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Newton Systems of Cofactor Type in Euclidean and Riemannian Spaces

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Abstract

We study second order ordinary differential equations of Newton type with integrals of motion that depend quadratically on the velocity. In particular, we introduce the class of cofactor pair systems, which admit two quadratic integrals of motion of a special form. It is shown that this implies that the system in fact admits a full set of Poisson commuting integrals of motion, and consequently is completely integrable. Methods are given for testing whether a given Newton system belongs to this class, and for constructing infinite families of cofactor pair systems. Several known result about separable potentials are included in the theory as special cases. As an application, it is shown how to extend the classical concept of Stäckel separability to a class of time-dependent potentials.

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1 Quasi-potential Newton systems

This thesis, which is a collection of three research papers [1, 2, 3], concerns systems of ordinary differential equations of Newton type (*Newton systems* or *Newton equations* for short)[1]

$$\ddot{q} = M(q), \qquad q \in R^n. \tag{1.1}$$

This is the type of equations obtained from Newton's law that mass times acceleration equals force, if it is assumed that forces depend only on position, and not on velocity or time. Such equations are in general nonlinear and highly nontrivial to solve. Most known results deal with the conservative case $\ddot{q} = -\nabla V$, when the force is derived from a potential $V(q)$. Here we study instead "quasi-potential" Newton systems[2]

$$\ddot{q} = -A(q)^{-1}\nabla W(q), \tag{1.2}$$

with a symmetric $n \times n$ matrix $A(q)$ satisfying the cyclic conditions

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0, \quad \text{for all } i, j, k = 1, \dots, n. \tag{1.3}$$

By construction, such a system admits an integral of motion[3] E which is "energy-like" in the sense of being quadratic in the velocity components $\dot{q}_i$:

$$E(q, \dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(q)\dot{q}_i\dot{q}_j + W(q) = \frac{1}{2}\dot{q}^T A(q)\,\dot{q} + W(q). \tag{1.4}$$

Notice that when $A = I$ (the identity matrix) this reduces to the classical conservative case, with E being the usual energy (kinetic energy plus potential energy).

2 Integrable Hamiltonian systems

We investigate such systems from the point of view of *integrability.* Naïvely speaking, a system is called integrable if it can be solved (more or less) explicitly. A more precise definition can be given in the framework of Hamiltonian mechanics. For a good introduction, see chapter 6 in Olver's book [5]. Roughly speaking,

[1] Dots denote derivatives with respect to time: $\dot{q} = dq/dt$ and $\ddot{q} = d^2q/dt^2$. We will use matrix notation, considering $q = (q_1, \dots, q_n)^T$ and $M(q) = (M_1(q), \dots, M_n(q))^T$ as a column vectors (T denotes the transpose of a matrix).

[2] The early papers [4, 1] have a factor $\frac{1}{2}$ in this formula instead of in the integral of motion E. The convention used here seems more in harmony with the tradition in mechanics and differential geometry.

[3] An *integral of motion* is a function of q and $\dot{q}$ which is constant along solutions of the system; if $q(t)$ is a solution, then $\frac{d}{dt}E(q(t), \dot{q}(t)) \equiv 0$. Other names: *first integral*, *constant of motion*, *invariant.*

if a system can be rewritten as a *Hamiltonian system* (see below) then it can in principle be solved provided that it admits *sufficiently many integrals of motion*, where "sufficiently many" means about half as many as would be needed for a system without the Hamiltonian structure.

A conservative Newton system $\ddot{q} = -\nabla V$ can be written as the following Hamiltonian system on the "phase space" R^{2n}:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -\nabla V(q) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}.$$

where $H(q,p) = \frac{1}{2}p^T p + V(q)$. This is the "standard" or "canonical" Hamiltonian form, with "n degrees of freedom." More generally, a Hamiltonian system is a system of the form $\dot{x} = \Pi(x)\frac{\partial H}{\partial x}$. Here x are coordinates on some phase space, while $\Pi(x)$ is a *Poisson matrix*, which means an antisymmetric matrix such that the *Poisson bracket* defined by $\{F,G\} = (\frac{\partial F}{\partial x})^T \Pi(x) \frac{\partial G}{\partial x}$ satisfies the Jacobi identity

$$\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$$

for all functions F, G, H on the phase space.

In the canonical case, the system is *completely integrable* (in the Liouville–Arnol'd sense) if it admits n functionally independent integrals of motion H_i that pairwise commute with respect to the Poisson bracket: $\{H_i, H_j\} = 0$. Liouville showed that in this situation the system can in principle be solved "by quadratures." One also has geometric information about the trajectories; they must wind periodically or quasi-periodically around n-dimensional tori in phase space (in the case that the surface of constant energy is compact), as was shown by Arnol'd [6]. Similar statements can be made in the case of nonstandard Hamiltonian structures.

A special class of integrable systems are given by *separable potentials*. This has to do with the Hamilton–Jacobi method, the details of which are beyond the scope of this introduction. The idea is that the solution of $\ddot{q} = -\nabla V(q)$ can be found if one is able to solve the *Hamilton–Jacobi equation*, a nonlinear PDE involving $V(q)$. The Hamilton–Jacobi equation is in general very difficult to solve, but for certain potentials V one can do it using separation of variables (after having changed to a suitable coordinate system). The so-called Stäckel conditions describe when separation in orthogonal coordinates is possible. First of all, there is a restriction on the coordinate system. In certain spaces (like R^n, S^n, H^n) the possible coordinate systems have been completely classified. In R^2 these are the cartesian, polar, elliptic, and parabolic coordinates. Secondly, the potential, when expressed in these new coordinates u, must take the form $V(u) = \sum_{i=1}^n f_i(u_i)/H_i^2$, where H_i^2 are the metric coefficients of the orthogonal coordinate system. For example, the potential V is separable in polar coordinates on R^2, with metric $ds^2 = 1^2 dr^2 + r^2 d\phi^2$, if and only if it has the form $V(r,\phi) = \frac{f_1(r)}{1^2} + \frac{f_2(\phi)}{r^2}$. If V is separable, then the system $\ddot{q} = -\nabla V$ is integrable, and the integrals of motion are all (at most) quadratic in $\dot{q}$ (or p).

