut,$$
$$c(t) = \sqrt{\frac{C_1}{C_0}} \frac{1}{e^{Ut} + \frac{e^{-Ut}}{UC_0}},$$

is a solution to the Degasperis–Procesi equation.

Proof. We look for weak solutions

$$u(x,t) = \begin{cases} a(t)e^{-x}, & x \geq B(t), \\ c(t)\left(e^x + e^{-x}\right), & x \leq B(t), \end{cases}$$

where $a(t)$ and $c(t)$ are positive continuous functions, such that u is continuous at the peak $B(t)$ for all times. We stick to the weak formulation given in [9], i.e., $u(x,t)$ must satisfy

$$\langle \left(1 - \partial_x^2\right) u_t + \left(4 - \partial_x^2\right) \partial_x \left(\frac{1}{2}u^2\right), \varphi \rangle = 0.$$

As before, u_t is piecewise continuous, so via integration by parts we find three conditions on u^+ and u^- at the peak, one of which is satisfied because of continuity. Eliminating $a(t)$, we end up with a system similar to (3.5), but not the same:

$$\frac{d}{dt}\left(ce^B\right) = 2c^2,$$
$$\frac{dB}{dt} = c\left(e^B + e^{-B}\right).$$

With $G(t)=c(t)e^{B(t)}$, $K(t)=\frac{e^{B(t)}}{c(t)}$, we get

$$\frac{dG}{dt}=\frac{2G}{K},$$
$$\frac{dK}{dt}=2GK.$$

Using the same method as before, we find a relation between K and G:

$$\frac{dK}{dG}=\frac{\frac{dK}{dt}}{\frac{dG}{dt}}=K^2 \implies \int \frac{dK}{K^2}=\int dG \implies -\frac{1}{K}=G+\text{constant}.$$

Let the constant be named $-U$. Since G and $\frac{1}{K}$ are nonnegative, $U=0$ only gives the trivial solution $a(t)=c(t)=0$. Assume $U\neq 0$. Then

$$\frac{dK}{dt}=2GK=2K\left(U-\frac{1}{K}\right) \implies \frac{dK}{dt}-2KU=-2,$$

which has the general solution

$$K=C_0e^{2Ut}+\frac{1}{U}.$$

This gives $G(t)$ via

$$\frac{dG}{dt}=\frac{2G}{K}=\frac{2G}{C_0e^{2Ut}+\frac{1}{U}} \implies G=\frac{C_1}{\frac{e^{-2Ut}}{UC_0}+1},$$

so we get

$$e^{B(t)}=\sqrt{GK}=\sqrt{C_1}\sqrt{\frac{C_0e^{2Ut}+\frac{1}{U}}{\frac{e^{-2Ut}}{UC_0}+1}}=\sqrt{C_0C_1e^{2Ut}}=\sqrt{C_0C_1}e^{Ut}$$
$$\implies B(t)=\ln\sqrt{C_0C_1}+Ut,$$

and

$$c(t)=\sqrt{\frac{G}{K}}=\sqrt{\frac{C_1}{\left(\frac{e^{-2Ut}}{UC_0}+1\right)\left(C_0e^{2Ut}+\frac{1}{U}\right)}}=\sqrt{\frac{C_1}{C_0e^{2Ut}\left(1+\frac{e^{-2Ut}}{UC_0}\right)^2}}=\sqrt{\frac{C_1}{C_0}}\frac{1}{e^{Ut}+\frac{e^{-Ut}}{UC_0}}.$$

This gives

$$a(t)=c(t)\left(1+e^{2B(t)}\right)=\sqrt{\frac{C_1}{C_0}}\left(\frac{1+C_0C_1e^{2Ut}}{e^{Ut}+\frac{e^{-Ut}}{UC_0}}\right).$$

□

Note that $B(t)$ here is defined for all times, so there is no peakon creation in this solution. We have found an unbounded piece-wise defined solution though. It is possible that a more general ansatz yields a solution with peakon creation in the DP case. It would also be interesting to investigate if one can find a solution with creation of so-called shockpeakons [11].

A Lie Symmetries

In this appendix we use the framework of symmetry groups, due to Lie, to construct transformations taking solutions of the Novikov equation (2.1) to other solutions. Similar results have been presented for the related Camassa–Holm equation in [2] and more recently for the Degasperis–Procesi equation in [15]. Note that computation of symmetry groups is quite cumbersome, so to find them explicitly, the Jets package in Maple is used. For more information on the Jets algorithm and how to use the package, see [12] and [1] respectively.

A.1 Definitions

Herein we will mainly use the notation employed in Olver's book [14], which also contains all details and proofs omitted in this section.

Let $X = \{\bar{x} = \left(x^1, \dots, x^p\right)\}$ and $U = \{\bar{u} = \left(u^1, \dots, u^q\right)\}$ be the spaces of independent and dependent variables, respectively, involved in a system of differential equations. The *n-th prolongation* of a scalar function u is defined as a tuple, denoted $u^{(n)}$, containing u and all its derivatives up to order n, where derivatives are arranged by order and then lexicographically. For example, with independent variables $x^1 = x, x^2 = t$ one gets $u^{(2)} = (u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$. Furthermore, we define for vector-valued functions

$$\bar{u}^{(n)} = \left((u^1)^{(n)}, \dots, (u^q)^{(n)}\right),$$

and set $U^{(n)} = \{\bar{u}^{(n)} \mid \bar{u} \in U\}$.

An n-th order system of differential equations can then be given as

$$\Delta_r\left(\bar{x}, \bar{u}^{(n)}\right) = 0, \quad r = 1, \dots, l, \tag{A.1}$$

where the system has *maximal rank* if the Jacobian $J_\Delta\left(\bar{x}, \bar{u}^{(n)}\right)$ has rank l for all points $\left(\bar{x}, \bar{u}^{(n)}\right)$ that are solutions to the system.

If G is a local group of transformations on $M \subset X \times U$ and $g \in G$, one defines the prolonged action $g^{(n)}$ on a point $\left(\bar{x}, \bar{u}^{(n)}\right) \in M^{(n)} \subset X \times U^{(n)}$ as transforming $\bar{x}$ and $\bar{u}$, and then re-evaluating derivatives. What we are looking for are *symmetry groups*, i.e., local groups of transformations on M such that their prolongations take solutions of the system (A.1) to other solutions.

To a one-parameter group G there corresponds an *infinitesimal generator* $\mathbf{v}$, which is a vector field defined on M, with the property that orbits of the group action are maximal integral curves of $\mathbf{v}$. Similarly, to an m-parameter group there corresponds a set of m infinitesimal generators $\mathbf{v}_1, \dots, \mathbf{v}_m$, which has the property that it is closed under taking Lie bracket, and that each infinitesimal generator corresponds to a generator of the group G.

We define the prolongation of an infinitesimal generator $\mathbf{v}$ of a group G to be the vector field $\mathbf{v}^{(n)}$, defined on $M^{(n)}$, which is the infinitesimal generator of the group $G^{(n)} := \{g^{(n)} | g \in G\}$. We want to give a formula for computing $\mathbf{v}^{(n)}$.

Let J be a multi-index of the form

$$J = (j_1, \ldots, j_k), \quad 1 \le j_k \le p, \quad 1 \le k \le n,$$

where p is the number of independent variables. Then one can introduce a compact notation for derivatives as

$$u_j^\alpha = \frac{\partial u^\alpha}{\partial x_j} \quad \text{and} \quad u_J^\alpha = \frac{\partial^k u^\alpha}{\partial x_{j_1} \cdots \partial x_{j_k}},$$

and we shall also use the notation

$$D_j \varphi(\bar{x}, \bar{u}) = \frac{\partial \varphi}{\partial x^j} + \sum_{\alpha=1}^{q} u_j^\alpha \frac{\partial \varphi}{\partial u^\alpha}$$

for total derivatives, and $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$ for multi-indices J.

The following theorem (Theorem 2.36 in [14]) gives the general formula for $\mathbf{v}^{(n)}$:

Theorem A.1. *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(\bar{x}, \bar{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(\bar{x}, \bar{u}) \frac{\partial}{\partial u^\alpha} \tag{A.2}$$

be a vector field on $M \subset X \times U$. Then

$$\mathbf{v}^{(n)} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \varphi_\alpha^J \left(\bar{x}, \bar{u}^{(n)}\right) \frac{\partial}{\partial u_J^\alpha}, \tag{A.3}$$

where the second sum is over all multi-indices J, and φ_α^J is given by

$$\varphi_\alpha^J \left(\bar{x}, \bar{u}^{(n)}\right) = D_J \left(\varphi_\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i} \right) + \sum_{i=1}^{p} \xi^i \frac{\partial u_J^\alpha}{\partial x^i}. \tag{A.4}$$

The next theorem (Theorem 2.31 in [14]) is the main tool for finding symmetry groups:

Theorem A.2. *Suppose*

$$\Delta_r \left(\bar{x}, \bar{u}^{(n)}\right) = 0, \quad r = 1, \ldots, l,$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, with infinitesimal generator $\mathbf{v}$*, and*

$$\mathbf{v}^{(n)} \left(\Delta_r \left(\bar{x}, \bar{u}^{(n)}\right)\right) = 0, \quad r = 1, \ldots, l, \quad \text{whenever} \quad \Delta\left(\bar{x}, \bar{u}^{(n)}\right) = 0,$$

then G is a symmetry group of the system.

Thus, the method for finding symmetry groups is to make the ansatz (A.2) for $\mathbf{v}$, prolong it using expressions (A.3) and (A.4), apply it as a differential operator to the system (A.1), and find the conditions for which this expression is zero. Then $\mathbf{v}$ is an infinitesimal generator of the symmetry group, so finding G is just a matter of exponentiating the vector field.

A.2 Using Jets

The computations required to determine **v** become increasingly more involved as the number of variables or the number of equations in the system grows. A semi-automatic process, called Jets, is used here to solve this problem. Jets is implemented in Maple, and it is well suited for dealing with large symbolic expressions appearing in the ansatz for $\mathbf{v}^{(n)}$. More concretely, what happens is the following:

Let **v** be defined as in (A.2). As a computational trick, define

$$Q^{\alpha}\left(\bar{x}, \bar{u}^{(1)}\right) = \varphi_{\alpha}(\bar{x}, \bar{u}) - \sum_{i=1}^{p} \xi^{i}(\bar{x}, \bar{u}) u_i^{\alpha}, \quad \alpha = 1, \dots, q.$$

We call $Q = (Q^1, \dots, Q^q)$ the *characteristic* of **v**. Note that one can recover **v** from Q through the relations

$$\begin{cases} \xi^{i}(\bar{x}, \bar{u}) = -\frac{\partial}{\partial u_i^{\alpha}} Q^{\alpha}, \\ \varphi_{\alpha}(\bar{x}, \bar{u}) = Q^{\alpha}\left(\bar{x}, \bar{u}^{(1)}\right) + \sum_{i=1}^{p} \xi^{i}(\bar{x}, \bar{u}) u_i^{\alpha}. \end{cases} \tag{A.5}$$

Jets is built to produce Q, so that we can recover **v** and exponentiate it to find the symmetry group.

The Novikov equation, as stated before, is

$$u_{xxt} - u_t = 4u^2 u_x - 3u u_x u_{xx} - u^2 u_{xxx}.$$

We note that this is just a single third-order partial differential equation, with two independent and one dependent variable. This means that one can drop the α's and the bar on $\bar{u}$ in the equations above. Also, let $x^1 = x$, $x^2 = t$, so that the ansatz for **v** becomes

$$\mathbf{v} = \xi^{x}(x, t, u) \frac{\partial}{\partial x} + \xi^{t}(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u},$$

and its third prolongation

$$\begin{aligned} \mathbf{v}^{(3)} = \mathbf{v} + \varphi^{x} \frac{\partial}{\partial u_x} + \varphi^{t} \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} + \\ + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}} + \varphi^{xxt} \frac{\partial}{\partial u_{xxt}} + \varphi^{xtt} \frac{\partial}{\partial u_{xtt}} + \varphi^{ttt} \frac{\partial}{\partial u_{ttt}}. \end{aligned}$$

If one wanted to do the work manually one would now compute the coefficients φ^x, etc., using Theorem A.1, apply $\mathbf{v}^{(3)}$ to the Novikov equation, and find conditions on the ξ's and φ. Instead, let's go with Jets, and study the characteristic

$$Q(x, t, u, u_x, u_t) = \varphi(x, t, u) - \xi^{x}(x, t, u) u_x - \xi^{t}(x, t, u) u_t.$$

With the following setup, Jets will generate all conditions for Q being the characteristic of the Novikov equation:

```
> read("Jets.s");
> coordinates([x,t], [u], 3);
> equation ('u_xxt' = u_t + 4*u^2*u_x - 3*u*u_x*u_xx - u^2*u_xxx);
> S := symmetries(u = Q);
> dependence(Q(x, t, u, u_t, u_x));
> unknowns(Q);
> run(S);
> dependence();
> S1 := clear(pds);
```

We find that Q depends on all variables in general, and must satisfy the following conditions:

$$\frac{\partial^2}{\partial t^2}Q = \frac{\partial^2}{\partial u_x^2}Q = \frac{\partial^2}{\partial u_t^2}Q = 0, \tag{A.6a}$$

$$\frac{\partial^2}{\partial t \partial x}Q = \frac{\partial^2}{\partial u_x \partial t}Q = \frac{\partial^2}{\partial u_t \partial x}Q = \frac{\partial^2}{\partial u_t \partial u_x}Q = 0, \tag{A.6b}$$

$$\left[\frac{\partial^2}{\partial u_t \partial t} - \frac{1}{u_t}\frac{\partial}{\partial t}\right] Q = 0, \tag{A.6c}$$

$$\left[\frac{\partial}{\partial u} + \frac{1}{u}\left(u_x \frac{\partial}{\partial u_x} + u_t \frac{\partial}{\partial u_t} - 1\right)\right] Q = 0, \tag{A.6d}$$

$$\left[\frac{\partial^2}{\partial u_x \partial x} + \frac{2}{u}\left(1 - u_x \frac{\partial}{\partial u_x} - u_t \frac{\partial}{\partial u_t}\right) - \frac{1}{u_t}\frac{\partial}{\partial t}\right] Q = 0, \tag{A.6e}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{2u_x}{u}\frac{\partial}{\partial x} + \frac{2(u^2 - u_x^2)}{u u_t}\frac{\partial}{\partial t} + \frac{4(u^2 - u_x^2)}{u^2}\left(u_x \frac{\partial}{\partial u_x} + u_t \frac{\partial}{\partial u_t} - 1\right)\right] Q = 0. \tag{A.6f}$$

It follows from (A.6a) and (A.6b) that the characteristic Q is a polynomial of first degree in both u_x and t, with no mixed terms, so one can split it into three parts, denoted Q_0, Q_1 and Q_2, that only depend on u, x and u_t, so that $Q = Q_0 u_x + Q_1 t + Q_2$. This simplifies the dependence of Q, so we run Jets again:

```
> Q := Q0*u_x + Q1*t + Q2;
> dependence(Q0(u, x, u_t), Q1(u, x, u_t), Q2(u, x, u_t));
> unknowns(Q0, Q1, Q2);
> run(S1);
> dependence();
> S2 := clear(pds);
```

This time, Jets is able to reduce the dependencies, so that Q_0 now only depends on x, while Q_1 only depends on u_t. However, Q_2 still depends on u, x, and u_t. The list of conditions is now more manageable:

$$\left(\frac{\partial^3}{\partial x^3}-4\frac{\partial}{\partial x}\right)Q_0=0, \tag{A.7a}$$

$$\left(\frac{\partial}{\partial u_t}-\frac{1}{u_t}\right)Q_1=0, \tag{A.7b}$$

$$\frac{u}{2}\frac{\partial^2}{\partial x^2}Q_0+\frac{\partial}{\partial x}Q_2=0, \tag{A.7c}$$

$$\frac{1}{2}\frac{\partial}{\partial x}Q_0-\frac{1}{2u_t}Q_1+\frac{\partial}{\partial u}Q_2=0, \tag{A.7d}$$

$$-\frac{u}{2u_t}\frac{\partial}{\partial x}Q_0+\frac{u}{2u_t^2}Q_1+\left(\frac{\partial}{\partial u_t}-\frac{1}{u_t}\right)Q_2=0. \tag{A.7e}$$

Now, conditions (A.7a) and (A.7b) imply that

$$Q_0=Q_{00}e^{2x}+Q_{01}e^{-2x}+Q_{02},$$
$$Q_1=Q_{10}u_t,$$

where Q_{00} up to Q_{10} are constants. Inserting these expressions into conditions (A.7c) through (A.7e) and solving for Q_2 gives

$$Q_2=-uQ_{00}e^{2x}+uQ_{01}e^{-2x}+\frac{u}{2}Q_{10}+u_tQ_{20},$$

where Q_{20} is also constant.

We conclude that the most general characteristic for the Novikov equation is

$$Q=\left(-ue^{2x}+u_xe^{2x}\right)Q_{00}+\left(ue^{-2x}+u_xe^{-2x}\right)Q_{01}+u_xQ_{02}+\left(\frac{1}{2}u+tu_t\right)Q_{10}+u_tQ_{20}. \tag{A.8}$$

Note that it has five degrees of freedom, which correspond to five different generators for the symmetry group. From the characteristic, we recover five infinitesimal generators, using (A.5).

$$\mathbf{v}_1=-\frac{\partial}{\partial x},$$
$$\mathbf{v}_2=-\frac{\partial}{\partial t},$$
$$\mathbf{v}_3=-\frac{\partial}{\partial t}+\frac{u}{2}\frac{\partial}{\partial u},$$
$$\mathbf{v}_4=-e^{2x}\frac{\partial}{\partial x}-e^{2x}u\frac{\partial}{\partial u},$$
$$\mathbf{v}_5=-e^{-2x}\frac{\partial}{\partial x}+e^{-2x}u\frac{\partial}{\partial u}.$$

Exponentiating the vector fields, we find the symmetry group of the Novikov equation.

Theorem A.3. *If $u = f(x,t)$ solves the Novikov equation (2.1), then so do*

$$\begin{aligned} u_1 &= f(x-\varepsilon, t), \\ u_2 &= f(x, t-\varepsilon), \\ u_3 &= e^{\varepsilon/2} f(x, te^{\varepsilon}), \\ u_4 &= \sqrt{1+2\varepsilon e^{2x}}\, f\left(-\frac{1}{2}\ln\left(e^{-2x}+2\varepsilon\right), t\right), \\ u_5 &= \sqrt{1+2\varepsilon e^{-2x}}\, f\left(\frac{1}{2}\ln\left(e^{2x}+2\varepsilon\right), t\right). \end{aligned}$$

It is easy to check the first three by inspecting the equation; the last two are best checked by computer.

Finally, while computing the Lie symmetries of the Novikov equation, we also did the same for its two-component generalization due to Geng–Xue [8]. While not directly relevant to this article, this might be a good place to mention the results. The Geng–Xue system is given by

$$\begin{cases} u_{xxt} - u_t = (u_x - u_{xxx})uv + 3(u - u_{xx})vu_x, \\ v_{xxt} - v_t = (v_x - v_{xxx})uv + 3(v - v_{xx})uv_x. \end{cases}$$

Proceeding with the help of Jets as before, we find the following symmetries.

Theorem A.4. *If*

$$\begin{cases} u = f(x,t), \\ v = g(x,t), \end{cases}$$

solves the Geng–Xue system (A.2), then so do

$$\begin{cases} u_1 = \sqrt{1+2\varepsilon e^{2x}}\, f\left(-\tfrac{1}{2}\ln\left(e^{-2x}+2\varepsilon\right), t\right), \\ v_1 = \sqrt{1+2\varepsilon e^{2x}}\, g\left(-\tfrac{1}{2}\ln\left(e^{-2x}+2\varepsilon\right), t\right), \end{cases}$$

$$\begin{cases} u_2 = \sqrt{1+2\varepsilon e^{-2x}}\, f\left(\tfrac{1}{2}\ln\left(e^{2x}+2\varepsilon\right), t\right), \\ v_2 = \sqrt{1+2\varepsilon e^{-2x}}\, g\left(\tfrac{1}{2}\ln\left(e^{2x}+2\varepsilon\right), t\right), \end{cases}$$

$$\begin{cases} u_3 = f(x-\varepsilon, t), \\ v_3 = g(x-\varepsilon, t), \end{cases} \qquad \begin{cases} u_4 = f(x, t-\varepsilon), \\ v_4 = g(x, t-\varepsilon), \end{cases}$$

$$\begin{cases} u_5 = f(x, te^{\varepsilon}), \\ v_5 = e^{\varepsilon} g(x, te^{\varepsilon}), \end{cases} \qquad \begin{cases} u_6 = e^{\varepsilon} f(x,t), \\ v_6 = e^{-\varepsilon} g(x,t). \end{cases}$$

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Paper II

Peakon–antipeakon solutions of the Novikov equation

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ABSTRACT. Certain nonlinear partial differential equations admit multi-soliton solutions in the form of a superposition of peaked waves, so-called peakons. The Camassa–Holm and Degasperis–Procesi equations are two well-known examples, and a more recent one is the Novikov equation, which has cubic nonlinear terms instead of quadratic. In this article we investigate multipeakon solutions of the Novikov equation, in particular interactions between peakons with positive amplitude and antipeakons with negative amplitude. The solutions are given by explicit formulas, which makes it possible to analyze them in great detail. As in the Camassa–Holm case, the slope of the wave develops a singularity when a peakon collides with an antipeakon, while the wave itself remains continuous and can be continued past the collision to provide a global weak solution. However, the Novikov equation differs in several interesting ways from other peakon equations, especially regarding asymptotics for large times. For example, peakons and antipeakons both travel to the right, making it possible for several peakons and antipeakons to travel together with the same speed and collide infinitely many times. Such clusters may exhibit very intricate periodic or quasi-periodic interactions. It is also possible for peakons to have the same asymptotic velocity but separate at a logarithmic rate; this phenomenon is associated with coinciding eigenvalues in the spectral problem coming from the Lax pair, and requires nontrivial modifications to the previously known solution formulas which assume that all eigenvalues are simple. To facilitate the reader's understanding of these multipeakon phenomena, we have included a particularly detailed description of the case with just one peakon and one antipeakon, and also made an effort to provide plenty of graphics for illustration.

1 Introduction

The Novikov equation

$$u_t - u_{xxt} = (3u_x u_{xx} - 4uu_x + uu_{xxx})\, u, \tag{1.1}$$

is an integrable nonlinear partial differential equation for $u = u(x,t)$, a function of the space and time coordinates. It was found by Vladimir Novikov [27, 19] in a search for integrable equations similar in form to the Camassa–Holm equation [5],

$$u_t - u_{xxt} = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \tag{1.2}$$

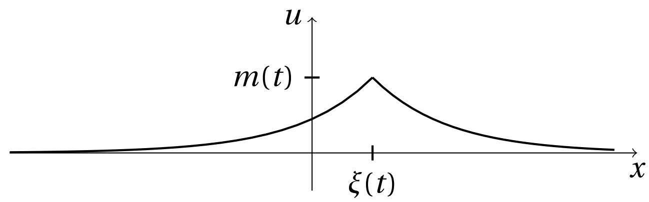

Figure 1: The shape of a single peakon $u(x,t) = m(t)\, e^{-|x-\xi(t)|}$ at a fixed value of t.

but with cubic nonlinearities instead of quadratic. The Camassa–Holm equation arises as a model for waves in shallow water, but we are not aware of any physical applications of Novikov's equation.

One of several remarkable features contributing to the fame of the Camassa–Holm equation is that it admits a particular class of weak solutions known as *peakons* (short for *peaked solitons*), which can be computed explicitly [1, 2]. Recall that solitons are localized wave pulses which can interact in a particle-like manner, and *peaked* here means that at some points the left and right derivatives of u with respect to x are unequal (but finite).

Likewise, Novikov's equation admits peakon solutions, and these are the subject of our study here. Before going into details, let us mention some closely related PDEs also having peakon solutions, such as the Degasperis–Procesi equation [10, 24, 25]

$$u_t - u_{xxt} = 3u_x u_{xx} - 4uu_x + uu_{xxx} \tag{1.3}$$

and the Geng–Xue equation [11, 26]

$$\begin{aligned} u_t - u_{xxt} &= (3u_x u_{xx} - 4u^2 u_x + u^2 u_{xxx})\, v, \\ v_t - v_{xxt} &= (3v_x v_{xx} - 4v^2 v_x + v^2 v_{xxx})\, u, \end{aligned} \tag{1.4}$$

an integrable two-component generalization of Novikov's equation.

In all these cases, the general expression for a single peakon is

$$u(x,t) = m(t)\, e^{-|x-\xi(t)|} = \begin{cases} m(t)\, e^{-x+\xi(t)}, & x \ge \xi(t), \\ m(t)\, e^{x-\xi(t)}, & x \le \xi(t), \end{cases} \tag{1.5}$$

where the functions $\xi(t)$ and $m(t)$ are suitably chosen in order to obtain a solution to the PDE in question. Thus, the shape of a peakon at any fixed time t is given by the function $e^{-|x|}$, multiplied by an amplitude factor $m(t)$, and translated so that it is centered at $x = \xi(t)$; see Figure 1. To distinguish the particular case $m(t) < 0$ we use the term *antipeakon*.

For the Camassa–Holm and Degasperis–Procesi equations, the single-peakon solution is a travelling wave: the amplitude $m(t) = c$ is a (nonzero) constant, and the position is $\xi(t) = \xi_0 + ct$, so that the wave moves with constant velocity c. In particular, a peakon with $c > 0$ moves to the right, but an antipeakon with $c < 0$ moves to the left.

Novikov's equation is different, because of its cubic nonlinearities; again the single-peakon solution is a travelling wave with constant amplitude $m(t) = c$, but the position

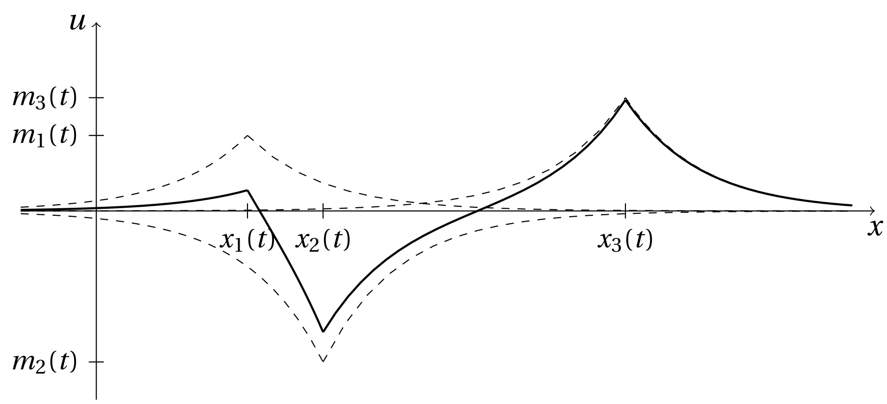

Figure 2: A multipeakon wave profile $u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x-x_k(t)|}$ with $n = 3$, for some fixed value of t. This picture shows the mixed peakon–antipeakon case: m_1 and m_3 are positive, while m_2 is negative. The dashed curves show the contributions from the individual terms, and the solid curve is the graph of $x \mapsto u(x,t)$. For Camassa–Holm and Degasperis–Procesi peakons, the instantaneous velocity $\dot{x}_k(t)$ of each peakon is given by the amplitude $u(x_k(t),t)$ of the wave at the corresponding position, but for Novikov peakons, the velocity is the *square* of the amplitude: $\dot{x}_k(t) = u(x_k(t),t)^2$.

is $\xi(t) = \xi_0 + c^2 t$. Thus, the velocity is $c^2 > 0$, so antipeakons and peakons both move to the right.

In what follows, we will study *multipeakons*, i.e., linear combinations of peakons and/or antipeakons; see Figure 2. A function of the form

$$u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x-x_k(t)|} \tag{1.6}$$

is a solution of Novikov's equation (in the precise sense given by Definition 2.1 below) if and only if the positions $x_k(t)$ and the amplitudes $m_k(t)$ satisfy the following system of $2n$ ordinary differential equations:

$$\begin{aligned} \dot{x}_k &= \left(\sum_{i=1}^{n} m_i\, e^{-|x_k - x_i|} \right)^2, \\ \dot{m}_k &= m_k \left(\sum_{i=1}^{n} m_i\, e^{-|x_k - x_i|} \right) \left(\sum_{j=1}^{n} m_j \operatorname{sgn}(x_k - x_j) e^{-|x_k - x_j|} \right), \end{aligned} \tag{1.7}$$

for $k = 1, \dots, n$. Here dots denote time derivatives, and $\operatorname{sgn}(0) = 0$. In shorthand notation, this system can be written as

$$\begin{aligned} \dot{x}_k &= u(x_k)^2, \\ \dot{m}_k &= -m_k u(x_k) \langle u_x(x_k) \rangle, \end{aligned} \tag{1.8}$$

where $u(x_k)$ denotes $u(x_k(t),t)$ evaluated using (1.6), and

$$\langle u_x(x_k) \rangle = \frac{1}{2} \left(u_x(x_k^-) + u_x(x_k^+) \right)$$

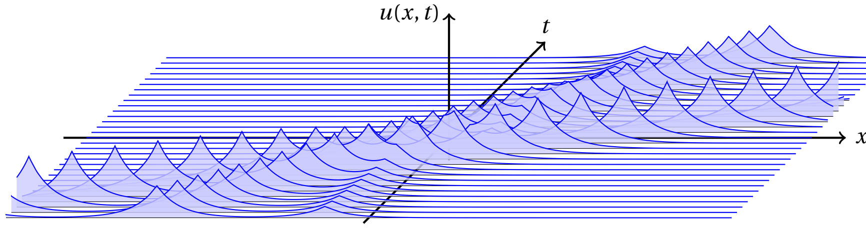

Figure 3: A three-peakon solution $u(x,t) = \sum_{k=1}^{3} m_k(t)\, e^{-|x-x_k(t)|}$ of the Novikov equation, plotted from the exact solution formulas in Theorem 2.7, which express the functions $(m_k(t))^2$ and $e^{2x_k(t)}$ as rational expressions in the exponentials e^{t/λ_1}, e^{t/λ_2} and e^{t/λ_3}. In this example, the parameters λ_k have the values $\lambda_1 = 1/5$, $\lambda_2 = 1/2$, $\lambda_3 = 4$. The solution formulas also contain three other parameters $b_k(0)$, here with the values $b_1(0) = 10^3$, $b_2(0) = 2$, $b_3(0) = 1$. The figure uses a parallel projection with the same scale on all axes, and the domain shown is $-16 \le x \le 16$ and $-5 \le t \le 5$, with the wave profile $u(x,t)$ sampled at equidistant times $t = n/3$, $n \in \mathbf{Z}$. The positions of the peakons satisfy $x_1(t) < x_2(t) < x_3(t)$ for all t, which is seen more clearly in Figure 4. For large $|t|$ the peakons are well separated and behave nearly like single-peakon solutions (travelling waves with constant velocity, equal to the square of the amplitude). For $t \ll 0$ the amplitudes are ordered with the tallest/fastest peakon to the left, $m_1(t) > m_2(t) > m_3(t)$, while for $t \gg 0$, after the interactions have taken place, it is the other way around, $m_1(t) < m_2(t) < m_3(t)$; see Figure 5.

is the average of the left and right x derivatives of $u(x,t)$ at $x = x_k(t)$.

Hone, Lundmark and Szmigielski [18] solved the peakon ODEs (1.7) explicitly, for arbitrary n, with initial values in the pure peakon sector, i.e., points $(x_1,\dots,x_n,m_1,\dots,m_n) \in \mathbf{R}^{2n}$ such that $x_1 < x_2 < \cdots < x_n$ and all $m_i > 0$. Given such initial data at time $t = 0$, the solutions were found to exist for all $t \in \mathbf{R}$, both forwards and backwards in time, and to remain in the pure peakon sector [18, Theorem 4.5]. In particular this means that the strict ordering $x_1 < x_2 < \cdots < x_n$ is automatically preserved by the evolution, so collisions $x_i = x_{i+1}$ never occur. As $t \to \pm\infty$ the peakons scatter, i.e., $x_{i+1} - x_i \to \infty$ for all i, and their amplitudes and velocities tend to constant values. In other words, as the peakons separate, each one of them is increasingly less influenced by the others, and therefore behaves asymptotically as a single-peakon solution, moving with constant amplitude and velocity. See Section 2.3 for a more detailed description, and Figures 3, 4 and 5 for illustrations.

Note that the pure antipeakon case where all $m_i < 0$ is essentially the same as the pure peakon case, since the system is invariant under the transformation

$$(x_1,x_2,\dots,x_n,m_1,m_2,\dots,m_n) \mapsto (x_1,x_2,\dots,x_n,-m_1,-m_2,\dots,-m_n).$$

In this article, we will consider the mixed peakon–antipeakon case, where it turns out that the solutions can exhibit a much more intricate behaviour. Collisions will occur, where $x_k = x_{k+1}$ for some k, and at these instants the amplitudes m_k and m_{k+1} blow

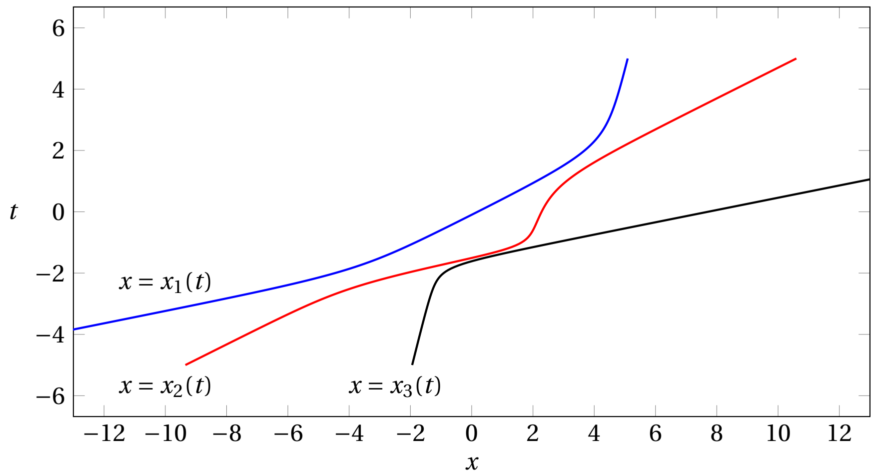

Figure 4: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the pure 3-peakon solution in Figure 3, with $\lambda_1 = 1/5$, $\lambda_2 = 1/2$ and $\lambda_3 = 4$. As $t \to \pm\infty$, the curves $x = x_k(t)$ approach certain straight lines of the form $x = t/\lambda_i + \text{const.}$, as explained in Theorem 2.9 and Remark 2.11. In other words, the peakons are asymptotically travelling with the constant velocities $1/\lambda_1 = 5$, $1/\lambda_2 = 2$ and $1/\lambda_3 = 1/4$.

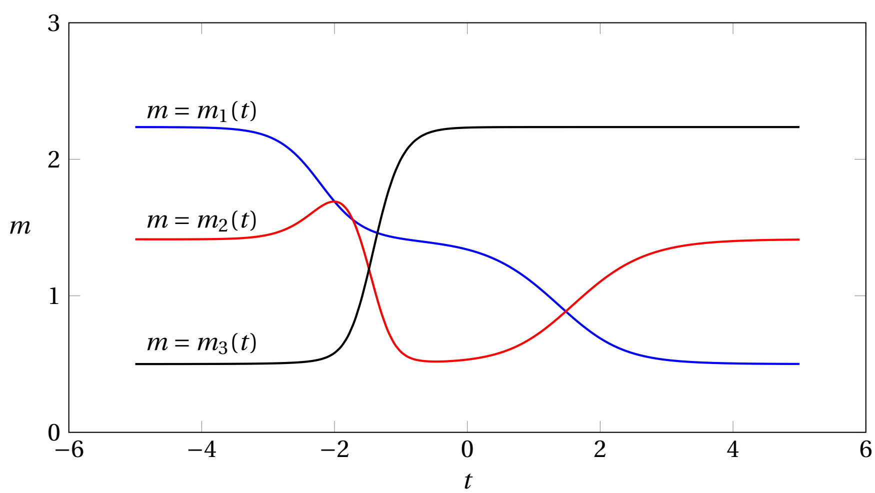

Figure 5: Graphs of the amplitudes $m_k(t)$ for the pure 3-peakon solution in Figure 3, with $\lambda_1 = 1/5$, $\lambda_2 = 1/2$ and $\lambda_3 = 4$. We see that $m_k(t) \to 1/\sqrt{\lambda_k}$ as $t \to -\infty$, in agreement with Theorem 2.9. The same limits ($\sqrt{5}$, $\sqrt{2}$ and $1/2$) are approached as $t \to +\infty$ as well, but in the opposite order; the tallest/fastest peakon has become the lowest/slowest, etc. Note that the asymptotic velocities $1/\lambda_k$ seen in Figure 4 are the squares of the asymptotic amplitudes $1/\sqrt{\lambda_k}$.

up. However, the function $u(x,t)$ remains well-behaved and can be continued past the collision, to provide a global solution. These collisions are similar to the ones occurring in Camassa–Holm peakon–antipeakon solutions (see Section 2.2), but the global behaviour of Novikov peakon–antipeakon solutions can be very much different: instead of all peakons scattering, each with its own velocity, there may be clusters of peakons travelling together with the same (average) velocity and colliding repeatedly in a periodic or quasi-periodic fashion. There are also borderline cases between scattering and clustering, where peakons separate at a logarithmic rate although their velocities tend to the same value. (The solution formulas for these borderline cases are particularly involved; the other cases are simply described by the extending the range of allowed parameter values in the formulas already known for pure peakon solutions.)

As an example of these phenomena, Figure 6 shows the positions $x_k(t)$ for a peakon–antipeakon solution with $n=5$. The solution is computed from the exact solution formulas described in Section 6, with parameter values

$$\lambda_1 = 1, \quad \lambda_2 = \overline{\lambda_3} = \frac{1}{1+i}, \quad \lambda_4 = \lambda_5 = 3, \qquad b_k(0) = 1 \quad \forall k. \tag{1.9}$$

The solution formulas are much too large to be written out in detail here, but can be computed using a computer algebra system. We think it is safe to say that a solution like this would be very difficult to discover by numerical integration of the differential equations, since the clustering only happens for very special values of the parameters, and numerical roundoff errors would unevitably disturb the delicate balance and cause the peakons to scatter.

2 Preliminaries

2.1 Weak solutions of the Novikov equation

We begin by explaining exactly in which sense peakons are weak solutions. There are various ways of defining weak solutions, all of which involve rewriting Novikov's equation (1.1) into a form which is equivalent if u is a smooth function, but also makes sense under weaker conditions, and then taking this new form as the definition. For example, (1.1) can be written as

$$(1-\partial_x^2)(u_t + u^2 u_x) + 3uu_x u_{xx} + 2u_x^3 + 3u^2 u_x = 0.$$

Taking convolution with $G(x) = \frac{1}{2}e^{-|x|}$ as the inverse of the operator $(1-\partial_x^2)$, this becomes a nonlocal evolution eqation,

$$u_t + u^2 u_x + G * (3uu_x u_{xx} + 2u_x^3 + 3u^2 u_x) = 0, \tag{2.1}$$

and one can consider functions u which satisfy this in a distributional sense (using test functions $\varphi(x,t)$ of two variables).

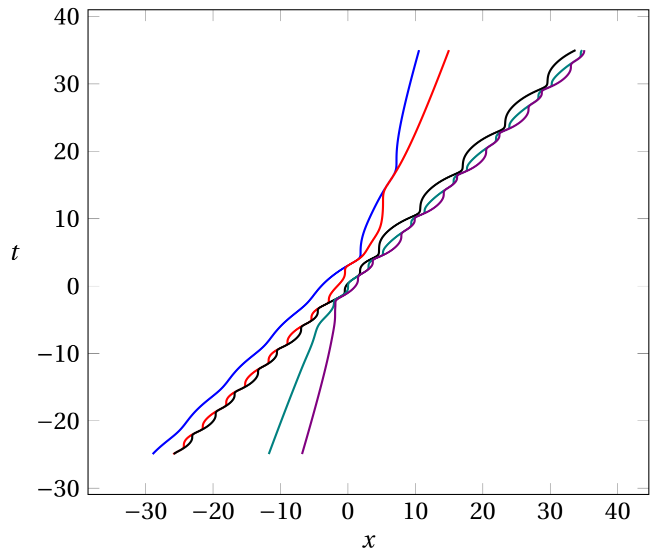

Figure 6: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the peakon–antipeakon solution of Novikov's equation with $n = 5$ and parameters as in (1.9). The parameters correspond to an asymptotic 3-cluster of peakons and antipeakons, and a pair of peakons with logarithmic separation. Note how the peakons switch roles, so that the asymptotic behaviour exhibited when $t \to -\infty$ is reversed after interactions.

However, the nonlocal convolution term can lead to quite tedious calculations if one wants to verify that a proposed solution really satisfies the definition, so another approach [18] is more convenient for dealing with peakons; rewrite (1.1) as

$$(1-\partial_x^2)u_t + (4-\partial_x^2)\partial_x\left(\tfrac{1}{3}u^3\right) + \partial_x\left(\tfrac{3}{2}uu_x^2\right) + \tfrac{1}{2}u_x^3 = 0, \tag{2.2}$$

leave the operator $1-\partial_x^2$ as it is (don't apply the inverse), and consider distributions in the x direction only, with time t entering as a parameter. Then computations with peakons will involve nothing more complicated than the one-dimensional Dirac delta distribution $\delta(x)$ and its distributional derivative $\delta'(x)$.

Definition 2.1. A (global) weak solution of the Novikov equation is a continuous function $u(x,t)$ such that:

1. For each $t \in \mathbf{R}$, the function $x \mapsto u(x,t)$ belongs to the Sobolev space $W^{1,3}_{\text{loc}}(\mathbf{R})$, i.e., the functions $x \mapsto u(x,t)^3$ and $x \mapsto u_x(x,t)^3$ are locally integrable.

   (It then follows from Hölder's inequality with conjugate indices 3 and $\frac{3}{2}$ that uu_x^2 is locally integrable as well. This means that u^3, u_x^3 and uu_x^2 can be interpreted as distributions in $\mathscr{D}'(\mathbf{R})$ for each fixed t.)

2. For almost every $t \in \mathbf{R}$, the time derivative $u_t(\cdot,t) \in \mathscr{D}'(\mathbf{R})$ exists as a distribution defined as the limit of a difference quotient,

$$\left\langle u_t(\cdot,t),\varphi\right\rangle = \lim_{\tau\to 0}\frac{\langle u(\cdot,t+\tau)-u(\cdot,t),\varphi\rangle}{\tau} = \lim_{\tau\to 0}\int_{\mathbf{R}}\frac{u(x,t+\tau)-u(x,t)}{\tau}\,\varphi(x)\,dx,$$

and equation (2.2) is satisfied in the space of distributions $\mathscr{D}'(\mathbf{R})$ for these values of t, i.e.,

$$\Big\langle (1-\partial_x^2)u_t + (4-\partial_x^2)\partial_x\left(\tfrac{1}{3}u^3\right) + \partial_x\left(\tfrac{3}{2}uu_x^2\right) + \tfrac{1}{2}u_x^3, \varphi \Big\rangle = 0 \tag{2.3}$$

for all test functions $\varphi(x) \in C_0^\infty(\mathbf{R})$ (i.e., smooth and with compact support).