3 Cofactor systems and cofactor pair systems

When studying arbitrary Newton system $\ddot{q} = M(q)$ we cannot use the standard Hamiltonian structure. However, it turns out that quasi-potential Newton systems admit a nonstandard Hamiltonian formulation with R^{2n+1} as phase space, if the matrix $A(q)$ has the form

$$A(q) = \operatorname{cof} G(q), \quad \text{where} \quad G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}. \tag{3.1}$$

Here cof denotes the cofactor matrix, $\operatorname{cof} X = (\det X) X^{-1}$. An integral of motion (1.4) with $A(q)$ of this form is said to be of *cofactor type*, and the corresponding system (1.2) is called a *cofactor system.* The Hamiltonian formulation considerably simplifies the study of integrability of cofactor systems. Thanks to the Liouville–Arnol'd theorem, the main problem is now to construct cofactor systems with a sufficient number of extra integrals of motion, in addition to the one which exists by assumption. The surprising answer is that it is enough to require *one* extra integral of motion $\widetilde{E}$, provided that it is of cofactor type too. Then there must in fact exist n integrals of motion, all quadratic in $\dot{q}$. The investigation of such *cofactor pair systems* occupies the major part of this work.

4 Some motivating examples

There are methods for constructing integrable Newton systems from stationary or restricted flows of soliton equations (see for example [7, 8, 9]). The interest in quasi-potential Newton equations arose when studying some examples found in this way.

Example 4.1. The fifth order Korteweg–de Vries (KdV) equation is the integrable PDE

$$u_t = \frac{1}{16}(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x.$$

A *stationary flow* is a time-independent solution $u(x)$, which clearly must satisfy the ODE $u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 = 16c$, where c is a constant of integration. This ODE is equivalent, under the substitution $q_1 = \frac{1}{4}u$, $q_2 = \frac{1}{16}u_{xx} + \frac{5}{32}u_x^2$, to the Newton system

$$\begin{aligned} \ddot{q}_1 &= -10q_1^2 + 4q_2, \\ \ddot{q}_2 &= 10q_1^3 - 20q_1q_2 + c. \end{aligned}$$

This Newton system is completely integrable, because it inherits two Hamiltonian formulations (one canonical and one non-canonical) from the KdV equation in a way which we will not go into here. Such a *bi-Hamiltonian formulation* is an effective formalism for producing Poisson commuting integrals of motion [10, 5]. There is a hierarchy of KdV equations, and the KdV equation of order $2n+1$ gives rise to an integrable Newton system in R^n, whose integrals of motion all depend quadratically on $\dot{q}$.

Example 4.2. Another soliton hierarchy is the Harry Dym hierarchy, whose second stationary flow

$$0 = \frac{1}{4}(u^{-1/2})_{xxx} - \kappa(u^{-1/2})_x,$$

where κ is a constant, can be integrated once and then written as the Newton system

$$\ddot{q}_1 = \kappa\, q_1 - q_2/q_1^5,$$
$$\ddot{q}_2 = 4\kappa\, q_2 - c,$$

by setting $u = q_1^{-4}$. One (non-canonical) Hamiltonian formulation was known for this system, but in contrast to all the other examples studied so far, no Lagrangian formulation could be found. (As is well-known from classical mechanics, a nonsingular Lagrangian system is equivalent to a canonical Hamiltonian system via the Legendre transformation.) In the search for a Lagrangian, Rauch–Wojciechowski accidentally discovered that the system is equivalent to the "quasi-Lagrangian" equations (note the change of sign compared to the usual Euler–Lagrange derivative)

$$\frac{\partial E}{\partial q_i} + \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_i} = 0, \quad i = 1, 2,$$

where

$$E = -q_2\dot{q}_1^2 + q_1\dot{q}_1\dot{q}_2 - \kappa q_1^2 q_2 + \frac{q_2^2}{2q_1^4} + \frac{c}{2}q_1^2 \tag{4.1}$$

is a quadratic integral of motion for the system.

Conservative Newton systems $\ddot{q} = -\nabla V$ with additional quadratic integrals of motion besides the energy have been studied for a long time, since this is closely connected to the question of *separability* of the potential V as we mentioned above. The basic results for the two-dimensional case, by Bertrand and Darboux, can be found in Whittaker's classical book [11, sec. 152].

The desire to understand example 4.2 better was the main motivation for studying Newton systems *without a potential* but still admitting a quadratic integral of motion. Such systems were initially referred to as "quasi-Lagrangian" systems, but this feature later turned out to be rather irrelevant. In particular, it has no counterpart when the theory is generalized to Riemannian manifolds. This is why we now prefer to speak about "quasi-potential" systems instead.

5 Overview of the papers

The first results, which were mainly restricted to the two-dimensional case, were published in a paper by Rauch–Wojciechowski, Marciniak, and the author [4].[4] There we showed (for $n = 2$) that any quasi-potential system admits a nonstandard Hamiltonian formulation, similar to the one known for example 4.2, and that any Newton system with two quadratic integrals of motion is completely integrable (bi-Hamiltonian, with two nonstandard Hamiltonian formulations). It is not obvious that any nontrivial such systems actually exist. The question is closely related to the existence of solutions of a certain second order linear PDE with polynomial coefficients, called the "fundamental equation." A recursion formula for constructing infinite families of such integrable systems (and solutions of the fundamental

[4]That paper is not included in this thesis for reasons of space, and since the results are superseded by Paper 1 and Paper 2.

equation) was given. As a special case (when one of the two quadratic integrals is just the usual energy) several known results concerning separable potentials were recovered. Finally, we showed that *driven* systems

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1), \\ \ddot{q}_2 &= M_2(q_1, q_2), \end{aligned}$$

admitting a quadratic integral of motion, can be integrated in a more concrete way by introducing a new type of separation coordinates. These coordinates are somewhat similar to the classical separation coordinates for separable potentials (elliptic coordinates and degenerations), but they are nonorthogonal and the coordinate curves are conics which are not confocal. The system (4.2) is an example of such a driven system (if we rename $q_1 \leftrightarrow q_2$).

In Paper 1 all of these results, except the ones about driven systems, were extended to the n-dimensional case. The main new insight was the following. For $n = 2$ the general solution of the cyclic conditions (1.3) can easily be found. In [4] it was written as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} a\,q_2^2 + b\,q_2 + \alpha & -a\,q_1q_2 - \frac{b}{2}q_1 - \frac{c}{2}q_2 + \frac{\beta}{2} \\ -a\,q_1q_2 - \frac{b}{2}q_1 - \frac{c}{2}q_2 + \frac{\beta}{2} & a\,q_1^2 + c\,q_1 + \gamma \end{pmatrix},$$

depending on the six parameters a, b, c, α, β, γ. However, to make the relevant structure emerge, it is better to rename the parameters as α, $2\beta_2$, $2\beta_1$, γ_{22}, $-2\gamma_{12}$, γ_{11}, so that the solution takes the form

$$\begin{aligned} A &= \begin{pmatrix} \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} & -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) \\ -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) & \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} \end{pmatrix} \\ &= \operatorname{cof} \begin{pmatrix} \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} & \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} \\ \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} & \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} \end{pmatrix}. \end{aligned}$$

From this it is seen that in two dimensions the general solution of the cyclic conditions has the form (3.1). For $n > 2$, it turns out that (3.1) still gives solutions to the cyclic conditions, although not the general solution. Consequently, the special role played by integrals of motion of cofactor type is not apparent until one considers the higher-dimensional case. Once this was realized, nearly all of the results from the case $n = 2$ could be generalized to $n > 2$, in many cases with a clearer structure and better notation.

The paper [4] and an earlier version of Paper 1 appeared in the author's "licentiate thesis" [12].

There remained the question of how to extend the results about driven systems to higher dimensions. This was not solved until recently, and the solution is presented in Paper 2. There it is shown that if a Newton system in R^{m+n} of the form

$$
\begin{aligned}
\ddot{y}_1 &= M_1(y_1, \dots, y_m),\\
&\vdots\\
\ddot{y}_m &= M_m(y_1, \dots, y_m),\\
\ddot{x}_1 &= -\frac{\partial V}{\partial x_1}(y_1, \dots, y_m; x_1, \dots, x_n),\\
&\vdots\\
\ddot{x}_n &= -\frac{\partial V}{\partial x_n}(y_1, \dots, y_m; x_1, \dots, x_n)
\end{aligned}
$$

admits a quadratic integral of motion of cofactor type, then it admits n additional integrals of motion, one of which involves only the variables y_i and is of cofactor type in these variables. Moreover, given any solution $y(t)$ of the "driving system" (i.e., the first m equations), the solution for $x(t)$ can be found by quadratures. The system $\ddot{x} = -\frac{\partial V}{\partial x}(y(t), x)$ can be seen as given by a time-dependent potential, and the method of solution as a natural extension of the classical Stäckel separability to the time-dependent case.

During the work on Paper 2 there appeared a preprint by Crampin and Sarlet (now published [13]), where the results from Paper 1 were given an invariant geometric formulation, valid not only in R^n but also on Riemannian manifolds. Their basic setup is briefly described in the beginning of Paper 3. We believe that the results of Paper 2 can also be generalized to this more general setting with minor modifications, but this has not been done yet.

Paper 3 concerns the question of constructing cofactor pair systems in the setting of Crampin and Sarlet, a topic not addressed in [13]. It is shown that the recursion formula holds virtually unchanged, and, what is more interesting, that it is only a special case of a "multiplication formula" which maps two given cofactor pair systems to a third one. The proofs are much simplified using the powerful formalism introduced by Crampin and Sarlet. As a special case, the multiplication formula contains the fact that the product of two holomorphic functions is again holomorphic. It is an interesting open question whether more of the classical function theory can be transferred to the setting of cofactor pair systems.

A question which is not resolved in these papers is whether all cofactor pair systems are separable in some sense. The answer to this question is yes, according to a recent preprint by Marciniak and Błaszak [14].

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Paper 1

Higher-dimensional integrable Newton systems with quadratic integrals of motion

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Abstract

Newton systems $\ddot{q} = M(q)$, $q \in R^n$, with integrals of motion quadratic in velocities, are considered. We show that if such a system admits two quadratic integrals of motion of so-called *cofactor type*, then it has in fact n quadratic integrals of motion and can be embedded into a $(2n+1)$-dimensional bi-Hamiltonian system, which under some non-degeneracy assumptions is completely integrable. The majority of these *cofactor pair* Newton systems are new, but they include also conservative systems with elliptic and parabolic separable potentials, as well as many integrable Newton systems previously derived from soliton equations. We explain the connection between cofactor pair systems and solutions of a certain system of second order linear PDEs (the *fundamental equations*), and use this to recursively construct infinite families of cofactor pair systems.