Remark 2.2. "Almost every" means except on a set of Lebesgue measure zero. The notation $\langle\cdot,\cdot\rangle$ means the action of a distribution (in x) on a test function; in particular $\langle f,\varphi\rangle = \int_{\mathbf{R}} f(x)\,\varphi(x)\,dx$ if f is a locally integrable function. Spatial derivatives ∂_x are taken in the distributional sense: $\langle T_x,\varphi\rangle = -\langle T,\varphi_x\rangle$. So the requirement is, to be explicit, that

$$\begin{aligned}\lim_{\tau\to 0}\int_{\mathbf{R}} &\frac{u(x,t+\tau)-u(x,t)}{\tau}\big(\varphi(x)-\varphi_{xx}(x)\big)\,dx \\ &+\int_{\mathbf{R}}\Big(\tfrac{1}{3}u^3(\varphi_{xxx}-4\varphi_x) - \tfrac{3}{2}uu_x^2\varphi_x + \tfrac{1}{2}u_x^3\varphi\Big)\,dx = 0\end{aligned} \tag{2.4}$$

for all test functions φ and almost all t.

Remark 2.3. In [18], equation (2.3) was required to hold for all $t \in \mathbf{R}$. This is fine for globally defined pure peakon solutions, but it turns out to fail at peakon–antipeakon collisions, which is why we have relaxed the condition to hold only for almost all values of t (which is rather natural if one compares to the approach with distributional solutions using test functions of two variables). At the exceptional values of time, we instead impose the extra requirement that $u(x,t)$ be continuous (as a function of two variables).

As mentioned in the introduction, it is readily verified that a function of the multipeakon form (1.6) is a weak solution in this sense, in an interval $t \in I$, if and only if it satisfies the ODEs (1.7) for $t \in I$. Pure peakon and pure antipeakon solutions are globally defined ($I = \mathbf{R}$), but for mixed peakon–antipeakon solutions we will see that there are instants $t = t_j$ (isolated and at most countably many) when one or several pairs of peakons collide ($x_k = x_{k+1}$), and the corresponding amplitudes m_k and m_{k+1} tend to plus or minus infinity; in other words, the solution of the ODEs blows up after finite time. Then one must glue solutions from intervals (t_1,t_2) and (t_2,t_3), say, so that $u(x,t)$ becomes continuous across the line $t = t_2$. The continuation after a collision will not be unique; indeed, it is a subtle question to impose additional conditions which will pick out a unique solution. This has been studied in depth for some other peakon equations, such as the Camassa–Holm and Degasperis–Procesi equations (see Section 2.2), but not yet for the Novikov equation, as far as we know. The n-peakon solutions of Novikov's equation that we will consider here are *conservative* in the sense that the energy integral $\int_{\mathbf{R}}(u^2+u_x^2)\,dx$ is preserved for all $t \in \mathbf{R}$ except at the instants of collision (cf. Sections 2.2 and 3), and also in the sense that the values of all the n constants of motion $\{H_k\}_{k=1}^n$ of the ODEs (1.7) (see Theorem 2.20) are preserved for all $t \in \mathbf{R}$ except for being undefined at the instants of collision.

2.2 Comparison with CH and DP peakon–antipeakon collisions

The question of peakon–antipeakon collisions has been very carefully studied in the case of the Camassa–Holm equation (1.2). To begin with, Beals, Sattinger and Szmigielski [1, 2] derived solution formulas for n-peakon solutions, initially under the assumption that all peakon amplitudes m_k are of the same sign. In this case, the solution $\{x_k(t), m_k(t)\}_{k=1}^n$ to the peakon ODEs exists for all times $t \in \mathbf{R}$ (and is given by completely explicitly known expressions in terms of elementary functions). The ODE solution formulas provide a globally defined (weak) solution

$$u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x-x_k(t)|} \tag{2.5}$$

to the Camassa–Holm PDE, with the property that the conditions $x_1 < x_2 < \cdots < x_n$ and $m_k > 0$ are preserved automatically by the evolution of the system. However, they also noticed that the formulas obtained are purely algebraic and apply equally well in the mixed peakon–antipeakon case where there are m_k of both signs present. In this case, as was already observed for $n = 2$ in the original Camassa–Holm article [5], it may happen that the solution to the peakon ODEs develops a singularity after finite time (some m_k tends to ∞ or $-\infty$).

The detailed analysis in [2] showed that the functions $x_k(t)$ given by the explicit solution formulas are defined for all $t \in \mathbf{R}$, and that the condition $x_1 < x_2 < \cdots < x_n$ holds except for finitely many values of t. At such an exceptional time $t = t_0$, one has $x_k(t_0) = x_{k+1}(t_0)$ for one or several values of k (although "triple collisions" $x_k(t_0) = x_{k+1}(t_0) = x_{k+2}(t_0)$ can never occur), and then also the functions $m_k(t)$ and $m_{k+1}(t)$ (as given by the solution formulas) have simple poles at $t = t_0$ for those values of k. From the point of view of the peakon ODEs, the explicit solution formulas provide an analytic continuation of the solution by going around the singularities in the complex t plane. Moreover, it was shown that even though some m_k are undefined at $t = t_0$, the function $u(x,t)$ in (2.5) remains bounded and has a well-defined limit as $t \to t_0$, so that it can be extended to a globally defined function (also denoted by $u(x,t)$) whose properties are described in detail in [2]; for example, as $t \to t_0^-$, it exhibits "wave breaking" in the sense that the derivative $u_x(x,t)$ tends to $-\infty$ on the shrinking interval $x_k(t) < x < x_{k+1}(t)$. However, from the point of view of the original PDE, it is not at all obvious that this function $u(x,t)$ really is a weak solution, and (if so) by what principle it has been continued past the singularity of the PDE at $t = t_0$ (where u_x blows up).

This question was addressed by Bressan and Constantin [3], who introduced a set of new variables (both dependent and independent) in terms of which the Camassa–Holm equation takes the form of a semilinear system with the property that solutions are globally defined in time. For example, one of the new dependent variables v is defined by $u_x = \tan\frac{v}{2}$; then, instead of having a singularity where u_x blows up, one can let v pass smoothly from one interval $((2n-1)\pi, (2n+1)\pi)$ into a neighbouring one. The global solution to their semilinear system translates back into a solution $u(x,t)$ of the original PDE which is *conservative* in the sense that the energy integral $\int_{-\infty}^{\infty}(u^2 + u_x^2)\,dx$ has the same value for almost all values of $t \in \mathbf{R}$. The exceptional times correspond to singularities where a finite energy contribution from large values of u_x^2 on a very short interval is "lost" as that interval momentarily shrinks to a point (but immediately reappears again afterwards). Bressan and Constantin showed how to keep track of such effects by augmenting the function u with an accompanying measure μ in such a way as to obtain a semigroup of solution couples (u, μ), and they also showed in detail how the $n = 2$ peakon–antipeakon solution (as given by the explicit solution formulas for all t) fits into their scheme.

In another paper [4], the same authors considered an alternative way of continuing solutions of the Camassa–Holm equation past singularities. In this *dissipative* scenario, the energy of every solution $u(x, t)$ is required to be a nonincreasing function of time, so a "lost" piece of energy is gone forever and can not reappear. In terms of peakons, this corresponds to a peakon and an antipeakon colliding and continuing afterwards as a single peakon (or antipeakon) of lower total energy (or even annihilating each other completely, if they have exactly equal strength). Since the number of peakons is not preserved (nor is the total energy), this is a different continuation of the solution to the PDE than that directly provided by the explicit n-peakon solution formulas in [2]. (However, those formulas provide a means to describe this situation explicitly as well, since one can piece together solutions with different values of n in different time intervals.)

Another way of resolving these questions has been proposed by Holden and Raynaud [14, 17], who also reformulate the Camassa–Holm equation as a semilinear system, but in a different set of variables. Their approach uses the "Lagrangian" philosophy of fluid dynamics, where one traces the flow of individual fluid elements; in particular, instead of the "unphysical" variable v of Bressan and Constantin, they use the total energy to the left of a given characteristic curve as one of the variables. They have gone to great lengths to rigorously show how the multipeakon solutions for general n satisfy the Camassa–Holm equation in their sense, both in the conservative and dissipative settings [15, 16].

The Degasperis–Procesi equation (1.3) displays a quite different behaviour. Here, the n-peakon solution formulas derived by Lundmark and Szmigielski [25] in the pure peakon case do provide solutions also in the mixed peakon–antipeakon case, but *only up until the time of the first collision.* At that point, the solution becomes discontinuous, so one is forced to leave the world of peakons, and instead consider so-called *shockpeakons.* See [23, 28, 29] for more about shockpeakons, and [6, 7] for discontinuous solutions of the DP equation in general.

When we now begin our study of peakon–antipeakon solutions of Novikov's equation, the situation is similar to when Beals, Sattinger and Szmigielski started their investigations of Camassa–Holm peakon–antipeakon solutions. The subtle questions of continuation of solutions of the Novikov PDE past singularities have not yet (to our knowledge) been studied as thoroughly as for the Camassa–Holm and Degasperis–Procesi equations, and even if that were the case, it would be beyond the scope of this paper to tackle all this at once. However, the explicit solution of the Novikov n-peakon ODEs is known in the pure peakon case (Theorem 2.7), and we can at least study the properties of the functions $x_k(t)$ and $m_k(t)$ given by these solution formulas, and the properties of the associated function $u(x,t) = \sum_k m_k e^{-|x-x_k|}$, when one relaxes the assumption that all m_k are of the same sign.

2.3 Explicit solution formulas for Novikov multipeakons

Here we recall some notation and results from the paper by Hone, Lundmark and Szmigielski [18], to which we refer for proofs and more detailed explanations. In particular, we will make heavy use of the explicit formulas derived in [18] for pure n-peakon solutions of Novikov's equation, i.e., global solutions of the peakon ODEs (1.7) which for all $t \in \mathbf{R}$ have the properties

$$x_1(t) < x_2(t) < \cdots < x_n(t), \qquad \text{all } m_i(t) > 0.$$

These solution formulas, quoted in Theorem 2.7 below, are stated in terms of certain symmetric functions of n constant parameters $(\lambda_1, \dots, \lambda_n)$ called *eigenvalues* and n time-dependent parameters $(b_1, \dots, b_n)$ called *residues*; the origin of this terminology will become clear in Theorem 2.15 below. Collectively we refer to the eigenvalues and residues as *spectral data* or *spectral variables*.

Definition 2.4. For $k \ge 0$, let $\binom{[1,n]}{k}$ denote the set of k-element subsets

$$I = \{i_1 < \cdots < i_k\}$$

of the integer interval $[1,n] = \{1,2,\dots,n\}$. For $I, J \in \binom{[1,n]}{k}$, let

$$\begin{aligned}
\Delta_I &= \Delta(\lambda_{i_1}, \dots, \lambda_{i_k}) = \prod_{i<j} (\lambda_i - \lambda_j), \\
\Gamma_I &= \Gamma(\lambda_{i_1}, \dots, \lambda_{i_k}) = \prod_{i<j} (\lambda_i + \lambda_j), \\
\Gamma_{I,J} &= \Gamma(\lambda_{i_1}, \dots, \lambda_{i_k}; \lambda_{j_1}, \dots, \lambda_{j_k}) = \prod_{1 \le p,q \le k} (\lambda_{i_p} + \lambda_{j_q}),
\end{aligned} \tag{2.6}$$

with the special cases $\Delta_\emptyset = \Gamma_\emptyset = \Delta_{\{i\}} = \Gamma_{\{i\}} = 1$. Furthermore, let

$$\lambda_I = \prod_{i \in I} \lambda_i, \qquad b_I = \prod_{i \in I} b_i,$$

with $\lambda_\emptyset = b_\emptyset = 1$.

Definition 2.5. Using the abbreviation

$$\Psi_I = \frac{\Delta_I^2}{\Gamma_I}, \tag{2.7}$$

let

$$T_k = \sum_{I \in \binom{[1,n]}{k}} \frac{\Psi_I b_I}{\lambda_I}, \qquad U_k = \sum_{I \in \binom{[1,n]}{k}} \Psi_I b_I, \qquad V_k = \sum_{I \in \binom{[1,n]}{k}} \Psi_I \lambda_I b_I, \tag{2.8}$$

for $1 \le k \le n$, let $U_0 = V_0 = T_0 = 1$, and let $U_k = V_k = T_k = 0$ if $k < 0$ or $k > n$. Moreover, let

$$\begin{aligned} W_k &= \begin{vmatrix} U_k & V_{k-1} \\ U_{k+1} & V_k \end{vmatrix} = U_k V_k - U_{k+1} V_{k-1}, \\ Z_k &= \begin{vmatrix} T_k & U_{k-1} \\ T_{k+1} & U_k \end{vmatrix} = T_k U_k - T_{k+1} U_{k-1}. \end{aligned} \tag{2.9}$$

Remark 2.6. In [25, Lemma 2.20], W_k is written as a sum of terms, each of which is positive if all λ_i and b_i are positive, and one obtains a corresponding formula for Z_k by changing b_i to b_i/λ_i everywhere. Thus $W_k > 0$ and $Z_k > 0$ in this case; in Theorem 5.8 we show that this remains true under much weaker assumptions on λ_i and b_i.

The following theorem summarizes the main results of [18]; see in particular Theorem 9.1 in that paper.

Theorem 2.7 (Explicit solution formulas)**.** *The formulas*

$$x_{n+1-k} = \frac{1}{2} \ln \frac{Z_k}{W_{k-1}}, \qquad m_{n+1-k} = \frac{\sqrt{Z_k W_{k-1}}}{U_k U_{k-1}} \qquad (k = 1, \dots, n) \tag{2.10}$$

define a bijection between the set of admissible spectral data

$$\mathscr{R} = \big\{ (\lambda_1, \dots, \lambda_n, b_1, \dots, b_n) \in \mathbf{R}^{2n} \,:\, 0 < \lambda_1 < \cdots < \lambda_n, \text{ all } b_i > 0 \big\} \tag{2.11}$$

and the pure peakon sector

$$\mathscr{P} = \big\{ (x_1, \dots, x_n, m_1, \dots, m_n) \in \mathbf{R}^{2n} \,:\, x_1 < \cdots < x_n, \text{ all } m_i > 0 \big\}. \tag{2.12}$$

Under this bijection, the Novikov peakon ODEs (1.7) *for* $\{x_k(t), m_k(t)\}_{k=1}^n$ *are equivalent to the following linear ODEs for the spectral data:*

$$\dot\lambda_k = 0, \qquad \dot b_k = \frac{b_k}{\lambda_k} \qquad (k = 1, \dots, n). \tag{2.13}$$

Consequently, any pure n-peakon solution of Novikov's equation is obtained by fixing constant parameters

$$\big(\lambda_1, \dots, \lambda_n, b_1(0), \dots, b_n(0)\big) \in \mathscr{R}$$

and defining $\{x_k(t), m_k(t)\}_{k=1}^n$ *via* (2.10)*, with* b_k *having the time dependence*

$$b_k(t) = b_k(0)\, e^{t/\lambda_k} \qquad (k = 1, \dots, n). \tag{2.14}$$

Example 2.8. The two-peakon solution ($n = 2$) of Novikov's equation is obtained by letting $b_1 = b_1(t) = b_1(0)\, e^{t/\lambda_1}$ and $b_2 = b_2(t) = b_2(0)\, e^{t/\lambda_2}$ in the formulas

$$\begin{aligned}
x_1(t) &= \frac{1}{2}\ln\frac{Z_2}{W_1} = \frac{1}{2}\ln\frac{\dfrac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\, b_1^2 b_2^2}{\lambda_1\, b_1^2 + \lambda_2\, b_2^2 + \dfrac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2}, \\
x_2(t) &= \frac{1}{2}\ln\frac{Z_1}{W_0} = \frac{1}{2}\ln\left(\frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right), \\
m_1(t) &= \frac{\sqrt{Z_2 W_1}}{U_2 U_1} = \frac{\left(\dfrac{(\lambda_1-\lambda_2)^4\, b_1^2 b_2^2}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\left(\lambda_1\, b_1^2 + \lambda_2\, b_2^2 + \dfrac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2\right)\right)^{1/2}}{\dfrac{(\lambda_1-\lambda_2)^2\, b_1 b_2}{\lambda_1+\lambda_2}\,(b_1+b_2)}, \\
m_2(t) &= \frac{\sqrt{Z_1 W_0}}{U_1 U_0} = \frac{\left(\dfrac{b_1^2}{\lambda_1} + \dfrac{b_2^2}{\lambda_2} + \dfrac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right)^{1/2}}{b_1+b_2}.
\end{aligned} \tag{2.15}$$

We return to these formulas and analyze them thoroughly in Section 4. For a larger example, see [18, Example 9.3], where the solution formulas for $n = 3$ are written out in detail.

The following [18, Theorem 9.4] is a fairly simple corollary to Theorem 2.7.

Theorem 2.9 (Asymptotics of pure peakon solutions)**.** *With spectral data in $\mathscr{R}$, there is a constant $\delta > 0$ depending on $\{\lambda_1, \dots, \lambda_n\}$ such that the asymptotic behaviour of the pure n-peakon solution as $t \to \pm\infty$ is described by the following formulas (for $k = 1, \dots, n$):*

$$\begin{aligned}
x_k(t) &= \frac{t}{\lambda_k} + \frac{1}{2}\ln\frac{b_k(0)^2}{\lambda_k} + \sum_{i=k+1}^{n} \ln\frac{(\lambda_i-\lambda_k)^2}{(\lambda_i+\lambda_k)\lambda_i} + \mathscr{O}\big(e^{\delta t}\big), \quad \text{as } t \to -\infty, \\
x_{n+1-k}(t) &= \frac{t}{\lambda_k} + \frac{1}{2}\ln\frac{b_k(0)^2}{\lambda_k} + \sum_{i=1}^{k-1} \ln\frac{(\lambda_i-\lambda_k)^2}{(\lambda_i+\lambda_k)\lambda_i} + \mathscr{O}\big(e^{-\delta t}\big), \quad \text{as } t \to +\infty,
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
m_k(t) &= \frac{1}{\sqrt{\lambda_k}} + \mathscr{O}\big(e^{\delta t}\big), \quad \text{as } t \to -\infty, \\
m_{n+1-k}(t) &= \frac{1}{\sqrt{\lambda_k}} + \mathscr{O}\big(e^{-\delta t}\big), \quad \text{as } t \to +\infty.
\end{aligned} \tag{2.17}$$

(Empty sums $\sum_{i=1}^{0}$ and $\sum_{i=n+1}^{n}$ are to be interpreted as zero.)

Remark 2.10. In [18], this theorem is formulated with $o(1)$ instead of $\mathscr{O}\big(e^{-\delta|t|}\big)$, but the stronger statement here follows easily from the proof in [18].

Remark 2.11. What Theorem 2.9 says is that "initially", for $t \ll 0$, the peakons are well separated, each of them behaving approximately like a travelling wave with constant velocity and amplitude; peakon number 1 (the leftmost one) is the fastest, with velocity $\dot{x}_1 \approx 1/\lambda_1$ and amplitude $\dot{m}_1 \approx 1/\sqrt{\lambda_1}$, peakon number 2 is the second fastest, with velocity $\dot{x}_2 \approx 1/\lambda_2$ and amplitude $\dot{m}_2 \approx 1/\sqrt{\lambda_2}$, and so on. As the faster peakons to the left catch up with the slower peakons to the right, there is a complicated nonlinear interaction with transfer of momentum from faster to slower peakons, and what emerges "finally", for $t \gg 0$, is again a train of peakons, each of which has almost constant velocity and amplitude, but now in the opposite order: peakon number n (the rightmost one) has the greatest velocity $\dot{x}_n \approx 1/\lambda_1$ and amplitude $\dot{m}_n \approx 1/\sqrt{\lambda_1}$, peakon number $n-1$ is the second fastest, with velocity $\dot{x}_{n-1} \approx 1/\lambda_2$ and amplitude $\dot{m}_{n-1} \approx 1/\sqrt{\lambda_2}$, and so on. Figures 3, 4 and 5 provide an illustration of this phenomenon for $n = 3$.

In this paper, we are going to use the pure-peakon solution formulas (2.10) to describe also mixed peakon–antipeakon solutions, simply by allowing spectral data in a bigger domain. When investigating which spectral data that can be permitted, it will be necessary to know some more details about how the map (2.10) from $\mathscr{R}$ to $\mathscr{P}$ was constructed in [18], so we proceed to explain this now. In fact, the story begins with the map in the opposite direction, from $\mathscr{P}$ to $\mathscr{R}$.

Definition 2.12. Given any point $(x_1, \dots, x_n, m_1, \dots, m_n)$ in $\mathbf{R}^{2n}$, and in particular given a point in the pure peakon sector $\mathscr{P}$, let

$$S_k(\lambda) = \begin{pmatrix} 1 - \lambda\, m_k^2 & -2\lambda\, m_k\, e^{-x_k} & -\lambda^2\, m_k^2\, e^{-2x_k} \\ m_k\, e^{x_k} & 1 & \lambda\, m_k\, e^{-x_k} \\ m_k^2\, e^{2x_k} & 2\, m_k\, e^{x_k} & 1 + \lambda m_k^2 \end{pmatrix} \tag{2.18}$$

for $k = 1, \dots, n$. In terms of these matrices, we define the polynomials $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ by

$$\begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} = S_n(\lambda) S_{n-1}(\lambda) \cdots S_1(\lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.19}$$

and the rational functions $\omega(\lambda)$ and $\zeta(\lambda)$, called the *Weyl functions*, by

$$\omega(\lambda) = -\frac{B(\lambda)}{A(\lambda)}, \qquad \zeta(\lambda) = -\frac{C(\lambda)}{A(\lambda)}. \tag{2.20}$$

From equation (4.13) in [18] we then have the following theorem:

Theorem 2.13. *The polynomial $A(\lambda)$ is the characteristic polynomial of the $n \times n$ matrix $TPEP$,*

$$A(\lambda) = \det(I - \lambda TPEP), \tag{2.21}$$

where the $n \times n$ matrices P, E and T are defined as

$$\begin{aligned} P &= \operatorname{diag}(m_1, m_2, \dots, m_n), \\ E &= (E_{jk}) = \big(e^{-|x_j - x_k|}\big), \\ T &= (T_{jk}) = \big(1 + \operatorname{sgn}(j-k)\big). \end{aligned} \tag{2.22}$$

Remark 2.14. The matrix T has the number 1 along the main diagonal, 0 everywhere above it, and 2 everywhere below it. The matrix E is symmetric, with $E_{jk} = e^{x_j - x_k}$ for $j \le k$ if we are in the pure peakon sector $\mathscr{P}$ where $x_1 < x_2 < \cdots < x_n$.

The next result involves showing a matrix is oscillatory, i.e., a totally nonnegative matrix, some power of which is totally positive. It is proved in Section A.2 and Theorem 6.1 in [18].

Theorem 2.15. *In the polynomial*

$$A(\lambda) = \det(I - \lambda TPEP) = 1 + \sum_{k=1}^{n} (-1)^k H_k \lambda^k,$$

each coefficient H_k is a homogeneous polynomial of degree $2k$ in the variables $\{m_1, \dots, m_n\}$, with coefficients that are polynomials in the variables $\{E_{ij}\}_{i<j}$.

*If $(x_1, \dots, x_n, m_1, \dots, m_n) \in \mathscr{P}$, then the matrix $TPEP$ is oscillatory, and therefore its **eigenvalues** $\{\lambda_1, \dots, \lambda_n\}$ are **positive** and **simple**:*

$$A(\lambda) = \det(I - \lambda TPEP) = \left(1 - \frac{\lambda}{\lambda_1}\right)\left(1 - \frac{\lambda}{\lambda_2}\right) \cdots \left(1 - \frac{\lambda}{\lambda_n}\right),$$

where we choose to number the eigenvalues in increasing order,

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

*Moreover, in this case the **residues** $\{b_1, \dots, b_k\}$ in the partial fraction expansion of the Weyl function ω,*

$$\omega(\lambda) = -\frac{B(\lambda)}{A(\lambda)} = \sum_{k=1}^{n} \frac{b_k}{\lambda - \lambda_k}, \tag{2.23}$$

*are **positive**.*

Equation (2.23) (implicitly) defines the forward spectral map, from peakon variables in $\mathscr{P}$ to spectral data in $\mathscr{R}$, and the explicit formulas (2.10) in Theorem 2.7 give the inverse map from $\mathscr{R}$ to $\mathscr{P}$.

Remark 2.16. The two Weyl functions ω and ζ are not independent, since they can be shown to satisfy the relation

$$\zeta(\lambda) + \zeta(-\lambda) + \omega(\lambda)\omega(-\lambda) = 0. \tag{2.24}$$

(See Section 6 in [18], and especially equation (6.4), together with the final paragraphs of Section 5.)

In the situation $(x_1, \dots, x_n, m_1, \dots, m_n) \in \mathscr{P}$, where the poles of $\omega(\lambda)$ are simple and positive, (2.23) and (2.24) imply that ζ is uniquely determined by ω as

$$\zeta(\lambda) = -\frac{C(\lambda)}{A(\lambda)} = \sum_{k=1}^{n} \frac{c_k}{\lambda - \lambda_k}, \qquad c_k = \sum_{m=1}^{n} \frac{b_k b_m}{\lambda_k + \lambda_m}.$$

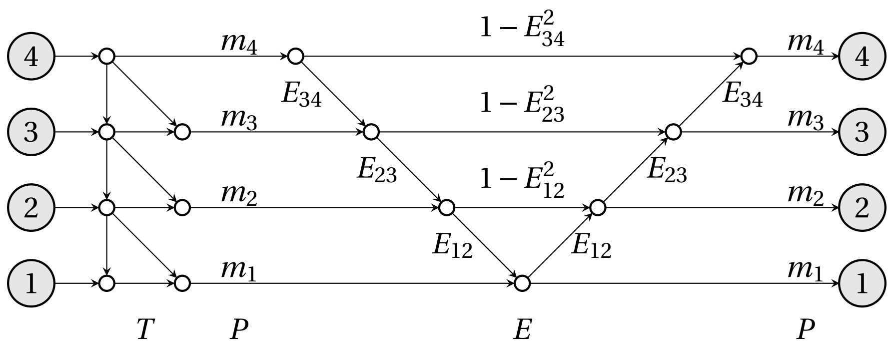

Figure 7: Planar network for the matrix $TPEP$ (in the case $n=4$). Unlabelled edges have weight 1.

Remark 2.17. The proof of Theorem 2.15 is based on the fact that the matrix $TPEP$ is the weighted path matrix of a planar network of the form illustrated in Figure 7. What this means is that the entry $(TPEP)_{ij}$ equals the sum over all paths through the network from source number i on the left to sink number j on the right, where each such path contributes the product of the weights on its edges. In the pure peakon case, where all the weights in the network are positive, this implies, via the famous path-counting lemma of Karlin–McGregor, Lindström, and Gessel–Viennot [20, 22, 12], that $TPEP$ is an oscillatory matrix. The network can also be used for computing H_k, also with the help of that lemma; H_k is the sum of the principal $k\times k$ minors of $TPEP$, and each minor can be computed as a sum over node-disjoint path families through the network. An alternative description is that H_k is the sum of *all* $k\times k$ minors (principal and non-principal) of the simpler matrix PEP. This follows by letting $X=PEP$ in Theorem 2.18; see [18, Theorem 4.1] and [13].

Theorem 2.18 ("The Canada Day Theorem")**.** *Define T as in* (2.22)*, and let X be any* ***symmetric*** *$n\times n$ matrix. Then, for every $k\in\{1,2,\dots,n\}$, the sum of the* ***principal*** *$k\times k$ minors of TX equals the sum of* ***all*** *$k\times k$ minors of X.*

Example 2.19. For $n=2$ we have $A(\lambda)=1-H_1\lambda+H_2\lambda^2$, where

$$\begin{aligned} H_1 &= m_1^2+m_2^2+2m_1m_2E_{12},\\ H_2 &= m_1^2m_2^2(1-E_{12}^2), \end{aligned} \tag{2.25}$$

and for $n=3$ we have $A(\lambda)=1-H_1\lambda+H_2\lambda^2-H_3\lambda^3$, where

$$\begin{aligned} H_1 &= m_1^2+m_2^2+m_3^2+2m_1m_2E_{12}+2m_1m_3E_{13}+2m_2m_3E_{23},\\ H_2 &= (1-E_{12}^2)\,m_1^2\,m_2^2+(1-E_{13}^2)\,m_1^2\,m_3^2+(1-E_{23}^2)\,m_2^2\,m_3^2\\ &\quad +2(E_{23}-E_{12}E_{13})\,m_1^2\,m_2\,m_3+2(E_{12}-E_{13}E_{23})\,m_1\,m_2\,m_3^2,\\ H_3 &= (1-E_{12}^2)(1-E_{23}^2)\,m_1^2\,m_2^2\,m_3^2. \end{aligned} \tag{2.26}$$

The time dependence of the spectral data induced by the peakon evolution, which is $\dot{\lambda}_k = 0$ and $\dot{b}_k = b_k/\lambda_k$ as we already stated in (2.13) above, is an immediate corollary of the following facts (see equations (4.5), (4.22), (4.23) and Theorem 4.7 in [18]):

Theorem 2.20. *If the positions $x_k(t)$ and amplitudes $m_k(t)$ are functions of time which satisfy the Novikov peakon ODEs* (1.7)*, then the corresponding (seemingly time-dependent) polynomial $A(\lambda)$ is in fact independent of t; its coefficients $\{H_k\}_{k=1}^n$ are functionally independent constants of motion of the Novikov peakon ODEs* (1.7)*. The Weyl function $\omega(\lambda)$ satisfies the ODE*

$$\dot{\omega}(\lambda) = \frac{\omega(\lambda) - \omega(0)}{\lambda}. \tag{2.27}$$

This concludes our summary of the results from [18].

3 Continuity of solutions at peakon–antipeakon collisions

Since we will be dealing with peakon–antipeakon collisions, let us begin our study by stating some general properties which the Novikov equation has in common with the Camassa–Holm equation. The basic fact is that these equations behave similarly at collisions, and their behaviour is rather nice compared to the Degasperis–Procesi equation, for example. This similarity between peakon–antipeakon collisions for CH and for Novikov is due to preservation of the H^1 norm in both cases. Indeed, the arguments in this section are probably well-known to readers familiar with the Camassa–Holm equation, but we include them here for completeness.

Lemma 3.1. *The integral*

$$I = \int_{\mathbf{R}} \frac{u^2 + u_x^2}{2}\, dx \tag{3.1}$$

is a conserved quantity for smooth decaying solutions to Novikov's equation (1.1).

Proof. If u is a solution to Novikov's equation

$$u_t - u_{xxt} = (3u_x u_{xx} - 4uu_x + uu_{xxx})\, u$$

such that u and all its derivatives vanish as $x \to \pm\infty$, then

$$\begin{aligned}
\frac{dI}{dt} &= \int_{\mathbf{R}} \frac{\partial}{\partial t}\left(\frac{u^2 + u_x^2}{2}\right) dx = \int_{\mathbf{R}} (uu_t + u_x u_{xt})\, dx \\
&= \int_{\mathbf{R}} \big((uu_{xt})_x + u(u_t - u_{xxt})\big)\, dx \\
&= 0 + \int_{\mathbf{R}} u^2 (3u_x u_{xx} - 4uu_x + uu_{xxx})\, dx \\
&= \int_{\mathbf{R}} (u^3 u_{xx} - u^4)_x\, dx = 0.
\end{aligned}$$

□

In the context of peakon solutions, the integral I reduces to a sum:

Lemma 3.2. *If* $u(x) = \sum_{k=1}^{n} m_k\, e^{-|x-x_k|}$ *then*

$$I = \int_{\mathbf{R}} \frac{u^2 + u_x^2}{2}\, dx = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j e^{-|x_i - x_j|}. \tag{3.2}$$

Proof. We have $I = \frac{1}{2}\langle u, u\rangle$, where $\langle u, v\rangle = \int_{\mathbf{R}} (uv + u_x v_x)\, dx$ denotes the H^1 inner product. For $x_i \leq x_j$ we compute

$$\begin{aligned}\left\langle e^{-|x-x_i|}, e^{-|x-x_j|}\right\rangle &= \int_{-\infty}^{x_i} (e^{x-x_i} e^{x-x_j} + e^{x-x_i} e^{x-x_j})\, dx \\ &\quad + \int_{x_i}^{x_j} (e^{x_i-x} e^{x-x_j} + (-1) e^{x_i-x} e^{x-x_j})\, dx \\ &\quad + \int_{x_j}^{\infty} (e^{x_i-x} e^{x_j-x} + (-1)^2 e^{x_i-x} e^{x_j-x})\, dx \\ &= e^{x_i-x_j} + 0 + e^{x_i-x_j} = 2e^{-|x_i-x_j|},\end{aligned}$$

and by symmetry this holds also for $x_i \geq x_j$. Now the claim follows from the bilinearity of the inner product. □

Corollary 3.3. *The integral I remains a conserved quantity also for peakon solutions of the Novikov equation.*

Proof. The sum on the right-hand side in (3.2) is a constant of motion for the Novikov peakon ODEs. Indeed, in terms of the matrices E, P and T defined in (2.22), that sum equals the sum of the elements in the matrix PEP, which according to Theorem 2.18 (or by simple inspection) is the trace of the matrix $TPEP$; this trace was named H_1 in Theorem 2.15. Thus $I = H_1$ in the peakon case, and according to Theorem 2.20, H_1 is a constant of motion for the Novikov peakon ODEs. □

Theorem 3.4. *If* $u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x-x_k(t)|}$ *is a Novikov multipeakon solution for* $t_0 < t < t_1$*, and if there is a collision* $x_k(t_1) = x_{k+1}(t_1)$ *at time* t_1*, then*

$$u(x_{k+1}(t), t) - u(x_k(t), t) \to 0 \quad \text{as } t \to t_1^-, \tag{3.3}$$

so that $u(x,t)$ *extends continuously to* $t = t_1$.

Proof. Using Hölder's inequality, together with the fact that the integral $I = H_1$ is conserved, we get

$$\begin{aligned}|u(x_{k+1}(t), t) - u(x_k(t), t)| &\leq \int_{x_k(t)}^{x_{k+1}(t)} |u_x(\xi, t)|\, d\xi \\ &\leq \sqrt{\int_{x_k(t)}^{x_{k+1}(t)} u_x^2(\xi, t)\, d\xi}\, \sqrt{\int_{x_k(t)}^{x_{k+1}(t)} 1^2\, d\xi} \\ &\leq \sqrt{2H_1}\sqrt{x_{k+1}(t) - x_k(t)},\end{aligned} \tag{3.4}$$

so if $x_{k+1}(t) - x_k(t) \to 0$, then $u(x_{k+1}(t), t) - u(x_k(t), t) \to 0$. □

Remark 3.5. The wave profile at the collision, $u(x,t_1)$, will still have the multipeakon shape seen in Figure 2, but with fewer peaks. Indeed, it must be an **R**-linear combination of e^x and e^{-x} in each interval $x_i(t_1) < x < x_{i+1}(t_1)$, and for $x < x_1(t_1)$ and $x > x_n(t_1)$ it is a multiple of e^x and e^{-x}, respectively. And since it is a continuous function of x, this means that it can be written as a multipeakon. We will see several examples of this, for example in Figure 10.

Remark 3.6. There may be several pairs of peakons colliding at the same time, but that doesn't affect the argument above. However, three consecutive peakons cannot collide with each other at the same time; this is Theorem 5.15.

(To be precise, this has only been proved under the assumption that all eigenvalues λ_k are simple. In the multiple-eigenvalue case, treated in Section 6, we have no general proof, only special cases. For example, Theorem 6.32 says that if $n = 3$ and $\lambda_1 = \lambda_2 = \lambda_3$, then a triple collision cannot happen.)

Remark 3.7. We will see that the amplitudes $m_k(t)$ and $m_{k+1}(t)$ blow up to $+\infty$ or $-\infty$ when $x_k(t) = x_{k+1}(t)$; they are meromorphic functions of t with a pole at the time of collision. (But unlike the CH case, this pole is not necessarily simple; see Theorem 5.16.)

What Theorem 3.4 says is that there must be cancellation between the corresponding terms in the sum $u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x-x_k(t)|}$ so that $u(x,t)$ remains bounded, and even continuous, at the collision. However, the derivative $u_x(x,t)$ does not remain bounded; see for example Figure 14, where the steepening of the wave between the colliding peakons is clearly visible.

4 Solutions of the Novikov equation with one peakon and one antipeakon

As we remarked already in the introduction, Novikov's equation has the property that both peakons and antipeakons move to the right, since $\dot{x}_k = u(x_k)^2$. This is a significant difference from the Camassa–Holm and Degasperis–Procesi equations where antipeakons move to the left (except when they are close to a positive peakon of larger amplitude), and this feature gives rise to some interesting new kinds of peakon–antipeakon interaction.

The behaviour of an n-peakon solution which is given by the explicit solution formulas from Theorem 2.7 is completely determined by the values of the spectral parameters $\{\lambda_k, b_k(0)\}_{k=1}^{n}$. Section 2.3 cites the results from [18] which describe what happens in the pure peakon case, and the more general case of mixed peakon–antipeakon solutions (with arbitrary n) given by these same formulas will be described in Section 5 below. There are also non-generic cases where some eigenvalues have multiplicity greater than one; this cannot happen for pure peakon solutions, and solutions of this kind are not covered by Theorem 2.7. The modified solution formulas which describe such solutions are derived in Section 6.

As a preparation for understanding the various phenomena that occur in the general multipeakon case, it is worthwhile spending some time on examining the $n = 2$ peakon–antipeakon interaction in detail, and this is the topic of Section 4. For comparison, we also give a thorough review of the properties of the pure two-peakon solution.

4.1 The governing ODEs

For $n = 2$, the ODE system (1.7) governing the dynamics of peakons for Novikov's equation takes the form

$$\begin{aligned} \dot{x}_1 &= m_1^2 + 2m_1 m_2 E_{12} + m_2^2 E_{12}^2, \\ \dot{x}_2 &= m_1^2 E_{12}^2 + 2m_1 m_2 E_{12} + m_2^2, \\ \dot{m}_1 &= -m_1^2 m_2 E_{12} - m_1 m_2^2 E_{12}^2, \\ \dot{m}_2 &= m_1^2 m_2 E_{12}^2 + m_1 m_2^2 E_{12}, \end{aligned} \tag{4.1}$$

where $E_{12} = e^{-|x_1 - x_2|}$; note that $0 < E_{12} < 1$. If we assume that $x_1(t) < x_2(t)$, then we can write $E_{12} = e^{x_1 - x_2}$; this will hold at least for all t in some interval around $t = 0$ if we impose initial data such that $x_1(0) < x_2(0)$. We make no assumptions on the signs of m_1 and m_2, although if $m_k(0) = 0$, then the ODEs force $m_k(t) = 0$ for all t, so that the corresponding term $m_k e^{-|x - x_k|}$ in $u(x, t)$ is absent. Let us therefore assume that m_1 and m_2 are nonzero.

4.2 Pure peakon (or pure antipeakon) solutions

Before we look at peakon–antipeakon solutions, let us recall what the situation is like when all the amplitudes m_k have the same sign. Such solutions are described by Theorem 2.7, and the special case $n = 2$ follows below. (We already stated these formulas in Example 2.8, but we repeat them here for convenience.)

Theorem 4.1 (Pure peakon solutions with $n = 2$)**.** *Given initial conditions* $x_1(0) < x_2(0)$, $m_1(0) > 0$, $m_2(0) > 0$, *there is a unique solution of* (4.1), *which is globally defined and satisfies* $x_1(t) < x_2(t)$, $m_1(t) > 0$, $m_2(t) > 0$ *for all* $t \in \mathbf{R}$. *The set of all such pure peakon solutions is parametrized by four constants* $0 < \lambda_1 < \lambda_2$, $b_1(0) > 0$, $b_2(0) > 0$, *given explicitly* *by the formulas*

$$
\begin{aligned}
x_1(t) &= \frac{1}{2}\ln \frac{\dfrac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\, b_1^2 b_2^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2}, \\
x_2(t) &= \frac{1}{2}\ln\left(\frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right), \\
m_1(t) &= \frac{\left(\dfrac{(\lambda_1-\lambda_2)^4 b_1^2 b_2^2}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2\right)\right)^{1/2}}{\dfrac{(\lambda_1-\lambda_2)^2 b_1 b_2}{\lambda_1+\lambda_2}\,(b_1+b_2)}, \\
m_2(t) &= \frac{\left(\dfrac{b_1^2}{\lambda_1} + \dfrac{b_2^2}{\lambda_2} + \dfrac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right)^{1/2}}{b_1+b_2},
\end{aligned}
\tag{4.2a}
$$

where

$$
b_1 = b_1(t) = b_1(0)\, e^{t/\lambda_1}, \qquad b_2 = b_2(t) = b_2(0)\, e^{t/\lambda_2}. \tag{4.2b}
$$

Corollary 4.2 (Pure antipeakon solutions with $n = 2$)**.** *With initial data* $x_1(0) < x_2(0)$*,* $m_1(0) < 0$*,* $m_2(0) < 0$*, there is a unique solution of* (4.1) *which is globally defined and satisfies* $x_1(t) < x_2(t)$*,* $m_1(t) < 0$*,* $m_2(t) < 0$ *for all* $t \in \mathbf{R}$*. The set of such pure antipeakon solutions is parametrized by the four constants* $0 < \lambda_1 < \lambda_2$*,* $b_1(0) < 0$*,* $b_2(0) < 0$*, and they are given explicitly by the same formulas as in Theorem 4.1.*

Proof. Given a solution of (4.1), we obtain another solution by keeping $x_1(t)$ and $x_2(t)$ but changing the sign of $m_1(t)$ and $m_2(t)$. In terms of the formulas (4.2), this is accomplished by keeping λ_1 and λ_2 but changing the sign of b_1 and b_2. □

Remark 4.3. Note that this differs from the Camassa–Holm and Degasperis–Procesi equations, where the pure antipeakon solutions are obtained by changing the signs of the eigenvalues λ_k, but leaving the residues b_k unchanged. When it comes to mixed peakon–antipeakon solutions we will see further differences. For CH, the eigenvalues are always real and simple, the residues are always positive, and the number of positive and negative eigenvalues equals the number of peakons and antipeakons, respectively. (This is proved in [2], except that they use the opposite sign convention for the eigenvalues.) For DP, the eigenvalues need neither be real nor simple in the mixed peakon–antipeakon case with $n \geq 3$, and at least for $n = 3$ the number of eigenvalues with positive and negative real part equals the number of peakons and antipeakons, respectively [28, 29]. Also for Novikov's equation, the eigenvalues can be complex or of multiplicity greater than one, but they must always have positive real part, as we shall see.