1 Introduction

Conservative Newton systems, i.e., systems of differential equations of the form

$$\ddot{q} = -\nabla V(q), \tag{1.1}$$

are of fundamental importance in classical mechanics. Here $q = (q_1, \ldots, q_n)^T$ are Cartesian coordinates on R^n, dots denote derivatives with respect to time t, and $\nabla = (\partial_1, \ldots, \partial_n)^T$ is the gradient operator. (We use ∂_i, or sometimes ∂_{q_i}, as an abbreviation for $\partial/\partial q_i$, and X^T denotes the transpose of a matrix X. Thus, we regard elements of R^n as column vectors. We will only consider systems on R^n, not on general manifolds.) A large mathematical machinery has been built up for integrating such systems. We will here quickly review some well-known facts. For a system of the form (1.1), the energy $E = \frac{1}{2}\dot{q}^T\dot{q} + V(q)$ is always an integral of motion. There are the standard Lagrangian and Hamiltonian formulations. The system is called *completely integrable* if it has n Poisson commuting integrals of motion, in which case the Liouville–Arnol'd theorem says (among other things) that it can, in principle, be integrated by quadrature. A powerful method for finding solutions analytically is the Hamilton–Jacobi method, which is applicable if the potential V is such that the Hamilton–Jacobi equation $\frac{1}{2}\sum_1^n(\partial_i F(q))^2 + V(q) = E$ can be solved by separation of variables in some suitable coordinate system. In that case the potential is said to be *separable*, and the n integrals of motion of the system will all depend quadratically on the momenta $p_i = \dot{q}_i$. It is known through the work of many people, beginning with classical results by

Stäckel, Levi-Civita and Eisenhart, that in R^n such separation can only occur in so-called generalized elliptic coordinates or some degeneration thereof. There exist criteria for determining if, and in that case in which system of coordinates, a given potential V is separable. For $n = 2$, the condition is that $V(q_1, q_2)$ must satisfy the Bertrand–Darboux equation [1, Sec. 152]

$$\begin{aligned} 0 =&(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12})(\partial_{22}V - \partial_{11}V) \\ &+ (\alpha(q_1^2 - q_2^2) + 2\beta_1 q_1 - 2\beta_2 q_2 + \gamma_{11} - \gamma_{22})\partial_{12}V \\ &- 3(\alpha q_2 + \beta_2)\partial_1 V + 3(\alpha q_1 + \beta_1)\partial_2 V \end{aligned} \tag{1.2}$$

for some constants α, β_1, β_2, γ_{12}, $\gamma_{11} - \gamma_{22}$, not all zero. Depending on the values of these parameters, the characteristic coordinates of the Bertrand–Darboux equation are either elliptic, polar, parabolic, or Cartesian coordinates, and this determines the coordinate system in which the Hamilton–Jacobi equation separates. The extra integral of motion is $F = (\alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22})\dot{q}_1^2 - 2(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12})\dot{q}_1\dot{q}_2 + (\alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11})\dot{q}_2^2 + k(q_1, q_2)$ for some function k. Similar results are known for $n > 2$. These will be described in section 7.

For general (nonconservative) Newton systems

$$\ddot{q} = M(q) \tag{1.3}$$

less is known. (In this article we use the term *Newton system* only for systems in which the right-hand side $M(q)$ does not depend on the velocity $\dot{q}$ or on time t.) In [2] we studied the class of systems of the form (1.3) which possess an "energy-like" integral of motion E which is quadratic in $\dot{q}_1, \ldots, \dot{q}_n$. The theory originated from the following example.

Example 1.1 (Harry Dym stationary flow). The system

$$\begin{aligned} \ddot{q}_1 &= \kappa q_1 - q_2/q_1^5, \\ \ddot{q}_2 &= 4\kappa q_2 - d, \end{aligned} \tag{1.4}$$

is equivalent, under the substitution $u = q_1^{-4}$, to the second stationary flow of the Harry Dym soliton hierarchy, and therefore it was suspected to be integrable in some sense. In addition to the integral of motion $F = \frac{1}{2}\dot{q}_2^2 - 2\kappa q_2^2 + dq_2$, which comes from the second equation alone, this system has another quadratic integral of motion

$$\begin{aligned} E =& -q_2\dot{q}_1^2 + q_1\dot{q}_1\dot{q}_2 - \kappa q_1^2 q_2 + \frac{q_2^2}{2q_1^4} + \frac{d}{2}q_1^2 \\ =& \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} -q_2 & q_1/2 \\ q_1/2 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} - \kappa q_1^2 q_2 + \frac{q_2^2}{2q_1^4} + \frac{d}{2}q_1^2 \\ =& \dot{q}^T A(q)\,\dot{q} + k(q). \end{aligned} \tag{1.5}$$

No Lagrangian formulation could be found for the system (1.4). However, it was discovered that it could be generated from E in a "quasi-Lagrangian" way by changing the minus sign in the Euler–Lagrange derivate δ to plus. Indeed, defining the quasi-Lagrangian operator $\delta^+ = (\delta_1^+, \ldots, \delta_n^+)^T$ by

$$\delta_i^+ E = \frac{\partial E}{\partial q_i} + \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_i}, \tag{1.6}$$

one finds immediately that the equation $0 = \delta^+ E$ yields

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_1^+ E \\ \delta_2^+ E \end{pmatrix} = 2 \begin{pmatrix} -q_2 & q_1/2 \\ q_1/2 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 - (\kappa q_1 - q_2/q_1^5) \\ \ddot{q}_2 - (4\kappa q_2 - d) \end{pmatrix},$$

which is clearly equivalent to (1.4). This proved to be a general feature of Newton systems with quadratic integrals of motion, so such systems were given the name *quasi-Lagrangian Newton systems*, or *QLN systems*. Expressed in terms of the matrix $A(q)$ and the function $k(q)$ in (1.5), the system (1.4) can be written

$$\ddot{q} = -\frac{1}{2} A(q)^{-1} \nabla k(q),$$

a result that also holds in general (see theorem 2.1 below). Clearly, this contains the conservative case (1.1) as the special case $A = I$ (identity matrix) and $k = 2V$.

The following nonstandard Hamiltonian formulation was found for the system (1.4):

$$\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -q_1/2 & p_1 \\ & 0 & -q_1/2 & -q_2 & p_2 \\ & & 0 & p_1/2 & \kappa q_1 - q_2/q_1^5 \\ & * & & 0 & 4\kappa q_2 - d \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \Pi \overline{\nabla} d, \tag{1.7}$$

where the star denotes entries determined by antisymmetry of the matrix Π, and $\overline{\nabla} = (\partial_{q_1}, \partial_{q_2}, \partial_{p_1}, \partial_{p_2}, \partial_d)^T$ is the gradient operator on the extended phase space $\mathcal{M} = R^5$. The last column in the matrix Π equals the Hamiltonian vector field determined by the function $H(q,p,d) = d$, while the other entries are chosen so that $\{f, g\} = (\overline{\nabla} f)^T \Pi \overline{\nabla} g$ defines a Poisson bracket (in particular, so that the Jacobi identity is satisfied). The quadratic integral of motion E is a Casimir of Π, i.e., $\Pi \overline{\nabla} E = 0$.

The results for the system in example 1.1 gave rise to a general theory of two-dimensional QLN systems, developed in [2]. It was shown that they all admit a nonstandard Hamiltonian formulation similar to (1.7). In general, unlike in example 1.1, the parameter d which is used as an extra phase space variable is not present from the start, but has to be introduced by adding terms linear in d to the right-hand side of the original Newton system, which can be recovered as the restriction of the Hamiltonian system to the hyperplane $d = 0$.

Of special interest are the integrability properties of two-dimensional QLN systems with *two* functionally independent quadratic integrals of motion, say $E = \dot{q}^T A(q)\, \dot{q} + k(q)$ and $F = \dot{q}^T B(q)\, \dot{q} + l(q)$. It was shown that such a system can be embedded into a completely integrable bi-Hamiltonian system in extended phase space, in the sense that the trajectories of the extended system on the hyperplane $d = 0$ coincide with the trajectories of the original QLN system; however, they are in general traversed at a different speed. The reason for this extra complication is that some care has to be taken in order to ensure that both integrals of motion of the QLN system really give rise to corresponding integrals of motion of the extended system. The Poisson structures for the bi-Hamiltonian system are in general both non-canonical.

Both E and F can be used for generating the Newton system, which leads to the equality $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k = -\frac{1}{2} B^{-1} \nabla l$. From $\nabla k = A B^{-1} \nabla l$ and the equality of mixed second order derivatives of k, one finds that l satisfies a certain second order linear PDE, whose coefficients depend on the entries of the matrices $A(q)$ and $B(q)$ (these entries are known to be quadratic polynomials of a certain form; see (3.1) below). Similarly one finds from $\nabla l = BA^{-1}\nabla k$ that k satisfies another PDE. A remarkable discovery made in [2] was that if one substitutes $k = K_1 \det A$ and $l = K_2 \det B$ in these equations, then K_1 and K_2 both satisfy *the same* second order linear PDE, which was named the *fundamental equation* associated with the matrices A and B. The coefficients in this equation are cubic polynomials in q, depending on the entries of A and B. It was shown that there is a one-to-one correspondence between fundamental equations and linear spans $\lambda A + \mu B$, which makes it possible to classify the types of systems that occur according to the polynomial degree of the matrices A and B. For example, when $B = I$ the fundamental equation reduces to the Bertrand–Darboux equation (1.2), which shows that this new class of system includes, but also significantly extends, the class of two-dimensional conservative systems with separable potentials. The fundamental equation was also used for constructing infinite families of integrable two-dimensional QLN system.

The aim of the present paper is to investigate what can be said in the n-dimensional case. In particular, we are interested in finding nonstandard Hamiltonian and bi-Hamiltonian formulations, similar to the ones in [2], which will allow us to show the integrability of (in general nonconservative) n-dimensional Newton systems with sufficiently many quadratic integrals of motions. The benefit of a Hamiltonian formulation is that only n integrals are needed, instead of $2n - 1$ as in the general case. The rather unexpected result of our investigations is that even in n dimensions the existence of just *two* quadratic integrals of motion implies integrability, provided these integrals are of what we call *cofactor type*. Any Newton system with two such integrals of motion must in fact have n quadratic integrals of motion of a certain structure. Such systems are the principal objects of study in this paper, and we call them *cofactor pair systems*. We give a simple method of testing if a given Newton system is a cofactor pair system, and show how any cofactor pair system can be embedded in a bi-Hamiltonian system in $(2n + 1)$-dimensional phase space. This bi-Hamiltonian system, whose Poisson structures are in general both non-canonical, is completely integrable under some mild non-degeneracy conditions, which explains in what sense cofactor pair systems can be considered integrable. We also find the analogue of the fundamental equation, which in this case is a system of $\binom{n}{2}$ second order linear PDEs, whose coefficients are cubic polynomials in q, and use this to recursively construct infinite families of cofactor pair systems.

This theory is richly illustrated by examples, and connects many different results obtained by other methods. In particular, we explain how n-dimensional separable potentials fit into this framework.

2 Quasi-Lagrangian Newton systems in n dimensions

In this section we review the basic facts about Newton systems with one or more quadratic integrals of motion. A characteristic feature of such a system is that it can easily be reconstructed from any of its quadratic integrals of motion $E =$

$\dot{q}^T A(q)\,\dot{q} + k(q)$, either via the quasi-Lagrangian equations $\delta^+ E = 0$ or directly as $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k$.