Remark 4.4. In Section 2.3, especially Theorem 2.15, we explain where formulas (4.2) come from. Let us recall what this construction looks like in the special case $n = 2$: the

Weyl function is defined as

$$\omega(\lambda) = -\frac{B(\lambda)}{A(\lambda)}, \tag{4.3}$$

where the polynomials $A(\lambda)$ and $B(\lambda)$ are given by equation (2.19) (cf. also (2.25)); with $E_{12} = e^{x_1 - x_2}$, one finds

$$B(\lambda) = (m_1 e^{x_1} + m_2 e^{x_2}) - \lambda m_1^2 m_2 e^{x_2} (1 - E_{12}^2) \tag{4.4}$$

and

$$A(\lambda) = 1 - H_1 \lambda + H_2 \lambda^2, \tag{4.5}$$

where

$$H_1 = m_1^2 + m_2^2 + 2 m_1 m_2 E_{12}, \qquad H_2 = m_1^2 m_2^2 (1 - E_{12}^2). \tag{4.6}$$

If $A(\lambda)$ has simple zeros λ_1 and λ_2, then the Weyl function has a partial fraction decomposition of the form

$$\omega(\lambda) = \frac{b_1}{\lambda - \lambda_1} + \frac{b_2}{\lambda - \lambda_2}. \tag{4.7}$$

By identifying coefficients we see that this holds if and only if

$$\begin{aligned}
\frac{1}{\lambda_1} + \frac{1}{\lambda_2} &= H_1 = m_1^2 + m_2^2 + 2 m_1 m_2 E_{12}, \\
\frac{1}{\lambda_1 \lambda_2} &= H_2 = m_1^2 m_2^2 (1 - E_{12}^2), \\
\frac{b_1}{\lambda_1} + \frac{b_2}{\lambda_2} &= m_1 e^{x_1} + m_2 e^{x_2}, \\
b_1 + b_2 &= \frac{e^{x_2}}{m_2},
\end{aligned} \tag{4.8}$$

and the formulas (4.2a) give the unique solution of this system of equations for the variables $\{x_1, x_2, m_1, m_2\}$.

Next, recall that the asymptotic behaviour of pure peakon solutions is fully described by Theorem 2.9 and Remark 2.11. Let us state the special case $n = 2$ explicitly, for later comparison with the peakon–antipeakon case:

Theorem 4.5. *For a Novikov pure two-peakon solution, given by* (4.2) *with positive simple eigenvalues* $0 < \lambda_1 < \lambda_2$ *and positive residues* $b_1(0)$ *and* $b_2(0)$*, there are no collisions, i.e.,* $x_1(t) < x_2(t)$ *for all* t*. With*

$$\delta = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} > 0, \tag{4.9}$$

the asymptotics as $t \to -\infty$ *are*

$$
\begin{aligned}
x_1(t) &= \frac{t}{\lambda_1} + \frac{1}{2}\ln\frac{b_1(0)^2}{\lambda_1} + \ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_2} + \mathcal{O}\big(e^{\delta t}\big),\\
x_2(t) &= \frac{t}{\lambda_2} + \frac{1}{2}\ln\frac{b_2(0)^2}{\lambda_2} + \mathcal{O}\big(e^{\delta t}\big),\\
m_1(t) &= \frac{1}{\sqrt{\lambda_1}} + \mathcal{O}\big(e^{\delta t}\big),\\
m_2(t) &= \frac{1}{\sqrt{\lambda_2}} + \mathcal{O}\big(e^{\delta t}\big),
\end{aligned}
\tag{4.10a}
$$

and as $t \to +\infty$ *we have*

$$
\begin{aligned}
x_1(t) &= \frac{t}{\lambda_2} + \frac{1}{2}\ln\frac{b_2(0)^2}{\lambda_2} + \ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1} + \mathcal{O}\big(e^{-\delta t}\big),\\
x_2(t) &= \frac{t}{\lambda_1} + \frac{1}{2}\ln\frac{b_1(0)^2}{\lambda_1} + \mathcal{O}\big(e^{-\delta t}\big),\\
m_1(t) &= \frac{1}{\sqrt{\lambda_2}} + \mathcal{O}\big(e^{-\delta t}\big),\\
m_2(t) &= \frac{1}{\sqrt{\lambda_1}} + \mathcal{O}\big(e^{-\delta t}\big).
\end{aligned}
\tag{4.10b}
$$

Remark 4.6. The Novikov equation is invariant with respect to translations in x and t, so we can reduce the number of parameters by two. Indeed, consider the new independent variables $\tilde{x} = x - x_0$ and $\tilde{t} = t - t_0$, and the corresponding dependent variables $\tilde{x}_k(\tilde{t}) = x_k(\tilde{t} + t_0) + x_0$ and $\tilde{m}_k(\tilde{t}) = m_k(\tilde{t} + t_0)$. Then $\{\tilde{x}_k(\tilde{t}), \tilde{m}_k(\tilde{t})\}$ are given by (4.2), with λ_1 and λ_2 unchanged, but with $b_k(t) = b_k(0)\, e^{t/\lambda_k}$ replaced by $\tilde{b}_k(\tilde{t}) = \tilde{b}_k(0)\, e^{\tilde{t}/\lambda_k}$ where $\tilde{b}_k(0) = b_k(0)\, e^{-x_0}\, e^{t_0/\lambda_k}$. Consequently we can make $\tilde{b}_1(0)$ and $\tilde{b}_2(0)$ take any positive values by a suitable choice of x_0 and t_0. Thus, λ_1 and λ_2 are the essential parameters in the two-peakon solution, while the values of $b_1(0)$ and $b_2(0)$ merely reflect the choice of origin in the coordinate system. (But for the n-peakon solution with $n \geq 3$, things are of course not this simple, since we can only simplify two of the residues b_k in this way, not all n of them.) As the next theorem shows, there is a particularly useful choice of parameters which reveals that the pure two-peakon solution always has a certain symmetry (which the n-peakon solution in general does not have); see Figure 8 for an illustration.

Theorem 4.7. *When*

$$
b_1(0) = b_2(0) = \frac{\sqrt{(\lambda_1+\lambda_2)\lambda_1\lambda_2}}{\lambda_2 - \lambda_1},
\tag{4.11}
$$

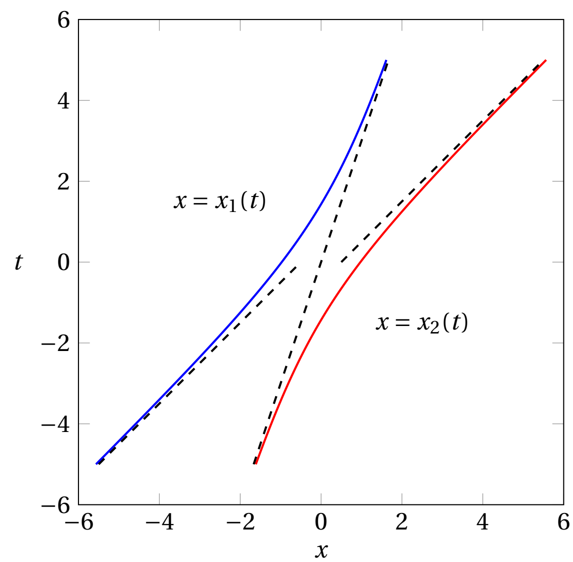

Figure 8: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the Novikov pure 2-peakon solution, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. The residues $b_1(0) = b_2(0) = \sqrt{3}$ are chosen as in Theorem 4.7, in order to make the solution symmetric with respect to the origin. According to (4.13), as $t \to -\infty$ the curves $x = x_1(t)$ and $x = x_2(t)$ approach the lines $x = t - \frac{1}{2}\ln 3$ and $x = t/3$, respectively, and (symmetrically) as $t \to +\infty$ the curves approach the lines $x = t/3$ and $x = t + \frac{1}{2}\ln 3$. The eigenvalue ratio $\lambda_2/\lambda_1 = 3$ is a bit special in that the asymptotic line for the slower peakon is the same in both time directions. In general there will be a shift for both peakons; always in the forward direction for the faster peakon, and forward/backward for the slower peakon if λ_2/λ_1 is greater/less than 3.

the pure two-peakon solution with $0 < \lambda_1 < \lambda_2$ *takes the following symmetric form:*

$$\begin{aligned}
x_1(t) &= -x_2(-t),\\
m_1(t) &= m_2(-t),\\
x_2(t) &= \frac{1}{2}\ln\left(\frac{e^{2t/\lambda_1}}{\lambda_1}+\frac{e^{2t/\lambda_2}}{\lambda_2}+\frac{4\,e^{t/\lambda_1+t/\lambda_2}}{\lambda_1+\lambda_2}\right)-\frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2},\\
m_2(t) &= \frac{1}{e^{t/\lambda_1}+e^{t/\lambda_2}}\left(\frac{e^{2t/\lambda_1}}{\lambda_1}+\frac{e^{2t/\lambda_2}}{\lambda_2}+\frac{4\,e^{t/\lambda_1+t/\lambda_2}}{\lambda_1+\lambda_2}\right)^{1/2}.
\end{aligned} \tag{4.12}$$

In this case, the asymptotics for the positions as $t \to \pm\infty$ *are*

$$\begin{aligned}
x_1(t) &= \frac{t}{\lambda_1}+\frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_2}+\mathcal{O}\big(e^{\delta t}\big), && \text{as } t\to-\infty,\\
x_2(t) &= \frac{t}{\lambda_2}-\frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1}+\mathcal{O}\big(e^{\delta t}\big), && \text{as } t\to-\infty,\\
x_1(t) &= \frac{t}{\lambda_2}+\frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1}+\mathcal{O}\big(e^{-\delta t}\big), && \text{as } t\to+\infty,\\
x_2(t) &= \frac{t}{\lambda_1}-\frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_2}+\mathcal{O}\big(e^{-\delta t}\big), && \text{as } t\to+\infty,
\end{aligned} \tag{4.13}$$

where $\delta = 1/\lambda_1 - 1/\lambda_2 > 0$.

4.3 Extending the solution formulas to describing peakon–antipeakon solutions

Next, we will show that the formulas for pure two-peakon solutions can also be used to describe solutions with one peakon and one antipeakon, i.e., solutions where m_1 and m_2 have opposite signs: $m_1\, m_2 < 0$. This is done simply by extending the range of allowed spectral data. The following theorem is a special case of the much more general results in Section 5, but in the spirit of this section, we will prove it here using simple direct arguments.

Theorem 4.8. *The formulas* (4.2) *in Theorem 4.1 provide solutions to the Novikov peakon ODEs* (4.1) *also in the case*

$$0 < \lambda_1 < \lambda_2, \qquad b_1(0) \in \mathbf{R}, \qquad b_2(0) \in \mathbf{R}, \qquad b_1(0)\, b_2(0) < 0, \tag{4.14}$$

and in the case

$$\lambda_1 = \overline{\lambda_2} \in \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0, \operatorname{Im}\lambda \neq 0\}, \qquad b_1(0) = \overline{b_2(0)} \in \mathbf{C}\setminus\{0\}. \tag{4.15}$$

These solutions have the property that $x_1(t) < x_2(t)$ *and* $m_1(t)\, m_2(t) < 0$ *for all* t*, except for a discrete set of values of* t *where* $b_1(t) + b_2(t) = 0$*. At those particular instants there* *is a collision:* $x_1(t) = x_2(t)$*, while* $m_1(t)$ *and* $m_2(t)$ *are undefined. Thus, according to the discussion in Section 3, the function* $u(x,t)$*, defined by the formula*

$$u(x,t) = m_1(t)\, e^{-|x-x_1(t)|} + m_2(t)\, e^{-|x-x_2(t)|}$$

for all t *such that* $b_1(t) + b_2(t) \neq 0$*, extends uniquely to a globally defined weak solution of Novikov's equation.*

Proof. Note first that the conditions above are preserved by the time evolution $b_k(t) = b_k(0)\, e^{t/\lambda_k}$, i.e., if (4.14) holds, then $b_k(t) \in \mathbf{R}$ and $b_1(t)\, b_2(t) < 0$ for all t, and if (4.15) holds, then $b_1(t) = \overline{b_2(t)} \in \mathbf{C} \setminus \{0\}$ for all t.

It is a purely algebraic fact that the functions defined by the formulas (4.2) satisfiy the ODEs (4.1) if E_{12} is interpreted as $e^{x_1 - x_2}$; in fact, we can take arbitrary complex constants $b_1(0) \neq 0$, $b_2(0) \neq 0$ and $\lambda_1 \neq \lambda_2$, if we choose some branches for the complex logarithm and square root functions appearing in (4.2a). Consequently, if the parameters $\{\lambda_k, b_k(0)\}$ are chosen such that the expressions inside the logarithms and the square roots are positive, and such that $x_1(t) < x_2(t)$, then (4.2) provides a solution to the actual peakon ODEs (4.1) where E_{12} is interpreted as $e^{-|x_1 - x_2|}$, so these are the conditions that we need to verify.

The positivity of the expressions inside logaritms and square roots is fairly obvious in both cases (4.14) and (4.15), except for the quantities

$$W_1 = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1 + \lambda_2} b_1 b_2$$

and

$$Z_1 = \frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1 + \lambda_2}\, b_1 b_2.$$

In the real case (4.14), we can see the positivity of W_1 from

$$W_1 = \lambda_1 \Big(b_1 + \frac{2\lambda_2}{\lambda_1 + \lambda_2}\, b_2 \Big)^2 + \lambda_2 \Big(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\, b_2 \Big)^2 > 0,$$

and in the complex-conjugate case (4.15) we have

$$\begin{aligned} W_1 &= \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \\ &= \lambda_1 b_1^2 + \overline{\lambda_1 b_1^2} + \frac{4\lambda_1 \overline{\lambda_1}}{\lambda_1 + \overline{\lambda_1}} b_1 \overline{b_1} \\ &= 2\,\mathrm{Re}(\lambda_1 b_1^2) + \frac{4|\lambda_1|^2}{2\,\mathrm{Re}(\lambda_1)}\, |b_1|^2 \\ &= \frac{2}{\mathrm{Re}(\lambda_1)} \Big(\mathrm{Re}(\lambda_1)\, \mathrm{Re}(\lambda_1 b_1^2) + |\lambda_1| \big|\lambda_1 b_1^2\big| \Big) > 0. \end{aligned}$$

For Z_1, just replace b_k by b_k/λ_k in these arguments.

The properties $x_1 < x_2$ and $m_1 m_2 < 0$ follow from

$$e^{2x_2} - e^{2x_1} = \frac{(b_1 + b_2)^4}{W_1} > 0 \qquad \text{if } b_1 + b_2 \neq 0,$$

and

$$-\operatorname{sgn}(m_1) = \operatorname{sgn}(m_2) = \operatorname{sgn}(b_1 + b_2).$$

As for the zeros of $b_1(t) + b_2(t)$, it is easy to see that in the real case there is exactly one zero, and in the complex-conjugate case $b_1 + b_2$ becomes zero periodically; see Theorem 4.12 and Remark 4.16 for details. □

Remark 4.9. Theorem 4.10 below shows that we cannot extend the range of allowed spectral data beyond this, i.e., for other values of $\{\lambda_k, b_k(0)\}$ the formulas (4.2) do not provide a solution to the peakon ODEs. There is one more type of solution, namely peakon–antipeakon solutions with $\lambda_1 = \lambda_2 > 0$, but in this case the solution formulas look different, as will be explained in Section 4.7 (and more generally in Section 6).

We shall return shortly to the question of how the peakon–antipeakon solution in Theorem 4.8 actually behave, but first we will sort out exactly when the real case (4.14) and the complex case (4.15) occur, in terms of the initital data $\{x_k(0), m_k(0)\}$.

4.4 Classification in terms of initial data

With just one peakon and one antipeakon, it is possible to give explicit conditions which classify the behaviour of the system not just in terms of spectral data as in Sections 4.2 and 4.3, but also in terms of the initial conditions for the positions $x_1(0) < x_2(0)$ and the amplitudes $m_1(0) \neq 0$ and $m_2(0) \neq 0$.

Given some initial data, the eigenvalues λ_1 and λ_2 are defined as the zeros of the polynomial $A(\lambda) = 1 - H_1\lambda + H_2\lambda^2$, where

$$\begin{aligned} H_1 &= m_1(0)^2 + m_2(0)^2 + 2m_1(0)m_2(0)E_{12}(0), \\ H_2 &= m_1(0)^2 m_2(0)^2 (1 - E_{12}(0)^2); \end{aligned}$$

see (4.5) and (4.6) above. Since $E_{12} = e^{-|x_1 - x_2|} = e^{x_1 - x_2}$ always satisfies $0 < E_{12} < 1$, we see that H_1 and H_2 are both positive, regardless of the signs of m_1 and m_2. This observation implies that the eigenvalues

$$\lambda_{1,2} = \frac{H_1 \pm \sqrt{H_1^2 - 4H_2}}{2H_2} \tag{4.16}$$

are located strictly in the right half of the complex plane:

$$\operatorname{Re} \lambda_{1,2} > 0. \tag{4.17}$$

Remember that in the pure peakon case one can say much more: λ_1 and λ_2 are positive and distinct. Now that we allow m_1 and m_2 to have opposite signs, we might also get a positive eigenvalue of multiplicity two, or a pair of complex conjugate eigenvalues. Let us state the exact conditions for this to happen.

Theorem 4.10. *For a Novikov two-peakon solution, the eigenvalues $\lambda_{1,2}$ are non-real if and only if $m_1 m_2 < 0$ (the mixed peakon–antipeakon case) and the amplitude ratio*

$$\rho = \left| \frac{m_1}{m_2} \right|$$

is such that the quantity

$$\sigma = \rho + \frac{1}{\rho}$$

satisfies

$$2 \le \sigma < \sqrt{8} \qquad \left(\iff \frac{1}{\sqrt{2}+1} < \rho < \sqrt{2}+1 \right) \tag{4.18}$$

and moreover the positions $x_1 < x_2$ are such that $E_{12} = e^{x_1 - x_2}$ satisfies

$$\tfrac{1}{4}\left(\sigma - \sqrt{8-\sigma^2}\right) < E_{12} < \tfrac{1}{4}\left(\sigma + \sqrt{8-\sigma^2}\right). \tag{4.19}$$

In this case, the real part of $\lambda_1 = \overline{\lambda_2}$ is positive, and $b_1 = \overline{b_2} \neq 0$.

The case with a double eigenvalue $\lambda_1 = \lambda_2 > 0$ occurs if and only if $m_1 m_2 < 0$ and

$$2 < \sigma < \sqrt{8} \qquad \left(\iff \frac{1}{\sqrt{2}+1} < \rho < \sqrt{2}+1 \quad \textit{and} \quad \rho \neq 1 \right) \tag{4.20}$$

and

$$E_{12} = \tfrac{1}{4}\left(\sigma - \sqrt{8-\sigma^2}\right) \qquad \textit{or} \qquad E_{12} = \tfrac{1}{4}\left(\sigma + \sqrt{8-\sigma^2}\right). \tag{4.21}$$

(In this case, the complementary spectral variables b_1 and b_2 are defined differently; see Section 4.7.)

Otherwise λ_1 and λ_2 are positive and distinct. In this case, if m_1 and m_2 are both positive or both negative, then b_1 and b_2 are both positive or both negative, respectively. If m_1 and m_2 have opposite signs, then b_1 and b_2 have opposite signs, and

$$-\operatorname{sgn}(m_1) = \operatorname{sgn}(m_2) = \operatorname{sgn}(b_1 + b_2) \tag{4.22}$$

holds.

See Figure 9 for an illustration of the conditions in this theorem.

Remark 4.11. Although σ and E_{12} are time-dependent, the eigenvalues λ_k are not. Thus if the conditions of Theorem 4.10 are satisfied at some time t, then they hold for all t (except of course at the instant of a collision, where m_1 and m_2 are undefined).

Proof of Theorem 4.10. From (4.16) it is clear that the eigenvalues are non-real if and only if

$$0 > H_1^2 - 4H_2 = (m_1^2 + m_2^2 + 2m_1 m_2 E_{12})^2 - 4m_1^2 m_2^2 (1 - E_{12}^2).$$

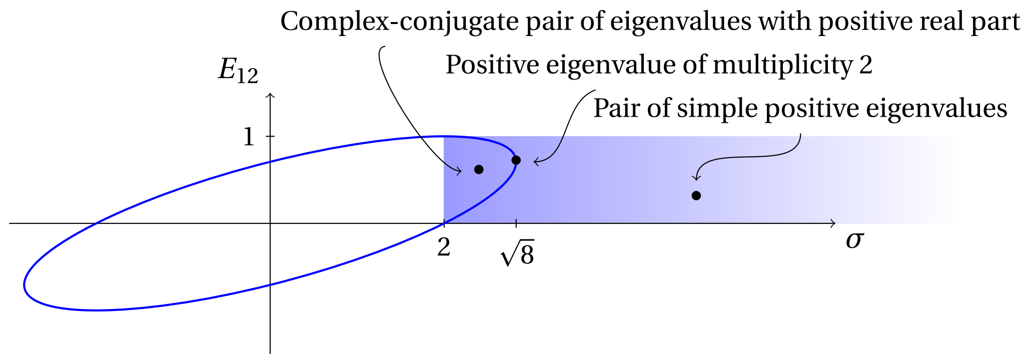

Figure 9: If m_1 and m_2 are of the same sign, the eigenvalues λ_1 and λ_2 in the two-peakon solution of the Novikov equation will always be positive and simple. But if m_1 and m_2 are of opposite signs, the character of the eigenvalues is determined by the conditions in Theorem 4.10, illustrated in this picture: the quantities $\sigma = |m_1/m_2| + |m_2/m_1|$ and $E_{12} = e^{x_1-x_2}$ automatically satisfy $\sigma \geq 2$ and $0 < E_{12} < 1$ (since $x_1 < x_2$ is assumed), and within this strip we distinguish the three cases that the point (σ, E_{12}) lies inside/on/outside the ellipse $E_{12} = \frac{1}{4}\bigl(\sigma \pm \sqrt{8-\sigma^2}\bigr)$, i.e., $(\frac{\sigma}{2} - E_{12})^2 + E_{12}^2 = 1$. We see that in order for non-real eigenvalues to occur, the amplitude ratio $\rho = |m_1/m_2|$ must be fairly close to 1, and exactly how much it is allowed to deviate from 1 depends on how far apart the peakons are.

Dividing by the positive factor $8m_1^2m_2^2$, one obtains the equivalent inequality

$$\begin{aligned} 0 &> \left(E_{12} + \frac{m_1^2+m_2^2}{4m_1m_2}\right)^2 + \frac{m_1^4+m_2^4-6m_1^2m_2^2}{16m_1^2m_2^2} \\ &= \left(E_{12} + \operatorname{sgn}(m_1m_2)\frac{\sigma}{4}\right)^2 + \frac{\sigma^2-8}{16}. \end{aligned} \tag{4.23}$$

A necessary condition for this inequality to have any solutions is that $\sigma^2 - 8 < 0$, i.e., $\sigma < \sqrt{8}$, which is the nontrivial half of condition (4.18); the other inequality $\sigma \geq 2$ is automatic, since $\rho + \rho^{-1} \geq 2$ holds for all $\rho > 0$ (with equality if and only if $\rho = 1$).

With this condition satisfied, one can solve for E_{12} in (4.23), to get

$$\tfrac{1}{4}\bigl(-\operatorname{sgn}(m_1m_2)\sigma - \sqrt{8-\sigma^2}\bigr) < E_{12} < \tfrac{1}{4}\bigl(-\operatorname{sgn}(m_1m_2)\sigma + \sqrt{8-\sigma^2}\bigr)$$

In the pure peakon (or pure antipeakon) case $m_1m_2 > 0$, the eigenvalues must be real according to the general results in [18], and we can also see directly that in this case the inequality above can't be satisfied: $E_{12} = e^{x_1-x_2} > 0$, but $\frac{1}{4}\bigl(-\sigma + \sqrt{8-\sigma^2}\bigr) \leq 0$ for $\sigma \leq 2 < \sqrt{8}$ (it's a decreasing function of σ which equals 0 when $\sigma = 2$).

In the mixed peakon–antipeakon case $m_1m_2 < 0$, the inequality reduces to (4.19); in this case there are indeed solutions, since the left-hand and right-hand sides are both in the interval $[0,1]$; see Figure 9.

The statement about double eigenvalues follows by repeating the same calculations with $H_1^2 - 4H_2 = 0$ instead of $H_1^2 - 4H_2 < 0$. The reason for excluding $\sigma = 2$ is that condition (4.21) becomes "$E_{12} = 0$ or $E_{12} = 1$" in this case, and this is impossible since $0 < E_{12} < 1$ always.

As for b_1 and b_2, they are given by the last two equations in (4.8), which form a linear system once λ_1 and λ_2 are known:

$$\begin{pmatrix} 1/\lambda_1 & 1/\lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} m_1 e^{x_1} + m_2 e^{x_2} \\ e^{x_2}/m_2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & -1/\lambda_2 \\ -1 & 1/\lambda_1 \end{pmatrix} \begin{pmatrix} m_1 e^{x_1} + m_2 e^{x_2} \\ e^{x_2}/m_2 \end{pmatrix}.$$

From this it is clear that $b_1 = \overline{b_2}$ if $\lambda_1 = \overline{\lambda_2}$, and we can also compute the expression $-\left(\frac{\lambda_2-\lambda_1}{(\lambda_1\lambda_2}\right)^2 b_1 b_2$ to be equal to

$$\begin{aligned}
&(m_1 e^{x_1} + m_2 e^{x_2})^2 - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)(m_1 e^{x_1} + m_2 e^{x_2}) \frac{e^{x_2}}{m_2} + \frac{1}{\lambda_1 \lambda_2} \left(\frac{e^{x_2}}{m_2}\right)^2 \\
&= (m_1 e^{x_1} + m_2 e^{x_2})^2 - H_1 (m_1 e^{x_1} + m_2 e^{x_2}) \frac{e^{x_2}}{m_2} + H_2 \left(\frac{e^{x_2}}{m_2}\right)^2 \\
&= m_1^2 e^{2x_1} + m_2^2 e^{2x_2} + 2 m_1 m_2 e^{x_1+x_2} - H_1 (m_1 e^{x_1} + m_2 e^{x_2}) \frac{e^{x_2}}{m_2} + m_1^2 (e^{2x_2} - e^{2x_1}) \\
&= m_2^2 e^{2x_2} + 2 m_1 m_2 e^{x_1+x_2} - H_1 (m_1 e^{x_1} + m_2 e^{x_2}) \frac{e^{x_2}}{m_2} + m_1^2 e^{2x_2} \\
&= H_1 e^{2x_2} - H_1 (m_1 e^{x_1} + m_2 e^{x_2}) \frac{e^{x_2}}{m_2} \\
&= -\frac{H_1 m_1 e^{x_1+x_2}}{m_2} \neq 0,
\end{aligned}$$

which shows that b_1 and b_2 are nonzero. The sign relations between $m_{1,2}$ and $b_{1,2}$ in the case of real simple eigenvalues can be seen from (4.2a), as already noted in the proof of Theorem 4.1. □

Next, we consider the dynamics of a peakon–antipeakon pair in these different cases separately.

4.5 Dynamics of a peakon–antipeakon pair in the case of positive simple eigenvalues

In this section we consider the case $0 < \lambda_1 < \lambda_2$, with b_1 and b_2 of opposite sign, so that we get a peakon–antipeakon solution (rather than a pure peakon or pure antipeakon solution as in Section 4.2 above). In terms of initial data $\{x_k(0), m_k(0)\}$, this case occurs when $m_1(0)\, m_2(0) < 0$ and the point

$$(\sigma, E_{12}) = \left(\left|\frac{m_1(0)}{m_2(0)}\right| + \left|\frac{m_2(0)}{m_1(0)}\right|, e^{x_1(0)-x_2(0)} \right)$$

lies outside of the ellipse in Figure 9.

Theorem 4.12. *For a Novikov peakon–antipeakon pair, given by* (4.2) *with positive simple eigenvalues* $0 < \lambda_1 < \lambda_2$ *and residues of opposite signs,* $b_1(0)\, b_2(0) < 0$*, there is exactly one collision:* $x_1(t) = x_2(t)$ *at* $t = t_0$*, where*

$$t_0 = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \ln\left(-\frac{b_2(0)}{b_1(0)}\right). \tag{4.24}$$

The asymptotics as $t \to \pm\infty$ *are given by the same formulas* (4.10) *as in the pure peakon case, except for the signs of* m_1 *and* m_2*:*

$$\begin{aligned} m_1(t) &= \frac{\operatorname{sgn} b_1(0)}{\sqrt{\lambda_1}} + \mathscr{O}\big(e^{\delta t}\big), \quad & m_2(t) &= \frac{\operatorname{sgn} b_2(0)}{\sqrt{\lambda_2}} + \mathscr{O}\big(e^{\delta t}\big), \quad & &\text{as } t \to -\infty, \\ m_1(t) &= \frac{\operatorname{sgn} b_2(0)}{\sqrt{\lambda_2}} + \mathscr{O}\big(e^{-\delta t}\big), & m_2(t) &= \frac{\operatorname{sgn} b_1(0)}{\sqrt{\lambda_1}} + \mathscr{O}\big(e^{-\delta t}\big), & &\text{as } t \to +\infty. \end{aligned} \tag{4.25}$$

Proof. By Theorem 4.8, collisions happen when

$$0 = b_1(t) + b_2(t) = b_1(0)\, e^{t/\lambda_1} + b_2(0)\, e^{t/\lambda_2}$$

i.e.,

$$e^{t/\lambda_1 - t/\lambda_2} = -b_2(0)/b_1(0) > 0,$$

which clearly has exactly one solution, given by (4.24).

The proof of the asymptotic formulas is nearly identical to the pure peakon case. For example, as $t \to +\infty$ we have

$$\frac{b_2}{b_1} = \frac{b_2(t)}{b_1(t)} = \frac{b_2(0)\, e^{t/\lambda_2}}{b_1(0)\, e^{t/\lambda_1}} = \frac{b_2(0)}{b_1(0)}\, e^{-\delta t} = \mathscr{O}\big(e^{-\delta t}\big),$$

so

$$\begin{aligned} x_2(t) &= \frac{1}{2} \ln\left(\frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1 + \lambda_2}\, b_1 b_2\right) \\ &= \frac{1}{2} \ln\left(\frac{b_1^2}{\lambda_1}\left(1 + \frac{4}{\lambda_1 + \lambda_2}\,\frac{b_2}{b_1} + \frac{\lambda_1}{\lambda_2}\,\frac{b_2^2}{b_1^2}\right)\right) \\ &= \frac{1}{2} \ln\left(\frac{b_1^2}{\lambda_1}\left(1 + \mathscr{O}\big(e^{-\delta t}\big) + \mathscr{O}\big(e^{-2\delta t}\big)\right)\right) \\ &= \frac{1}{2} \ln \frac{b_1(0)^2\, e^{2t/\lambda_1}}{\lambda_1} + \frac{1}{2} \ln\left(1 + \mathscr{O}\big(e^{-\delta t}\big)\right) \\ &= \frac{t}{\lambda_1} + \frac{1}{2} \ln \frac{b_1(0)^2}{\lambda_1} + \mathscr{O}\big(e^{-\delta t}\big). \end{aligned}$$

The only difference is that we have to take the sign of b_k into account when simplifying $(b_k^2)^{1/2}$ in the expressions for m_1 and m_2. □

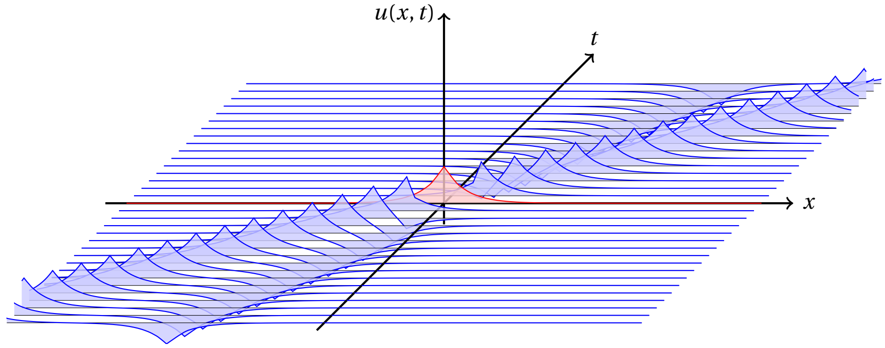

Figure 10: The graph of $u(x,t)$ for the peakon–antipeakon solution in Theorem 4.13. The eigenvalues are real and simple, $\lambda_1 = 1/2$ and $\lambda_2 = 1$, so the asymptotic velocities are 2 (for the peakon) and 1 (for the antipeakon). The residues, $b_1(0) = -b_2(0) = \sqrt{3}$, are chosen to place the collision at the origin. At the instant of collision, the two peakons merge into the single peak $u(x,0) = 2\,e^{-|t|}$, indicated with red in the picture. The projection is parallel, the scale is the same on all axes, and the domain shown is $-15 \le x \le 15$ and $-8 \le t \le 8$, with the wave profile $u(x,t)$ sampled at equidistant times $t = n/2$, $n \in \mathbf{Z}$.

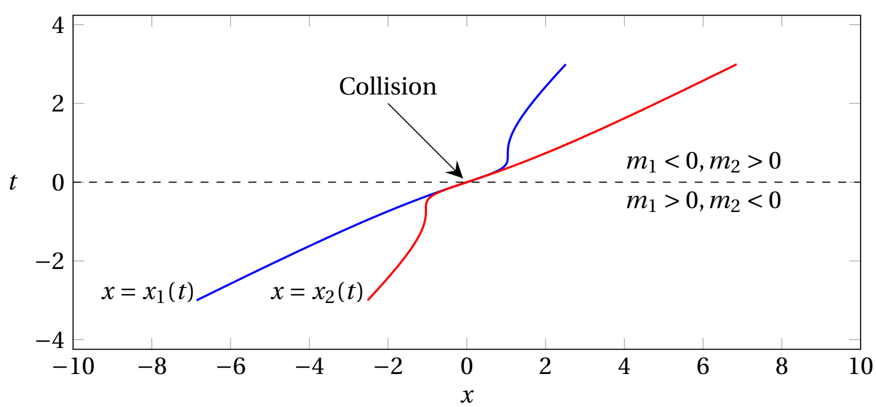

Figure 11: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the peakon–antipeakon solution in Figure 10 with $\lambda_1 = 1/2$, $\lambda_2 = 1$ and $b_1(0) = -b_2(0) = \sqrt{3}$. Note how the curves approach the straight lines $x = t/\lambda_k + \text{const.}$ given by (4.28) as $t \to \pm\infty$, and that both curves are tangent to the line $x = (1/\lambda_1 + 1/\lambda_2)\,t = 3t$ at the origin, in agreement with (4.29).

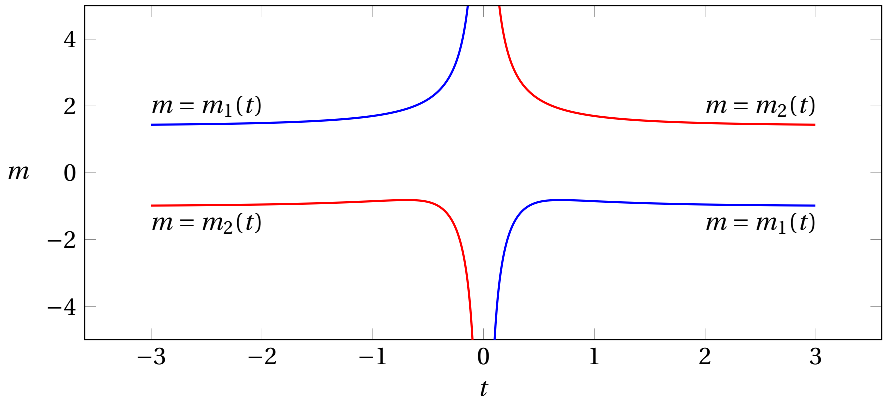

Figure 12: Graphs of the amplitudes $m_k(t)$ for the peakon–antipeakon solution in Figures 10 and 11. At the instant of collision, $t = 0$, both $m_1(t)$ and $m_2(t)$ blow up, but they do it in such a way that the sum $m_1(t) + m_2(t)$, and hence the wave elevation $u = m_1 e^{-|x-x_1|} + m_2 e^{-|x-x_2|}$, tends to a finite limit.

Just like in Remark 4.6, we can rescale $b_1(0)$ and $b_2(0)$ as we like (except altering their signs), by making a suitable translation of the (x, t) coordinate system. To see clearer what happens at the collision, it is useful to place the origin of the coordinate system at the site of the collision; this corresponds to one of the two choices

$$b_1(0) = -b_2(0) = \pm \frac{\sqrt{(\lambda_1 + \lambda_2)\lambda_1\lambda_2}}{\lambda_2 - \lambda_1} = \pm \frac{\sqrt{\left(1 + \frac{\lambda_2}{\lambda_1}\right)\lambda_2}}{\frac{\lambda_2}{\lambda_1} - 1}.$$

Changing the sign here merely flips the signs of m_1 and m_2, so it is enough to study one case. The following theorem describes the situation where m_1 starts out positive and m_2 negative; see Figures 10, 11 and 12 for illustrations.

Theorem 4.13. *When*

$$b_1(0) = -b_2(0) = \frac{\sqrt{(\lambda_1 + \lambda_2)\lambda_1\lambda_2}}{\lambda_2 - \lambda_1}, \tag{4.26}$$

the Novikov $n = 2$ peakon–antipeakon solution with $0 < \lambda_1 < \lambda_2$ takes the following symmetric form:

$$\begin{aligned} x_1(t) &= -x_2(-t), \\ m_1(t) &= m_2(-t), \\ x_2(t) &= \frac{1}{2} \ln\left(\frac{e^{2t/\lambda_1}}{\lambda_1} + \frac{e^{2t/\lambda_2}}{\lambda_2} - \frac{4\,e^{t/\lambda_1 + t/\lambda_2}}{\lambda_1 + \lambda_2}\right) - \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)\lambda_1\lambda_2}, \\ m_2(t) &= \frac{1}{e^{t/\lambda_1} - e^{t/\lambda_2}} \left(\frac{e^{2t/\lambda_1}}{\lambda_1} + \frac{e^{2t/\lambda_2}}{\lambda_2} - \frac{4\,e^{t/\lambda_1 + t/\lambda_2}}{\lambda_1 + \lambda_2}\right)^{1/2}. \end{aligned} \tag{4.27}$$

In this case, the asymptotics for the positions as $t \to \pm\infty$ are

$$\begin{aligned}
x_1(t) &= \frac{t}{\lambda_1} + \frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_2} + \mathcal{O}\big(e^{\delta t}\big), \quad \text{as } t \to -\infty,\\
x_2(t) &= \frac{t}{\lambda_2} - \frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1} + \mathcal{O}\big(e^{\delta t}\big), \quad \text{as } t \to -\infty,\\
x_1(t) &= \frac{t}{\lambda_2} + \frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1} + \mathcal{O}\big(e^{-\delta t}\big), \quad \text{as } t \to +\infty,\\
x_2(t) &= \frac{t}{\lambda_1} - \frac{1}{2}\ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_2} + \mathcal{O}\big(e^{-\delta t}\big), \quad \text{as } t \to +\infty,
\end{aligned} \tag{4.28}$$

where $\delta = 1/\lambda_1 - 1/\lambda_2 > 0$. Moreover, as $t \to 0$,

$$\begin{aligned}
x_1(t) &= \Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)t + \frac{1}{3\lambda_1\lambda_2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)t^3 - \frac{1}{4\lambda_1\lambda_2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)^2 t^4 + \mathcal{O}(t^5),\\
x_2(t) &= \Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)t + \frac{1}{3\lambda_1\lambda_2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)t^3 + \frac{1}{4\lambda_1\lambda_2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)^2 t^4 + \mathcal{O}(t^5),\\
m_1(t) &= \frac{-1/t}{\sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}} + \frac{1}{2}\sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}} + \mathcal{O}(t),\\
m_2(t) &= \frac{1/t}{\sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}} + \frac{1}{2}\sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}} + \mathcal{O}(t),
\end{aligned} \tag{4.29}$$

so in particular

$$x_2(t) - x_1(t) = x_2(t) + x_2(-t) = \frac{1}{2\lambda_1\lambda_2}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)^2 t^4 + \mathcal{O}(t^6) \tag{4.30}$$

and

$$m_1(t) + m_2(t) = m_2(-t) + m_2(t) = \sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}} + \mathcal{O}(t^2). \tag{4.31}$$

At the collision, the wave profile u takes the shape of a single peakon with positive amplitude,

$$u(x,0) := \lim_{t\to 0} u(x,t) = \sqrt{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}\, e^{-|x|}. \tag{4.32}$$

Proof. This is straightforward computation; just insert (4.26) into (4.2) and (4.10), and then calculate the Maclaurin/Laurent expansions of (4.27). The last claim (4.32) follows since

$$\begin{aligned}
u(x_1(t),t) &= m_1(t) + m_2(t)\, e^{x_1(t)-x_2(t)}\\
&= \big(m_1(t)+m_2(t)\big) - \underbrace{m_2(t)\big(1-e^{x_1(t)-x_2(t)}\big)}_{=\mathcal{O}(t^3)}
\end{aligned} \tag{4.33}$$

and

$$
\begin{aligned}
u(x_2(t),t) &= m_1(t)\,e^{x_1(t)-x_2(t)} + m_2(t) \\
&= \big(m_1(t)+m_2(t)\big) - \underbrace{m_1(t)\big(1-e^{x_1(t)-x_2(t)}\big)}_{=\mathcal{O}(t^3)}
\end{aligned}
\tag{4.34}
$$

both have the same limit as $m_1(t)+m_2(t)$ at $t=0$, i.e., the both tend to $(1/\lambda_1+1/\lambda_2)^{1/2}$. □

Remark 4.14. Similar calculations show that the derivative $u_x(x,t)$ behaves like $m_2(t)-m_1(t)$ in the interval $x_1(t)<x<x_2(t)$, i.e., like a positive constant times $1/t$. So the slope between the peakons tends to $-\infty$ as the collision approaches, and comes back from $+\infty$ afterwards (or the other way around, for the upside-down solution $u(x,t)$ with the opposite sign.) This steepening is seen in Figure 10, and also in the figures for the other cases below, especially Figure 14. However, we will see in Section 5.2 that $m_2(t)-m_1(t)$ may have a pole of higher order if $n\geq 3$, leading to somewhat different behaviour.

4.6 Dynamics in the case of non-real eigenvalues

We now turn to the case of a complex conjugated pair of eigenvalues $\lambda_1=\overline{\lambda_2}$ with positive real parts and nonzero imaginary parts. In terms of initial data $\{x_k(0),m_k(0)\}$, this case occurs when $m_1(0)\,m_2(0)<0$ and the point

$$
(\sigma,E_{12}) = \left(\left|\frac{m_1(0)}{m_2(0)}\right| + \left|\frac{m_2(0)}{m_1(0)}\right|, e^{x_1(0)-x_2(0)}\right)
$$

lies inside the ellipse in Figure 9.

According to Theorem 4.8, the peakon solution $\{x_k(t),m_k(t)\}$ is given by the usual formulas (4.2), but it is useful to rewrite these formulas in terms of real quantities, as follows:

Theorem 4.15. *Define the constants*

$$
\alpha>0,\qquad \beta>0,\qquad \psi\in(0,\pi/2),\qquad B>0,\qquad \varphi\in(-\pi,\pi]
$$

by

$$
\frac{1}{\lambda_1}=\alpha+i\beta,\qquad \frac{1}{\lambda_2}=\alpha-i\beta,\qquad \psi=\arg(\alpha+i\beta) \tag{4.35}
$$

and

$$
b_1(0)=Be^{i\varphi},\qquad b_2(0)=Be^{-i\varphi}. \tag{4.36}
$$

Moreover, let

$$
T(t)=\beta t+\varphi,\qquad K=\tfrac{1}{2}\ln\frac{2B^2\beta^2}{\alpha},\qquad L=-\tfrac{1}{2}\ln(\sin\psi)>0. \tag{4.37}
$$

In terms of these quantities, the Novikov $n = 2$ peakon–antipeakon solution with non-real eigenvalues becomes

$$
\begin{aligned}
x_1(t) &= \alpha t + K - L - \frac{1}{2}\ln\frac{1+\cos\psi\cos(2T(t)-\psi)}{\sin\psi},\\
x_2(t) &= \alpha t + K + L + \frac{1}{2}\ln\frac{1+\cos\psi\cos(2T(t)+\psi)}{\sin\psi},\\
m_1(t) &= \frac{-1}{\cos T(t)}\sqrt{\frac{\alpha^2+\beta^2}{2\alpha}\big(1+\cos\psi\cos(2T(t)-\psi)\big)},\\
m_2(t) &= \frac{1}{\cos T(t)}\sqrt{\frac{\alpha^2+\beta^2}{2\alpha}\big(1+\cos\psi\cos(2T(t)+\psi)\big)}.
\end{aligned}
\tag{4.38}
$$

Proof. This is just computation. We have

$$
\begin{aligned}
b_1(t) &= b_1(0)\,e^{t/\lambda_1} = Be^{i\varphi}e^{t(\alpha+i\beta)} = Be^{\alpha t}e^{iT(t)},\\
b_2(t) &= b_2(0)\,e^{t/\lambda_2} = \overline{b_1(t)} = Be^{\alpha t}e^{-iT(t)}
\end{aligned}
$$

(so b_1 and b_2 move in spirals in the complex plane) and

$$
\begin{aligned}
\alpha &= (\alpha^2+\beta^2)^{\frac{1}{2}}\cos\psi,\\
\beta &= (\alpha^2+\beta^2)^{\frac{1}{2}}\sin\psi,\\
\lambda_{1,2} &= (\alpha\pm i\beta)^{-1} = (\alpha^2+\beta^2)^{-\frac{1}{2}}\,e^{\mp i\psi}.
\end{aligned}
\tag{4.39}
$$

Hence,

$$
U_1 = b_1 + b_2 = 2\,\mathrm{Re}\;b_1 = 2Be^{\alpha t}\cos T(t)
$$

and

$$
\begin{aligned}
U_2 = \Psi_{12}b_1b_2 &= \frac{(\lambda_1-\lambda_2)^2\,b_1b_2}{\lambda_1+\lambda_2} = \frac{\big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\big)^2|b_1|^2}{\big(\frac{1}{\lambda_2}+\frac{1}{\lambda_1}\big)\frac{1}{\lambda_1\lambda_2}}\\
&= \frac{(-2i\beta)^2B^2e^{2\alpha t}}{2\alpha(\alpha^2+\beta^2)} = \frac{-2\beta^2B^2e^{2\alpha t}}{\alpha(\alpha^2+\beta^2)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
W_1 &= \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}\,b_1b_2\\
&= 2\,\mathrm{Re}(\lambda_1 b_1^2) + \frac{4|b_1|^2}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}\\
&= 2\,\mathrm{Re}\left(\frac{B^2e^{2\alpha t}e^{2iT(t)}}{(\alpha^2+\beta^2)^{\frac{1}{2}}\,e^{i\psi}}\right) + \frac{4B^2e^{2\alpha t}}{2\alpha}\\
&= \frac{2B^2e^{2\alpha t}}{\alpha}\left(\frac{\alpha}{(\alpha^2+\beta^2)^{\frac{1}{2}}}\;\mathrm{Re}\Big(e^{i(2T(t)-\psi)}\Big)+1\right)\\
&= \frac{2B^2e^{2\alpha t}}{\alpha}\big(1+\cos\psi\,\cos(2T(t)-\psi)\big),
\end{aligned}
$$

and a similar calculation gives

$$\begin{aligned} Z_1 &= \frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1+\lambda_2} b_1 b_2 \\ &= \frac{2B^2 e^{2\alpha t}(\alpha^2+\beta^2)}{\alpha} \big(1+\cos\psi\cos(2T(t)+\psi)\big). \end{aligned}$$

Finally,

$$Z_2 = \frac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2} b_1^2 b_2^2 = \frac{U_2^2}{|\lambda_1|^2} = \left(\frac{-2\beta^2 B^2 e^{2\alpha t}}{\alpha(\alpha^2+\beta^2)}\right)^2 (\alpha^2+\beta^2) = \frac{4\beta^4 B^4 e^{4\alpha t}}{\alpha^2(\alpha^2+\beta^2)}.$$

Now the result is obtained by inserting all this into the two-peakon solution formulas (4.2):

$$x_1(t) = \frac{1}{2}\ln\frac{Z_2}{W_1}, \quad x_2(t) = \frac{1}{2}\ln\frac{Z_1}{W_0}, \quad m_1(t) = \frac{\sqrt{Z_2 W_1}}{U_2 U_1}, \quad m_2(t) = \frac{\sqrt{Z_1 W_0}}{U_1 U_0}.$$

(Recall that $U_0 = W_0 = 1$ by definition.) □

Remark 4.16. The interpretation of this theorem is that the quantities

$$x_1(t) - (\alpha t + K - L) \qquad \text{and} \qquad x_2(t) - (\alpha t + K + L)$$

oscillate with period π/β between the values $\pm M$, where

$$M = \tfrac{1}{2}\ln\tfrac{1+\cos\psi}{\sin\psi} = -\tfrac{1}{2}\ln\tfrac{1-\cos\psi}{\sin\psi} > L > 0.$$

(Since $1-\cos\psi > 1-\cos^2\psi = \sin^2\psi$.) In other words, the motion of peakon number 1 consists of an oscillation of amplitude M and period π/β, overlaid on a drift with constant speed along the straight line $x = \alpha t + K - L$, and similarly peakon number 2 oscillates around the line $x = \alpha t + K + L$. We can thus consider the couple as a whole to be centered on the line $x = \alpha t + K$. The oscillations occur in such a way that $x_1(t) \le x_2(t)$ for all t, with equality exactly for those values of t where $0 = \cos T(t) = \cos(\beta t + \varphi)$:

$$t = \frac{1}{\beta}\Big(-\varphi + \frac{\pi}{2} + n\pi\Big), \qquad n \in \mathbf{Z}. \tag{4.40}$$

This follows since we already know that collisions occur precisely when $0 = U_1 = b_1 + b_2 = 2\,\mathrm{Re}\, b_1 = 2Be^{\alpha t}\cos T(t)$, and it can also be seen using a bit of trigonometric manipulation:

$$\begin{aligned} x_2(t) - x_1(t) &= \Big(\alpha t + K + L + \frac{1}{2}\ln\frac{1+\cos\psi\cos(2T(t)+\psi)}{\sin\psi}\Big) \\ &\quad - \Big(\alpha t + K - L - \frac{1}{2}\ln\frac{1+\cos\psi\cos(2T(t)-\psi)}{\sin\psi}\Big) \\ &= \frac{1}{2}\ln\Big(1 + \frac{2\cos^2 T(t)\cos^2\psi}{\sin^4\psi}\Big(1+\cos(T(t)+\psi)\cos(T(t)-\psi)\Big)\Big) \\ &\ge 0, \end{aligned} \tag{4.41}$$

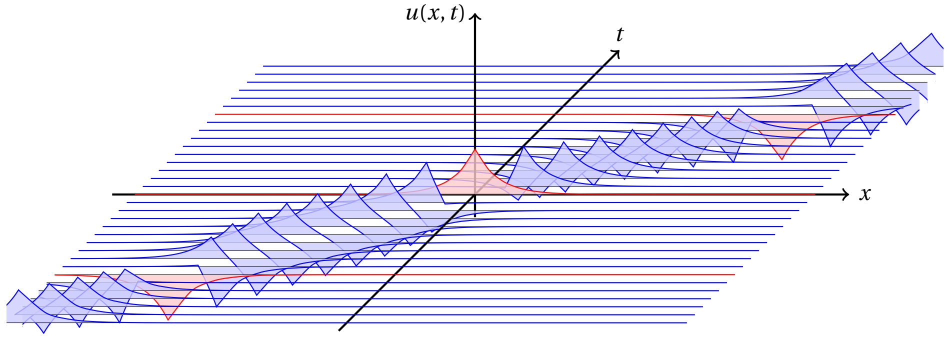

Figure 13: The graph of $u(x,t)$ for the peakon–antipeakon solution in Theorem 4.17. The eigenvalues are $\lambda_1 = \overline{\lambda_2} = 1/(\alpha + i\beta)$ with $\alpha = 2$ and $\beta = 5/\pi$, so that the peakon–antipeakon couple travels together with the overall velocity $\alpha = 2$, and the time between collisions is $\pi/\beta = 5$. The coefficients $b_1(0) = \overline{b_2(0)} = -iB$ with $B = 5/\pi$ are chosen to place one of the collisions at the origin; see Figure 14 for a closeup of this collision. Thus collisions occur when $t = 5n$, $n \in \mathbf{Z}$, and are marked with red. The domain shown is $-15 \le x \le 15$ and $-8 \le t \le 8$, with the wave profile $u(x,t)$ sampled at $t = n/2$, $n \in \mathbf{Z}$.

with equality if and only if $\cos T(t) = 0$. One also sees from (4.38) that the amplitudes $m_1(t)$ and $m_2(t)$ are $2\pi/\beta$-periodic; during one such period of length $2\pi/\beta$ there are two collisions, where m_1 and m_2 blow up and change their sign. In between the collisions, one amplitude is positive and the other one is negative, as it should be. (We are looking at peakon–antipeakon solutions, after all.)

If we want to take a closer look at what happens at a collision, we can place one of them at the origin by a suitable choice of the parameters B and φ determining $b_1(0)$ and $b_2(0)$; this also shows the symmetry of the solution. See Figures 13, 14, 15 and 16 for illustrations.

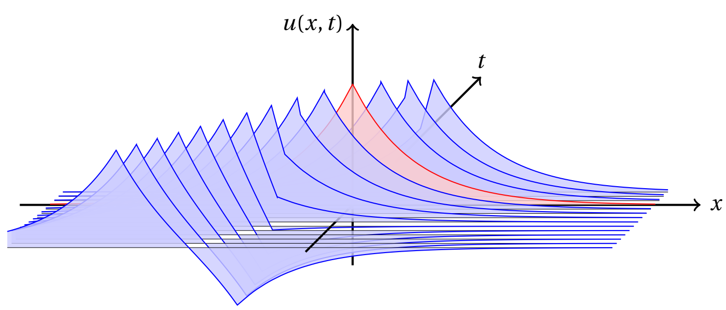

Figure 14: Closeup of the collision at the origin in Figure 13. Collisions occur periodically, and alternate between looking like this and being turned upside-down. The domain shown is $-5 \le x \le 5$ and $-1 \le t \le 3/10$, with the wave profile $u(x,t)$ sampled at $t = n/10$, $n \in \mathbf{Z}$.

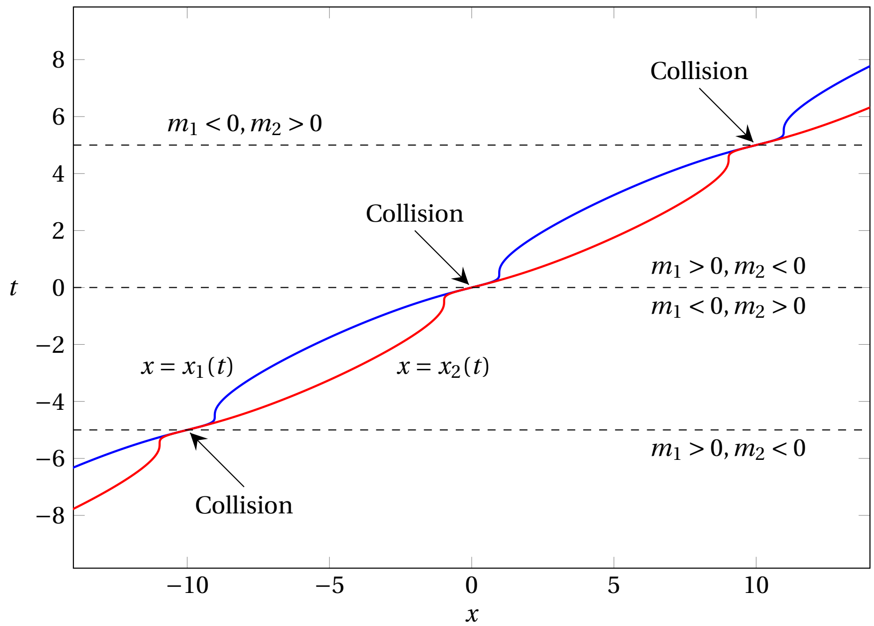

Figure 15: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the peakon–antipeakon solution with $\lambda_1 = \overline{\lambda_2} = 1/(2 + \frac{5}{\pi} i)$ in Figure 13.

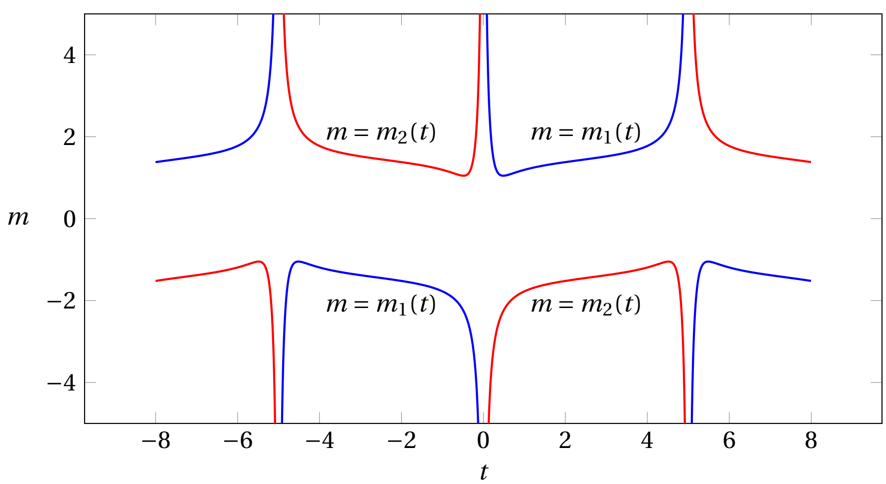

Figure 16: Graphs of the amplitudes $m_k(t)$ for the peakon–antipeakon solution with $\lambda_1 = \overline{\lambda_2} = 1/(2 + \frac{5}{\pi} i)$ in Figure 13.

Theorem 4.17. *When*

$$B = \frac{1}{\sqrt{2\alpha}\,\tan\psi} = \frac{\sqrt{\alpha/2}}{\beta}, \qquad \varphi = -\frac{\pi}{2}, \tag{4.42}$$

the peakon–antipeakon solution with non-real eigenvalues $\lambda_{1,2} = 1/(\alpha \pm i\beta)$, *where* $\alpha > 0$ *and* $\beta > 0$, *takes the following symmetric form:*

$$\begin{aligned} x_1(t) &= -x_2(-t), \\ m_1(t) &= m_2(-t), \\ x_2(t) &= \alpha t + \frac{1}{2}\ln\frac{1-\cos\psi\cos(2\beta t+\psi)}{\sin^2\psi}, \\ m_2(t) &= \frac{1}{\sin(\beta t)}\sqrt{\frac{\alpha^2+\beta^2}{2\alpha}\big(1-\cos\psi\cos(2\beta t+\psi)\big)}, \end{aligned} \tag{4.43}$$

where $\psi = \arg(\alpha + i\beta) \in (0,\pi/2)$. *Collisions occur when* $t = n\pi/\beta$, $n \in \mathbf{Z}$. *As* $t \to 0$,

$$\begin{aligned} x_1(t) &= 2\alpha t - \tfrac{2}{3}\alpha(\alpha^2+\beta^2)\,t^3 - \alpha^2(\alpha^2+\beta^2)\,t^4 + \mathcal{O}(t^5), \\ x_2(t) &= 2\alpha t - \tfrac{2}{3}\alpha(\alpha^2+\beta^2)\,t^3 + \alpha^2(\alpha^2+\beta^2)\,t^4 + \mathcal{O}(t^5), \\ m_1(t) &= \frac{-1/t}{\sqrt{2\alpha}} + \sqrt{\frac{\alpha}{2}} + \mathcal{O}(t), \\ m_2(t) &= \frac{1/t}{\sqrt{2\alpha}} + \sqrt{\frac{\alpha}{2}} + \mathcal{O}(t), \end{aligned} \tag{4.44}$$

so in particular

$$x_2(t) - x_1(t) = x_2(t) + x_2(-t) = 2\alpha^2(\alpha^2+\beta^2)\,t^4 + \mathcal{O}(t^6) \tag{4.45}$$

and

$$m_1(t) + m_2(t) = m_2(-t) + m_2(t) = \sqrt{2\alpha} + \mathcal{O}(t^2). \tag{4.46}$$

At the collision, the wave profile u takes the shape of a single peakon with positive amplitude,

$$u(x,0) := \lim_{t\to 0} u(x,t) = \sqrt{2\alpha}\,e^{-|x|}. \tag{4.47}$$

Proof. It is a routine matter to just substitute the parameter values and compute the series expansion. The claim about $u(x,0)$ is proved exactly like in Theorem 4.13. □

Remark 4.18. The value of B was chosen in order to make $K = \frac{1}{2}\ln(2B^2\beta^2/\alpha)$ vanish. Taking $\varphi = \frac{\pi}{2}$ instead of $\varphi = -\frac{\pi}{2}$ puts the other kind of collision at the origin; it looks exactly the same except for being turned upside down (the signs of m_1 and m_2 are flipped).

4.7 Dynamics in the case of a positive double eigenvalue

The final (and non-generic) case in our investigation of a single peakon–antipeakon pair occurs when λ_1 and λ_2 coincide, say

$$\lambda_1 = \lambda_2 = \mu > 0.$$

In terms of initial data $\{x_k(0), m_k(0)\}$, this happens when $m_1(0)\, m_2(0) < 0$ and the point

$$(\sigma, E_{12}) = \left(\left| \frac{m_1(0)}{m_2(0)} \right| + \left| \frac{m_2(0)}{m_1(0)} \right|, e^{x_1(0)-x_2(0)} \right)$$

lies precisely on the ellipse in Figure 9, on the borderline between the real and complex cases.

In this case, the spectral data cannot be defined as in (2.23), since the Weyl function does not have simple poles. Instead the partial fraction expansion takes the form

$$\omega(\lambda) = \frac{a_1}{\lambda - \mu} + \frac{a_2}{(\lambda - \mu)^2}$$

for some time-dependent coefficients a_1 and a_2. As it turns out, the solution formulas look nicer if stated in terms of $b_1 = a_1$ and $b_2 = a_2/\mu$, so the partial fraction expansion that we are actually going to work with is

$$\omega(\lambda) = \frac{b_1}{\lambda - \mu} + \frac{\mu\, b_2}{(\lambda - \mu)^2}. \tag{4.48}$$

Although we have continued to use the names b_1 and b_2 (since this will turn out convenient in the general case studied in Section 6), these coefficents are not on an equal footing here, and the solution formulas will no longer be symmetric with respect to the interchange of the indices 1 and 2. Also, the time dependence of b_1 will be more complicated than before; see (4.51) below.

In Section 6 we derive general peakon solution formulas, valid for any n and for eigenvalues of arbitrary multiplicities, by applying a limiting procedure to the solution formulas that we already know for the case of simple eigenvalues. But here we take a more direct approach, and derive the solution for the particular case $n = 2$ by revisiting the inverse spectral problem, and solving it for a Weyl function of the new form (4.48).

Theorem 4.19. *The general Novikov $n = 2$ peakon–antipeakon solution corresponding to an eigenvalue of multiplicity two is parametrized by spectral data consisting of three constants:*

$$\mu > 0, \quad b_1(0) \in \mathbf{R}, \quad b_2(0) \in \mathbf{R} \setminus \{0\}. \tag{4.49}$$

The solution formulas are

$$x_1(t) = \tfrac{1}{2} \ln Q_1, \quad x_2(t) = \tfrac{1}{2} \ln Q_2, \quad m_1(t) = P_1 \sqrt{Q_1}, \quad m_2(t) = P_2 \sqrt{Q_2}, \tag{4.50a}$$

where

$$
\begin{aligned}
Q_1 &= \frac{b_2^4}{4\mu\,(b_1^2 + b_1 b_2 + \frac{1}{2} b_2^2)}, &\qquad P_1 &= \frac{-2(b_1^2 + b_1 b_2 + \frac{1}{2} b_2^2)}{b_1 b_2^2},\\
Q_2 &= \frac{b_1^2 - b_1 b_2 + \frac{1}{2} b_2^2}{\mu}, &\qquad P_2 &= \frac{1}{b_1},
\end{aligned}
\tag{4.50b}
$$

and where the coefficients b_1 *and* b_2 *from the Weyl function* (4.48) *have the time dependence*

$$
b_1(t) = \Bigl(b_1(0) - b_2(0)\,\tfrac{t}{\mu}\Bigr) e^{t/\mu}, \qquad b_2(t) = b_2(0)\, e^{t/\mu}. \tag{4.51}
$$

Proof. The time dependence of the Weyl function is given by the same differential equation (2.27) as before, but it will induce a different time dependence for b_1 and b_2:

$$
\begin{aligned}
\frac{\dot b_1}{\lambda - \mu} + \frac{\mu\, \dot b_2}{(\lambda-\mu)^2} = \dot\omega(\lambda) &= \frac{\omega(\lambda) - \omega(0)}{\lambda}\\
&= \frac{1}{\lambda}\left(\left(\frac{b_1}{\lambda-\mu} + \frac{\mu\, b_2}{(\lambda-\mu)^2}\right) - \left(\frac{b_1}{-\mu} + \frac{b_2}{\mu}\right)\right)\\
&= \frac{(b_1 - b_2)/\mu}{\lambda - \mu} + \frac{b_2}{(\lambda-\mu)^2}.
\end{aligned}
$$

This gives the equations

$$
\dot b_1 = \frac{b_1 - b_2}{\mu}, \qquad \dot b_2 = \frac{b_2}{\mu}, \tag{4.52}
$$

whose solution in terms of initial data $b_1(0)$ and $b_2(0)$ is given by (4.51).

Next, identifying coefficients in $\omega(\lambda) = -B(\lambda)/A(\lambda) = b_1/(\lambda-\mu) + \mu\, b_2/(\lambda-\mu)^2$, like in Remark 4.4, we find the equations determining x_1, x_2, m_1 and m_2:

$$
\begin{aligned}
\frac{2}{\mu} &= H_1 = m_1^2 + m_2^2 + 2 m_1 m_2 E_{12}, &\qquad \frac{b_1 + b_2}{\mu} &= m_1 e^{x_1} + m_2 e^{x_2},\\
\frac{1}{\mu^2} &= H_2 = m_1^2 m_2^2 (1 - E_{12}^2), &\qquad b_1 &= \frac{e^{x_2}}{m_2}.
\end{aligned}
\tag{4.53}
$$

This system has a structure similar to (4.8), but it is slightly different. Note that the first and the second equation here are not independent, as $H_1^2 - 4H_2 = 0$ is a necessary requirement for $\lambda_1 = \lambda_2$, but it is also due to this supplementary condition that we can describe the solution in this case using only three parameters $\{\mu, b_1, b_2\}$ rather than four. In any case, solving for $\{x_k, m_k\}$, one finds the formulas (4.50).

We prove later in Theorem 6.23 that each highest-order coefficient associated to an eigenvalue in the Weyl function is nonzero, which in this case means that $b_2(0) \neq 0$ (and hence $b_2(t) \neq 0$ for all t). But we can give a direct argument here: from (4.53) we have

$$
b_2 = \frac{2(m_1 e^{x_1} + m_2 e^{x_2})}{H_1} - \frac{e^{x_2}}{m_2} = \frac{(m_2^2 - m_1^2)\, e^{x_2}}{H_1 m_2},
$$

and this cannot be zero, since if $m_1 = -m_2 = k \neq 0$, then

$$
H_1^2 - 4H_2 = 8k^4 E_{12}(E_{12} - 1) \neq 0,
$$

in contradiction to the assumption that we are in the double eigenvalue case. □

Remark 4.20. Actually, if we compute the spectral data $\{\mu, b_1(0), b_2(0)\}$ from given initial data $\{x_k(0), m_k(0)\}$ at time $t = 0$, it is clear from the relation $b_1 = e^{x_2}/m_2$ that $b_1(0)$ will be nonzero as well, and not just $b_2(0)$. But it is still natural to allow $b_1(0) = 0$ in the general solution; this simply corresponds to the case when the collision happens to occur at $t = 0$, as the next theorem shows.

Corollary 4.21. *In the case of a double eigenvalue, the peakon and the antipeakon collide exactly once, namely when*

$$t = \mu \frac{b_1(0)}{b_2(0)}. \tag{4.54}$$

The asymptotics as $t \to \pm\infty$ are given by

$$\begin{aligned}
x_1(t) &= \frac{t}{\mu} - \frac{1}{2}\ln\frac{2t^2}{\mu^2} + \frac{1}{2}\ln\frac{b_2(0)^2}{2\mu} + \mathcal{O}(1/t),\\
x_2(t) &= \frac{t}{\mu} + \frac{1}{2}\ln\frac{2t^2}{\mu^2} + \frac{1}{2}\ln\frac{b_2(0)^2}{2\mu} + \mathcal{O}(1/t),\\
m_1(t) &= \frac{1}{\sqrt{\mu}}\operatorname{sgn}(t)\operatorname{sgn}(b_2(0))\bigl(1 + \mathcal{O}(1/t)\bigr),\\
m_2(t) &= -\frac{1}{\sqrt{\mu}}\operatorname{sgn}(t)\operatorname{sgn}(b_2(0))\bigl(1 + \mathcal{O}(1/t)\bigr).
\end{aligned} \tag{4.55}$$

In particular the distance between the peakons tends to infinity at a logarithmic rate:

$$x_2(t) - x_1(t) = \ln\frac{2t^2}{\mu^2} + \mathcal{O}(1/t), \quad \textit{as } t \to \pm\infty. \tag{4.56}$$

Proof. Since $b_2(t) \neq 0$ for all t, a collision happens exactly when $b_1(t) = 0$, i.e., when

$$b_1(t) = b_1(0) - b_2(0)\,\tfrac{t}{\mu} = 0,$$

which proves the first claim. As for the asymptotics, the solution formulas from Theorem 4.19 say that

$$\begin{aligned}
x_1(t) &= \frac{1}{2}\ln\frac{\frac{1}{2\mu}\,b_2(0)^4\,e^{2t/\mu}}{2\bigl(b_1(0) - b_2(0)\,\frac{t}{\mu}\bigr)^2 + 2\bigl(b_1(0) - b_2(0)\,\frac{t}{\mu}\bigr)\,b_2(0) + b_2(0)^2}\\
&= \frac{1}{2}\ln e^{2t/\mu} + \frac{1}{2}\ln\left(\frac{\frac{1}{2\mu}\,b_2(0)^4}{2\bigl(b_2(0)\,\frac{t}{\mu}\bigr)^2\bigl(1 + \mathcal{O}(1/t)\bigr)}\right)\\
&= \frac{t}{\mu} - \frac{1}{2}\ln\frac{2t^2}{\mu^2} + \frac{1}{2}\ln\frac{b_2(0)^2}{2\mu} + \mathcal{O}(1/t),
\end{aligned}$$

as claimed; similarly for $x_2(t)$, $m_1(t)$ and $m_2(t)$. □

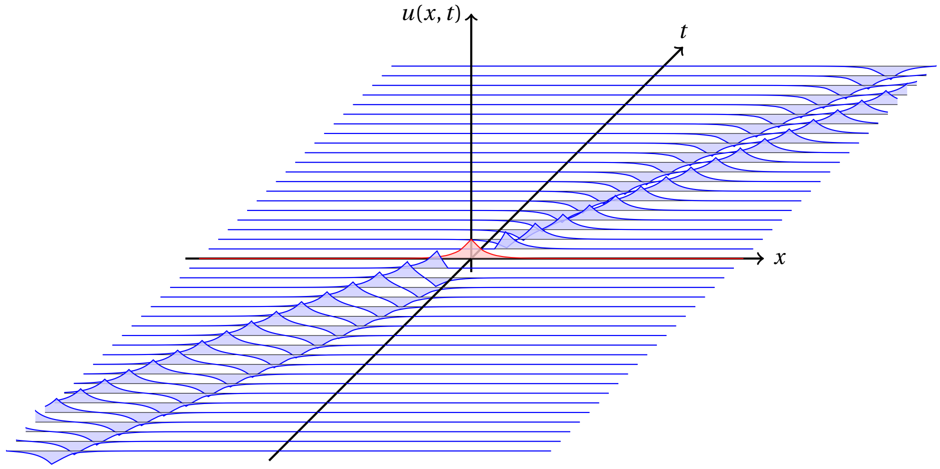

Figure 17: The graph of $u(x,t)$ for the peakon–antipeakon solution in Theorem 4.23, with coinciding eigenvalues $\lambda_1 = \lambda_2 = \mu$. Here $\mu = 1$, and the parameters $b_1(0) = 0$ and $b_2(0) = -\sqrt{2}$ are chosen to place the collision at the origin. The domain shown is $-20 \le x \le 20$ and $-20 \le t \le 20$, with the wave profile $u(x,t)$ sampled at $t \in \mathbf{Z}$.

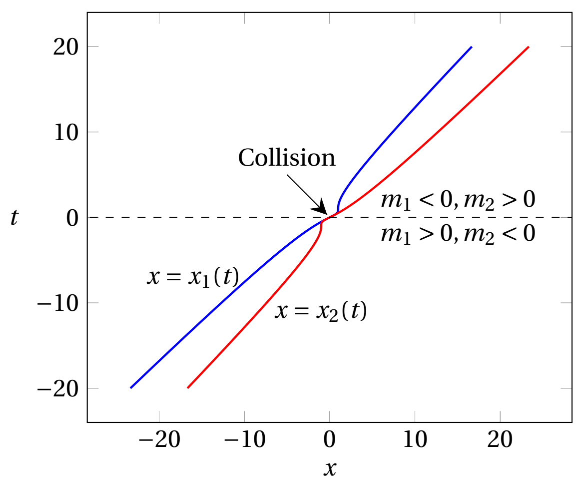

Figure 18: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x,t)-plane for the peakon–antipeakon solution with a double eigenvalue $\mu = 1$ in Figure 17. As $t \to \pm\infty$, the peakon and the antipeakon both travel roughly with the velocity $1/\mu$, but there is also a additional term making them separate slowly; the distance $x_2(t) - x_1(t)$ grows like $\ln(2t^2/\mu^2)$.

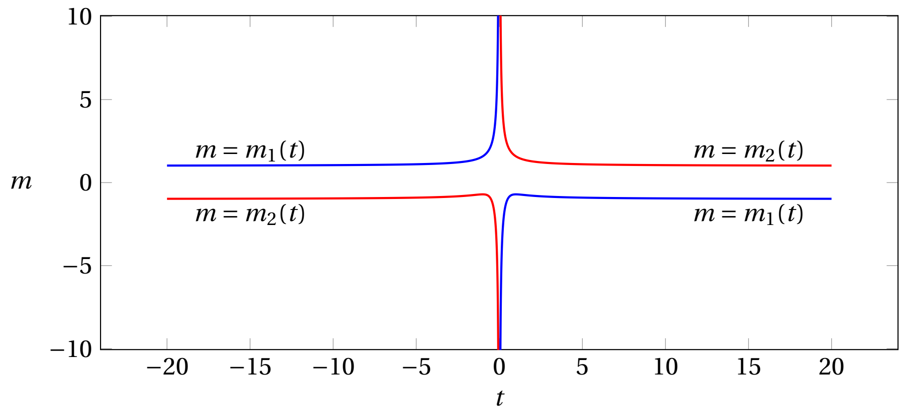

Figure 19: Graphs of the amplitudes $m_k(t)$ for the peakon–antipeakon solution with a double eigenvalue $\mu = 1$ in Figure 17.

Remark 4.22. As before, changing the sign of both b_1 and b_2 simply amounts to changing the sign of both m_1 and m_2, and we can change b_1 and b_2 as we like (except altering the sign of b_2) by translating the (x, t) coordinate system, so the eigenvalue μ is the only essential parameter in this case. The choice $b_1(0) = 0$ and $b_2(0) = \pm\sqrt{2\mu}$ makes the collision take place at the origin, and (not surprisingly) it also shows that the solution is symmetric. One of these two particular solutions is described in the following theorem, and illustrated in Figures 17, 18 and 19.

Theorem 4.23. *When*

$$b_1(0) = 0, \qquad b_2(0) = -\sqrt{2\mu}, \tag{4.57}$$

the Novikov $n = 2$ peakon–antipeakon solution with a double eigenvalue $\lambda_1 = \lambda_2 = \mu > 0$ takes the following symmetric form:

$$\begin{aligned} x_1(t) &= -x_2(-t), \\ m_1(t) &= m_2(-t), \\ x_2(t) &= \frac{t}{\mu} + \frac{1}{2}\ln\left(\frac{2t^2}{\mu^2} + \frac{2t}{\mu} + 1\right), \\ m_2(t) &= \frac{1}{t}\sqrt{\frac{\mu}{2}}\sqrt{\frac{2t^2}{\mu^2} + \frac{2t}{\mu} + 1}, \end{aligned} \tag{4.58}$$

or more compactly written using the abbreviation $\tau = t/\mu$:

$$\begin{aligned} x_2(t) &= \tau + \frac{1}{2}\ln\left(2\tau^2 + 2\tau + 1\right), \\ m_2(t) &= \frac{\sqrt{2\tau^2 + 2\tau + 1}}{\tau\sqrt{2\mu}}. \end{aligned} \tag{4.59}$$

As $t \to 0$,

$$\begin{aligned} x_1(t) &= \frac{2}{\mu}\,t - \frac{2}{3\mu^3}\,t^3 - \frac{1}{\mu^4}\,t^4 + \mathcal{O}(t^5), \\ x_2(t) &= \frac{2}{\mu}\,t - \frac{2}{3\mu^3}\,t^3 + \frac{1}{\mu^4}\,t^4 + \mathcal{O}(t^5), \\ m_1(t) &= -\sqrt{\frac{\mu}{2}}\,\frac{1}{t} + \frac{1}{\sqrt{2\mu}} + \mathcal{O}(t), \\ m_2(t) &= \sqrt{\frac{\mu}{2}}\,\frac{1}{t} + \frac{1}{\sqrt{2\mu}} + \mathcal{O}(t), \end{aligned} \tag{4.60}$$

so in particular

$$x_2(t) - x_1(t) = x_2(t) + x_2(-t) = \frac{2}{\mu^4}\,t^4 + \mathcal{O}(t^6) \tag{4.61}$$

and

$$m_1(t) + m_2(t) = m_2(-t) + m_2(t) = \sqrt{\frac{2}{\mu}} + \mathcal{O}(t^2). \tag{4.62}$$

At the collision, the wave profile u takes the shape of a single peakon with positive amplitude,

$$u(x,0) := \lim_{t\to 0} u(x,t) = \sqrt{\frac{2}{\mu}}\,e^{-|x|}. \tag{4.63}$$

Proof. This is a straight-forward computation: just insert

$$b_1(t) = \sqrt{\tfrac{2}{\mu}}\,t\,e^{t/\mu}, \qquad b_2(t) = -\sqrt{2\mu}\,e^{t/\mu}$$

into the solution formulas, and so on. □

Remark 4.24. Note that

$$\frac{2t^2}{\mu^2} + \frac{2t}{\mu} + 1 = 2\Big(\frac{t}{\mu} + \frac{1}{2}\Big)^2 + \frac{1}{2} > 0$$

for all t, so there are no problems with the logarithms or square roots. Note also that $m_1(t)$ and $m_2(t)$ are actually algebraic functions of t in the double eigenvalue case; there are no exponentials present.

5 Peakon–antipeakon solutions with arbitrary n and only simple eigenvalues

Some of the theorems in the previous section generalize well to the case with n peakons and antipeakons in any order, while others are much more difficult. For example, the map to spectral data requires us to find the roots of a polynomial of degree n, where the coefficients H_k are complicated expressions in $\{x_k, m_k\}$. Thus it would be very difficult, and perhaps not very useful, to explicitly classify the behaviour of multipeakon solutions in terms of the initial conditions $\{x_k(0), m_k(0)\}$ for $n \ge 3$.

Instead, the aim of this section is to investigate the range of possible spectral data, show that the multipeakon solution formulas in Theorem 2.7 are valid for any simple eigenvalues λ_k with positive real part and any nonzero residues b_k, draw some conclusions about collisions, and finally to derive the formulas for the asymptotics as $t \to \pm\infty$. We will see that the asymptotics for Novikov peakon–antipeakon solutions can be *much* more complicated than for any other peakon equation known to us. In fact, merely *stating* the asymptotic formulas requires the exact formulas for the general multipeakon solution. This is due to the fact that when some of the eigenvalues have nonzero imaginary part, it is the real parts of the reciprocals of those eigenvalues that determine the corresponding asymptotic velocities, and the eigenvalues can be placed so that there are an arbitrary number of peakons, say m, with the same asymptotic velocity. And to describe the asymptotics of such an m-cluster within an n-peakon solution, we need the exact formulas for the general m-peakon solution.

5.1 Spectral data and formulas for peakon–antipeakon solutions

We remind the reader about Definition 2.12, which to any given point

$$(x_1, \dots, x_n, m_1, \dots, m_n) \in \mathbf{R}^{2n}$$

associates certain polynomials $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ with real coefficients, and also defines the Weyl functions

$$\omega(\lambda) = -\frac{B(\lambda)}{A(\lambda)}, \qquad \zeta(\lambda) = -\frac{C(\lambda)}{A(\lambda)}.$$

Also recall the notation $\mathscr{P}$ for the set of pure peakon configurations (those with $x_1 < \dots < x_n$ and all $m_i > 0$) and $\mathscr{R}$ for the corresponding set of spectral data ($0 < \lambda_1 < \dots < \lambda_n$ and all $b_i > 0$) determined by the partial fraction decomposition

$$\omega(\lambda) = \sum_{k=1}^{n} \frac{b_k}{\lambda - \lambda_k}.$$

This formula defines "the spectral map", a bijection from $\mathscr{P}$ to $\mathscr{R}$, whose inverse map from $\mathscr{R}$ back to $\mathscr{P}$ is given explicitly by the formulas for the n-peakon solution; see Theorems 2.7 and 2.15.

We will now extend the notation to cover mixed multipeakon configuration (where $x_1 < \dots < x_n$, and all m_i are nonzero but may be either positive or negative), and investigate the nature of the spectral data in this case. Unfortunately our results are not quite as complete as we would wish, but there are still many things that can be established.

Definition 5.1. Let

$$\widehat{\mathscr{P}} = \left\{(x_1, \dots, x_n, m_1, \dots, m_n) \in \mathbf{R}^{2n} \,:\, x_1 < \dots < x_n, \text{ all } m_i \neq 0\right\} \tag{5.1}$$

denote the set of n-peakon configurations, and let

$$\widehat{\mathscr{P}}_{\mathrm{s}} \subset \widehat{\mathscr{P}} \tag{5.2}$$

be the subset of configurations such that the corresponding polynomial $A(\lambda)$ has only *simple* roots $\lambda_1, \dots, \lambda_n$.

Remark 5.2. The hats on $\widehat{\mathscr{P}}$ and $\widehat{\mathscr{P}}_{\mathrm{s}}$ are reminders that we are not necessarily in the pure peakon case anymore, and the subscript "s" stands for "simple".

Remark 5.3. The set $\widehat{\mathscr{P}}$ is clearly open in $\mathbf{R}^{2n}$, and so is $\widehat{\mathscr{P}}_{\mathrm{s}}$, since we are removing a closed set from $\widehat{\mathscr{P}}$, namely the zero level set of the discriminant of $A(\lambda)$.

(Recall that the discriminant of $A(\lambda)$, which up to a conventional constant factor is the resultant of $A(\lambda)$ and its derivative $A'(\lambda)$, is a polynomial expression in the coefficients $\{H_k\}_{k=1}^n$, hence a continuous function of $\{x_k, m_k\}_{k=1}^n$, and it vanishes if and only if $A(\lambda)$ and $A'(\lambda)$ have any roots in common, i.e., if and only if $A(\lambda)$ has roots with multiplicity greater than one.)

The condition that $A(\lambda)$ has simple roots ensures that the partial fraction decomposition of $\omega(\lambda)$ still has the form above, so that we get an extended spectral map taking a peakon configuration

$$(x_1, \dots, x_n, m_1, \dots, m_n) \in \widehat{\mathscr{P}}_{\mathrm{s}}$$

to a tuple

$$(\lambda_1, \dots, \lambda_n, b_1, \dots, b_n) \in \mathbf{C}^{2n},$$

where all λ_k are distinct, and where any non-real λ_k and b_k come in complex-conjugated pairs since $\omega(\lambda)$ has real coefficients. Of course, this tuple is only well-defined up to relabeling of the indices $\{1, \dots, n\}$, so to be precise we should think of the map as taking values in the quotient space $\mathbf{C}^{2n}/S_n$, where the symmetric group acts by permuting the indices. Since the coefficients of $\omega(\lambda)$ depend continuously on the variables $\{x_k, m_k\}_{k=1}^n$, this extended spectral map is continuous, and we want to investigate its range.

This extended spectral map is only defined on $\widehat{\mathscr{P}}_{\mathrm{s}}$, not on all of $\widehat{\mathscr{P}}$, since the residues b_k are not defined on $\widehat{\mathscr{P}} \setminus \widehat{\mathscr{P}}_{\mathrm{s}}$. However, the eigenvalues λ_k are defined for any peakon configuration in $\widehat{\mathscr{P}}$, and they are the topic of our first theorem.

Theorem 5.4. *For any peakon configuration in $\widehat{\mathscr{P}}$, the roots of $A(\lambda)$ lie in the closed half plane* $\operatorname{Re}(\lambda) \ge 0$.

Remark 5.5. We believe that the roots must actually lie in the open half plane $\operatorname{Re}(\lambda) > 0$, but unfortunately we have not been able to prove this in general. At least we saw in (4.17) that it is true in the case of one peakon–antipeakon pair ($n = 2$). The solutions that will be studied in the remainder of this paper are all associated with eigenvalues with strictly positive real part, but we cannot at present rule out the possibility that there might be *other* solutions, where some λ_k lie on the imaginary axis. (Note, however, that $\lambda = 0$ is ruled out by the fact that $A(0) = 1$.)

Proof of Theorem 5.4. Recall that the eigenvalues λ_k in the spectral data, i.e., the roots of $A(\lambda)$, are the eigenvalues of the matrix $TPEP$ defined in Theorem 2.13. From the planar network in Figure 7, we can easily find the kth principal minor of the matrix PEP. In fact, according to the Gessel–Viennot lemma, calculating the minor is the same as finding the sum of the weights of all nonintersecting path families connecting sources 1 through k with sinks 1 through k. In this network, there is only one such family for each k, namely the one where each path from source l to sink l is a straight horizontal path. The total weight of a family is the product of the weights of all paths included in the family, so we find that the principal minor is equal to

$$m_1^2 \cdots m_k^2 \prod_{l=1}^{k-1} \left(1 - E_{l(l+1)}^2\right).$$

This expression is positive regardless of the signs of m_k, which shows that PEP is a symmetric, positive definite matrix.