Theorem 2.1 (Newton systems with quadratic integrals). *Let*

$$E(q,\dot{q}) = \dot{q}^T A(q)\,\dot{q} + k(q) = \sum_{i,j=1}^{n} A_{ij}(q)\,\dot{q}_i\dot{q}_j + k(q), \tag{2.1}$$

where $A^T = A$. Then E is an integral of motion for the Newton system $\ddot{q} = M(q)$ if and only if

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0, \quad \textit{for all } i,\ j,\ k\ = 1,\dots,n, \tag{2.2}$$

and

$$2A(q)M(q) + \nabla k(q) = 0. \tag{2.3}$$

So if $\det A(q) \neq 0$*, then the system can be reconstructed from its integral of motion E as*

$$\ddot{q} = M(q) = -\frac{1}{2}A(q)^{-1}\nabla k(q), \tag{2.4}$$

which is equivalent to the system of quasi-Lagrangian equations $\delta^+ E = 0$ defined by (1.6)*.*

Proof. E is an integral of motion if and only if

$$\dot{E} = \sum_i (2(A\ddot{q})_i + \partial_i k)\dot{q}_i + \frac{1}{3}\sum_{i,j,k}(\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij})\dot{q}_i\dot{q}_j\dot{q}_k$$

vanishes identically, which proves the first statement. Moreover,

$$\delta_i^+ E = 2(A\ddot{q})_i + \partial_i k + \sum_{j,k}(\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij})\dot{q}_j\dot{q}_k,$$

so that $\delta^+ E = 2AM + \nabla k$ if (2.2) holds. □

Definition 2.2 (QLN system). A Newton system of the form (2.4) in theorem 2.1, or, in other words, a Newton system with a quadratic integral of motion $E = \dot{q}^T A(q)\,\dot{q} + k(q)$ with $\det A \neq 0$, will be called a *quasi-Lagrangian Newton system*, or *QLN system*.

Definition 2.3 (Cyclic conditions). The system (2.2) of linear first order PDEs $\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0$ will be referred to as the *cyclic conditions* for the matrix $A(q)$.

Remark 2.4 (Killing tensor). The cyclic conditions, with covariant instead of partial derivatives, are the equation for a second order Killing tensor on a Riemannian manifold (i.e., a tensor A_{ij} such that $A_{ij}\dot{q}^i\dot{q}^j$ is an integral of motion of the geodesic equations). Consequently, in our case we could speak of Killing tensors on R^n with the Euclidean metric. However, most of the time we will simply refer to "matrices satisfying the cyclic conditions."

Remark 2.5. That a Newton system $\ddot{q} = M(q)$ can be reconstructed from one of its integrals of motion was known already to Bertrand, whose method is not restricted to quadratic integrals [1, Sec. 151]. The quasi-Lagrangian formulation, however, was noticed only recently—it was first published in [3], and further theory was developed in [2]. It is at present unclear whether it has any geometric or similar significance, or if it is just an algebraic property. For example, unlike the ordinary Lagrange equations which admit arbitrary point transformations, the quasi-Lagrangian equations are only invariant under affine changes of variables. In any case, "QLN system" is a convenient designation for "velocity-independent Newton system, in general not conservative, with a nondegenerate quadratic integral of motion," and the notation $\delta^+ E = 0$ is also useful.

Remark 2.6. From theorem 2.1 it follows that if a Newton system $\ddot{q} = M(q)$ has two (or more) quadratic integrals of motion, say $E = \dot{q}^T A(q)\,\dot{q} + k(q)$ and $F = \dot{q}^T B(q)\,\dot{q} + l(q)$, then any of them can be used to reconstruct the system as long as the matrix is nonsingular. Thus,

$$M = -\frac{1}{2} A^{-1} \nabla k = -\frac{1}{2} B^{-1} \nabla l. \tag{2.5}$$

If B is singular but not A, then $B + \lambda A$ is nonsingular for some $\lambda \in R$, so we can replace F with $F + \lambda E$ to give

$$M = -\frac{1}{2} A^{-1} \nabla k = -\frac{1}{2} \big(B + \lambda A\big)^{-1} \nabla \big(l + \lambda k\big).$$

3 Matrices satisfying the cyclic conditions

A QLN system $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k$ is completely determined by the matrix $A(q)$, which has to satisfy the cyclic conditions (2.2), and the arbitrary function $k(q)$, which plays the role of a "potential." In order to understand the class of QLN systems, it is essential to determine what a matrix A satisfying the cyclic conditions looks like. The general solution of the cyclic conditions is known (see for instance [4], which also gives results about Killing tensors on general manifolds). Since we will not use these results in full, we merely outline the general structure in proposition 3.1 below, which shows that the entries of A must be quadratic polynomials in q. The main purpose of this section is to introduce a special class of solutions that will be central in what follows: cofactor matrices of *elliptic coordinates matrices.*

Proposition 3.1. *If the symmetric matrix $A(q)$ satisfies the cyclic conditions* (2.2)*, then*

1. *For all i, j, k and l,*

$$\partial_{ij} A_{kl} = \partial_{kl} A_{ij} = const.$$

   In particular, each matrix entry $A_{ij}(q)$ is a polynomial of degree at most two.

2. *A_{ii} is independent of q_i for all i.*

3. *For $i \neq j$, A_{ij} contains no q_i^2 or q_j^2 terms.*

Proof. Taking $i = j = k$ the cyclic conditions read $3\partial_i A_{ii} = 0$, so A_{ii} does not depend on q_i. For $k = i \neq j$ we have $\partial_j A_{ii} + 2\,\partial_i A_{ij} = 0$, which shows that $\partial_i A_{ij}$ is independent of q_i. Thus, A_{ij} is linear in q_i and, by symmetry, in q_j. Finally, the stated relationship between the second derivatives follows from

$$\begin{aligned} 2(\partial_{kl} A_{ij} - \partial_{ij} A_{kl}) = &\, \partial_l(\partial_k A_{ij} + \partial_i A_{jk} + \partial_j A_{ki}) \\ &+ \partial_k(\partial_l A_{ij} + \partial_i A_{jl} + \partial_j A_{li}) \\ &- \partial_i(\partial_j A_{kl} + \partial_k A_{lj} + \partial_l A_{jk}) \\ &- \partial_j(\partial_i A_{kl} + \partial_k A_{li} + \partial_l A_{ik}) \\ = &\, 0, \end{aligned}$$