Next, we study the effect of multiplying PEP by the matrix T from the left. Split T in its symmetric and skewsymmetric parts, shown here in the case $n = 3$,

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} =: R + S, \tag{5.3}$$

and note that the symmetric part R is positive semidefinite: the corresponding quadratic form is $x^t R x = (\sum x_i)^2$. Thus, for any symmetric positive definite real matrix Q and for any vector u with complex conjugate $\overline{u}$,

$$\begin{aligned} 0 &\le 2\overline{u}^T Q^T R Q u = \overline{u}^T Q^T (T + T^T) Q u \\ &= \overline{u}^T Q^T T Q u + \overline{u}^T Q^T T^T Q u. \end{aligned}$$

Letting $Q = PEP$ and u be an eigenvector of $TPEP$ with eigenvalue λ, we get

$$\begin{aligned} 0 &\le \overline{u}^T (PEP)^T T (PEP) u + \overline{u}^T (PEP)^T T^T (PEP) u \\ &= \overline{u}^T (PEP)^T (TPEPu) + (TPEP\overline{u})^T (PEP) u \\ &= \overline{u}^T (PEP)^T \lambda u + (\overline{\lambda}\overline{u})^T (PEP) u. \end{aligned}$$

Since PEP is symmetric,

$$0 \le (\lambda + \overline{\lambda}) \overline{u}^T PEPu = 2\,\mathrm{Re}(\lambda) \overline{u}^T PEPu,$$

and since PEP is positive definite, it follows that $\mathrm{Re}\,\lambda \ge 0$. □

Now we focus our attention on the case when the eigenvalues have positive real part.

Definition 5.6. Let $\widehat{\mathscr{R}}_{\mathrm{s}} \subset \mathbf{C}^{2n}/S_n$ denote the set of all spectral data

$$(\lambda_1, \dots, \lambda_n, b_1, \dots, b_n)$$

satisfying the following conditions:

- The eigenvalues λ_k are simple ($\lambda_i \neq \lambda_j$ if $i \neq j$) and located strictly in the right half of the complex plane ($\operatorname{Re}\lambda_k > 0$), and non-real eigenvalues only exist in complex-conjugated pairs.
- All residues b_k are nonzero, and they too come in conjugated pairs: if $\lambda_i = \overline{\lambda_j}$ then $b_i = \overline{b_j}$.

Remark 5.7. Spectral data coming from a peakon configuration in $\widehat{\mathscr{P}}_s$ automatically satisfy all these conditions except $\operatorname{Re}\lambda_k > 0$ and $b_k \neq 0$. We will prove in Theorem 6.23 that if a peakon configuration is such that $\operatorname{Re}\lambda_k > 0$ for all k, then in fact for each eigenvalue the highest corresponding coefficient in the partial fraction expansion of $\omega(\lambda)$ must be nonzero. In the case of simple eigenvalues, this just means that $b_k \neq 0$ for all k, so this requirement is actually redundant in Definition 5.6. (We don't give a separate proof here, since the argument would not really be any simpler than in the general case of multiple eigenvalues.)

Theorem 5.8. *For spectral data in $\widehat{\mathscr{R}}_s$, the functions W_k and Z_k (see Definition 2.5) are strictly positive (for $0 \leq k \leq n$).*

Proof. We first prove the theorem for positive simple eigenvalues. Study the $n \times n$ matrix M with elements $M_{ij} = \frac{1}{\lambda_i + \lambda_j}$. Each $k \times k$ minor M_{IJ} is a Cauchy determinant, and is equal to $\frac{\Delta_I \Delta_J}{\Gamma_{I,J}}$ in the notation of (2.6). The minors are all positive, in particular the principal minors, so M is positive definite. Thus the exterior product $\Lambda^k(M)$, which has as its elements all minors of M of a given size k, is also positive definite. This follows from the fact that $\Lambda^k(M)$ has eigenvalues that consist of all possible products of k eigenvalues of M.

From equations (8.7) and (7.3) in [18] we know that the expression Z_k can be calculated in terms of a bimoment determinant, from which one can get that

$$Z_k = 2^k \sum_{I,J} \frac{(\Delta_I)^2 (\Delta_J)^2}{\Gamma_{I,J}} b_I b_J.$$

This is equal to the quadratic form $x^T \Lambda^k(M) x$, evaluated for the nonzero vector x with elements $x_I = \Delta_I b_I$. Since $\Lambda^k(M)$ is positive definite, it follows that $Z_k > 0$. Since W_k can be obtained by replacing b_I with $b_I \lambda_I$ in the formula for Z_k, we have $W_k > 0$ as well.

The proof for the non-real case is a bit trickier. Let us first study the simplest case, i.e., $Z_1 = x^T M x$. Note that one can write

$$Z_1 = x^T M P \overline{x}$$

for some permutation matrix P, since conjugating x is the same as switching the places of each non-real component $x_i = b_i$ and its conjugate $\overline{b_i}$. The matrix P should thus be chosen to fix the rows in x containing real components, and swapping rows containing conjugated pairs.

Consider the product MP, which is just M with some columns swapped. Note that P fixes the columns with indices belonging to real eigenvalues, and swaps the other columns pairwise, in such a way that

$$(MP)_{ij} = \frac{1}{\lambda_i + \overline{\lambda_j}}.$$

This matrix is Hermitian, as the elements $(MP)_{ii}$ on the diagonal are real, and

$$(MP)_{ij} = 1/(\lambda_i + \overline{\lambda_j}) = \overline{1/(\overline{\lambda_i} + \lambda_j)} = \overline{(MP)_{ji}}, \qquad i \neq j.$$

The leading principal minors of the Cauchy matrix MP are of the form

$$\frac{\prod_{1\le i<j\le m}(\lambda_i - \lambda_j) \prod_{1\le i<j\le m}(\overline{\lambda_i} - \overline{\lambda_j})}{\prod_{1\le i,j\le m}(\lambda_i + \overline{\lambda_j})}, \qquad 1 \le m \le n.$$

The numerator is clearly real and positive, since the two products are conjugates of each other. In the denominator, there are different kinds of factors. For $i = j$, the factors simplify to $2\operatorname{Re}\lambda_i$, which is positive. For each factor $(\lambda_i + \overline{\lambda_j})$ where $i \neq j$, its conjugate is also in the product, so pairing them in this way, one sees that the denominator is also positive. Thus, Sylvester's criterion shows that Z_1 is a positive definite form.

Now define $\overline{J}$ to be the index set corresponding to the complex conjugates of the eigenvalues with indices in J. We note that the sum

$$Z_k = 2^k \sum_{I,J} \frac{(\Delta_I)^2(\Delta_J)^2}{\Gamma_{I,J}} b_I b_J$$

is over all k-subsets of n, so we can replace J with $\overline{J}$ and change the order of summation to get

$$Z_k = 2^k \sum_{I,J} \frac{(\Delta_I)^2(\Delta_{\overline{J}})^2}{\Gamma_{I,\overline{J}}} b_I b_{\overline{J}}.$$

Since $\Delta_{\overline{J}} = \pm\overline{\Delta_J}$, we have $(\Delta_{\overline{J}})^2 = (\overline{\Delta_J})^2$. Also, note that $b_{\overline{J}} = \overline{b_J}$. Thus we can write

$$Z_k = 2^k \sum_{I,J} \frac{\Delta_I \overline{\Delta_J}}{\Gamma_{I,\overline{J}}} (\Delta_I b_I)(\overline{\Delta_J b_J}),$$

so that $Z_k = x^T \Lambda^k (MP)\overline{x}$ is the kth exterior power of Z_1. It thus follows that any Z_k is positive definite. Again, W_k can be obtained by replacing b_I with $b_I \lambda_I$ in the formula for Z_k, so the argument works for W_k as well. □

Now recall the peakon solution formulas from Theorem 2.7,

$$x_{n+1-k} = \frac{1}{2} \ln \frac{Z_k}{W_{k-1}}, \qquad m_{n+1-k} = \frac{\sqrt{Z_k W_{k-1}}}{U_k U_{k-1}}, \tag{5.4}$$

which are known to map positive spectral data in $\mathscr{R}$ bijectively to pure peakon configurations in $\mathscr{P}$. What Theorem 5.8 shows is that at least the expressions for the positions x_k are well-defined also for spectral data in in the larger domain $\widehat{\mathscr{R}}_{\mathrm{s}}$. And we can say a bit more:

Theorem 5.9. *With spectral data in $\widehat{\mathscr{R}}_{\mathrm{s}}$ the formulas for x_k in* (5.4) *preserve the ordering:*

$$x_1 \le x_2 \le \cdots \le x_n. \tag{5.5}$$

The equality $x_k = x_{k+1}$ holds if and only if $U_{n-k} = 0$.

Proof. We compute the difference

$$e^{2x_{n+1-k}} - e^{2x_{n-k}} = \frac{Z_k}{W_{k-1}} - \frac{Z_{k+1}}{W_k} = \frac{Z_k W_k - Z_{k+1} W_{k-1}}{W_{k-1} W_k} = \frac{U_k^4}{W_{k-1} W_k}, \tag{5.6}$$

using [18, Corollary 8.4] in the final step. From Theorem 5.8 we know that $W_{k-1} > 0$ and $W_k > 0$ for all spectral data in $\widehat{\mathscr{P}}_{\mathrm{s}}$, and clearly also $U_k^4 \ge 0$, and the claims follow. □

Let us give a name to the set of spectral data for which we have strict inequalities, $x_1 < \cdots < x_n$.

Definition 5.10. Let $\widehat{\mathscr{R}}_{\mathrm{nc}}$ be the subset of $\widehat{\mathscr{R}}_{\mathrm{s}}$ defined by the condition that the quantities $U_1, \dots, U_{n-1}$ are all nonzero.

Remark 5.11. The subscript "nc" in $\widehat{\mathscr{R}}_{\mathrm{nc}}$ stands for "no collisions".

Remark 5.12. Recall that the U_k are polynomials in $b_1, \dots, b_n$ with coefficients Ψ_I that are rational functions in $\lambda_1, \dots, \lambda_n$; see Definition 2.5. With spectral data in $\widehat{\mathscr{R}}_{\mathrm{s}}$, there is no division by zero in these rational functions Ψ_I, since then the factors $\lambda_i + \lambda_j$ in the denominators all have positive real part. Thus the U_k are continuous nonconstant functions on $\widehat{\mathscr{R}}_{\mathrm{s}}$, so $\widehat{\mathscr{R}}_{\mathrm{nc}}$ is an open nonemtpy subset of $\widehat{\mathscr{R}}_{\mathrm{s}}$.

Theorem 5.13. *The formulas* (5.4) *map spectral data in $\widehat{\mathscr{R}}_{\mathrm{nc}}$ injectively to peakon configurations in $\widehat{\mathscr{P}}_{\mathrm{s}}$.*

Proof. It is clear from Theorems 5.8 and 5.9, and the definition of $\widehat{\mathscr{R}}_{\mathrm{nc}}$, that the formulas produce values $\{x_k, m_k\}_{k=1}^n$ satisfying $x_1 < \cdots < x_n$ and $m_k > 0$ for all k. In other words, they map given spectral data $\Lambda \in \widehat{\mathscr{R}}_{\mathrm{nc}}$ to a configuration $X \in \widehat{\mathscr{P}}$.

Next, remember the expressions for the coefficient H_k in the polynomial

$$A(\lambda) = 1 - H_1\lambda + H_2\lambda^2 - \cdots + (-1)^n H_n \lambda^n$$

in terms of $\{x_k, m_k\}_{k=1}^n$; see Theorem 2.15, Remark 2.17 and Example 2.19. We know from the pure peakon case, where $\mathscr{P}$ and $\mathscr{R}$ are in bijection, that if we insert (5.4) into the formula for H_k, we obtain the kth elementary symmetric function in the variables $\{1/\lambda_1, \dots, 1/\lambda_n\}$. And since this is a purely algebraic fact (an identity between rational functions), it must hold here as well. This means that the polynomial $A(\lambda)$ associated to the configuration $X = \{x_k, m_k\}$ coming from the spectral data $\Lambda = \{\lambda_k, b_k\}$ actually has precisely these numbers λ_k as its roots. By the same reasoning, inserting (5.4) into the corresponding formulas for the coefficients of the polynomial $B(\lambda)$, we find that the Weyl function $\omega(\lambda) = -B(\lambda)/A(\lambda)$ becomes exactly what it must be in order that the residues associated to X be the numbers b_k that we started with.

So the map $\Lambda \to X$ is indeed injective, with the spectral map as its inverse, and thus we must have $X \in \widehat{\mathscr{P}}_{\mathrm{s}}$, since the λ_k in $\Lambda \in \widehat{\mathscr{R}}_{\mathrm{nc}}$ are simple by definition. □

We can now show that the peakon solution formulas from Theorem 2.7 remain valid also for spectral data in $\widehat{\mathscr{R}}_{\mathrm{s}}$, so that we can use them for studying peakon–antipeakon solutions.

Theorem 5.14. *Let*

$$\big(\lambda_1, \dots, \lambda_n, b_1(0), \dots, b_n(0)\big)$$

be initial spectral data in the set $\widehat{\mathscr{R}}_{\mathrm{s}}$, i.e., such that the eigenvalues are simple with positive real parts, all residues $b_i(0)$ are nonzero, and any non-real eigenvalues and residues come in complex-conjugated pairs. Then the time-dependent spectral data

$$\big(\lambda_1, \dots, \lambda_n, b_1(t), \dots, b_n(t)\big), \qquad b_k(t) = b_k(0)\, e^{t/\lambda_k},$$

lie in the "no-collision" subset $\widehat{\mathscr{R}}_{\mathrm{nc}} \subset \widehat{\mathscr{R}}_{\mathrm{s}}$ for all $t \in \mathbf{R} \setminus T$, where T is a (finite or countably infinite) discrete set of values of t where $U_{n-k}(t)$ becomes zero for at least one index k. The formulas (5.4)*,*

$$x_{n+1-k}(t) = \frac{1}{2} \ln \frac{Z_k(t)}{W_{k-1}(t)}, \qquad m_{n+1-k}(t) = \frac{\sqrt{Z_k(t)\, W_{k-1}(t)}}{U_k(t)\, U_{k-1}(t)},$$

with the time dependence given by letting $b_k = b_k(t)$, yield a peakon configuration

$$\big(x_1(t), \dots, x_n(t), m_1(t), \dots, m_n(t)\big)$$

which for all $t \in \mathbf{R} \setminus T$ lies in $\widehat{\mathscr{P}}_{\mathrm{s}}$ and satisfies the Novikov n-peakon ODEs (1.7)*. If $t_c \in T$, and k is an index such that $U_{n-k}(t_c) = 0$, then there is a collision: $x_k(t_c) = x_{k+1}(t_c)$, while $m_k(t)$ and $m_{k+1}(t)$ blow up as $t \to t_c$. The function*

$$u(x,t) = \sum_{k=1}^{n} m_k(t)\, e^{-|x - x_k(t)|}, \qquad t \in \mathbf{R} \setminus T,$$

extends continuously to $t \in \mathbf{R}$ and provides a global weak solution of Novikov's equation, in the sense of Section 2.1.

Proof. There is actually not much left to prove. Regarding the time dependence for the Weyl function induced by the Novikov peakon ODEs, there is no difference from the pure peakon case. The functions $U_k(t)$, meaning U_k with $b = b_k(t)$, are entire function of t (considered as a complex variable), hence they must have discrete zeros in the complex plane, at most countably many, and T consists of all such zeros that happen to lie on the real axis. All the rest follows from the earlier theorems in this section and the discussion of collisions in Section 3. □

5.2 More about collisions

In Section 3, we saw that the Novikov equation is fairly well-behaved when it comes to peakon–antipeakon collisions; even though $u_x(x,t)$ blows up at a collision, the solution $u(x,t)$ itself can be continued past the collision as a continuous function. The continuation is not unique; here we only consider the "conservative" continuation given by the explicit peakon formulas as described in Theorem 5.14 above. These formulas now allow us to give a more detailed description of what collisions may look like in general. (We have already seen some examples for $n=2$ in Section 4.)

Theorem 5.15. *The solutions in Theorem 5.14, with spectral data in $\widehat{\mathscr{R}}_{\mathrm{s}}$, do not allow triple collisions. In other words, there is no time t such that $x_k(t) = x_{k+1}(t) = x_{k+2}(t)$.*

Proof. The two adjacent peakons $x_{n+1-k}(t)$ and $x_{n-k}(t)$ collide precisely when $U_k(t)^4 = 0$. But from Theorem 5.8 and the definition (2.9) of W_k we get

$$0 < W_k = U_k V_{k+1} - U_{k+1} V_{k-1},$$

so U_k and U_{k+1} cannot both be zero at the same time. □

Theorem 5.16. *For the solutions in Theorem 5.14, if $t = t_c \in T$ is the time of a collision $x_k = x_{k+1}$, then*

$$x_{k+1}(t) - x_k(t) = \mathscr{O}((t-t_c)^4), \qquad \text{as } t \to t_c. \tag{5.7}$$

Generically speaking, the distance does vanish as $x_{k+1}(t) - x_k(t) \sim \alpha(t-t_c)^4$ for some constant $\alpha > 0$, but it is also possible (if $n \geq 3$) to have collisions where the distance tends to zero even faster. The amplitudes $m_k(t)$ and $m_{k+1}(t)$ have a pole at $t = t_c$, generically of order 1, but higher-order poles are also possible.

Proof. Let $r = n-k$. The distance $x_{k+1} - x_k = x_{n+1-r} - x_{n-r}$ behaves like $1/2$ times $e^{2x_{n+1-r}} - e^{2x_{n-r}}$, which by (5.6) has a zero of order at least four at $t = t_0$, since U_r and W_r are analytic functions of t with $U_r = 0$ and $W_r > 0$ at $t = t_c$. For the amplitudes, we have

$$m_{n+1-r} = \frac{\sqrt{Z_r W_{r-1}}}{U_r U_{r-1}}, \qquad m_{n-r} = \frac{\sqrt{Z_{r+1} W_r}}{U_{r+1} U_r},$$

where $U_{r\pm1}$ must be nonzero when $U_r = 0$ (by Theorem 5.15), so m_k and m_{k+1} both have a pole with the same order as the zero of U_r, i.e., of U_{n-k}.

The existance of a case where U_r has a zero of multiplicity greater than one is shown in Examples 5.18 and 5.19 below. □

Remark 5.17. Equation (5.7) implies that the curves $x = x_k(t)$ and $x = x_{k+1}(t)$ lie very close to each other near a collision, and from a plot of these curves alone it is difficult to see exactly when the collision takes place.

As a comparison, we may recall that in a Camassa–Holm peakon–antipeakon collision [2], the distance always behaves like $\alpha(t-t_c)^2$ with $\alpha > 0$, and the poles of $m_k(t)$ and $m_{k+1}(t)$ are always simple. And in Degasperis–Procesi collisions [23, 28, 29], the peakons approach each other transversally, not tangentially, leading to shockpeakon formation.

Example 5.18. If $n = 3$, then the collision $x_2 = x_3$ occurs precisely when $U_1 = 0$, and

$$\begin{aligned} U_1(t) &= b_1(t) + b_2(t) + b_3(t) \\ &= b_1(0)\, e^{t/\lambda_1} + b_2(0)\, e^{t/\lambda_2} + b_3(0)\, e^{t/\lambda_3} \\ &= b_1(0) + b_2(0) + b_3(0) + \Big(\frac{b_1(0)}{\lambda_1} + \frac{b_2(0)}{\lambda_2} + \frac{b_3(0)}{\lambda_3}\Big) t + \mathcal{O}(t^2). \end{aligned} \tag{5.8}$$

Hence, the condition for such a collision to occur at $t = t_c = 0$ is that $\sum_1^3 b_k(0) = 0$, and if the additional condition $\sum_1^3 b_k(0)/\lambda_k = 0$ holds as well, this will be a "higher-order collision" with $x_2(t) - x_1(t) = \mathcal{O}(t^8)$. There are many ways of satisfying these conditions; one example is

$$\lambda_1 = \frac{1}{4}, \quad \lambda_2 = \frac{1}{3}, \quad \lambda_3 = \frac{1}{2}, \qquad b_1(0) = b_3(0) = 1, \quad b_2(0) = -2. \tag{5.9}$$

In this case the amplitudes $m_1(t)$ and $m_2(t)$ have double poles at $t = 0$, and hence they keep their signs across the collision. Since the peakons approach each other even more closely than usual, and also the steepening of the slope u_x between them is faster than usual it is difficult to make a good picture of the wave profile $u(x,t)$ where one can actually see what happens, so we must leave the visualization to the reader's imagination. (Recall that $u_x \sim m_2 - m_1$ between the peakons, according to Remark 4.14, so it keeps its sign across the collision too.)

Example 5.19. Another example of spectral data fulfilling the conditions for a higher-order collision at $t = 0$,

$$\sum_1^3 b_k(0) = \sum_1^3 \frac{b_k(0)}{\lambda_k} = 0,$$

is

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{1 + i\beta}, \quad \lambda_3 = \frac{1}{1 - i\beta}, \qquad b_1(0) = -2, \quad b_2(0) = b_3(0) = 1, \tag{5.10}$$

for any $\beta > 0$. In this case the solution will consist of a 3-peakon cluster (see Section 5.3 below) moving with overall velocity 1, with collisions occuring periodically among the peakons. In particular,

$$\begin{aligned} U_1(t) &= b_1(t) + b_2(t) + b_3(t) \\ &= b_1(0)\, e^{t/\lambda_1} + b_2(0)\, e^{t/\lambda_2} + \overline{b_2(0)\, e^{t/\lambda_2}} \\ &= -2e^t + 2\,\mathrm{Re}\, e^{(1+i\beta)t}) \\ &= -2e^t(1 - \cos\beta t), \end{aligned}$$

so peakons number 2 and 3 will collide periodically, at $t = 2\pi n/\beta$, $n \in \mathbf{Z}$, experience higher-order collisions where $m_2(t)$ and $m_3(t)$ have double poles, like in Example 5.18 above.

5.3 Peakon clusters

In Section 4.6 we saw what the 2-peakon solution looks like in the complex case

$$\lambda_1 = \overline{\lambda_2} = \frac{1}{\alpha + i\beta}, \qquad \alpha > 0, \quad \beta > 0.$$

Namely, the peakons form a pair which travels together with the velocity α, and on top of this linear drift there are oscillations with the period $2\pi/\beta$, with two collisions $x_1 = x_2$ occuring during each period. Between the collisions, one of the amplitudes m_1 and m_2 is positive and the other one negative, and at each collision they blow up to $\pm\infty$ and interchange their signs. See in particular the illustrations in Figures 13, 14, 15 and 16.

This generalizes to arbitrary n as follows:

Theorem 5.20. *An n-peakon solution given by Theorem 5.14 with spectral data in $\widehat{\mathscr{R}}_{\mathrm{s}}$ satisfying*

$$\operatorname{Re}\frac{1}{\lambda_1} = \cdots = \operatorname{Re}\frac{1}{\lambda_n} = \alpha > 0 \tag{5.11}$$

will consist of a cluster of peakons and antipeakons, where the whole cluster travels with overall velocity α, with the individual peakons and antipeakons oscillating and colliding on top of this linear drift.

Proof. Notice that if $I = \{i_1 < i_2 < \cdots < i_k\}$, then

$$\begin{aligned} b_I(t) &= b_{i_1}(0)\, e^{t/\lambda_{i_1}} \cdots b_{i_k}(0)\, e^{t/\lambda_{i_k}} \\ &= b_I(0) \exp\Bigl(\frac{1}{\lambda_{i_1}} + \cdots + \frac{1}{\lambda_{i_k}}\Bigr) t \\ &= b_I(0)\, e^{k\alpha t}\, e^{itC_I}, \end{aligned} \tag{5.12}$$

where $C_I = \sum_{i\in I} \operatorname{Im}\frac{1}{\lambda_i}$ is a real constant depending on I. Thus the quantities U_k, V_k and T_k will all equal $e^{k\alpha t}$ times some bounded (constant or oscillating) factor, and the claim then follows easily from the peakon solution formulas (5.4). □

Example 5.21. Figure 20 shows a 4-peakon cluster with

$$\lambda_1 = \overline{\lambda_2} = \frac{1}{1+i}, \qquad \lambda_3 = \overline{\lambda_4} = \frac{1}{1+2i} \tag{5.13}$$

and

$$b_1(0) = \overline{b_2(0)} = 1, \qquad b_3(0) = \overline{b_4(0)} = 1. \tag{5.14}$$

Here the imaginary parts of $1/\lambda_k$ are commensurable, so the two corresponding oscillations have a common period, and the whole 4-peakon solution consists of a 2π-periodic oscillating motion overlaid on the cluster's overall linear drift with velocity 1.

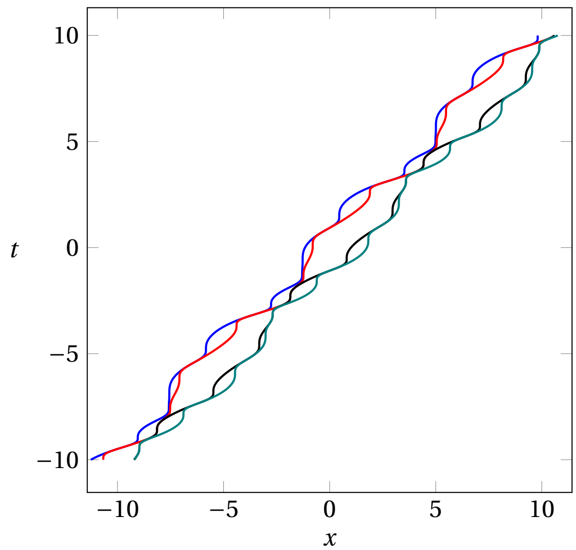

Figure 20: Positions $x = x_k(t)$ for the 4-peakon cluster with periodic oscillations described in Example 5.21.

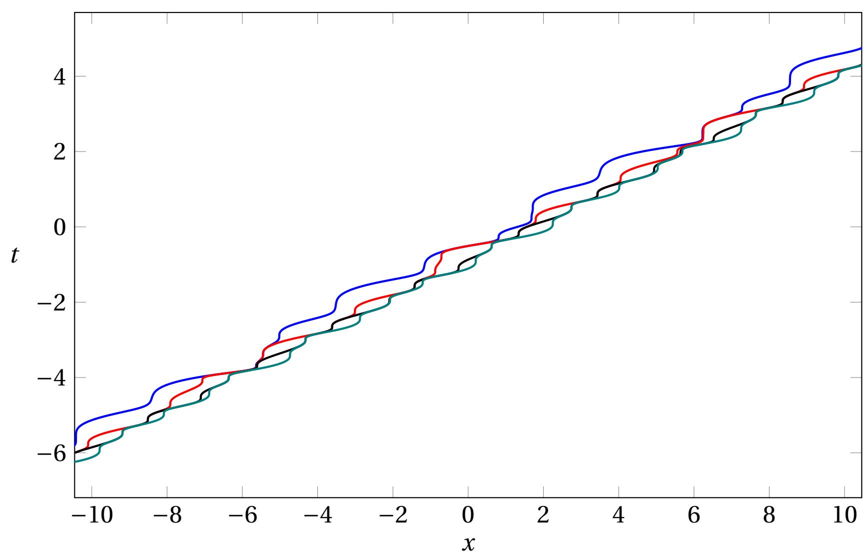

Figure 21: Positions $x = x_k(t)$ for the 4-peakon cluster with quasi-periodic oscillations described in Example 5.22.

Example 5.22. Figure 21 shows a 4-peakon cluster with

$$\lambda_1 = \overline{\lambda_2} = \frac{1}{2+i}, \qquad \lambda_3 = \overline{\lambda_4} = \frac{1}{2+2\pi i} \tag{5.15}$$

and

$$b_1(0) = \overline{b_2(0)} = 1, \qquad b_3(0) = \overline{b_4(0)} = 1. \tag{5.16}$$

Here the imaginary parts of $1/\lambda_k$ are incommensurable, and the two oscillations with the corresponding frequencies give the whole solution a quasi-periodic character, but with an overall linear drift with velocity 2.

5.4 Asymptotics

In this section we derive the asymptotics as $t \to \pm\infty$ for the Novikov peakon–antipeakon solutions given by Theorem 5.14, where it is assumed that the n eigenvalues $\lambda_1, \dots, \lambda_n$ are all of multiplicity one.

The eigenvalues are assumed to lie strictly in the right half-plane, and are either real or come in complex-conjugated pairs. Consider the n numbers $1/\lambda_k$, also in the right half-plane, and group them by their real parts. We will show that for each such group of, say, m numbers with a common real part σ, one can observe as $t \to \infty$ (or $t \to -\infty$) a cluster of m peakons that separate from the other peakons, travel together with speed σ, and interact among themselves like they would do if there were no other peakons present, i.e., according to the solution formulas for the m-peakon solution.

Remark 5.23. This phenomenon is somewhat reminiscent of what is called "waltzing peakons" in [8]. Note, however, that they are studying a *pair* of cross-coupled Camassa–Holm-like PDEs, where the "waltzing" takes place between coupled peakons living in the two different components $u(x,t)$ and $v(x,t)$. In our case, all the waltzing interactions happen within one single function $u(x,t)$.

Definition 5.24. If I and J are sets of indices, let

$$\psi_{I,J} = \prod_{i\in I,\, j\in J} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}. \tag{5.17}$$

In particular, if $J = \{j\}$ is a singleton set, we write

$$\psi_{I,j} = \prod_{i\in I} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}. \tag{5.18}$$

Theorem 5.25 (Asymptotics)**.** *Fix $\sigma > 0$ such that*

$$S = \{i \in [1,n] : \mathrm{Re}(1/\lambda_i) = \sigma\}$$

is nonempty (with $|S| > 0$ elements), and let

$$R = \{i \in [1,n] : \mathrm{Re}(1/\lambda_i) > \sigma\}$$

(with $|R| \geq 0$ elements). For $i \in S$, let

$$b_i' = \frac{\Psi_{R,i}\, b_i}{\lambda_R} \qquad \textit{and} \qquad b_I' = b_{i_1}' \dots b_{i_k}' = \frac{\Psi_{R,I}\, b_I}{\lambda_R^k},$$

and let U_k' denote the expression with the same form as U_k but computed only with indices from the set S, and with b_i' instead of b_i:

$$U_k' = \sum_{I \in \binom{S}{k}} \Psi_I b_I',$$

and similarly for T_k', V_k', W_k', Z_k'. Then for $k = 1, \dots, |S|$, the peakon solution given by Theorem 5.14 satisfies

$$x_{n+1-(|R|+k)} - \frac{1}{2} \ln \frac{Z_k'}{W_{k-1}'} \to 0, \qquad \textit{as } t \to +\infty,$$
$$m_{n+1-(|R|+k)} - \frac{\sqrt{Z_k' W_{k-1}'}}{U_k' U_{k-1}'} \to 0, \qquad \textit{as } t \to +\infty.$$

Similarly, if

$$L = \{ i \in [1, n] : \mathrm{Re}(1/\lambda_i) < \sigma \}$$

(with $|L| \geq 0$ elements), and

$$U_k'' = \sum_{I \in \binom{S}{k}} \Psi_I b_I'', \qquad \textit{where} \quad b_i'' = \frac{\Psi_{L,i}\, b_i}{\lambda_L},$$

etc., then for $k = 1, \dots, |S|$,

$$x_{|L|+k} - \frac{1}{2} \ln \frac{Z_k''}{W_{k-1}''} \to 0, \qquad \textit{as } t \to -\infty,$$
$$m_{|L|+k} - \frac{\sqrt{Z_k'' W_{k-1}''}}{U_k'' U_{k-1}''} \to 0, \qquad \textit{as } t \to -\infty.$$

Proof. We'll show only the statement regarding $t \to +\infty$, since the case $t \to -\infty$ is analogous.

The growth rate of $b_i(t) = b_i(0)\, e^{t/\lambda_i}$ as $t \to \infty$ increases with $\mathrm{Re}(1/\lambda_i)$. Consequently all the b_i with $i \in R$ grow faster than those with $i \in S$, which in turn all grow equally fast ($\sim e^{\sigma t}$), which is faster than those with $i \in L$. So in $U_{|R|+k}$ with $0 \leq k \leq |S|$, the dominant contribution as $t \to +\infty$ comes from those products b_I where I contains all the indices in R together with k indices from S (and none from L); in formulas, $I = R \cup J$ where $J \in \binom{S}{k}$. All these terms grow like $e^{\mu t}$ where $\mu = k\sigma + \sum_{i \in R} \mathrm{Re}(1/\lambda_i)$. The other terms (if there are any) grow more slowly: if $k < |S|$ there are terms where at least one index i is chosen from S instead of from R, which decreases the coefficient of t in the exponent by at least $\min_{i\in R} \operatorname{Re}(1/\lambda_i) - \sigma$, and if L is nonempty there are terms where at least one i is chosen from L instead of from S or R, and this decreases the coefficient by at least $\sigma - \max_{i\in L} \operatorname{Re}(1/\lambda_i)$. Letting δ denote the smallest of these two numbers, we see that the non-dominant terms have growth rate at most $e^{(\mu-\delta)t}$. Thus,

$$\begin{aligned} U_{|R|+k} &= \sum_{I \in \binom{[1,n]}{|R|+k}} \Psi_I b_I \\ &= \sum_{J\in\binom{S}{k}} \Psi_{R\cup J} b_{R\cup J} + \mathcal{O}(e^{(\mu-\delta)t}) \\ &= \sum_{J\in\binom{S}{k}} \Psi_R \Psi_{R,J} \Psi_J\, b_R b_J + \mathcal{O}(e^{(\mu-\delta)t}) \\ &= \lambda_R^k \Psi_R b_R \sum_{J\in\binom{S}{k}} \Psi_J\, b'_J + \mathcal{O}(e^{(\mu-\delta)t}) \\ &= \lambda_R^k \Psi_R b_R U'_k + \mathcal{O}(e^{(\mu-\delta)t}). \end{aligned}$$

Similarly, one finds that

$$\begin{aligned} T_{|R|+k} &= \lambda_R^{k-1} \Psi_R b_R T'_k + \mathcal{O}(e^{(\mu-\delta)t}), \\ V_{|R|+k} &= \lambda_R^{k+1} \Psi_R b_R V'_k + \mathcal{O}(e^{(\mu-\delta)t}), \end{aligned}$$

so that asymptotically as $t \to \infty$,

$$\begin{aligned} W_{|R|+k} - \lambda_R^{2k+1} \Psi_R^2 b_R^2 W'_k \to 0, \\ Z_{|R|+k} - \lambda_R^{2k-1} \Psi_R^2 b_R^2 Z'_k \to 0, \end{aligned}$$

which gives the desired formulas:

$$\begin{aligned} x_{n+1-(|R|+k)} &= \frac{1}{2} \ln \frac{Z_{|R|+k}}{W_{|R|+k-1}} \\ &= \frac{1}{2} \ln \frac{\lambda_R^{2k-1} \Psi_R^2 b_R^2 Z'_k + o(1)}{\lambda_R^{2(k-1)+1} \Psi_R^2 b_R^2 W'_{k-1} + o(1)} = \frac{1}{2} \ln \frac{Z'_k}{W'_{k-1}} + o(1), \end{aligned}$$

and similarly for $m_{n+1-(|R|+k)}$. □

Example 5.26. Figure 22 shows the positions $x_k(t)$ for a peakon–antipeakon solution with $n = 6$, with eigenvalues

$$\lambda_1 = \overline{\lambda_2} = \frac{1}{1+i}, \qquad \lambda_3 = \frac{1}{2}, \qquad \lambda_4 = \overline{\lambda_5} = \frac{1}{2+3i}, \qquad \lambda_6 = \frac{1}{4} \tag{5.19}$$

and residues

$$b_1(0) = b_2(0) = b_3(0) = b_4(0) = b_5(0) = b_6(0) = 1. \tag{5.20}$$

The location of the eigenvalues (see Figure 23) is chosen in order to obtain (asymptotically as $t \to \pm\infty$) one lone peakon with velocity 4, one 3-peakon cluster with velocity 2, and one pair with velocity 1.

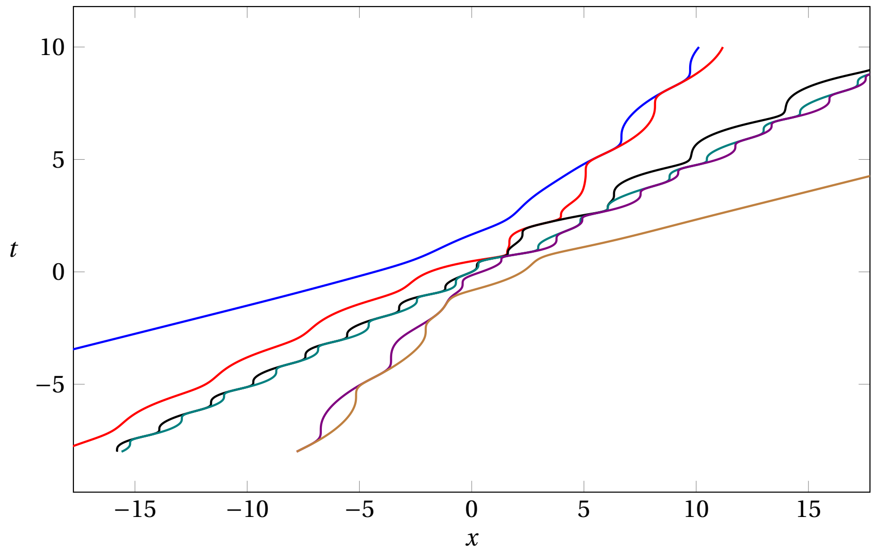

Figure 22: Positions $x = x_k(t)$ for the $n = 6$ peakon–antipeakon solution of Novikov's equation in Example 5.26. The eigenvalues λ_k are given by (5.19), as illustrated in Figure 23, and the residues $b_k(0)$ are all equal to 1. As $t \to -\infty$, the peakons are seen to separate into a fast single peakon (velocity 4), an intermediate-speed 3-peakon cluster (velocity 2), and a slow pair (velocity 1), and likewise as $t \to +\infty$, but with the grouping taking place in the opposite order. (For example, peakons number 5 and 6 pair up as $t \to -\infty$, but as as $t \to +\infty$ it is peakons number 1 and 2 that are forming a pair.)

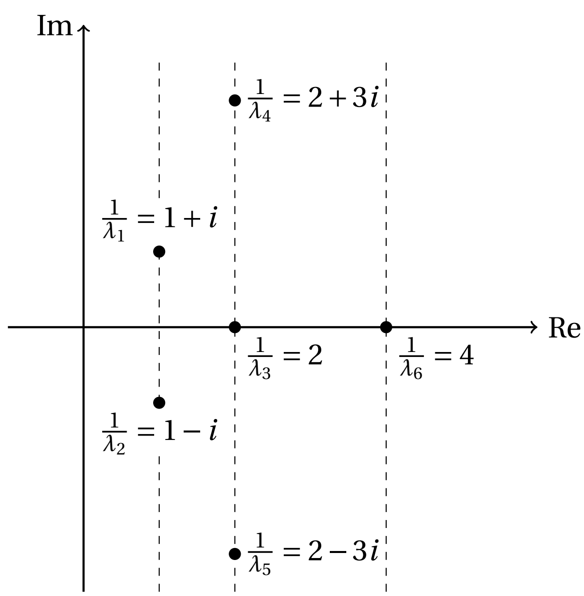

Figure 23: Location of the reciprocal eigenvalues $1/\lambda_k$ in the complex plane, for the 6-peakon solution in Figure 22. They are situated on the lines $\mathrm{Re}(1/\lambda) \in \{1, 2, 4\}$, and these values are the asymptotic velocities of (respectively) the pair, the 3-peakon cluster, and the single peakon seen as $t \to \pm\infty$.

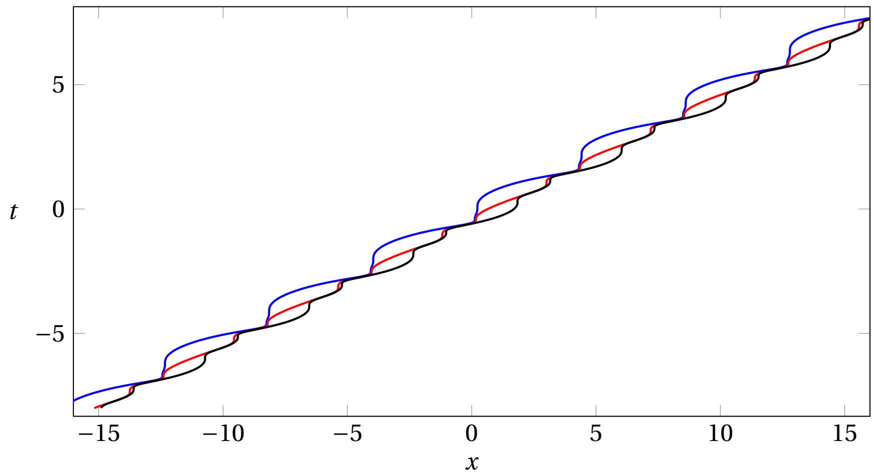

Figure 24: For the 6-peakon solution in Figure 22, the 3-peakon clusters seen as $t \to +\infty$ and as $t \to -\infty$ are associated with the three eigenvalues $\lambda_3 = \frac{1}{2}$, $\lambda_4 = \frac{1}{2+3i}$ and $\lambda_5 = \frac{1}{2-3i}$ satisfying $\operatorname{Re}\frac{1}{\lambda} = 2$. This picture shows $x = x_k(t)$ for the 3-peakon solution obtained using these eigenvalues together with the unmodified residues $b_3(0) = b_4(0) = b_5(0) = 1$, and clearly it looks like neither of the two 3-peakon clusters in the 6-peakon solution. In particular, one cannot obtain the correct asympotics for those clusters just by taking this picture and translating it in the (x, t) plane. (Cf. the case of real eigenvalues, where the peakons asymptotically travel in straight lines, and the only difference between $t \to -\infty$ and $t \to +\infty$ is such a shift.) Instead we must use the modified residues $b'_k(0)$ and $b''_k(0)$, as shown in Figures 25 and 26.

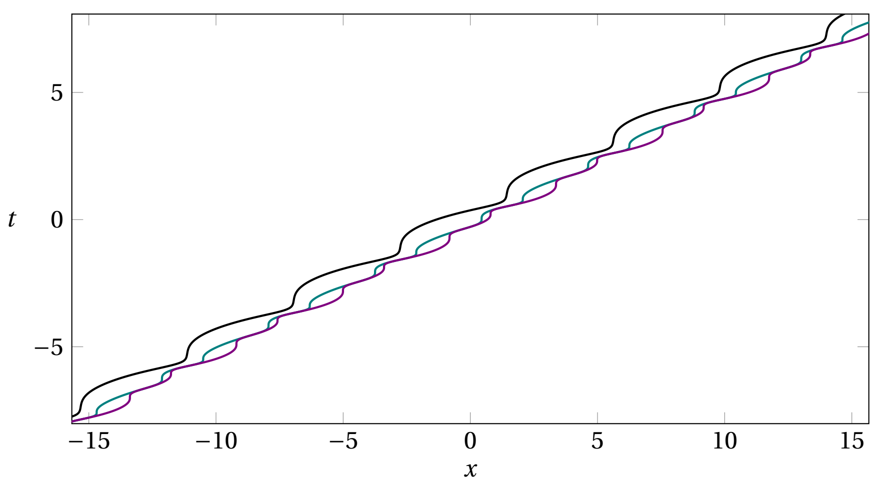

Figure 25: Positions $x = x_k(t)$ for the 3-peakon solution obtained using the eigenvalues $\lambda_{3,4,5}$ from (5.19) together with the modified residues $b'_{3,4,5}(0)$ from (5.21). This 3-peakon solution looks just like the 3-peakon cluster seen in Figure 22 as $t \to +\infty$, in agreement with Theorem 5.25.

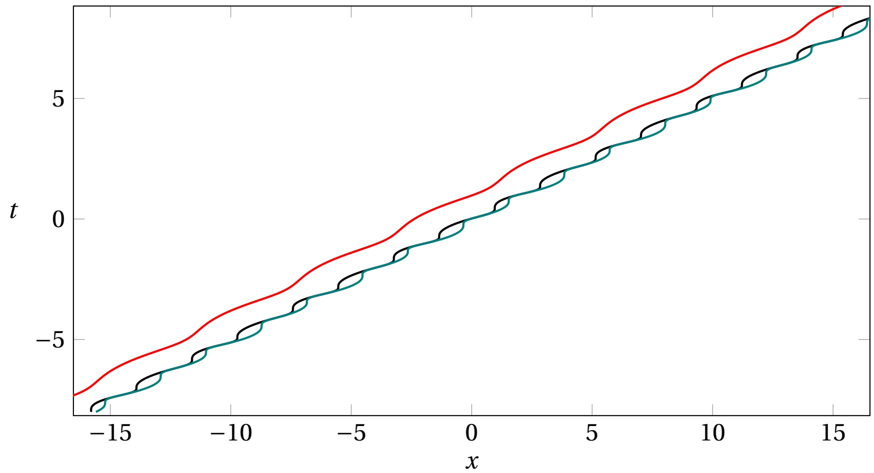

Figure 26: Positions $x = x_k(t)$ for the 3-peakon solution obtained using the eigenvalues $\lambda_{3,4,5}$ from (5.19) together with the modified residues $b''_{3,4,5}(0)$ from (5.22). This 3-peakon solution looks just like the 3-peakon cluster seen in Figure 22 as $t \to -\infty$, in agreement with Theorem 5.25.

To illustrate Theorem 5.25, we shall consider the asymptotics of the 3-peakon cluster, which is associated with the eigenvalues such that $\mathrm{Re}(1/\lambda_k) = 2$, namely those λ_k whose index k belongs to the set

$$S = \{3,4,5\}.$$

To the left of $1/\lambda_3$, $1/\lambda_4$ and $1/\lambda_5$ in Figure 23 we have $1/\lambda_1$ and $1/\lambda_2$, and to the right there is $1/\lambda_6$, so

$$L = \{1,2\}, \qquad R = \{6\}.$$

As Figure 24 illustrates, we do not get the correct description of the asymptotics just by taking the 3-peakon solution with spectral data consisting of λ_3, λ_4, λ_5 together with the original residues $b_3(0) = b_4(0) = b_5(0) = 1$. Instead, according to Theorem 5.25, we get the right asymptotics as $t \to +\infty$ by using the modified residues $b'_{3,4,5}$ defined by

$$b'_i = \frac{\Psi_{R,i}\, b_i}{\lambda_R} = \frac{(\lambda_6 - \lambda_i)^2}{\lambda_6(\lambda_6 + \lambda_i)}\, b_i = \frac{(1 - \frac{\lambda_i}{\lambda_6})^2}{1 + \frac{\lambda_i}{\lambda_6}}\, b_i,$$

i.e.,

$$\begin{aligned} b'_3(0) &= \frac{(1-\frac{4}{2})^2}{1+\frac{4}{2}}\, b_3(0) = \frac{1}{3}, \\ b'_4(0) = \overline{b'_5(0)} &= \frac{(1-\frac{4}{2+3i})^2}{1+\frac{4}{2+3i}}\, b_4(0) = \frac{-101+28i}{195}. \end{aligned} \tag{5.21}$$

As we can see in Figure 25, the 3-peakon solution with these spectral data agrees very well with the 3-peakon cluster seen in the full 6-peakon solution for $t \geq 5$ (roughly).

Similarly, the correct asymptotics for the 3-peakon cluster as $t \to -\infty$ is obtained using the modified residues $b''_{3,4,5}$ defined by

$$b''_i = \frac{\Psi_{L,i}\, b_i}{\lambda_L} = \frac{(\lambda_1-\lambda_i)^2(\lambda_2-\lambda_i)^2}{\lambda_1\lambda_2(\lambda_1+\lambda_i)(\lambda_2+\lambda_i)}\, b_i = \frac{(1-\frac{\lambda_i}{\lambda_1})^2(1-\frac{\lambda_i}{\lambda_2})^2}{(1+\frac{\lambda_i}{\lambda_1})(1+\frac{\lambda_i}{\lambda_2})}\, b_i,$$

i.e.,

$$\begin{aligned} b''_3(0) &= \frac{(1-\frac{1+i}{2})^2(1-\frac{1-i}{2})^2}{(1+\frac{1+i}{2})(1+\frac{1-i}{2})}\, b_3(0) = \frac{1}{10}, \\ b''_4(0) = \overline{b''_5(0)} &= \frac{(1-\frac{1+i}{2+3i})^2(1-\frac{1-i}{2+3i})^2}{(1+\frac{1+i}{2+3i})(1+\frac{1-i}{2+3i})}\, b_4(0) = \frac{283+1506i}{4225}. \end{aligned} \tag{5.22}$$

The 3-peakon solution with these spectral data is shown in Figure 26, and there is perfect agreement between this picture and the asymptotic 3-peakon cluster.

6 Peakon–antipeakon solutions with arbitrary n and eigenvalues of any multiplicity

Finally, we turn to the most complicated case, when the eigenvalues are allowed to have any multiplicity. This makes the bookkeeping a bit more complicated, so we begin by defining some notation.

6.1 Notation

Given a peakon configuration $\{x_1,\dots,x_n,m_1,\dots,m_n\}$ with $x_1<\dots x_n$ and all $m_k\neq 0$, the polynomials $A(\lambda)$ and $B(\lambda)$ is defined by the same formula (2.19) as before (and $A(\lambda)$ is also given by (2.21)), and the Weyl function is still $\omega(\lambda) = -B(\lambda)/A(\lambda)$. Also as before, the eigenvalues are defined as the zeros of $A(\lambda)$, they are still denoted by $\lambda_1,\dots,\lambda_n$, and they still are time-independent since the coefficients of $A(\lambda)$ are constants of motion for the Novikov peakon ODEs. But now we no longer impose the condition that $A(\lambda)$ has simple zeros; in other words, we allow the possibility that some of the eigenvalues λ_k may have the same numerical value.

Definition 6.1. We suppose throughout Section 6 that there are J distinct values $\mu_1,\dots,\mu_J$ among the eigenvalues $\lambda_1,\dots,\lambda_n$, occuring with multiplicities $d_1,\dots,d_J$, respectively, where $d_1+\dots+d_J = n$. Numbering coinciding eigenvalues in sequence, we thus have

$$\begin{aligned} \lambda_1 = \dots = \lambda_{d_1} &= \mu_1, \\ \lambda_{d_1+1} = \dots = \lambda_{d_1+d_2} &= \mu_2, \\ \lambda_{d_1+d_2+1} = \dots = \lambda_{d_1+d_2+d_3} &= \mu_3, \\ &\vdots \\ \lambda_{n-d_J+1} = \dots = \lambda_n &= \mu_J. \end{aligned} \tag{6.1}$$

Definition 6.2. Partition the index set $\{1,\dots,n\}$ into J integer intervals $\mathscr{J}_1,\dots,\mathscr{J}_J$ with lengths $d_1,\dots,d_J$, in the obvious way:

$$\begin{aligned}
\mathscr{J}_1 &= [1, d_1],\\
\mathscr{J}_2 &= [d_1+1, d_1+d_2],\\
\mathscr{J}_3 &= [d_1+d_2+1, d_1+d_2+d_3],\\
&\;\vdots\\
\mathscr{J}_J &= [n-d_J+1, n].
\end{aligned} \tag{6.2}$$

(Hence if $k \in \mathscr{J}_a$, then $\lambda_k = \mu_a$.) To allow for an easy labelling of the elements of each $\mathscr{J}_j$, set

$$\delta_j = \sum_{i=1}^{j-1} d_i \qquad (\text{with } \delta_1 = 0), \tag{6.3}$$

so that

$$\mathscr{J}_j = [\delta_j+1, \delta_j+d_j] = \{\delta_j + k : 1 \le k \le d_j\}. \tag{6.4}$$

Definition 6.3. Define integers $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$, as follows: let α_k be the subscript a of the index group $\mathscr{J}_a$ to which k belongs, so that we have

$$\lambda_k = \mu_{\alpha_k}, \tag{6.5}$$

and let $\beta_k = k - \delta_{\alpha_k} - 1$, so that $\beta_k = 0,1,2,\dots$ according to which index k is within that group, i.e.,

$$\begin{aligned}
\alpha_k &= 1, \quad \beta_k = k-\delta_1-1 = k-1, && \text{if } k \in \mathscr{J}_1,\\
\alpha_k &= 2, \quad \beta_k = k-\delta_2-1 = k-d_1-1, && \text{if } k \in \mathscr{J}_2,\\
\alpha_k &= 3, \quad \beta_k = k-\delta_3-1 = k-d_1-d_2-1, && \text{if } k \in \mathscr{J}_3,
\end{aligned} \tag{6.6}$$

and so on.

As for the Weyl function, its partial fraction decomposition will no longer be given by (2.23), since its poles are not necessarily simple anymore. Instead it takes the following form:

Definition 6.4. Write the Weyl function $\omega(\lambda) = -B(\lambda)/A(\lambda)$ as

$$\omega(\lambda) = \sum_{k=1}^{n} \frac{a_k}{(\lambda-\lambda_k)^{\beta_k+1}} = \sum_{j=1}^{J} \sum_{k=1}^{d_j} \frac{a_{\delta_j+k}}{(\lambda-\mu_j)^k}, \tag{6.7}$$

where $a_1,\dots,a_n$ are some (time-dependent) coefficients. In the proofs, it will be convenient to work with these coefficients a_k, but the final formulas will become more homogeneous if written using another set of coefficients,

$$b_k = \frac{a_k}{\lambda_k^{\beta_k}}. \tag{6.8}$$

In terms of these coefficients b_k, (6.7) of course becomes

$$\omega(\lambda) = \sum_{k=1}^{n} \frac{\lambda_k^{\beta_k} b_k}{(\lambda-\lambda_k)^{\beta_k+1}} = \sum_{j=1}^{J} \sum_{k=1}^{d_j} \frac{\mu_j^{k-1} b_{\delta_j+k}}{(\lambda-\mu_j)^k}. \tag{6.9}$$

Example 6.5. Suppose the multiplicities of the eigenvalues are

$$d_1 = 5, \qquad d_2 = 3, \qquad d_3 = 1, \qquad d_4 = 1.$$

Then $n = 5+3+1+1 = 10$ and the indices $\{1,2,\dots,10\}$ are partitioned into the four sets

$$\mathscr{J}_1 = \{1,2,3,4,5\}, \qquad \mathscr{J}_2 = \{6,7,8\}, \qquad \mathscr{J}_3 = \{9\}, \qquad \mathscr{J}_4 = \{10\},$$

with the offsets $\delta_1 = 0$, $\delta_2 = 5$, $\delta_3 = 8$, $\delta_4 = 9$, and we have

$$\begin{aligned}
&\lambda_1 = \mu_1, \quad \alpha_1 = 1, \quad \beta_1 = 0, &\qquad& \lambda_6 = \mu_2, \quad \alpha_6 = 2, \quad \beta_6 = 0,\\
&\lambda_2 = \mu_1, \quad \alpha_2 = 1, \quad \beta_2 = 1, && \lambda_7 = \mu_2, \quad \alpha_7 = 2, \quad \beta_7 = 1,\\
&\lambda_3 = \mu_1, \quad \alpha_3 = 1, \quad \beta_3 = 2, && \lambda_8 = \mu_2, \quad \alpha_8 = 2, \quad \beta_8 = 2,\\
&\lambda_4 = \mu_1, \quad \alpha_4 = 1, \quad \beta_4 = 3, && \lambda_9 = \mu_3, \quad \alpha_9 = 3, \quad \beta_9 = 0,\\
&\lambda_5 = \mu_1, \quad \alpha_5 = 1, \quad \beta_5 = 4, && \lambda_{10} = \mu_4, \quad \alpha_{10} = 4, \quad \beta_{10} = 0.
\end{aligned}$$

The Weyl function takes the form

$$\begin{aligned}
\omega(\lambda) &= \left(\frac{a_1}{\lambda-\mu_1} + \frac{a_2}{(\lambda-\mu_1)^2} + \frac{a_3}{(\lambda-\mu_1)^3} + \frac{a_4}{(\lambda-\mu_1)^4} + \frac{a_5}{(\lambda-\mu_1)^5}\right)\\
&\quad + \left(\frac{a_6}{\lambda-\mu_2} + \frac{a_7}{(\lambda-\mu_2)^2} + \frac{a_8}{(\lambda-\mu_2)^3}\right) + \frac{a_9}{\lambda-\mu_3} + \frac{a_{10}}{\lambda-\mu_4}\\
&= \left(\frac{b_1}{\lambda-\mu_1} + \frac{\mu_1 b_2}{(\lambda-\mu_1)^2} + \frac{\mu_1^2 b_3}{(\lambda-\mu_1)^3} + \frac{\mu_1^3 b_4}{(\lambda-\mu_1)^4} + \frac{\mu_1^4 b_5}{(\lambda-\mu_1)^5}\right)\\
&\quad + \left(\frac{b_6}{\lambda-\mu_2} + \frac{\mu_2 b_7}{(\lambda-\mu_2)^2} + \frac{\mu_2^2 b_8}{(\lambda-\mu_2)^3}\right) + \frac{b_9}{\lambda-\mu_3} + \frac{b_{10}}{\lambda-\mu_4}.
\end{aligned}$$

Next, we define a generalization $\widetilde{\Psi}_I^{(s)}$ of the function Ψ_I in (2.7). Here s is an integer (in fact we will use only $s \in \{-1,0,1\}$), and the symbol

$$I = (i_1,\dots,i_k) \in [1,n]^k$$

now denotes an ordered k-tuple (where $1 \le k \le n$) of indices with values in $[1,n]$. In particular, repetitions will be allowed, i.e., several indices may have the same value, and there is no assumption about the indices being sorted in increasing order. (Earlier we had only distinct indices, and could therefore view $I = \{i_1 < \dots < i_k\}$ as a subset of $[1,n]$.)

Definition 6.6. Suppose $I = (i_1,\dots,i_k) \in [1,n]^k$. With

$$\Psi(z_1,\dots,z_k) = \frac{\Delta(z_1,\dots,z_k)^2}{\Gamma(z_1,\dots,z_k)} = \prod_{1\le a<b\le k} \frac{(z_a-z_b)^2}{z_a+z_b},$$

let

$$\widetilde{\Psi}_I^{(s)} = \frac{\left[\left(\frac{\partial}{\partial z_1}\right)^{\beta_{i_1}} \cdots \left(\frac{\partial}{\partial z_k}\right)^{\beta_{i_k}} (z_1 \cdots z_k)^s \, \Psi(z_1, \dots, z_k) \right]_{z_1 = \lambda_{i_1}, \dots, z_k = \lambda_{i_k}}}{\beta_{i_1}! \cdots \beta_{i_k}!}. \tag{6.10}$$

Remark 6.7. For convenience, we simply write

$$\widetilde{\Psi}_I = \widetilde{\Psi}_I^{(0)}$$

in the special case $s = 0$. And in concrete calculations, like in the examples in the later sections, we will write simply $\widetilde{\Psi}_{213}$ instead of $\widetilde{\Psi}_{(2,1,3)}$, etc.

Remark 6.8. If all eigenvalues are simple or, more generally, if I only contains indices i with $\beta_i = 0$, then

$$\widetilde{\Psi}_I^{(s)} = (\lambda_{i_1} \cdots \lambda_{i_k})^s \, \Psi(\lambda_{i_1}, \dots, \lambda_{i_k}),$$

so in this case it agrees with our previous quantity $\lambda_I^s \Psi_I$ if all the indices in I are distinct, and equals zero if there are repetitions in I. But in other cases, $\widetilde{\Psi}_I^{(s)}$ may be nonzero even if I contains repetitions.

Remark 6.9. It is clear that $\widetilde{\Psi}_I$ is invariant under permutations of the indices, i.e., $\widetilde{\Psi}_I = \widetilde{\Psi}_{I'}$ if $I = (i_1, \dots, i_k)$ and $I' = (i_{\pi(1)}, \dots, i_{\pi(k)})$ for some permutation $\pi \in S_k$. Thus, one may always write $\widetilde{\Psi}_I$ with $i_1 \le \cdots \le i_k$.

Example 6.10. Given the eigenvalues in Example 6.5, let us compute $\widetilde{\Psi}_{137}^{(1)}$, i.e. $\widetilde{\Psi}_I^{(s)}$ with $I = \{1, 3, 7\}$ and $s = 1$. The eigenvalues λ_1 and λ_3 are the first and the third in the group of eigenvalues that are equal to μ_1, and this is kept track of by the numbers $\beta_1 = 0$ and $\beta_3 = 2$. And λ_7 was the second of the eigenvalues equal to μ_2, so $\beta_7 = 1$. The index set I has three elements, so we are supposed to form the three-variable function

$$(z_1 z_2 z_3)^s \, \Psi(z_1, z_2, z_3) = \frac{z_1 z_2 z_3 \, (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2}{(z_1 + z_2)(z_1 + z_3)(z_2 + z_3)},$$

and differentiate it β_1 times with respect to z_1, β_3 times with respect to z_2, and β_7 times with respect to z_3. This is a routine matter for a computer algebra system, although it would be quite tedious to do it by hand:

$$\begin{aligned}
&\left(\frac{\partial}{\partial z_1}\right)^0 \left(\frac{\partial}{\partial z_2}\right)^2 \left(\frac{\partial}{\partial z_3}\right)^1 \frac{z_1 z_2 z_3 \, (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2}{(z_1 + z_2)(z_1 + z_3)(z_2 + z_3)} \\
&= 2 z_1 (z_1 - z_3) \Big(z_1^6 z_2^4 - 6 z_1^5 z_2^5 + 6 z_1^3 z_2^7 + 3 z_1^2 z_2^8 + 4 z_1^6 z_2^3 z_3 - 9 z_1^5 z_2^4 z_3 \\
&\quad + 6 z_1^3 z_2^6 z_3 - 12 z_1^2 z_2^7 z_3 - 9 z_1 z_2^8 z_3 + 6 z_1^6 z_2^2 z_3^2 + 24 z_1^5 z_2^3 z_3^2 + 19 z_1^4 z_2^4 z_3^2 \\
&\quad + 3 z_1^3 z_2^5 z_3^2 - 51 z_1^2 z_2^6 z_3^2 - 39 z_1 z_2^7 z_3^2 - 6 z_2^8 z_3^2 - 12 z_1^6 z_2 z_3^3 + 18 z_1^5 z_2^2 z_3^3 \\
&\quad + 28 z_1^4 z_2^3 z_3^3 + 45 z_1^3 z_2^4 z_3^3 + 3 z_1^2 z_2^5 z_3^3 - 39 z_1 z_2^6 z_3^3 - 15 z_2^7 z_3^3 - 3 z_1^6 z_3^4 \\
&\quad - 6 z_1^5 z_2 z_3^4 - 42 z_1^4 z_2^2 z_3^4 - 24 z_1^3 z_2^3 z_3^4 + 97 z_1^2 z_2^4 z_3^4 + 54 z_1 z_2^5 z_3^4 + 3 z_1^5 z_3^5 \\
&\quad - 84 z_1^3 z_2^2 z_3^5 - 14 z_1^2 z_2^3 z_3^5 + 69 z_1 z_2^4 z_3^5 + 30 z_2^5 z_3^5 + 15 z_1^4 z_3^6 \\
&\quad - 9 z_1^3 z_2 z_3^6 - 45 z_1^2 z_2^2 z_3^6 - 3 z_1 z_2^3 z_3^6 + 6 z_2^4 z_3^6 + 9 z_1^3 z_3^7 \\
&\quad - 9 z_1^2 z_2 z_3^7 - 9 z_1 z_2^2 z_3^7 - 3 z_2^3 z_3^7 \Big) (z_1 + z_2)^{-3} (z_1 + z_3)^{-2} (z_2 + z_3)^{-4}.
\end{aligned}$$

To obtain $\widetilde{\Psi}^{(1)}_{137}$ we now substitute $z_1 = \lambda_1$, $z_2 = \lambda_3$ and $z_3 = \lambda_7$ into this expression, in other words $z_1 = z_2 = \mu_1$ and $z_3 = \mu_2$, and divide by $\beta_1!\beta_2!\beta_3! = 0!2!1! = 2$. The result is

$$\widetilde{\Psi}^{(1)}_{137} = \frac{\mu_1(\mu_1-\mu_2)^3(\mu_1^2 - 6\mu_1\mu_2 - 3\mu_2^2)}{2(\mu_1+\mu_2)^3}.$$

Example 6.11. For an example showing that $\widetilde{\Psi}^{(s)}_I$ can be nonzero even if I contains repetitions, consider $\widetilde{\Psi}_{33}$ with the data from Example 6.5. Since $\beta_3 = 2$, we compute

$$\left(\frac{\partial}{\partial z_1}\right)^2 \left(\frac{\partial}{\partial z_2}\right)^2 \frac{(z_1-z_2)^2}{z_1+z_2} = \frac{16(z_1^2 - 4z_1z_2 + z_2^2)}{(z_1+z_2)^5},$$

substitute $z_1 = z_2 = \lambda_3 = \mu_1$ and divide by $\beta_3!\beta_3! = 2!2! = 4$, to obtain

$$\widetilde{\Psi}_{33} = -\frac{1}{4\mu_1^3}.$$

Note that the computation of $\widetilde{\Psi}_{kk}$ is basically the same whenever $\beta_k = 2$. For example, since $\beta_8 = 2$ and $\lambda_8 = \mu_2$, we also have

$$\widetilde{\Psi}_{88} = -\frac{1}{4\mu_2^3}.$$

Definition 6.12. For $I = (i_1,\dots,i_k) \in [1,n]^k$, write

$$a_I = a_{i_1}\cdots a_{i_k} = \lambda_{i_1}^{\beta_{i_1}} \cdots \lambda_{i_k}^{\beta_{i_k}} b_{i_1}\cdots b_{i_k} = \lambda_I^{\beta_I} b_I. \tag{6.11}$$

With this notation, together with the symbol $\Psi^{(s)}_I$, we can now extend the definition of U_k to the case of multiple eigenvalues.

Definition 6.13. Let

$$\widetilde{U}_k = \frac{1}{k!}\sum_{I\in[1,n]^k} \widetilde{\Psi}_I a_I = \frac{1}{k!}\sum_{I\in[1,n]^k} \widetilde{\Psi}_I \lambda_I^{\beta_I} b_I, \tag{6.12}$$

with the same conventions for degenerate cases as in Definition 2.5 ($\widetilde{U}_0 = 1$, etc.).

Remark 6.14. If all eigenvalues are simple, $\widetilde{U}_k$ reduces to the old U_k: all terms where I contains repetitions vanish, leaving $k!$ nonzero terms in the sum, coming from the $k!$ permutations of the distinct numbers $(i_1,\dots,i_k)$, and each of these $k!$ terms equals the old $\Psi_I b_I$ (with I viewed as a set).

Remark 6.15. There is redundancy in the sum (6.12) even in cases where $\widetilde{\Psi}_I$ can be nonzero when I contains repetitions, since any term $\widetilde{\Psi}_I a_I$ remains unchanged under permutations of the indices. So one can save some work by summing only over sorted tuples $I = (i_1,\dots,i_k)$ with $i_1 \le \cdots \le i_k$, if one takes into account how many times each term occurs in the full sum (6.12). Suppose the sorted k-tuple I contains r distinct numbers occuring $q_1, q_2, \dots, q_r$ times:

$$\underbrace{i_1 = \cdots = i_{q_1}}_{q_1 \text{ repetitions}} < \underbrace{i_{q_1+1} = \cdots = i_{q_1+q_2}}_{q_2 \text{ repetitions}} < \cdots .$$

Then there are

$$\frac{k!}{q_1!\, q_2! \cdots q_r!}$$

tuples I' with the same value $\widetilde{\Psi}_{I'} a_{I'} = \widetilde{\Psi}_I a_I$. Thus, if we let $\chi_I = q_1!\, q_2! \cdots q_r!$, we can write (6.12) in the more economical way

$$\widetilde{U}_k = \sum_{I \text{ sorted}} \frac{\widetilde{\Psi}_I a_I}{\chi_I} = \sum_{I \text{ sorted}} \frac{\widetilde{\Psi}_I \lambda_I^{\beta_I} b_I}{\chi_I}. \tag{6.13}$$

(In particular, $\chi_I = 1$ if all indices in I are distinct, since $q_1 = \cdots = q_k = 1$ in that case.)

Example 6.16. If $n = 3$, then

$$\begin{aligned} \widetilde{U}_3 = {} & \tfrac{1}{6}(\widetilde{\Psi}_{111} a_1^3 + \widetilde{\Psi}_{222} a_2^3 + \widetilde{\Psi}_{333} a_3^3) \\ & + \tfrac{1}{2}(\widetilde{\Psi}_{112} a_1^2 a_2 + \widetilde{\Psi}_{122} a_1 a_2^2 + \widetilde{\Psi}_{113} a_1^2 a_3 \\ & \qquad + \widetilde{\Psi}_{133} a_1 a_3^2 + \widetilde{\Psi}_{223} a_2^2 a_3 + \widetilde{\Psi}_{233} a_2 a_3^2) \\ & + \widetilde{\Psi}_{123} a_1 a_2 a_3, \end{aligned} \tag{6.14}$$

In the case of simple eigenvalues, only the last term survives, and we recover the old expression

$$\widetilde{U}_3 = \widetilde{\Psi}_{123} a_1 a_2 a_3 = \Psi_{123} b_1 b_2 b_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3. \tag{6.15}$$

In the case with one simple and one double eigenvalue, $\lambda_1 = \mu_1 \neq \lambda_2 = \lambda_3 = \mu_2$, it also turns out that only one term survives, but this time it is $\widetilde{\Psi}_{133}$ that is nonzero:

$$\widetilde{U}_3 = \tfrac{1}{2} \widetilde{\Psi}_{133}\, a_1 a_3^2 = -\frac{(\mu_1 - \mu_2)^4}{2\mu_2^3 (\mu_1 + \mu_2)}\, a_1 a_3^2 = -\frac{(\mu_1 - \mu_2)^4}{2\mu_2 (\mu_1 + \mu_2)}\, b_1 b_3^2 \tag{6.16}$$

And for a triple eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = \mu$ (see Section 6.4), the sum also reduces to one term, but now it is $\widetilde{\Psi}_{333}$ that is nonzero:

$$\widetilde{U}_3 = \tfrac{1}{6} \widetilde{\Psi}_{333}\, a_3^3 = -\frac{a_3^3}{8\mu^3} = -\frac{\mu^3 b_3^3}{8}. \tag{6.17}$$

(This example is perhaps a little misleading, since $\widetilde{U}_k$ for the maximal value $k = n$ is particularly simple. In general, $\widetilde{U}_k$ can be a much more complicated expression than the old U_k.)

Similarly, we extend the definitions of V_k and T_k by using $\widetilde{\Psi}_I^{(1)}$ and $\widetilde{\Psi}_I^{(-1)}$ instead of just $\widetilde{\Psi}_I = \widetilde{\Psi}_I^{(0)}$.

Definition 6.17. Let

$$\begin{aligned} \widetilde{V}_k &= \frac{1}{k!} \sum_{I \in [1,n]^k} \widetilde{\Psi}_I^{(1)} a_I = \frac{1}{k!} \sum_{I \in [1,n]^k} \widetilde{\Psi}_I^{(1)} \lambda_I^{\beta_I} b_I, \\ \widetilde{T}_k &= \frac{1}{k!} \sum_{I \in [1,n]^k} \widetilde{\Psi}_I^{(-1)} a_I = \frac{1}{k!} \sum_{I \in [1,n]^k} \widetilde{\Psi}_I^{(-1)} \lambda_I^{\beta_I} b_I, \end{aligned} \tag{6.18}$$

and

$$\widetilde{W}_k = \widetilde{U}_k \widetilde{V}_k - \widetilde{U}_{k+1} \widetilde{V}_{k-1}, \qquad \widetilde{Z}_k = \widetilde{T}_k \widetilde{U}_k - \widetilde{T}_{k+1} \widetilde{U}_{k-1}. \tag{6.19}$$

Finally, to describe the time dependence of the coefficients a_k and b_k, we will need a certain sequence of polynomials p_k.

Definition 6.18. Let the polynomials p_k be defined, for $k \geq 0$, by

$$p_k(1/z) = \frac{z^k}{e^{1/z}} \left(\frac{\partial}{\partial z} \right)^k e^{1/z}. \tag{6.20}$$

Example 6.19. One easily computes

$$\begin{aligned} p_0(w) &= 1, \\ p_1(w) &= -w, \\ p_2(w) &= w^2 + 2w, \\ p_3(w) &= -(w^3 + 6w^2 + 6w), \\ p_4(w) &= w^4 + 12w^3 + 36w^2 + 24w, \\ p_5(w) &= -(w^5 + 20w^4 + 120w^3 + 240w^2 + 120w), \end{aligned}$$

and so on. For example, $p_2(w) = w^2 + 2w$ since

$$\left(\frac{\partial}{\partial z} \right)^2 e^{1/z} = \frac{\partial}{\partial z} \left(-\frac{1}{z^2} e^{1/z} \right) = \frac{2}{z^3} e^{1/z} + \left(-\frac{1}{z^2} \right)^2 e^{1/z} = \underbrace{\left(\frac{1}{z^2} + \frac{2}{z} \right)}_{p_2(1/z)} \frac{e^{1/z}}{z^2}.$$

Proposition 6.20. *The polynomials p_k satisfy the recurrence relation*

$$\begin{aligned} p_0(w) &= 1, \\ p_{k+1}(w) &= -(w + k)\, p_k(w) - w\, p_k'(w), \qquad \text{for } k \geq 0, \end{aligned} \tag{6.21}$$

and are given explicitly by

$$\begin{aligned} p_0(w) &= 1, \\ p_k(w) &= (-1)^k \sum_{r=1}^{k} \binom{k-1}{r-1} \frac{k!}{r!} w^r, \qquad \text{for } k \geq 1. \end{aligned} \tag{6.22}$$

In particular, $p_0(0) = 1$*, but* $p_k(0) = 0$ *for* $k \ge 1$*. Moreover,*

$$\left(\frac{\partial}{\partial z}\right)^k e^{t/z} = \frac{p_k(t/z)\, e^{t/z}}{z^k}. \tag{6.23}$$

We omit the proof; it is fairly straightforward to verify that (6.22) satisfies the recurrence (6.21), which in turn follows immediately from the definition (6.20), and equation (6.23) is a direct consequence of (6.20) and the chain rule. Several different proofs that (6.20) $\iff$ (6.22) can be found in [9], which also contains more information about the coefficients

$$L(k,r) = \binom{k-1}{r-1}\frac{k!}{r!} = \binom{k}{r}\binom{k-1}{r-1}(k-r)!,$$

which are known as the Lah numbers and, for example, connect rising and falling factorial powers:

$$x^{\overline{k}} = \sum_{r=1}^{k} L(k,r)\, x^{\underline{r}}, \qquad x^{\underline{k}} = \sum_{r=1}^{k} (-1)^{k-r} L(k,r)\, x^{\overline{r}}.$$

6.2 Multipeakon solution formulas

With the notation of Section 6.1 in place, we finally come to the solution formulas for the Novikov peakon ODEs (1.7) in the case where eigenvalues may have multiplicity greater than one. As will be apparent, the notation is designed in order to make the solution formulas look just the same as before, when written in abbreviated form. However, the time dependence of the coefficients b_k is a bit more complicated than just $b_k(t) = b_k(0)\, e^{t/\lambda_k}$, and the expressions $\widetilde{U}_k$ (etc.) can be quite a lot more involved than the old U_k (etc.), if one actually writes out the details.

Before stating the results, let us recall some notation from Section 5.1. There $\widehat{\mathcal{P}}$ denoted the set of all peakon configurations with $x_1 < \cdots < x_n$ and all $m_k \neq 0$ (Definition 5.1). To any such configuration, we associated eigenvalues $\lambda_1, \dots, \lambda_n$, namely the zeros of the polynomial $A(\lambda)$, but only for configurations in the subset $\widehat{\mathcal{P}}_{\mathrm{s}}$ such that these eigenvalues are simple did we define the residues b_k in the Weyl function $\omega(\lambda)$. We also defined certain sets $\widehat{\mathcal{R}}_{\mathrm{s}}$ and $\widehat{\mathcal{R}}_{\mathrm{nc}}$ of spectral data (Definitions 5.6 and 5.10) associated to configurations in $\widehat{\mathcal{P}}_{\mathrm{s}}$.

Now that we have extended the notation to include the general kind of partial fraction expansion (6.9),

$$\omega(\lambda) = \sum_{k=1}^{n} \frac{\lambda_k^{\beta_k}\, b_k}{(\lambda - \lambda_k)^{\beta_k + 1}} = \sum_{j=1}^{J} \sum_{k=1}^{d_j} \frac{\mu_j^{k-1}\, b_{\delta_j + k}}{(\lambda - \mu_j)^k},$$

we can also extend the notion of the spectral map to all of $\widehat{\mathcal{P}}$: for *every* peakon configuration we can talk about its associated eigenvalues $\lambda_1, \dots, \lambda_n$ and coefficients $b_1, \dots, b_n$. If these eigenvalues take J distinct values μ_j, as in Section 6.1, then the spectral data are only defined up to a relabeling of the indices $\{1, \dots, J\}$ (we permute the eigenvalues μ_j, and then permute the corresponding coefficients $b_{\delta_j + k}$ in the same way), so we can think of the spectral map as taking its value in the quotient space $\mathbf{C}^{2n}/S_J$ at such a peakon configuration. Note that even though we have continued to use the notation b_k for the coefficients in the Weyl function, it is definitely *not* the case that this extended spectral map is in any sense continuous at a point where some eigenvalues coincide.

Since $\omega(\lambda)$ has real coefficients, any non-real spectral data must occur in complex-conjugated pairs. We also know from Theorem 5.4 that $\operatorname{Re}\lambda_k \geq 0$ always.

Definition 6.21. Let $\widehat{\mathscr{R}}$ denote the set of spectral data $\{\lambda_k, b_k\}_{k=1}^n$ satisfying the following conditions:

- The eigenvalues λ_k (taking the J distinct values μ_j with multiplicities d_j) are located strictly in the right half of the complex plane ($\operatorname{Re}\lambda_k > 0$), and non-real eigenvalues only exist in complex-conjugated pairs.
- The corresponding coefficients b_k also come in conjugated pairs: if $\mu_i = \overline{\mu_j}$, then $b_{\delta_i+k} = \overline{b_{\delta_j+k}}$ for $1 \leq k \leq d_j$.
- For each μ_j, the corresponding highest coefficient is nonzero:

$$b_{\delta_j+d_j} \neq 0, \qquad \text{for } 1 \leq j \leq J. \tag{6.24}$$

Remark 6.22. We will state our formulas for n-peakon solutions given that the spectral data are in $\widehat{\mathscr{R}}$. We believe that this is the most general kind of spectral data that can occur, but unfortunately we haven't been able to rigorously rule out the possibility that there might be other solutions with some $\operatorname{Re}\lambda_k = 0$; cf. Remark 5.5.

The next theorem shows that the requirement about highest coefficients being nonzero is actually redundant in Definition 6.21.

Theorem 6.23. *If*

$$(x_1, \ldots, x_n, m_1, \ldots, m_n) \in \widehat{\mathscr{P}}$$

is a peakon configuration whose associated eigenvalues $\{\lambda_k\}_{k=1}^n$ *all satisfy* $\operatorname{Re}\lambda_k > 0$*, then* (6.24) *holds automatically.*

Proof. Here we make use of both Weyl functions $\omega(\lambda)$ and $\zeta(\lambda)$ in (2.20). Since they have the same denominator, their partial fraction expansions have the same structure,

$$\omega(\lambda) = -\frac{B(\lambda)}{A(\lambda)} = \frac{b_{\delta_j+1}}{\lambda-\mu_j} + \cdots + \frac{b_{\delta_j+d_j}}{(\lambda-\mu_j)^{d_j}} + \text{other terms},$$
$$\zeta(\lambda) = -\frac{C(\lambda)}{A(\lambda)} = \frac{c_{\delta_j+1}}{\lambda-\mu_j} + \cdots + \frac{c_{\delta_j+d_j}}{(\lambda-\mu_j)^{d_j}} + \text{other terms}.$$

Studying the coefficient of $(\lambda-\mu_j)^{-d_j}$ in the relation (2.24),

$$\zeta(\lambda) + \zeta(-\lambda) + \omega(\lambda)\omega(-\lambda) = 0,$$

using that fact that there can be no pole in $-\mu_j$ if all eigenvalues have positive real part, one finds that if $b_{\delta_j+d_j}=0$ then also $c_{\delta_j+d_j}=0$.

Furthermore,

$$b_{\delta_j+d_j} = \lim_{\lambda\to\mu_j} (\lambda-\mu_j)^{d_j} \frac{-B(\lambda)}{A(\lambda)} = -\frac{B(\mu_j)}{\widehat{A}(\mu_j)},$$

where $\widehat{A}(\mu_j)\neq 0$, and similarly

$$c_{\delta_j+d_j} = \lim_{\lambda\to\mu_j} (\lambda-\mu_j)^{d_j} \frac{-C(\lambda)}{A(\lambda)} = -\frac{C(\mu_j)}{\widehat{A}(\mu_j)}.$$

Thus, if $b_{\delta_j+d_j}$ were zero, the vector $(A(\mu_j),B(\mu_j),C(\mu_j))^T$ would also be zero, but this is impossible due to the definition (2.19), since the matrices $S_k(\lambda)$ all have determinant 1.

□

Now we can state our last major results. The proofs will be given in Section 6.6, after we have looked at some examples.

Theorem 6.24. *The time dependence induced by the Novikov equation on the Weyl function* (6.9)*,*

$$\omega(\lambda;t) = \sum_{k=1}^{n} \frac{\lambda_k^{\beta_k}\, b_k(t)}{(\lambda-\lambda_k)^{\beta_k+1}} = \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{\mu_j^{k-1}\, b_{\delta_j+k}(t)}{(\lambda-\mu_j)^k}, \tag{6.25}$$

is given by

$$\begin{aligned}\omega(\lambda;t) &= \sum_{s=1}^{n} \frac{\lambda_s^{\beta_s}\, b_s(0)}{\beta_s!} \left[\left(\frac{\partial}{\partial z}\right)^{\beta_s} \frac{e^{t/z}}{\lambda-z}\right]_{z=\lambda_s} \\ &= \sum_{j=1}^{J}\sum_{r=0}^{d_j-1} \frac{\mu_j^r\, b_{\delta_j+1+r}(0)}{r!} \left[\left(\frac{\partial}{\partial z}\right)^{r} \frac{e^{t/z}}{\lambda-z}\right]_{z=\mu_j},\end{aligned} \tag{6.26}$$

which is equivalent to the following time dependence for the coefficients:

$$b_{\delta_j+k}(t) = \sum_{q=0}^{d_j-k} \frac{b_{\delta_j+k+q}(0)}{q!}\, p_q(t/\mu_j)\, e^{t/\mu_j}, \tag{6.27}$$

for $1\le j\le J$ *and* $1\le k\le d_j$*.*

Theorem 6.25. *Given spectral data in* $\widehat{\mathcal{R}}$*, the formulas*

$$x_{n+1-k}(t) = \frac{1}{2}\ln \frac{\widetilde{Z}_k(t)}{\widetilde{W}_{k-1}(t)}, \qquad m_{n+1-k}(t) = \frac{\sqrt{\widetilde{Z}_k(t)\,\widetilde{W}_{k-1}(t)}}{\widetilde{U}_k(t)\,\widetilde{U}_{k-1}(t)}, \tag{6.28}$$

for $1\le k\le n$*, with the time dependence given by letting* $b_i=b_i(t)$ *as in* (6.27)*, give a solution of the Novikov peakon ODEs* (1.7) *in every time interval where* $\widetilde{U}_k(t)\neq 0$*,* $\widetilde{W}_k(t)>0$ *and* $\widetilde{Z}_k(t)>0$ *for all k. .*

Remark 6.26. From the proof, where these formulas are obtained as a limiting case of the simple-eigenvalue solution formulas where $W_k > 0$ and $Z_k > 0$ always, it follows that $\widetilde{W}_k(t) \geq 0$ and $\widetilde{Z}_k(t) \geq 0$ must hold. We believe that the inequality remains strict here, but we don't have a proof. Provided that it is true, the same facts as before will be valid: a collision $x_k(t) = x_{k+1}(t)$ will take place exactly when $\widetilde{U}_{n-k}(t) = 0$, there will be no triple collisions, and the solution $u(x,t)$ will extend to a global weak solution.

And finally, for convenience, a theorem which can make the computations a bit shorter:

Theorem 6.27. *The quantities $\{\widetilde{V}_1, \dots, \widetilde{V}_n\}$ can be obtained from $\{\widetilde{U}_1, \dots, \widetilde{U}_n\}$ by making the substitutions*

$$b_{\delta_j+k} \mapsto \mu_j \sum_{r=0}^{\min(d_j-k,1)} b_{\delta_j+k+r}, \tag{6.29}$$

for $1 \leq j \leq J$ and $1 \leq k \leq d_j$, and similarly $\{\widetilde{T}_1, \dots, \widetilde{T}_n\}$ is obtained from $\{\widetilde{U}_1, \dots, \widetilde{U}_n\}$ via the substitutions

$$b_{\delta_j+k} \mapsto \frac{1}{\mu_j} \sum_{r=0}^{d_j-k} (-1)^r b_{\delta_j+k+r}. \tag{6.30}$$

Remark 6.28. We have not attempted to prove a general theorem about asymptotics. Already the case with simple eigenvalues (Theorem 5.25) was fairly complicated, and the formulas for multiple eigenvalues are much more involved. We will, however, look at some concrete special cases in the next section, to give an idea of the phenomena that may occur.

Remark 6.29. Also for the Degasperis–Procesi equation (1.3), eigenvalues of multiplicity greater than one may appear in the context of peakon–antipeakon solutions [28, 29]. The methods used here work exactly the same in that case too, so those solutions are given by the old DP peakon solution formulas, but with the usual U_k (etc.) replaced by the quantities with tilde defined above:

$$x_{n+1-k} = \ln \frac{\widetilde{U}_k}{\widetilde{V}_{k-1}}, \qquad m_{n+1-k} = \frac{\widetilde{U}_k^2 \widetilde{V}_{k-1}^2}{\widetilde{W}_k \widetilde{W}_{k-1}}. \tag{6.31}$$

But for the DP equation, even in the cases with complex or multiple eigenvalues, we don't observe the rich dynamics displayed by Novikov peakon–antipeakon solutions, for the reason that the peakon and antipeakon are not resurrected after a collision, but instead merge and form a shockpeakon, as described in Section 2.2. Thus, the formulas (6.31) cease to be relevant after the first collision, since the solution after that time cannot be described by the peakon ansatz (1.6). (To be precise, (6.31) only gives the solution under the assumption that $\lambda_i + \lambda_j \neq 0$ for all i and j. Already with all eigenvalues simple, the case where some $\lambda_i + \lambda_j$ vanishes must be dealt with separately. An example of this is the completely symmetric $n = 2$ peakon–antipeakon collision [23], but that is another story.)

6.3 Example revisited: $n = 2$, one double eigenvalue

In Section 4.7 we studied the case $n = 2$ with one double eigenvalue,

$$\lambda_1 = \lambda_2 = \mu > 0.$$

Let us see how the solution formulas (4.50) and (4.51) obtained there follow as a special case of Theorem 6.25.