and they are constant since

$$\begin{aligned} 3\partial_{klm} A_{ij} &= \partial_{kl}\partial_m A_{ij} + \partial_{lm}\partial_k A_{ij} + \partial_{km}\partial_l A_{ij} \\ &= \partial_{kl}(-\partial_i A_{mj} - \partial_j A_{im}) + \partial_{lm}(-\partial_i A_{kj} - \partial_j A_{ik}) \\ &\quad + \partial_{km}(-\partial_i A_{lj} - \partial_j A_{il}) \\ &= -\frac{1}{2}\Big[\partial_{ik}(\partial_l A_{mj} + \partial_m A_{lj}) + \partial_{il}(\partial_k A_{mj} + \partial_m A_{kj}) \\ &\quad + \partial_{kj}(\partial_l A_{im} + \partial_m A_{il}) + \partial_{lj}(\partial_k A_{im} + \partial_m A_{ik}) \\ &\quad + \partial_{mi}(\partial_l A_{kj} + \partial_k A_{lj}) + \partial_{mj}(\partial_l A_{ik} + \partial_k A_{il})\Big] \\ &= \frac{1}{2}\Big[\partial_{ik}\partial_j A_{lm} + \partial_{il}\partial_j A_{km} + \partial_{kj}\partial_i A_{ml} \\ &\quad + \partial_{lj}\partial_i A_{km} + \partial_{mi}\partial_j A_{lk} + \partial_{mj}\partial_i A_{lk}\Big] \\ &= \partial_{ij}(\partial_k A_{lm} + \partial_l A_{mk} + \partial_m A_{kl}) \\ &= 0. \end{aligned}$$

□

With the help of these facts it is possible to find the general solution of (2.2) for any given n. For $n = 2$, it is

$$\begin{aligned} A &= \begin{pmatrix} \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} & -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) \\ -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) & \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} \end{pmatrix} \\ &= \operatorname{cof}\begin{pmatrix} \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} & \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} \\ \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} & \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} \end{pmatrix}, \end{aligned} \tag{3.1}$$

which depends on the six parameters α, β_1, β_2, γ_{11}, γ_{12} and γ_{22}. The choice of notation will be made clear below; see in particular remark 3.9. For $n = 3$ the general solution already involves 20 parameters, and in general the number of parameters is $n(n+1)^2(n+2)/12$ [4].

Now we will focus on some special types of solutions of the cyclic conditions. First, we note that there is the following simple method of producing new solutions from a given one.

Proposition 3.2 (Change of variables). *If $A(q)$ satisfies the cyclic conditions (2.2), then so does $S^T A(Sq + v)S$, for any constant matrix $S \in R^{n\times n}$ and vector $v \in R^n$.*

Proof. This is easily verified directly using the chain rule. Alternatively, one can first verify that the quasi-Lagrangian equations are invariant under affine changes of variables $q = Sr + v$, which means that the Newton system $\delta^+ E(q, \dot{q}) = 0$ expressed in the new variables r is the Newton system generated by the integral of motion $E = \dot{q}^T A(q)\,\dot{q} + k(q)$ when expressed in r and $\dot{r}$:

$$E(r, \dot{r}) = (S\dot{r})^T A(Sr + v)\,(S\dot{r}) + k(Sr + v).$$

Thus, by theorem 2.1, $S^T A(Sr+v)S$ must satisfy the cyclic conditions (expressed in the r variables). □

There is a class of matrices satisfying the cyclic conditions, that will be very important in what follows: cofactor matrices of *elliptic coordinates matrices.* Let us remind the reader that the cofactor (or adjoint) matrix $\operatorname{cof} X$ of a quadratic matrix X is the matrix whose (i,j) entry is the cofactor of X_{ji} in $\det X$, so that $X \operatorname{cof} X = (\det X) I$. Elliptic coordinates matrices and their cofactor matrices appear in a natural way when trying to find a Hamiltonian formulation for QLN systems, as will be seen in theorem 4.1 in the next section.

Definition 3.3 (Elliptic coordinates matrix). A symmetric $n \times n$-matrix $G(q)$ whose entries are quadratic polynomials in q of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij} \tag{3.2}$$

will be called an *elliptic coordinates matrix.* Using matrix multiplication, $G(q)$ can be written

$$G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \text{where } \alpha \in R,\ \beta \in R^n,\ \gamma = \gamma^T \in R^{n \times n}.$$

(Let us emphasize, for clarity, that we consider elements in R^n as column vectors. Thus, qq^T is an $n \times n$-matrix, not to be confused with the scalar $q^T q = \sum q_i^2$.)

The reason for the terminology is that the eigenvalues $u_1(q), \ldots, u_n(q)$ of an elliptic coordinates matrix (under some assumptions) determine a change of variables from Cartesian coordinates q to generalized elliptic coordinates u, which will be of interest when discussing separable potentials (see section 7). For the moment, we are only interested in the following remarkable property of such matrices:

Theorem 3.4 (Cofactor matrix). *If $G(q)$ is an elliptic coordinates matrix, then $A(q) = \operatorname{cof} G(q)$ satisfies the cyclic conditions* (2.2).

Proof. To begin with, we note that A is symmetric, since G is symmetric. Now let $N(q) = \alpha q + \beta$. Differentiating G we find, using the Kronecker delta notation,

$$\partial_k G_{ij} = \alpha(\delta_{ki} q_j + q_i \delta_{kj}) + \beta_i \delta_{kj} + \beta_j \delta_{ki} = \delta_{ki} N_j + \delta_{kj} N_i, \tag{3.3}$$

or, in matrix notation,

$$\partial_k G = (N e_k^T + e_k N^T), \tag{3.4}$$

where $e_k = (0, \ldots, 1, \ldots, 0)^T$ is the k'th standard basis vector of R^n.