The Weyl function is

$$\omega(\lambda) = \frac{a_1}{\lambda - \mu} + \frac{a_2}{(\lambda-\mu)^2} = \frac{b_1}{\lambda-\mu} + \frac{\mu\, b_2}{(\lambda-\mu)^2} \tag{6.32}$$

where $b_1 = a_1$ and $b_2 = \lambda_2 a_2 = \mu a_2$, with time dependence given by Theorem 6.24:

$$\begin{aligned}
\omega(\lambda) &= b_1(0)\left[\frac{e^{t/z}}{\lambda - z}\right]_{z=\lambda_1} + \mu\, b_2(0)\left[\frac{\partial}{\partial z}\frac{e^{t/z}}{\lambda - z}\right]_{z=\lambda_2} \\
&= \frac{b_1(0)\, e^{t/\mu}}{\lambda-\mu} + \mu\, b_2(0)\left(\frac{(-t/\mu^2)\, e^{t/\mu}}{\lambda-\mu} + \frac{e^{t/\mu}}{(\lambda-\mu)^2}\right) \\
&= \frac{\big(b_1(0) - b_2(0)\, t/\mu\big)e^{t/\mu}}{\lambda-\mu} + \frac{\mu\, b_2(0)\, e^{t/\mu}}{(\lambda-\mu)^2} \\
&= \frac{b_1(t)}{\lambda-\mu} + \frac{\mu\, b_2(t)}{(\lambda-\mu)^2},
\end{aligned} \tag{6.33}$$

so that $b_1(t) = \big(b_1(0) - b_2(0)\, t/\mu\big)e^{t/\mu}$ and $b_2(t) = b_2(0)\, e^{t/\mu}$, in agreement with (4.51). We compute

$$\begin{aligned}
\widetilde{\Psi}_1 &= [1]_{z_1=\lambda_1} = 1, \\
\widetilde{\Psi}_2 &= \left[\frac{\partial}{\partial z_2} 1\right]_{z_2=\lambda_2} = 0, \\
\widetilde{\Psi}_{11} &= \left[\frac{(z_1-z_2)^2}{z_1+z_2}\right]_{z_1=z_2=\lambda_1} = 0, \\
\widetilde{\Psi}_{12} &= \left[\frac{\partial}{\partial z_2}\frac{(z_1-z_2)^2}{z_1+z_2}\right]_{z_1=\lambda_1, z_2=\lambda_2} = \left[\frac{-(z_1-z_2)(3z_1+z_2)}{(z_1+z_2)^2}\right]_{z_1=z_2=\mu} = 0, \\
\widetilde{\Psi}_{22} &= \left[\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\frac{(z_1-z_2)^2}{z_1+z_2}\right]_{z_1=z_2=\lambda_2} = \left[\frac{-8 z_1 z_2}{(z_1+z_2)^3}\right]_{z_1=z_2=\mu} = -\frac{1}{\mu},
\end{aligned} \tag{6.34}$$

which implies that

$$\begin{aligned}
\widetilde{U}_1 &= 1\cdot a_1 + 0\cdot a_2 = b_1, \\
\widetilde{U}_2 &= \frac{1}{2!}\left(0\cdot b_1^2 + 2\cdot 0\cdot b_1 b_2 + \frac{-1}{\mu}\, a_2^2\right) = -\frac{\mu\, b_2^2}{2}.
\end{aligned} \tag{6.35}$$

Similarly,

$$\begin{aligned}
\widetilde{\Psi}_1^{(1)} &= [z_1]_{z_1=\lambda_1} = \mu, \\
\widetilde{\Psi}_2^{(1)} &= \left[\frac{\partial}{\partial z_1} z_1\right]_{z_1=\lambda_2} = 1, \\
\widetilde{\Psi}_{11}^{(1)} &= \left[\frac{z_1 z_2 (z_1 - z_2)^2}{z_1 + z_2}\right]_{z_1=z_2=\lambda_1} = 0, \\
\widetilde{\Psi}_{12}^{(1)} &= \left[\frac{\partial}{\partial z_2} \frac{z_1 z_2 (z_1 - z_2)^2}{z_1 + z_2}\right]_{z_1=\lambda_1, z_2=\lambda_2} \\
&= \left[\frac{z_1 (z_1 - z_2)(z_1^2 - 3 z_1 z_2 - 2 z_2^2)}{(z_1 + z_2)^2}\right]_{z_1=z_2=\mu} = 0, \\
\widetilde{\Psi}_{22}^{(1)} &= \left[\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \frac{z_1 z_2 (z_1 - z_2)^2}{z_1 + z_2}\right]_{z_1=z_2=\lambda_2} \\
&= \left[\frac{2(z_1^4 - 6 z_1^2 z_2^2 + z_4^2)}{(z_1 + z_2)^3}\right]_{z_1=z_2=\mu} = -\mu,
\end{aligned} \tag{6.36}$$

so that

$$\begin{aligned}
\widetilde{V}_1 &= \mu \cdot a_1 + 1 \cdot a_2 = \mu (b_1 + b_2), \\
\widetilde{V}_2 &= \frac{1}{2!}\left(0 \cdot b_1^2 + 2 \cdot 0 \cdot b_1 b_2 + (-\mu)\, a_2^2\right) = -\frac{\mu^3 b_2^2}{2},
\end{aligned} \tag{6.37}$$

and

$$\begin{aligned}
\widetilde{\Psi}_1^{(-1)} &= \left[\frac{1}{z_1}\right]_{z_1=\lambda_1} = \frac{1}{\mu}, \\
\widetilde{\Psi}_2^{(-1)} &= \left[\frac{\partial}{\partial z_1} \frac{1}{z_1}\right]_{z_1=\lambda_2} = -\frac{1}{\mu^2}, \\
\widetilde{\Psi}_{11}^{(-1)} &= \left[\frac{(z_1 - z_2)^2}{z_1 z_2 (z_1 + z_2)}\right]_{z_1=z_2=\lambda_1} = 0, \\
\widetilde{\Psi}_{12}^{(-1)} &= \left[\frac{\partial}{\partial z_2} \frac{(z_1 - z_2)^2}{z_1 z_2 (z_1 + z_2)}\right]_{z_1=\lambda_1, z_2=\lambda_2} \\
&= \left[\frac{-z_1 (z_1 - z_2)(z_1 + 3 z_2)}{z_2^2 (z_1 + z_2)^2}\right]_{z_1=z_2=\mu} = 0, \\
\widetilde{\Psi}_{22}^{(-1)} &= \left[\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \frac{(z_1 - z_2)^2}{z_1 z_2 (z_1 + z_2)}\right]_{z_1=z_2=\lambda_2} = \left[\frac{-8}{(z_1 + z_2)^3}\right]_{z_1=z_2=\mu} = -\frac{1}{\mu^3},
\end{aligned} \tag{6.38}$$

so that

$$\begin{aligned}
\widetilde{T}_1 &= \frac{1}{\mu} \cdot a_1 + \frac{-1}{\mu^2} \cdot a_2 = \frac{b_1 - b_2}{\mu}, \\
\widetilde{T}_2 &= \frac{1}{2!}\left(0 \cdot b_1^2 + 2 \cdot 0 \cdot b_1 b_2 + \frac{-1}{\mu^3}\, a_2^2\right) = -\frac{b_2^2}{2\mu},
\end{aligned} \tag{6.39}$$

From this we get

$$\begin{aligned}
\widetilde{W}_1 &= \widetilde{U}_1 \widetilde{V}_1 - \widetilde{U}_2 = b_1 \cdot \mu(b_1+b_2) - \frac{-\mu b_2^2}{2} = \mu\left(b_1^2 + b_1 b_2 + \tfrac{1}{2} b_2^2\right), \\
\widetilde{Z}_1 &= \widetilde{T}_1 \widetilde{U}_1 - \widetilde{T}_2 = \frac{b_1 - b_2}{\mu} \cdot b_1 - \frac{-b_2^2}{2\mu} = \frac{b_1^2 - b_1 b_2 + \frac{1}{2} b_2^2}{\mu}, \\
\widetilde{Z}_2 &= \widetilde{T}_2 \widetilde{U}_2 = \frac{-b_2^2}{2\mu} \cdot \frac{-\mu b_2^2}{2} = \frac{b_2^4}{4},
\end{aligned} \tag{6.40}$$

and hence

$$\begin{aligned}
e^{2x_1} &= \frac{\widetilde{Z}_2}{\widetilde{W}_1} = \frac{b_2^2/4}{\mu\left(b_1^2 + b_1 b_2 + \frac{1}{2} b_2^2\right)} = Q_1, \\
e^{2x_2} &= \widetilde{Z}_1 = \frac{b_1^2 - b_1 b_2 + \frac{1}{2} b_2^2}{\mu} = Q_2, \\
m_1 e^{-x_1} &= \frac{\widetilde{W}_1}{\widetilde{U}_2 \widetilde{U}_1} = \frac{\mu\left(b_1^2 + b_1 b_2 + \frac{1}{2} b_2^2\right)}{-\frac{1}{2}\mu b_2^2 \cdot b_1} = P_1, \\
m_2 e^{-x_2} &= \frac{1}{\widetilde{U}_1} = \frac{1}{b_1} = P_2,
\end{aligned} \tag{6.41}$$

in agreement with (4.50).

As an alternative to the computation of $\widetilde{\Psi}_I^{(\pm 1)}$ above, we could have used Theorem 6.27 and obtained $\widetilde{V}_k$ and $\widetilde{T}_k$ from $\widetilde{U}_k$ by making the substitions

$$b_1 \mapsto \mu(b_1 + b_2), \qquad b_2 \mapsto \mu b_2$$

and

$$b_1 \mapsto \frac{b_1 - b_2}{\mu}, \qquad b_2 \mapsto \frac{b_2}{\mu},$$

respectively.

6.4 Example: $n = 3$, one triple eigenvalue

Let us now consider the case $n = 3$ with one triple eigenvalue,

$$\lambda_1 = \lambda_2 = \lambda_3 = \mu > 0.$$

The coefficients in the Weyl function

$$\omega(\lambda) = \frac{b_1}{\lambda - \mu} + \frac{\mu b_2}{(\lambda - \mu)^2} + \frac{\mu^2 b_3}{(\lambda - \mu)^3} \tag{6.42}$$

have the time dependence

$$\begin{aligned}
b_1(t) &= \sum_{r=0}^{2} \frac{b_{1+r}(0)}{r!}\, p_r(\tfrac{t}{\mu})\, e^{t/\mu_j} = \Big(b_1(0) - b_2(0)\,\tfrac{t}{\mu} + \tfrac{1}{2} b_3(0)\left(\tfrac{t^2}{\mu^2} + 2\tfrac{t}{\mu}\right)\Big) e^{t/\mu}, \\
b_2(t) &= \sum_{r=0}^{1} \frac{b_{2+r}(0)}{r!}\, p_r(\tfrac{t}{\mu})\, e^{t/\mu_j} = \Big(b_2(0) - b_3(0)\,\tfrac{t}{\mu}\Big) e^{t/\mu}, \\
b_3(t) &= \sum_{r=0}^{0} \frac{b_{3+r}(0)}{r!}\, p_r(\tfrac{t}{\mu})\, e^{t/\mu_j} = b_3(0)\, e^{t/\mu},
\end{aligned} \tag{6.43}$$

where the highest coefficient is nonzero: $b_3(0) \neq 0$. We find after some computation that

$$\widetilde{U}_1 = b_1, \qquad \widetilde{V}_1 = \mu(b_1 + b_2), \qquad \widetilde{T}_1 = \frac{b_1 - b_2 + b_3}{\mu}, \tag{6.44}$$

$$\begin{aligned}
\widetilde{U}_2 &= \mu\left(\frac{b_1 b_3}{2} - \frac{b_2^2}{2} + \frac{b_2 b_3}{4} - \frac{b_3^2}{8}\right), \\
\widetilde{V}_2 &= \mu^3\left(\frac{b_1 b_3}{2} - \frac{b_2^2}{2} - \frac{b_2 b_3}{4} - \frac{3b_3^2}{8}\right), \\
\widetilde{T}_2 &= \frac{1}{\mu}\left(\frac{b_1 b_3}{2} - \frac{b_2^2}{2} + \frac{3b_2 b_3}{4} - \frac{3b_3^2}{8}\right),
\end{aligned} \tag{6.45}$$

and

$$\widetilde{U}_3 = -\frac{\mu^3 b_3^3}{8}, \qquad \widetilde{V}_3 = -\frac{\mu^6 b_3^3}{8}, \qquad \widetilde{T}_3 = -\frac{b_3^3}{8}. \tag{6.46}$$

(Note that once we have computed $\widetilde{U}_k$, we can use Theorem 6.27 to obtain $\widetilde{V}_k$ and $\widetilde{T}_k$, by making the substitions

$$b_1 \mapsto \mu(b_1 + b_2), \qquad b_2 \mapsto \mu(b_2 + b_3), \qquad b_3 \mapsto \mu b_3$$

and

$$b_1 \mapsto \frac{b_1 - b_2 + b_3}{\mu}, \qquad b_2 \mapsto \frac{b_2 - b_3}{\mu}, \qquad b_3 \mapsto \frac{b_3}{\mu},$$

respectively.) This gives

$$\begin{aligned}
\widetilde{W}_1 &= \widetilde{U}_1 \widetilde{V}_1 - \widetilde{U}_2 \\
&= \mu\left(b_1^2 + b_1 b_2 - \frac{b_1 b_3}{2} + \frac{b_2^2}{2} - \frac{b_2 b_3}{4} + \frac{b_3^2}{8}\right), \\
\widetilde{W}_2 &= \widetilde{U}_2 \widetilde{V}_2 - \widetilde{U}_3 \widetilde{V}_1 \\
&= \mu^4\Big(\frac{b_1^2 b_3^2}{4} + \frac{b_2^4}{4} - \frac{b_1 b_2^2 b_3}{2} + \frac{3b_3^4}{64} + \frac{3b_2^2 b_3^2}{16} - \frac{b_1 b_3^3}{8} + \frac{b_2 b_3^3}{16}\Big)
\end{aligned} \tag{6.47}$$

and

$$\begin{aligned}
\widetilde{Z}_1 &= \widetilde{T}_1 \widetilde{U}_1 - \widetilde{T}_2 \\
&= \frac{1}{\mu}\left(b_1^2 - b_1 b_2 + \frac{b_1 b_3}{2} + \frac{b_2^2}{2} - \frac{3 b_2 b_3}{4} + \frac{3 b_3^2}{8}\right), \\
\widetilde{Z}_2 &= \widetilde{T}_2 \widetilde{U}_2 - \widetilde{T}_3 \widetilde{U}_1 \\
&= \frac{b_1^2 b_3^2}{4} + \frac{b_2^4}{4} - \frac{b_1 b_2^2 b_3}{2} + \frac{3 b_3^4}{64} + \frac{7 b_2^2 b_3^2}{16} \\
&\quad + \frac{b_1 b_2 b_3^2}{2} - \frac{b_2^3 b_3}{2} - \frac{3 b_2 b_3^3}{16} - \frac{b_1 b_3^3}{8}, \\
\widetilde{Z}_3 &= \widetilde{T}_3 \widetilde{U}_3 = \frac{\mu^3 b_3^6}{64}.
\end{aligned} \tag{6.48}$$

In terms of these quantities, the solution is given by

$$\begin{aligned}
e^{2x_1} &= \frac{\widetilde{Z}_3}{\widetilde{\widetilde{W}}_2}, \qquad & e^{2x_2} &= \frac{\widetilde{Z}_2}{\widetilde{\widetilde{W}}_1}, \qquad & e^{2x_3} &= \widetilde{Z}_1, \\
m_1 e^{-x_1} &= \frac{\widetilde{\widetilde{W}}_2}{\widetilde{U}_3 \widetilde{U}_2}, & m_2 e^{-x_2} &= \frac{\widetilde{\widetilde{W}}_1}{\widetilde{U}_2 \widetilde{U}_1}, & m_3 e^{-x_3} &= \frac{1}{\widetilde{U}_1}.
\end{aligned} \tag{6.49}$$

with the time dependence (6.43).

Example 6.30. An example of a solution with a triple eigenvalue is plotted in Figures 27 and 28. The parameters are

$$\mu = 1, \qquad b_1(0) = b_2(0) = 0, \qquad b_3(0) = -1. \tag{6.50}$$

It is straightforward to determine the asymptotics as $t \to \pm\infty$ by factoring out the dominant term $\frac{1}{2} b_3(0) \frac{t^2}{\mu^2} e^{t/\mu}$ (which appears in $b_1(t)$) from the numerators and denominators in (6.49). We obtain the following result, which can be compared to the asymptotics (4.55) for the case of a double eigenvalue.

Theorem 6.31. *The three-peakon solution with a triple eigenvalue $\mu > 0$ satisfies, as $t \to \pm\infty$,*

$$\begin{aligned}
x_1(t) &= \frac{t}{\mu} + \frac{1}{2} \ln \frac{b_3(0)^2}{4\mu} - \frac{1}{2} \ln \frac{t^4}{\mu^4} + \mathcal{O}(1/t), \\
x_1(t) &= \frac{t}{\mu} + \frac{1}{2} \ln \frac{b_3(0)^2}{4\mu} + \mathcal{O}(1/t), \\
x_3(t) &= \frac{t}{\mu} + \frac{1}{2} \ln \frac{b_3(0)^2}{4\mu} + \frac{1}{2} \ln \frac{t^4}{\mu^4} + \mathcal{O}(1/t), \\
m_1(t) &= \frac{1}{\sqrt{\mu}} \operatorname{sgn}(b_3(0)) \big(1 + \mathcal{O}(1/t)\big), \\
m_2(t) &= -\frac{1}{\sqrt{\mu}} \operatorname{sgn}(b_3(0)) \big(1 + \mathcal{O}(1/t)\big), \\
m_3(t) &= \frac{1}{\sqrt{\mu}} \operatorname{sgn}(b_3(0)) \big(1 + \mathcal{O}(1/t)\big).
\end{aligned} \tag{6.51}$$

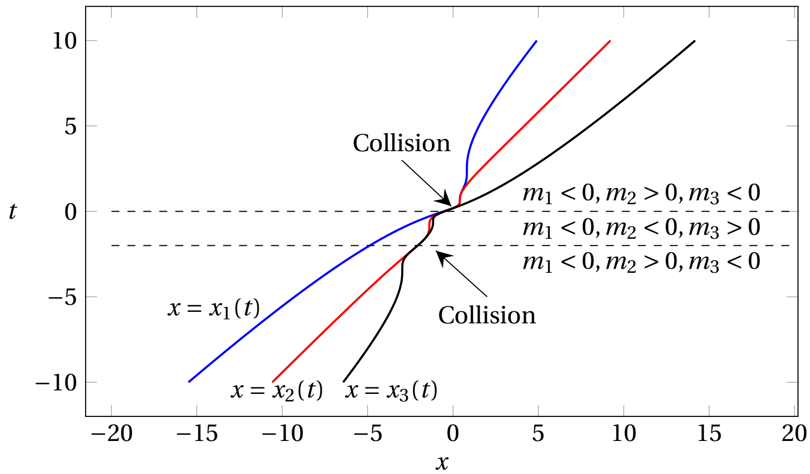

Figure 27: Spacetime plot showing the locations $x_k(t)$ of the peakons in the (x, t)-plane for the peakon–antipeakon solution with a triple eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = \mu$. The parameters used here are $\mu = 1$ and $b_1(0) = b_2(0) = 0$, $b_3(0) = -1$. In this case, there are no collisions between x_1 and x_2, but x_2 and x_3 collide twice (when $t = -2$ and $t = 0$, according to Theorem 6.32).

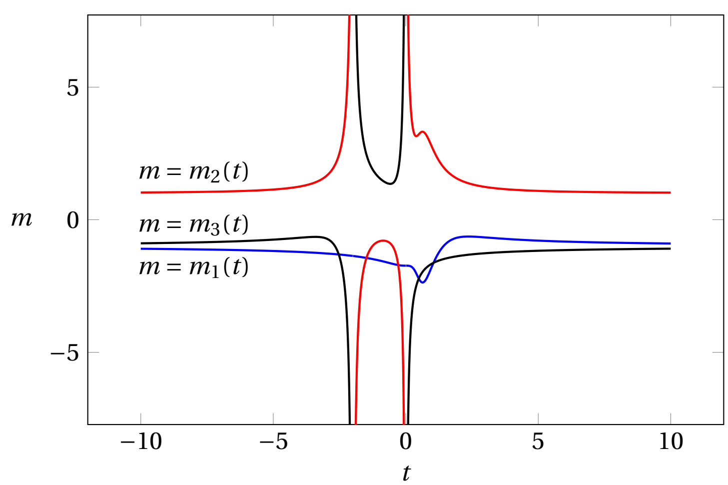

Figure 28: Graphs of the amplitudes $m_k(t)$ for the peakon–antipeakon solution with a triple eigenvalue $\mu = 1$ in Figure 27.

It is also easy to determine exactly when collisions occur:

Theorem 6.32. *For the three-peakon solution with a triple eigenvalue* $\mu > 0$*, the collision* $x_1(t_0) = x_2(t_0)$ *occurs if and only if*

$$\frac{t_0^2}{\mu^2} - \left(1 + \frac{2b_2(0)}{b_3(0)}\right)\frac{t_0}{\mu} + \left(\frac{1}{2} + \frac{2b_2(0)^2}{b_3(0)^2} - \frac{b_2(0)}{b_3(0)} - \frac{2b_1(0)}{b_3(0)}\right) = 0, \tag{6.52}$$

and the collision $x_2(t_0) = x_3(t_0)$ *occurs if and only if*

$$\frac{t_0^2}{\mu^2} + 2\left(1 - \frac{b_2(0)}{b_3(0)}\right)\frac{t_0}{\mu} + \frac{2b_1(0)}{b_3(0)} = 0. \tag{6.53}$$

Consequently, there can be at most two collisions of each kind. Moreover, there can be no triple collision, i.e., there is no t_0 *such that* $x_1(t_0) = x_2(t_0) = x_3(t_0)$.

Proof. We know from (5.6) that

$$x_1 = x_2 \iff \widetilde{U}_2 = 0, \qquad x_2 = x_3 \iff \widetilde{U}_1 = 0.$$

Inserting the time dependence (6.43) into (6.44) and (6.45) yields

$$\widetilde{U}_1(t) = \tfrac{1}{2}e^{t/\mu}\left(b_3(0)\,\tfrac{t^2}{\mu^2} + 2\left(b_3(0) - b_2(0)\right)\tfrac{t}{\mu} + 2b_1(0)\right)$$

and

$$\begin{aligned}\widetilde{U}_2(t) = -\tfrac{\mu}{8}e^{2t/\mu}\Big(&2b_3(0)^2\,\tfrac{t^2}{\mu^2} - \left(2b_3(0)^2 + 4b_2(0)b_3(0)\right)\tfrac{t}{\mu} \\ &+ b_3(0)^2 + 4b_2(0)^2 - 2b_2(0)b_3(0) - 4b_1(0)b_3(0)\Big),\end{aligned}$$

and the first claim follows.

Regarding triple collisions, note from (6.47) that

$$\frac{\widetilde{W}_1}{\mu} = \frac{\widetilde{U}_1\widetilde{V}_1 - \widetilde{U}_2}{\mu} = \left(b_1 + \tfrac{1}{2}b_2 - \tfrac{1}{4}b_3\right)^2 + \tfrac{1}{4}b_2^2 + \tfrac{1}{16}b_3^2 > 0,$$

since $b_3 \neq 0$ always, hence $\widetilde{U}_1$ and $\widetilde{U}_2$ cannot vanish simultaneously. □

Example 6.33. It may happen that a pair collides exactly once; for example, if $b_1(0) = \frac{1}{2}$, $b_2(0) = 0$ and $b_3(0) = 1$, then $\widetilde{U}_1(t) = \frac{1}{2}e^{t/\mu}\left(\frac{t}{\mu} - 1\right)^2$, so that the collision $x_2(t_0) = x_3(t_0)$ occurs for $t_0 = \mu$ only. In that case, the collision will be of higher order, like in Example 5.18. The amplitudes m_2 and m_3 keep their signs at the collision (which they must, in order not to violate the sign conditions dictated by the asymptotics), behaving like $(t - t_0)^{-2}$ rather than $(t - t_0)^{-1}$, and the distance $x_3 - x_2$ behaves like $(t - t_0)^8$ rather than $(t - t_0)^4$.

Example 6.34. There is a particular choice of the coefficients $b_k(0)$ which gives a solution that is symmetric with respect to the origin, namely

$$b_1(0) = \frac{3\sqrt{\mu}}{16}, \qquad b_2(0) = \frac{\sqrt{\mu}}{2}, \qquad b_3(0) = 2\sqrt{\mu}. \tag{6.54}$$

(And, as usual, also with the opposite sign on each $b_k(0)$, which amounts to flipping the sign of $u(x,t)$.) These numbers are found by first choosing $b_3(0)$ to kill the constant term $\ln\big(b_3(0)^2/4\mu\big)$ in the asymptotic formulas (6.51), and then computing $b_1(0)$ and $b_2(0)$ so that the polynomial in (6.53) equals the one in (6.52) with $-t_0$ instead of t_0, making the collisions take place symmetrically with respect to $t = 0$. This symmetric solution takes the following form, if we use the abbreviation $\tau = t/\mu$:

$$\begin{aligned}
x_1(t) &= -x_3(-t),\\
m_1(t) &= m_3(-t),\\
x_2(t) &= -x_2(-t) = \tau + \tfrac{1}{2}\ln\frac{\tau^4 - \tau^3 + \frac{9}{8}\tau^2 + \frac{9}{16}\tau + \frac{81}{256}}{\tau^4 + \tau^3 + \frac{9}{8}\tau^2 - \frac{9}{16}\tau + \frac{81}{256}},\\
m_2(t) &= m_2(-t)\\
&= \frac{\sqrt{\tau^4 - \tau^3 + \frac{9}{8}\tau^2 + \frac{9}{16}\tau + \frac{81}{256}}\,\sqrt{\tau^4 + \tau^3 + \frac{9}{8}\tau^2 - \frac{9}{16}\tau + \frac{81}{256}}}{-\sqrt{\mu}\left(\tau^4 - \frac{15}{8}\tau^2 + \frac{9}{256}\right)},\\
x_3(t) &= \tau + \tfrac{1}{2}\ln\left(\tau^4 + 5\tau^3 + \tfrac{65}{8}\tau^2 + \tfrac{59}{16}\tau + \tfrac{257}{256}\right),\\
m_3(t) &= \frac{\sqrt{\tau^4 + 5\tau^3 + \frac{65}{8}\tau^2 + \frac{59}{16}\tau + \frac{257}{256}}}{\sqrt{\mu}\left(\tau^2 + \frac{3}{2}\tau + \frac{3}{16}\right)}.
\end{aligned} \tag{6.55}$$

6.5 Example: $n = 4$, two double eigenvalues

Next, we look at the case $n = 4$ with two double eigenvalues,

$$\lambda_1 = \lambda_2 = \mu_1, \qquad \lambda_3 = \lambda_4 = \mu_2.$$

The Weyl function is

$$\omega(\lambda) = \frac{b_1}{\lambda - \mu_1} + \frac{\mu b_2}{(\lambda - \mu_1)^2} + \frac{b_3}{\lambda - \mu_2} + \frac{\mu b_4}{(\lambda - \mu_2)^2},$$

but let us write $c_1 = b_3$ and $c_2 = b_4$ to emphasize the symmetry:

$$\omega(\lambda) = \frac{b_1}{\lambda - \mu_1} + \frac{\mu b_2}{(\lambda - \mu_1)^2} + \frac{c_1}{\lambda - \mu_2} + \frac{\mu c_2}{(\lambda - \mu_2)^2}. \tag{6.56}$$

The coefficients have the time dependence

$$
\begin{aligned}
b_1(t) &= \Bigl(b_1(0) - b_2(0)\,\tfrac{t}{\mu_1}\Bigr) e^{t/\mu_1}, \quad & b_2(t) &= b_2(0)\, e^{t/\mu_1},\\
c_1(t) &= \Bigl(c_1(0) - c_2(0)\,\tfrac{t}{\mu_2}\Bigr) e^{t/\mu_2}, \quad & c_2(t) &= c_2(0)\, e^{t/\mu_2},
\end{aligned}
\tag{6.57}
$$

where the highest coefficient associated to each eigenvalue is nonzero: $b_2(0) \neq 0$ and $c_2(0) \neq 0$. There are two possibilities: the real case with $0 < \mu_1 < \mu_2$, or the complex case with $\mu_2 = \overline{\mu_1}$ (with nonzero imaginary part). In the complex case, we also have the constraints $c_1(0) = \overline{b_1(0)}$ and $c_2(0) = \overline{b_2(0)}$, since the Weyl function must be real. The solution formulas (which apply in both cases) are

$$
\begin{aligned}
e^{2x_1} &= \frac{\widetilde{Z}_4}{\widetilde{\widetilde{W}}_3}, & e^{2x_2} &= \frac{\widetilde{Z}_3}{\widetilde{\widetilde{W}}_2}, & e^{2x_3} &= \frac{\widetilde{Z}_2}{\widetilde{\widetilde{W}}_1}, & e^{2x_4} &= \widetilde{Z}_1,\\
m_1\, e^{-x_1} &= \frac{\widetilde{W}_3}{\widetilde{U}_4\,\widetilde{U}_3}, & m_2\, e^{-x_2} &= \frac{\widetilde{W}_2}{\widetilde{U}_3\,\widetilde{U}_2}, & m_3\, e^{-x_3} &= \frac{\widetilde{W}_1}{\widetilde{U}_2\,\widetilde{U}_1}, & m_4\, e^{-x_4} &= \frac{1}{\widetilde{U}_1},
\end{aligned}
\tag{6.58}
$$

where

$$
\begin{aligned}
\widetilde{U}_1 &= b_1 + c_1,\\
\widetilde{V}_1 &= \mu_1\,(b_1 + b_2) + \mu_2\,(c_1 + c_2),\\
\widetilde{T}_1 &= \frac{b_1 - b_2}{\mu_1} + \frac{c_1 - c_2}{\mu_2},
\end{aligned}
\tag{6.59}
$$

$$
\begin{aligned}
\widetilde{U}_2 &= -\frac{\mu_1\, b_2^2}{2} - \frac{\mu_2\, c_2^2}{2} + \frac{(\mu_1-\mu_2)^2}{\mu_1+\mu_2}\, b_1\, c_1 + \frac{\mu_1(\mu_1-\mu_2)(\mu_1+3\mu_2)}{(\mu_1+\mu_2)^2}\, b_2\, c_1\\
&\quad + \frac{\mu_2(\mu_2-\mu_1)(3\mu_1+\mu_2)}{(\mu_1+\mu_2)^2}\, b_1\, c_2 - \frac{8\mu_1^2\mu_2^2}{(\mu_1+\mu_2)^3}\, b_2\, c_2,\\
\widetilde{V}_2 &= -\frac{\mu_1\,(\mu_1\, b_2)^2}{2} - \frac{\mu_2\,(\mu_2\, c_2)^2}{2} + \frac{(\mu_1-\mu_2)^2}{\mu_1+\mu_2}\, \mu_1\,\mu_2 (b_1+b_2)(c_1+c_2)\\
&\quad + \frac{\mu_1(\mu_1-\mu_2)(\mu_1+3\mu_2)}{(\mu_1+\mu_2)^2}\, \mu_1\,\mu_2\, b_2 (c_1+c_2)\\
&\quad + \frac{\mu_2(\mu_2-\mu_1)(3\mu_1+\mu_2)}{(\mu_1+\mu_2)^2}\, \mu_1\,\mu_2 (b_1+b_2)\, c_2\\
&\quad - \frac{8\mu_1^2\mu_2^2}{(\mu_1+\mu_2)^3}\, \mu_1\,\mu_2\, b_2\, c_2,\\
\widetilde{T}_2 &= -\frac{\mu_1\,(b_2/\mu_1)^2}{2} - \frac{\mu_2\,(c_2/\mu_2)^2}{2} + \frac{(\mu_1-\mu_2)^2}{\mu_1+\mu_2}\, \frac{(b_1-b_2)(c_1-c_2)}{\mu_1\,\mu_2}\\
&\quad + \frac{\mu_1(\mu_1-\mu_2)(\mu_1+3\mu_2)}{(\mu_1+\mu_2)^2}\, \frac{b_2(c_1-c_2)}{\mu_1\,\mu_2}\\
&\quad + \frac{\mu_2(\mu_2-\mu_1)(3\mu_1+\mu_2)}{(\mu_1+\mu_2)^2}\, \frac{(b_1-b_2)c_2}{\mu_1\,\mu_2} - \frac{8\mu_1^2\mu_2^2}{(\mu_1+\mu_2)^3}\, \frac{b_2\, c_2}{\mu_1\,\mu_2},
\end{aligned}
\tag{6.60}
$$

$$
\begin{aligned}
\widetilde{U}_3 &= -\frac{(\mu_1-\mu_2)^4}{2(\mu_1+\mu_2)^2}\left(\mu_1\, b_2^2\, c_1 + \mu_2\, b_1\, c_2^2\right) \\
&\quad + \frac{\mu_1\,\mu_2\,(\mu_1-\mu_2)^3}{(\mu_1+\mu_2)^3}\Big((3\mu_1+\mu_2)\, b_2^2\, c_2 - (\mu_1+3\mu_2)\, b_2\, c_2^2\Big), \\
\widetilde{V}_3 &= -\frac{(\mu_1-\mu_2)^4}{2(\mu_1+\mu_2)^2}\left(\mu_1\,\mu_1^2\,\mu_2\, b_2^2\,(c_1+c_2) + \mu_2\,\mu_1\,\mu_2^2\,(b_1+b_2)\, c_2^2\right) \\
&\quad + \frac{\mu_1\,\mu_2\,(\mu_1-\mu_2)^3}{(\mu_1+\mu_2)^3}\Big((3\mu_1+\mu_2)\,\mu_1^2\,\mu_2\, b_2^2\, c_2 - (\mu_1+3\mu_2)\,\mu_1\,\mu_2^2\, b_2\, c_2^2\Big), \\
\widetilde{T}_3 &= -\frac{(\mu_1-\mu_2)^4}{2(\mu_1+\mu_2)^2}\left(\mu_1\,\frac{b_2^2(c_1-c_2)}{\mu_1^2\,\mu_2} + \mu_2\,\frac{(b_1-b_2)c_2^2}{\mu_1\,\mu_2^2}\right) \\
&\quad + \frac{\mu_1\,\mu_2\,(\mu_1-\mu_2)^3}{(\mu_1+\mu_2)^3}\Big((3\mu_1+\mu_2)\,\frac{b_2^2\, c_2}{\mu_1^2\,\mu_2} - (\mu_1+3\mu_2)\,\frac{b_2\, c_2^2}{\mu_1\,\mu_2^2}\Big), \\
\widetilde{U}_4 &= \frac{\mu_1\,\mu_2\,(\mu_1-\mu_2)^8}{4(\mu_1+\mu_2)^4}\, b_2^2\, c_2^2, \\
\widetilde{T}_4 &= \frac{(\mu_1-\mu_2)^8}{4(\mu_1+\mu_2)^4}\,\frac{b_2^2\, c_2^2}{\mu_1^2\,\mu_2^2}, \\
\widetilde{V}_4 &= \frac{\mu_1\,\mu_2\,(\mu_1-\mu_2)^8}{4(\mu_1+\mu_2)^4}\,\mu_1^2\,\mu_2^2\, b_2^2\, c_2^2,
\end{aligned}
\tag{6.61}
$$

and where $\widetilde{W}_k$ and $\widetilde{Z}_k$ are given by the usual expressions (6.19). Here we have used Theorem 6.27 to obtain $\widetilde{V}_k$ from $\widetilde{U}_k$ through the substitutions

$$
b_1 \mapsto \mu_1(b_1+b_2), \qquad b_2 \mapsto \mu_1 b_2, \qquad c_1 \mapsto \mu_2(c_1+c_1), \qquad c_2 \mapsto \mu_2 c_2,
$$

and $\widetilde{T}_k$ from $\widetilde{U}_k$ via

$$
b_1 \mapsto \frac{b_1-b_2}{\mu_1}, \qquad b_2 \mapsto \frac{b_2}{\mu_1}, \qquad c_1 \mapsto \frac{c_1-c_2}{\mu_2}, \qquad c_2 \mapsto \frac{c_2}{\mu_2}.
$$

Example 6.35. A plot of the real case with parameters

$$
\mu_1 = 1, \quad \mu_2 = 3, \quad b_1(0) = 0, \quad b_2(0) = 1, \quad c_1(0) = 0, \quad c_2(0) = 1 \tag{6.62}
$$

is shown in Figure 29. Asymptotically we see two peakon–antipeakon pairs travelling with the velocities $1/\mu_1$ and $1/\mu_2$, and hence scattering from each other at a linear rate, but also with the two peakons within each pair separating at a logarithmic rate.

Example 6.36. A plot of the complex case with parameters

$$
\mu_1 = \overline{\mu_2} = \frac{1}{\alpha+i\beta} = \frac{1}{1+i}, \qquad b_1(0) = \overline{c_1(0)} = 1, \qquad b_2(0) = \overline{c_2(0)} = 1 \tag{6.63}
$$

is shown in Figure 30. Asymptotically we see two peakon–antipeakon pairs, both oscillating with the same period $2\pi/\beta = 2\pi$, and both travelling with a velocity approaching

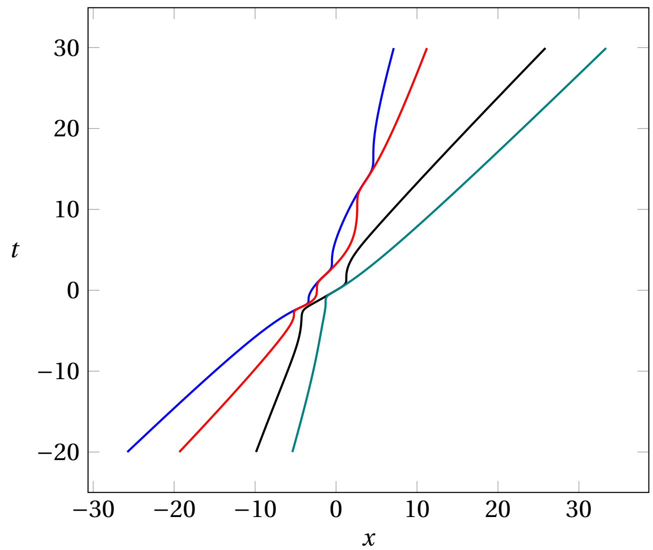

Figure 29: Positions $x = x_k(t)$ for the $n = 4$ peakon–antipeakon solution with two positive double eigenvalues $0 < \mu_1 < \mu_2$ in Example 6.35.

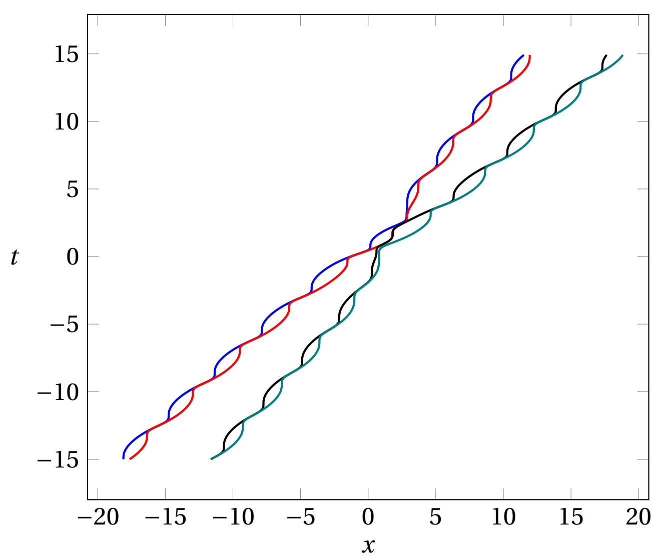

Figure 30: Positions $x = x_k(t)$ for the $n = 4$ peakon–antipeakon solution with two complex double eigenvalues $\mu_1 = \overline{\mu_2}$ in Example 6.36.

$1/\alpha = 1$, but separating from each other at a logarithmic rate. (As seen in the picture, the solutions for $x_k(t)$ asymptotically have π-periodic oscillations, but remember this only represents half a period of the full solution, since the same oscillation is repeated again with m_1 and m_2 having the opposite signs; compare with the 2-peakon case in Section 4.6.)

6.6 Proofs

We finish by giving the proofs of the claims in Section 6.2. The peakon solution formulas for the multiple-eigenvalue case will be obtained from the known solution formulas for the simple-eigenvalue case (Theorem 5.14) via a limiting procedure. The setup is as in Section 6.1; we have n eigenvalues λ_k with J distinct numerical values μ_j and multiplicities d_j, the coefficient α_k is defined by $\lambda_k = \mu_{\alpha_k}$, the integer interval $\mathscr{J}_j = [\delta_j + 1, \delta_j + d_j]$ contains those indices k for which $\alpha_k = j$, and the coefficient $\beta_k = k - \delta_{\alpha_k} - 1 \in \{0, 1, \dots\}$ keeps track of whether λ_k is the first/second/etc. eigenvalue with the value μ_{α_k}.

We now consider (for some fixed $\varepsilon > 0$ to begin with) the simple-eigenvalue n-peakon solution with ε-dependent spectral data defined as follows.

Definition 6.37. Let

$$\lambda_k(\varepsilon) = \mu_{\alpha_k} + \varepsilon \omega_{\alpha_k}^{\beta_k}, \qquad \text{for } 1 \le k \le n, \tag{6.64}$$

where

$$\omega_j = \exp(2\pi i / d_j), \qquad \text{for } 1 \le j \le J \tag{6.65}$$

are roots of unity.

Remark 6.38. An equivalent way of writing this, using the offsets δ_j, is

$$\lambda_{\delta_j + 1 + r}(\varepsilon) = \mu_j + \varepsilon \omega_j^r, \tag{6.66}$$

for $1 \le j \le J$ and $0 \le r \le d_j - 1$. Obviously, for $\varepsilon = 0$, $\lambda_k(\varepsilon)$ reduce to the λ_k in our multiple-eigenvalue setup, but for $\varepsilon > 0$ each *multiple* eigenvalue $\lambda_{a+1} = \cdots = \lambda_{a+d} = \mu$ of multiplicity d is replaced by d *simple* eigenvalues $\lambda_{a+1}(\varepsilon), \dots, \lambda_{a+d}(\varepsilon)$ equally spaced on a circle of radius ε around μ. (And if $\lambda_k = \mu$ is a simple eigenvalue already, it is moved to $\lambda_k(\varepsilon) = \mu + \varepsilon$. We might as well leave it where it is, but then we would have to implement that as an exception from the general formula (6.64), which would be inconvenient.) There might be accidental overlap for certain isolated values of ε, but for small enough $\varepsilon > 0$ the eigenvalues $\lambda_k(\varepsilon)$ are indeed simple (and have positive real part).

Definition 6.39. For $\varepsilon > 0$ small enough, so that the eigenvalues $\lambda_k(\varepsilon)$ are simple, let

$$b_k(t; \varepsilon) = \frac{1}{d_{\alpha_k}} \left(\sum_{s \in \mathscr{J}_{\alpha_k}} \frac{a_s(0)}{\left(\varepsilon \omega_{\alpha_k}^{\beta_k}\right)^{\beta_s}} \right) e^{t/\lambda_k(\varepsilon)}, \qquad \text{for } 1 \le k \le n, \tag{6.67}$$

where $a_1(0), \dots, a_n(0)$ are some constants fixed in advance.

Remark 6.40. Another way of writing this is

$$b_{\delta_j+1+r}(t;\varepsilon) = \frac{1}{d_j}\left(\sum_{s=0}^{d_j-1} \frac{a_{\delta_j+1+s}(0)}{\left(\varepsilon\omega_j^r\right)^s}\right) \exp \frac{t}{\mu_j + \varepsilon\omega_j^r}, \tag{6.68}$$

for $1 \le j \le J$ and $0 \le r \le d_j - 1$.

Remark 6.41. It is convenient to refer to $\{\lambda_k(\varepsilon), b_k(t;\varepsilon)\}$ as the "perturbed spectral data", and similarly for other quantities depending on ε, but then we are only referring to the fact that $\lambda_k(\varepsilon)$ is a small perturbation of λ_k. We emphasize that $b_k(t;\varepsilon)$ does not have a limit as $\varepsilon \to 0$, because of ε in the denominator.

In particular, later on we will start with some functions $b_k(t)$, let $a_k(0) = \lambda_k^{\beta_k} b_k(0)$, and use these constants $a_k(0)$ to define $b_k(t;\varepsilon)$. Then, despite the notation, the functions $b_k(t;\varepsilon)$ will not at all be small perturbations of the original functions $b_k(t)$. (But there will be other relations between them that justify using the same symbol b_k for both.)

Example 6.42. With the same numbers as in Example 6.5, the perturbed eigenvalues are

$$\begin{aligned}
\lambda_1(\varepsilon) &= \mu_1 + \varepsilon, & \lambda_6(\varepsilon) &= \mu_2 + \varepsilon,\\
\lambda_2(\varepsilon) &= \mu_1 + \varepsilon\,\omega_1, & \lambda_7(\varepsilon) &= \mu_2 + \varepsilon\,\omega_2,\\
\lambda_3(\varepsilon) &= \mu_1 + \varepsilon\,\omega_1^2, & \lambda_8(\varepsilon) &= \mu_2 + \varepsilon\,\omega_2^2,\\
\lambda_4(\varepsilon) &= \mu_1 + \varepsilon\,\omega_1^3, & \lambda_9(\varepsilon) &= \mu_3 + \varepsilon,\\
\lambda_5(\varepsilon) &= \mu_1 + \varepsilon\,\omega_1^4, & \lambda_{10}(\varepsilon) &= \mu_4 + \varepsilon,
\end{aligned}$$

where

$$\omega_1 = e^{2\pi i/5}, \qquad \omega_2 = e^{2\pi i/3} \qquad (\text{and } \omega_3 = \omega_4 = 1),$$

and the residues are

$$
\begin{aligned}
b_1(t;\varepsilon) &= \frac{1}{5}\left(\frac{a_1(0)}{\left(\varepsilon\omega_1^0\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega_1^0\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega_1^0\right)^2} + \frac{a_4(0)}{\left(\varepsilon\omega_1^0\right)^3} + \frac{a_5(0)}{\left(\varepsilon\omega_1^0\right)^4}\right) e^{t/\lambda_1(\varepsilon)},\\
b_2(t;\varepsilon) &= \frac{1}{5}\left(\frac{a_1(0)}{\left(\varepsilon\omega_1^1\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega_1^1\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega_1^1\right)^2} + \frac{a_4(0)}{\left(\varepsilon\omega_1^1\right)^3} + \frac{a_5(0)}{\left(\varepsilon\omega_1^1\right)^4}\right) e^{t/\lambda_2(\varepsilon)},\\
b_3(t;\varepsilon) &= \frac{1}{5}\left(\frac{a_1(0)}{\left(\varepsilon\omega_1^2\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega_1^2\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega_1^2\right)^2} + \frac{a_4(0)}{\left(\varepsilon\omega_1^2\right)^3} + \frac{a_5(0)}{\left(\varepsilon\omega_1^2\right)^4}\right) e^{t/\lambda_3(\varepsilon)},\\
b_4(t;\varepsilon) &= \frac{1}{5}\left(\frac{a_1(0)}{\left(\varepsilon\omega_1^3\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega_1^3\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega_1^3\right)^2} + \frac{a_4(0)}{\left(\varepsilon\omega_1^3\right)^3} + \frac{a_5(0)}{\left(\varepsilon\omega_1^3\right)^4}\right) e^{t/\lambda_4(\varepsilon)},\\
b_5(t;\varepsilon) &= \frac{1}{5}\left(\frac{a_1(0)}{\left(\varepsilon\omega_1^4\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega_1^4\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega_1^4\right)^2} + \frac{a_4(0)}{\left(\varepsilon\omega_1^4\right)^3} + \frac{a_5(0)}{\left(\varepsilon\omega_1^4\right)^4}\right) e^{t/\lambda_5(\varepsilon)},
\end{aligned}
$$

$$
\begin{aligned}
b_6(t;\varepsilon) &= \frac{1}{3}\left(\frac{a_6(0)}{\left(\varepsilon\omega_2^0\right)^0} + \frac{a_7(0)}{\left(\varepsilon\omega_2^0\right)^1} + \frac{a_8(0)}{\left(\varepsilon\omega_2^0\right)^2}\right) e^{t/\lambda_6(\varepsilon)},\\
b_7(t;\varepsilon) &= \frac{1}{3}\left(\frac{a_6(0)}{\left(\varepsilon\omega_2^1\right)^0} + \frac{a_7(0)}{\left(\varepsilon\omega_2^1\right)^1} + \frac{a_8(0)}{\left(\varepsilon\omega_2^1\right)^2}\right) e^{t/\lambda_7(\varepsilon)},\\
b_8(t;\varepsilon) &= \frac{1}{3}\left(\frac{a_6(0)}{\left(\varepsilon\omega_2^2\right)^0} + \frac{a_7(0)}{\left(\varepsilon\omega_2^2\right)^1} + \frac{a_8(0)}{\left(\varepsilon\omega_2^2\right)^2}\right) e^{t/\lambda_8(\varepsilon)},
\end{aligned}
$$

$$
b_9(t;\varepsilon) = a_9(0)\, e^{t/\lambda_9(\varepsilon)},
$$

$$
b_{10}(t;\varepsilon) = a_{10}(0)\, e^{t/\lambda_{10}(\varepsilon)}.
$$

Remark 6.43. The requirement that $b_i(t;\varepsilon) = \overline{b_j(t;\varepsilon)}$ if $\lambda_i = \overline{\lambda_j}$ is fulfilled if the coefficients $a_k(0)$ form suitable complex-conjugate pairs. Moreover, if the highest-numbered $a_k(0)$ associated with each eigenvalue is nonzero, then all $b_k(t;\varepsilon)$ will be nonzero for all sufficiently small $\varepsilon > 0$. Thus, if the spectral data $\{\lambda_k, \lambda_k^{\beta_k} a_k(0)\}$ is in $\widehat{\mathscr{R}}$ and ε is small enough, then the perturbed spectral data $\{\lambda_k(\varepsilon), b_k(t;\varepsilon)\}$ will be in $\widehat{\mathscr{R}}_{\mathrm{s}}$ for all t.

An example should make the first point clear. Suppose μ_1 is a triple eigenvalue with partner $\mu_2 = \overline{\mu_1}$. Then $\omega_1 = \omega_2 = e^{2\pi i/3}$, which we simply call ω, and the perturbed eigenvalues $\lambda_k(\varepsilon)$ lie as illustrated in Figure 31. Note that they pair up with a "backwards" numbering: the conjugates of $\lambda_{1,2,3}(\varepsilon)$ are $\lambda_{4,6,5}(\varepsilon)$, in that order. Considering for example the index pair $\{2,6\}$, we have

$$
\begin{aligned}
b_2(t;\varepsilon) &= \frac{1}{3}\left(\frac{a_1(0)}{\left(\varepsilon\omega^1\right)^0} + \frac{a_2(0)}{\left(\varepsilon\omega^1\right)^1} + \frac{a_3(0)}{\left(\varepsilon\omega^1\right)^2}\right) e^{t/\lambda_2(\varepsilon)},\\
b_6(t;\varepsilon) &= \frac{1}{3}\left(\frac{a_4(0)}{\left(\varepsilon\omega^2\right)^0} + \frac{a_5(0)}{\left(\varepsilon\omega^2\right)^1} + \frac{a_6(0)}{\left(\varepsilon\omega^2\right)^2}\right) e^{t/\lambda_6(\varepsilon)},
\end{aligned}
\tag{6.69}
$$

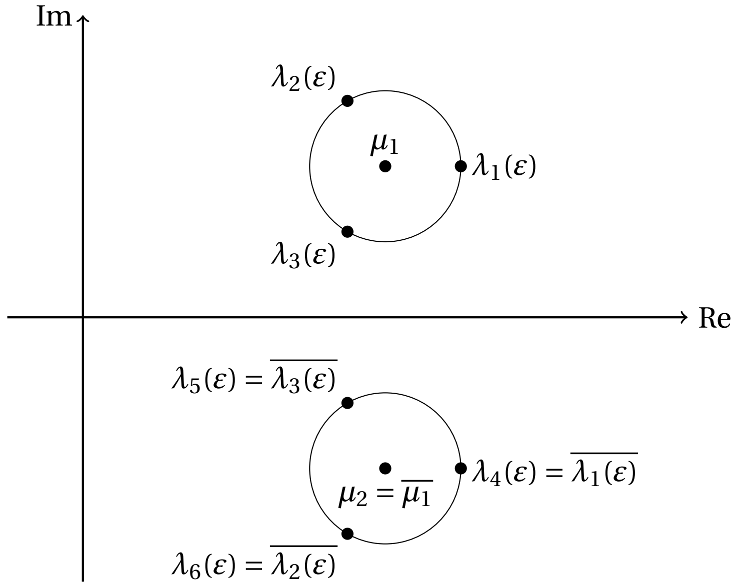

Figure 31: Illustration for Remark 6.43.

and, since $\omega^2 = \overline{\omega^1}$ and $\lambda_6(\varepsilon) = \overline{\lambda_2(\varepsilon)}$, we see that indeed $b_6(t;\varepsilon) = \overline{b_2(t;\varepsilon)}$ provided that

$$a_4(0) = \overline{a_1(0)}, \qquad a_5(0) = \overline{a_2(0)}, \qquad a_6(0) = \overline{a_3(0)}. \tag{6.70}$$

This condition also makes things work out correctly for the other index pairs. Note that the coefficients $a_k(0)$, unlike the eigenvalues, should be paired up using the natural numbering: the conjugates of $a_{1,2,3}(0)$ are $a_{4,5,6}(0)$, in that order.

Remark 6.44. The residues $b_k(t;\varepsilon)$ are obviously undefined for $\varepsilon = 0$, because of division by zero. But what we will show is that the Weyl function

$$\omega(\lambda;t;\varepsilon) = \sum_{k=1}^{n} \frac{b_k(t;\varepsilon)}{\lambda - \lambda_k(\varepsilon)}$$

has a finite limit as $\varepsilon \to 0$ (which is moreover of the right form for a partial fraction expansion with poles of arbitrary multiplicity), and likewise for all the quantities $U_k(t;\varepsilon)$, $W_k(t;\varepsilon)$, etc., in the n-peakon solution formulas with parameters $\lambda_k(\varepsilon)$ and $b_k(t;\varepsilon)$. These formulas satisfy the n-peakon ODEs for every sufficiently small $\varepsilon > 0$, and since all quantities involved depends analytically on ε, with removable singularities at $\varepsilon = 0$, the limiting formulas obtained as $\varepsilon \to 0$ must also satisfy the same ODEs. These limiting formulas thus give the solution for the multiple-eigenvalue setup that we started with.

To see what happens to $\omega(\lambda;t;\varepsilon)$ as $\varepsilon \to 0$, we need the following simple lemma, where $f^{(r)}(z)$ denotes the rth derivative of f with respect to z.

Lemma 6.45. *Suppose that $f(z)$ is analytic and that $\omega = \exp(2\pi i/d)$ for some integer $d \geq 1$. Then*

$$\frac{1}{d}\sum_{s=0}^{d-1} \frac{f(z+\varepsilon\omega^s)}{(\omega^s)^r} = \varepsilon^r \left(\frac{f^{(r)}(z)}{r!} + \mathscr{O}(\varepsilon^d) \right) \tag{6.71}$$

for $r = 0, \dots, d-1$.

Proof. The formula for a geometric sum shows that

$$\frac{1}{d}\sum_{s=0}^{d-1} (\omega^{q-r})^s = \begin{cases} 1, & \text{if } q-r \text{ is a multiple of } d, \\ 0, & \text{otherwise.} \end{cases}$$

Writing out the Taylor series of f, switching the order of summation, and using the fact just mentioned, we find

$$\begin{aligned} \frac{1}{d}\sum_{s=0}^{d-1} \frac{f(z+\varepsilon\omega^s)}{(\omega^s)^r} &= \frac{1}{d}\sum_{s=0}^{d-1} \frac{1}{(\omega^s)^r}\left(\sum_{q=0}^{\infty} \frac{(\varepsilon\omega^s)^q}{q!} f^{(q)}(z)\right) \\ &= \sum_{q=0}^{\infty}\left(\frac{1}{d}\sum_{s=0}^{d-1}(\omega^{q-r})^s\right)\frac{\varepsilon^q}{q!} f^{(q)}(z) \\ &= \frac{\varepsilon^r}{r!} f^{(r)}(z) + \frac{\varepsilon^{r+d}}{(r+d)!} f^{(r+d)}(z) + \frac{\varepsilon^{r+2d}}{(r+2d)!} f^{(r+2d)}(z) + \dots \\ &= \frac{\varepsilon^r}{r!} f^{(r)}(z) + \mathcal{O}(\varepsilon^{r+d}), \end{aligned}$$

as claimed. □

Theorem 6.46. *As* $\varepsilon \to 0$*, the Weyl function*

$$\omega(\lambda; t; \varepsilon) = \sum_{k=1}^{n} \frac{b_k(t;\varepsilon)}{\lambda - \lambda_k(\varepsilon)}, \tag{6.72}$$

with $\lambda_k(\varepsilon)$ *and* $b_k(t;\varepsilon)$ *given by Definitions 6.37 and 6.39, tends to*

$$\begin{aligned} \omega(\lambda; t) &:= \sum_{j=1}^{J}\sum_{r=0}^{d_j-1} \frac{a_{\delta_j+1+r}(0)}{r!}\left[\left(\frac{\partial}{\partial z}\right)^r \frac{e^{t/z}}{\lambda - z}\right]_{z=\mu_j} \\ &= \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{a_{\delta_j+k}(t)}{(\lambda-\mu_j)^k}, \end{aligned} \tag{6.73}$$

where

$$a_{\delta_j+k}(t) = \sum_{q=0}^{d_j-k} \frac{a_{\delta_j+k+q}(0)}{q!\,\mu_j^q}\, p_q(t/\mu_j)\, e^{t/\mu_j}, \tag{6.74}$$

for $1 \le j \le J$ *and* $1 \le k \le d_j$*. (Recall the polynomials* $p_k(w)$ *from Definition 6.18.)*

Remark 6.47. Since $p_0(0) = 1$ and $p_q(0) = 0$ for $q \ge 1$, setting $t = 0$ in the right-hand side of (6.74) does give the constant $a_{\delta_j+k}(0)$, so the notation is consistent.

Proof of Theorem 6.46. Insert the expressions (6.66) and (6.68) for $\lambda_k(\varepsilon)$ and $b_k(t;\varepsilon)$ into the Weyl function, change the order of summation, and apply Lemma 6.45 with $f(z) = e^{t/z}/(\lambda - z)$:

$$\begin{aligned}
\omega(\lambda;t;\varepsilon) &= \sum_{k=1}^{n} \frac{b_k(t;\varepsilon)}{\lambda - \lambda_k(\varepsilon)} \\
&= \sum_{j=1}^{J} \sum_{s=0}^{d_j-1} \frac{b_{\delta_j+1+s}(t;\varepsilon)}{\lambda - \lambda_{\delta_j+1+s}(\varepsilon)} \\
&= \sum_{j=1}^{J} \sum_{s=0}^{d_j-1} \frac{\frac{1}{d_j}\left(\sum_{r=0}^{d_j-1} \frac{a_{\delta_j+1+r}(0)}{\left(\varepsilon\omega_j^s\right)^r}\right) e^{t/(\mu_j+\varepsilon\omega_j^s)}}{\lambda - (\mu_j + \varepsilon\omega_j^s)} \\
&= \sum_{j=1}^{J} \sum_{r=0}^{d_j-1} \frac{a_{\delta_j+1+r}(0)}{\varepsilon^r} \left(\frac{1}{d_j} \sum_{s=0}^{d_j-1} \frac{1}{\left(\omega_j^s\right)^r} \frac{e^{t/(\mu_j+\varepsilon\omega_j^s)}}{\lambda - (\mu_j+\varepsilon\omega_j^s)} \right) \\
&= \sum_{j=1}^{J} \sum_{r=0}^{d_j-1} \frac{a_{\delta_j+1+r}(0)}{\varepsilon^r} \left(\frac{\varepsilon^r}{r!} \left[\left(\frac{\partial}{\partial z}\right)^r \frac{e^{t/z}}{\lambda - z} \right]_{z=\mu_j} + \mathcal{O}(\varepsilon^{r+d_j}) \right) \\
&= \sum_{j=1}^{J} \sum_{r=0}^{d_j-1} \frac{a_{\delta_j+1+r}(0)}{r!} \left[\left(\frac{\partial}{\partial z}\right)^r \frac{e^{t/z}}{\lambda - z} \right]_{z=\mu_j} + \mathcal{O}(\varepsilon),
\end{aligned}$$

which gives the first expression for $\omega(\lambda, t)$ in (6.73) when $\varepsilon \to 0$. To see that this equals the second expression in (6.73), it is enough to investigate what happens for each j, so let us simplify the notation a little and write d, μ, δ instead of d_j, μ_j, δ_j, and just A_r instead of $a_{\delta_j+1+r}(0)$. Then we find, using the Leibniz rule for the rth derivative of a product, and then formula (6.23) for the derivatives of $e^{1/z}$, that

$$\begin{aligned}
&\sum_{r=0}^{d-1} \frac{A_r}{r!} \left[\left(\frac{\partial}{\partial z}\right)^r \frac{e^{t/z}}{\lambda - z} \right]_{z=\mu} \\
&= \sum_{r=0}^{d-1} \sum_{k=1}^{r+1} \frac{A_r}{r!} \binom{r}{k-1} \left[\left(\left(\frac{\partial}{\partial z}\right)^{r-(k-1)} e^{t/z} \right) \left(\left(\frac{\partial}{\partial z}\right)^{k-1} \frac{1}{\lambda - z} \right) \right]_{z=\mu} \\
&= \sum_{k=1}^{d} \sum_{r=k-1}^{d-1} \frac{A_r}{r!} \binom{r}{k-1} \left[\frac{p_{r+1-k}(t/z)\, e^{t/z}}{z^{r+1-k}} \, \frac{(k-1)!}{(\lambda - z)^k} \right]_{z=\mu} \\
&= \sum_{k=1}^{d} \sum_{r=k-1}^{d-1} \frac{A_r}{(r+1-k)!} \, \frac{p_{r+1-k}(t/\mu)\, e^{t/\mu}}{\mu^{r+1-k}} \, \frac{1}{(\lambda - \mu)^k} \\
&= \sum_{k=1}^{d} \frac{1}{(\lambda - \mu)^k} \left(\sum_{r=k-1}^{d-1} \frac{A_r/\mu^{r+1-k}}{(r+1-k)!} \, p_{r+1-k}(t/\mu)\, e^{t/\mu} \right) \\
&= \sum_{k=1}^{d} \frac{1}{(\lambda - \mu)^k} \left(\sum_{q=0}^{d-k} \frac{A_{q+k-1}}{q!\,\mu^q} \, p_q(t/\mu)\, e^{t/\mu} \right),
\end{aligned}$$

and thus

$$\omega(\lambda;t) = \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{1}{(\lambda-\mu_j)^k}\left(\sum_{q=0}^{d_j-k} \frac{a_{\delta_j+1+(q+k-1)}(0)}{q!\,\mu_j^q}\, p_q(t/\mu_j)\, e^{t/\mu_j}\right),$$

as claimed. □

Proof of Theorem 6.24 (time dependence of $\omega(\lambda)$). Given spectral data

$$\{\lambda_k, b_k(0)\}_{k=1}^n \in \widehat{\mathscr{R}},$$

we want to find the time evolution of $b_k(t)$ in the Weyl function (6.25). Let

$$a_k(0) = \lambda_k^{\beta_k} b_k(0), \tag{6.75}$$

and form perturbed spectral data $\{\lambda_k(\varepsilon), b_k(t;\varepsilon)\}_{k=1}^n \in \widehat{\mathscr{R}}_{\mathrm{s}}$ as in Definitions 6.37 and 6.39.

Now the perturbed spectral data have the correct time evolution (known from the simple-eigenvalue case) in order to make the corresponding perturbed Weyl function $\omega(\lambda;t;\varepsilon)$ satisfy the governing ODE for all sufficiently small $\varepsilon > 0$:

$$\dot\omega(\lambda;t;\varepsilon) = \frac{\omega(\lambda;t;\varepsilon)-\omega(0;t;\varepsilon)}{\lambda}.$$

And according to Theorem 6.46 we have

$$\omega(\lambda;t;\varepsilon) \to \omega(\lambda;t) = \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{a_{\delta_j+k}(t)}{(\lambda-\mu_j)^k}, \qquad \text{as } \varepsilon \to 0,$$

with $\{a_k(t)\}_{k=1}^n$ given by (6.74). The limiting Weyl function $\omega(\lambda;t)$ must satisfy the ODE as well,

$$\dot\omega(\lambda;t) = \frac{\omega(\lambda;t)-\omega(0;t)}{\lambda},$$

and for $t=0$ it agrees with (6.25), by (6.75):

$$\omega(\lambda;0) = \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{a_{\delta_j+k}(0)}{(\lambda-\mu_j)^k} = \sum_{j=1}^{J}\sum_{k=1}^{d_j} \frac{\mu_j^{k-1}\, b_{\delta_j+k}(0)}{(\lambda-\mu_j)^k}.$$

This means that $\omega(\lambda;t)$ *is* the Weyl function (6.25) that we are looking for, so that the functions $b_k(t)$ in (6.25) are given by

$$\begin{aligned} b_{\delta_j+k}(t) &= \frac{a_{\delta_j+k}(t)}{\mu_j^{k-1}} \\ &= \frac{1}{\mu_j^{k-1}} \sum_{q=0}^{d_j-k} \frac{a_{\delta_j+k+q}(0)}{q!\,\mu_j^q}\, p_q(t/\mu_j)\, e^{t/\mu_j} \\ &= \frac{1}{\mu_j^{k-1}} \sum_{q=0}^{d_j-k} \frac{\mu_j^{k+q-1}\, b_{\delta_j+k+q}(0)}{q!\,\mu_j^q}\, p_q(t/\mu_j)\, e^{t/\mu_j} \\ &= \sum_{q=0}^{d_j-k} \frac{b_{\delta_j+k+q}(0)}{q!}\, p_q(t/\mu_j)\, e^{t/\mu_j}, \end{aligned}$$

for $1 \le j \le J$ and $1 \le k \le d_j$. This is equation (6.27). □

Alternative proof of Theorem 6.24. It is also possible to derive the time dependence for the coefficients in the Weyl function in a more conventional way. Simply inserting the partial fraction expansion (6.25) into the governing ODE

$$\dot{\omega}(\lambda;t) = \frac{\omega(\lambda;t) - \omega(0;t)}{\lambda}$$

and identifying coefficients, one obtains a coupled linear system of ODEs for $\{b_k(t)\}_{k=1}^n$. With just one eigenvalue μ of multiplicity d, this system is

$$\dot{b}_k = \frac{1}{\mu} \sum_{s=k}^{d} (-1)^{s-k} b_s, \qquad 1 \le k \le d, \tag{6.76}$$

and with several eigenvalues μ_j, we get one such system for each j. One way of solving this (triangular) system is by switching to the new time variable $\tau = t/\mu$, writing the system's coefficient matrix as $A = I + N$ where N is nilpotent, and computing $e^{A\tau} = e^{\tau} e^{N\tau}$ via a (finite) Maclaurin expansion. The solution obtained in this way turns out to agree with (6.27), of course. We omit the details. □

Next we want to investigate what happens to U_k, V_k and T_k as we perform the same limiting procedure. For this we need a few more simple facts; first, the generalization of Lemma 6.45 to functions of k variables, where we will use the notation

$$f^{(r_1,\dots,r_k)}(z_1,\dots,z_k) = \left(\frac{\partial}{\partial z_1}\right)^{r_1} \cdots \left(\frac{\partial}{\partial z_k}\right)^{r_k} f(z_1,\dots,z_k).$$

Lemma 6.48. *Suppose $f(z_1,\dots,z_k)$ is analytic and that $\omega_j = \exp(2\pi i/d_j)$ for some integers $d_1,\dots,d_k \ge 1$. Then*

$$\begin{aligned} &\frac{1}{d_1 \cdots d_k} \sum_{s_1=0}^{d_1-1} \cdots \sum_{s_k=0}^{d_k-1} \frac{f(z_1 + \varepsilon_1 \omega_1^{s_1}, \dots, z_k + \varepsilon_k \omega_k^{s_k})}{\left(\omega_1^{s_1}\right)^{r_1} \cdots \left(\omega_k^{s_k}\right)^{r_k}} \\ &= \varepsilon_1^{r_1} \cdots \varepsilon_k^{r_k} \left(\frac{f^{(r_1,\dots,r_k)}(z_1,\dots,z_k)}{r_1! \cdots r_k!} + \mathcal{O}(\varepsilon_1^{d_1}) + \cdots + \mathcal{O}(\varepsilon_k^{d_k}) \right). \end{aligned} \tag{6.77}$$

for $0 \le r_1 < d_1, \dots, 0 \le r_k < d_k$. In particular, with $\varepsilon_1 = \cdots = \varepsilon_k = \varepsilon$:

$$\begin{aligned} &\frac{1}{d_1 \cdots d_k} \sum_{s_1=0}^{d_1-1} \cdots \sum_{s_k=0}^{d_k-1} \frac{f(z_1 + \varepsilon \omega_1^{s_1}, \dots, z_k + \varepsilon \omega_k^{s_k})}{\left(\omega_1^{s_1}\right)^{r_1} \cdots \left(\omega_k^{s_k}\right)^{r_k}} \\ &= \varepsilon^{r_1 + \cdots + r_k} \left(\frac{f^{(r_1,\dots,r_k)}(z_1,\dots,z_k)}{r_1! \cdots r_k!} + \mathcal{O}(\varepsilon^{\min(d_1,\dots,d_k)}) \right). \end{aligned} \tag{6.78}$$

Proof. The proof is virtually the same as in the one-variable case. For example, with two variables:

$$
\begin{aligned}
&\frac{1}{d_1 d_2}\sum_{s_1=0}^{d_1-1}\sum_{s_2=0}^{d_2-1}\frac{f(z_1+\varepsilon_1\omega_1^{s_1}, z_2+\varepsilon_e\omega_2^{s_2})}{\left(\omega_1^{s_1}\right)^{r_1}\left(\omega_2^{s_2}\right)^{r_2}} \\
&= \frac{1}{d_1 d_2}\sum_{s_1=0}^{d_1-1}\sum_{s_2=0}^{d_2-1}\frac{1}{\left(\omega_1^{s_1}\right)^{r_1}\left(\omega_2^{s_2}\right)^{r_2}}\left(\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\frac{\left(\varepsilon_1\omega_1^{s_1}\right)^{q_1}\left(\varepsilon_2\omega_2^{s_2}\right)^{q_2}}{q_1!\,q_2!}f^{(q_1,q_2)}(z_1,z_2)\right) \\
&= \sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\left(\frac{1}{d_1}\sum_{s_1=0}^{d_1-1}\left(\omega_1^{q_1-r_1}\right)^{s_1}\right)\left(\frac{1}{d_2}\sum_{s_2=0}^{d_2-1}\left(\omega^{q_2-r_2}\right)^{s_2}\right)\frac{\varepsilon_1^{q_1}\,\varepsilon_2^{q_2}}{q_1!\,q_2!}f^{(q_1,q_2)}(z,z) \\
&= \sum_{\substack{q_1\in r_1+d_1\mathbf{N}\\ q_2\in r_2+d_2\mathbf{N}}}\frac{\varepsilon_1^{q_1}\,\varepsilon_2^{q_2}}{q_1!\,q_2!}f^{(q_1,q_2)}(z,z) \\
&= \varepsilon_1^{r_1}\varepsilon_2^{r_2}\left(\frac{f^{(r_1,r_2)}(z_1,z_2)}{r_1!r_2!}+\mathcal{O}(\varepsilon_1^{d_1})+\mathcal{O}(\varepsilon_2^{d_2})\right),
\end{aligned}
$$

as claimed. □

Lemma 6.49. *For analytic functions $f(z_1,\dots,z_k)$ and $g_1(z),\dots,g_k(z)$, the identity*

$$
\begin{aligned}
&\frac{1}{r_1!\dots r_k!}\left(\frac{\partial}{\partial z_1}\right)^{r_1}\cdots\left(\frac{\partial}{\partial z_k}\right)^{r_k}\Big[f(z_1,\dots,z_k)\,g_1(z_1)\cdots g_k(z_k)\Big] \\
&= \sum_{q_1=0}^{r_1}\cdots\sum_{q_k=0}^{r_k}\frac{f^{(r_1-q_1,\dots,r_k-q_k)}(z_1,\dots,z_k)}{(r_1-q_1)!\cdots(r_k-q_k)!}\,\frac{g_1^{(q_1)}(z_1)}{q_1!}\cdots\frac{g_k^{(q_k)}(z_k)}{q_k!}
\end{aligned}
\tag{6.79}
$$

holds.

Proof. This is just repeated application (once for each variable z_k) of the Leibniz rule for the rth derivative of a product, $D^r(fg)=\sum_{q=0}^{r}\frac{r!}{(q-r)!\,q!}(D^{r-q}f)(D^q g)$. □

Theorem 6.50. *Suppose $\{\lambda_k, b_k(t)\}_{k=1}^n$ are spectral data in $\widehat{\mathcal{R}}$ with the time dependence of $b_k(t)$ given by* (6.27)*:*

$$
b_{\delta_j+k}(t)=\sum_{q=0}^{d_j-k}\frac{b_{\delta_j+k+q}(0)}{q!}\,p_q(t/\mu_j)\,e^{t/\mu_j}.
$$

Let $a_k(0)=\lambda_k^{\beta_k}\,b_k(0)$, define $\{\lambda_k(\varepsilon), b_k(t;\varepsilon)\}\in\widehat{\mathcal{R}}_{\mathrm{s}}$ as in Definitions 6.37 and 6.39, and let $U_k(t;\varepsilon)$ denote the quantity U_k computed using the perturbed spectral data $\{\lambda_k(\varepsilon), b_k(t;\varepsilon)\}$. Then

$$
U_k(t;\varepsilon)\to\widetilde{U}_k(t),\qquad \text{as } \varepsilon\to 0,
$$

where $\widetilde{U}_k(t)$ denotes the quantity $\widetilde{U}_k$ computed using the spectral data $\{\lambda_k, b_k(t)\}$ that we started with. Similarly, $V_k(t;\varepsilon)\to\widetilde{V}_k(t)$ and $T_k(t;\varepsilon)\to\widetilde{T}_k(t)$.

Proof. Write the perturbed U_k as

$$\begin{aligned} U_k(t;\varepsilon) &= \sum_{I\in\binom{[1,n]}{k}} \Psi_I(\varepsilon)\, b_I(t;\varepsilon) \\ &= \frac{1}{k!}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} \Psi\big(\lambda_{i_1}(\varepsilon),\dots,\lambda_{i_k}(\varepsilon)\big)\, b_{i_1}(t;\varepsilon)\cdots b_{i_k}(t;\varepsilon). \end{aligned}$$

Here we have used that $\Psi(z_1,\dots,z_n) = \Delta(z_1,\dots,z_n)^2/\Gamma(z_1,\dots,z_n)$ is a symmetric function which vanishes whenever two z_i coincide. Since we have partitioned the interval $[1,n]$ into the subintervals $\mathscr{J}_1,\dots,\mathscr{J}_J$, we can split any sum over the "cube" $[1,n]^k$ into sums over "cuboids", i.e., products of the intervals $\mathscr{J}_j$:

$$\sum_{(i_1,\dots,i_k)\in[1,n]^k} F(i_1,\dots,i_k) = \sum_{(j_1,\dots,j_k)\in[1,J]^k}\Bigg(\sum_{(i_1,\dots,i_k)\in\mathscr{J}_{j_1}\times\dots\times\mathscr{J}_{j_k}} F(i_1,\dots,i_k)\Bigg)$$

Taking one such cuboid in the sum for $U_k(t;\varepsilon)$ (by fixing some $(j_1,\dots,j_k)\in[1,J]^k$) and inserting the expressions (6.66) and (6.68) for the perturbed eigenvalues and residues, we get

$$\begin{aligned} &\sum_{(i_1,\dots,i_k)\in\mathscr{J}_{j_1}\times\dots\times\mathscr{J}_{j_k}} \Psi\big(\lambda_{i_1}(\varepsilon),\dots,\lambda_{i_k}(\varepsilon)\big)\, b_{i_1}(t;\varepsilon)\cdots b_{i_k}(t;\varepsilon) \\ &= \sum_{s_1=0}^{d_{j_1}-1}\cdots\sum_{s_k=0}^{d_{j_k}-1} \Psi\big(\lambda_{\delta_{j_1}+1+s_1}(\varepsilon),\dots,\lambda_{\delta_{j_k}+1+s_k}(\varepsilon)\big)\, b_{\delta_{j_1}+1+s_1}(t;\varepsilon)\cdots b_{\delta_{j_k}+1+s_k}(t;\varepsilon) \\ &= \sum_{s_1=0}^{d_{j_1}-1}\cdots\sum_{s_k=0}^{d_{j_k}-1} \Psi\big(\mu_{j_1}+\varepsilon\omega_{j_1}^{s_1},\dots,\mu_{j_k}+\varepsilon\omega_{j_k}^{s_k}\big) \\ &\qquad\times \frac{1}{d_{j_1}}\Bigg(\sum_{r_1=0}^{d_{j_1}-1}\frac{a_{\delta_{j_1}+1+r_1}(0)}{\big(\varepsilon\omega_{j_1}^{s_1}\big)^{r_1}}\Bigg)\exp\frac{t}{\mu_{j_1}+\varepsilon\omega_{j_1}^{r_1}} \\ &\qquad\times\dots\times \frac{1}{d_{j_k}}\Bigg(\sum_{r_k=0}^{d_{j_k}-1}\frac{a_{\delta_{j_k}+1+r_k}(0)}{\big(\varepsilon\omega_{j_k}^{s_k}\big)^{r_k}}\Bigg)\exp\frac{t}{\mu_{j_k}+\varepsilon\omega_{j_k}^{r_k}} \\ &= \sum_{r_1=0}^{d_{j_1}-1}\cdots\sum_{r_k=0}^{d_{j_k}-1} \frac{a_{\delta_{j_1}+1+r_1}(0)\cdots a_{\delta_{j_k}+1+r_k}(0)}{\varepsilon^{r_1+\dots+r_k}} \\ &\qquad\times\Bigg(\frac{1}{d_{j_1}\cdots d_{j_k}}\sum_{s_1=0}^{d_{j_1}-1}\cdots\sum_{s_k=0}^{d_{j_k}-1}\frac{\widehat{\Psi}\big(\mu_{j_1}+\varepsilon\omega_{j_1}^{s_1},\dots,\mu_{j_k}+\varepsilon\omega_{j_k}^{s_k};t\big)}{\big(\omega_{j_1}^{s_1}\big)^{r_1}\cdots\big(\omega_{j_k}^{s_k}\big)^{r_k}}\Bigg), \end{aligned}$$

where

$$\widehat{\Psi}(z_1,\dots,z_k;t) = \Psi(z_1,\dots,z_k)\, e^{t/z_1}\cdots e^{t/z_k}.$$

By Lemma 6.48, this equals

$$\sum_{r_1=0}^{d_{j_1}-1}\cdots\sum_{r_k=0}^{d_{j_k}-1} a_{\delta_{j_1}+1+r_1}(0)\cdots a_{\delta_{j_k}+1+r_k}(0)\Bigg(\frac{\widehat{\Psi}^{(r_1,\dots,r_k)}(\mu_{j_1},\dots,\mu_{j_k};t)}{r_1!\cdots r_k!}+\mathscr{O}(\varepsilon)\Bigg).$$

Now we use Lemma 6.49 and formula (6.23),

$$\left(\frac{\partial}{\partial z}\right)^k e^{t/z} = \frac{p_k(t/z)\, e^{t/z}}{z^k},$$

to compute

$$\frac{\widehat{\Psi}^{(r_1,\ldots,r_k)}(\mu_{j_1},\ldots,\mu_{j_k};t)}{r_1!\cdots r_k!}$$
$$= \sum_{q_1=0}^{r_1}\cdots\sum_{q_k=0}^{r_k} \frac{\Psi^{(r_1-q_1,\ldots,r_k-q_k)}(\mu_{j_1},\ldots,\mu_{j_k})}{(r_1-q_1)!\cdots(r_k-q_k)!}\, \frac{p_{q_1}(\frac{t}{\mu_{j_1}})\, e^{t/\mu_{j_1}}}{q_1!\,\mu_{j_1}^{q_1}} \cdots \frac{p_{q_1}(\frac{t}{\mu_{j_1}})\, e^{t/\mu_{j_1}}}{q_1!\,\mu_{j_1}^{q_1}}$$

When we insert this back into the sum we are investigating, we will obtain k double sums of the form

$$\sum_{r=0}^{d-1}\sum_{q=0}^{r} F(r,q),$$

which can be rewritten as

$$\sum_{R=0}^{d-1}\sum_{q=0}^{d-1-R} F(R+q,q).$$

Hence, we find that the sum over the cuboid becomes, up to a term $\mathcal{O}(\varepsilon)$,

$$\sum_{R_1=0}^{d_{j_1}-1}\cdots\sum_{R_k=0}^{d_{j_k}-1}\sum_{q_1=0}^{d_{j_1}-1-R_1}\cdots\sum_{q_k=0}^{d_{j_k}-1-R_k} a_{\delta_{j_1}+1+R_1+q_1}(0)\cdots a_{\delta_{j_k}+1+R_k+q_k}(0)$$
$$\times \frac{\Psi^{(R_1,\ldots,R_k)}(\mu_{j_1},\ldots,\mu_{j_k})}{R_1!\cdots R_k!}\, \frac{p_{q_1}(\frac{t}{\mu_{j_1}})\, e^{t/\mu_{j_1}}}{q_1!\,\mu_{j_1}^{q_1}} \cdots \frac{p_{q_1}(\frac{t}{\mu_{j_1}})\, e^{t/\mu_{j_1}}}{q_1!\,\mu_{j_1}^{q_1}}$$
$$= \sum_{R_1=0}^{d_{j_1}-1}\cdots\sum_{R_k=0}^{d_{j_k}-1} \frac{\Psi^{(R_1,\ldots,R_k)}(\mu_{j_1},\ldots,\mu_{j_k})}{R_1!\cdots R_k!}$$
$$\times\left(\sum_{q_1=0}^{d_{j_1}-1-R_1} \frac{\mu_{j_1}^{R_1+q_1}\, b_{\delta_{j_1}+1+R_1+q_1}(0)}{q_1!\,\mu_{j_1}^{q_1}}\right)$$
$$\times\cdots\times\left(\sum_{q_k=0}^{d_{j_k}-1-R_k} \frac{\mu_{j_k}^{R_k+q_k}\, b_{\delta_{j_k}+1+R_k+q_k}(0)}{q_k!\,\mu_{j_k}^{q_k}}\right)$$
$$= \sum_{R_1=0}^{d_{j_1}-1}\cdots\sum_{R_k=0}^{d_{j_k}-1} \frac{\Psi^{(R_1,\ldots,R_k)}(\mu_{j_1},\ldots,\mu_{j_k})}{R_1!\cdots R_k!}\, \mu_{j_1}^{R_1} b_{\delta_{j_1}+1+R_1}(t)\cdots \mu_{j_k}^{R_k} b_{\delta_{j_k}+1+R_k}(t)$$
$$= \sum_{I\in\mathcal{J}_{j_1}\times\cdots\times\mathcal{J}_{j_k}} \widetilde{\Psi}_I \lambda_I^{\beta_I} b_I(t).$$

And since $U_k(t;\varepsilon)$ was $1/k!$ times the sum over all the cuboids, it equals

$$U_k(t;\varepsilon) = \frac{1}{k!}\sum_{I\in[1,n]^k} \widetilde{\Psi}_I \lambda_I^{\beta_I} b_I(t) + \mathcal{O}(\varepsilon) = \widetilde{U}(t) + \mathcal{O}(\varepsilon),$$

and the claim follows.

For V_k and T_k, the computation is almost the same, except that the function to which Lemma 6.48 is applied is

$$(z_1 \cdots z_k)^{\pm 1} \widehat{\Psi}(z_1, \dots, z_k)$$

instead of just $\widehat{\Psi}(z_1, \dots, z_k)$, so when applying Lemma 6.49 in the next step, we take derivatives of

$$(z_1 \cdots z_k)^{\pm 1} \Psi(z_1, \dots, z_k)$$

instead of $\Psi(z_1, \dots, z_k)$. Consequently we obtain almost the same expression in the end, the only difference being that we get $\widetilde{\Psi}_I^{(\pm 1)}$ instead of $\widetilde{\Psi}_I$, hence $\widetilde{V}_k(t)$ and $\widetilde{T}_k(t)$ instad of $\widetilde{U}_k(t)$. □

Proof of Theorem 6.25 (solution formulas). As explained in Remark 6.44, the solution formulas for the multiple-eigenvalue case are the result of applying our limiting argument to the solution formulas from the simple-eigenvalue case,

$$x_{n+1-k}(t) = \frac{1}{2} \ln \frac{Z_k(t)}{W_{k-1}(t)}, \qquad m_{n+1-k}(t) = \frac{\sqrt{Z_k(t)\, W_{k-1}(t)}}{U_k(t)\, U_{k-1}(t)},$$

so the result follows directly from Theorem 6.50 which we just proved: we replace $U_k(t)$ by $U_k(t;\varepsilon)$ and let $\varepsilon \to 0$, which results in $\widetilde{U}_k(t)$, and similarly for the other quantities. □

Proof of Theorem 6.27 (substitutions). Split the sums defining $\widetilde{V}_k$ and $\widetilde{T}_k$ into sums over "cuboids" as in the proof of Theorem 6.50. Then, in each such smaller sum, compute $\widetilde{\Psi}_I^{(\pm 1)}$ by applying Lemma 6.49 with $g(z_1, \dots, z_k) = \Psi(z_1, \dots, z_k)$ and $g_1(z) = \dots g_k(z) = z^{\pm 1}$, using of course

$$\frac{1}{k!} \left(\frac{\partial}{\partial z} \right)^k z = \begin{cases} z, & k = 0, \\ 1, & k = 1, \\ 0, & k \geq 2, \end{cases} \qquad \frac{1}{k!} \left(\frac{\partial}{\partial z} \right)^k z^{-1} = \frac{(-1)^k}{z^{k+1}}.$$

This results in an expression which looks like $\widetilde{U}_k$, since we now have derivatives of $\widetilde{\Psi}_I$ instead of $\widetilde{\Psi}_I^{(\pm 1)}$, but with additional terms as described by the substitution rules (6.29) and (6.30), respectively. □

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