Next, we show show that

$$\nabla \det G = 2AN, \tag{3.5}$$

a formula that also will be useful elsewhere in this article. For ease of notation, let us show the case $n = 3$:

$$
\begin{aligned}
\partial_1 \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ G_{12} & G_{22} & G_{23} \\ G_{13} & G_{23} & G_{33} \end{vmatrix} = \begin{vmatrix} 2N_1 & G_{12} & G_{13} \\ N_2 & G_{22} & G_{23} \\ N_3 & G_{23} & G_{33} \end{vmatrix} + \begin{vmatrix} G_{11} & N_2 & G_{13} \\ G_{12} & 0 & G_{23} \\ G_{13} & 0 & G_{33} \end{vmatrix} + \begin{vmatrix} G_{11} & G_{12} & N_3 \\ G_{12} & G_{22} & 0 \\ G_{13} & G_{23} & 0 \end{vmatrix} \\
=&(2N_1A_{11} + N_2A_{12} + N_3A_{13}) + N_2A_{12} + N_3A_{13} \\
=&2[AN]_1,
\end{aligned}
$$

and similarly for the other ∂_k. The notation $[AN]_1$ means, of course, the first entry in the column vector AN. It is obvious that a similar calculation can be made for any n, which proves (3.5).

Thus, differentiating the identity $AG = (\det G)I$, we obtain

$$(\partial_k A)G + A(Ne_k^T + e_kN^T) = 2[AN]_kI.$$

After multiplying this by A from the right, we extract from the (i, j) entry

$$(\det G)\partial_k A_{ij} = 2[AN]_kA_{ij} - [AN]_iA_{kj} - [AN]_jA_{ik}. \tag{3.6}$$

Summing cyclically we obtain

$$(\det G)(\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) = 0.$$

The left-hand side of this equation is a polynomial in q, whose coefficients are polynomials in the parameters α, β_r and γ_{rs}, while the right-hand side vanishes identically. Since $\det G$ is not identically zero as a function of these parameters, as can be seen by taking $\alpha = 0$, $\beta = 0$ and γ nonsingular, we conclude that the sum in parentheses must vanish identically. In other words, A satisfies the cyclic conditions for any values of α, β and γ (even such values that make $\det G = 0$). □

Remark 3.5. This theorem implies, by proposition 3.1, that the cofactors of $G(q)$ are polynomials in q of degree at most two. This is a rather surprising fact, since one could expect them to have degree 2^{n-1}, being determinants of $(n-1)\times(n-1)$-matrices of quadratic polynomials. What happens is that all the terms of degree higher than two cancel due to the special structure of G. Similarly, since $\det G$ is the cofactor of the lower right entry in an elliptic coordinates matrix of size $(n + 1) \times (n + 1)$, it must also be a quadratic polynomial. However, checking this by direct calculation is already for $n = 3$ a quite formidable task!

We can use theorem 3.4 to produce a "cofactor chain" of matrices satisfying the cyclic conditions. Such chains will be very useful later on.

Proposition 3.6. *Let $G(q) = \alpha qq^T + q\beta^T + \beta q^T + \gamma$ and $\widetilde{G}(q) = \widetilde{\alpha}qq^T + q\widetilde{\beta}^T + \widetilde{\beta}q^T + \widetilde{\gamma}$ be elliptic coordinates matrices. Then the matrices $A^{(0)}, \ldots, A^{(n-1)}$ defined by*

$$A_\mu = \operatorname{cof}(G + \mu\widetilde{G}) = \sum_{i=0}^{n-1} A^{(i)}\mu^i \tag{3.7}$$

all satisfy the cyclic conditions (2.2).

Proof. $G+\mu\widetilde{G}$ is an elliptic coordinates matrix, with $\alpha+\mu\widetilde{\alpha}$ instead of α and so on. By theorem 3.4, A_μ satisfies the cyclic conditions for all μ. These being linear equations, it follows that the coefficients at different powers of μ in A_μ each must satisfy the cyclic conditions. □

Remark 3.7. Note that $A^{(0)} = \operatorname{cof} G$ and $A^{(n-1)} = \operatorname{cof}\widetilde{G}$, but that the interjacent matrices $A^{(1)},\dots,A^{(n-2)}$ in general are impossible to write as cofactor matrices of elliptic coordinates matrices.

Obviously, we can obtain even larger variation if we form linear combinations of more than two elliptic coordinates matrices. For example, in $\operatorname{cof}(G+\mu G'+\lambda G'')$ the coefficient at each different power $\mu^i\lambda^j$ will satisfy the cyclic conditions. However, we have not found any particular use for this. Combinations of two matrices, on the other hand, are absolutely fundamental for the construction of integrable Newton systems in section 5, as the following example indicates.

Example 3.8 (KdV stationary flow). Define elliptic coordinates matrices G and $\widetilde{G}$ by

$$\alpha=0,\ \beta=\begin{pmatrix}0\\0\\1\end{pmatrix},\gamma=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix};\quad \widetilde{\alpha}=0,\ \widetilde{\beta}=\begin{pmatrix}0\\0\\0\end{pmatrix},\widetilde{\gamma}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Then

$$\begin{aligned}\operatorname{cof}(G+\mu\widetilde{G}) &= \operatorname{cof}\left[\begin{pmatrix}0&-1&q_1\\-1&0&q_2\\q_1&q_2&2q_3\end{pmatrix}+\mu\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\right]\\ &= \begin{pmatrix}-q_2^2&q_1q_2+2q_3&-q_2\\q_1q_2+2q_3&-q_1^2&-q_1\\-q_2&-q_1&-1\end{pmatrix}\\ &\quad+\mu\begin{pmatrix}2q_3&q_2&-q_1\\q_2&-2q_1&-1\\-q_1&-1&0\end{pmatrix}+\mu^2\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\\ &= A^{(0)}+\mu A^{(1)}+\mu^2A^{(2)}.\end{aligned}$$

According to proposition 3.6, the matrices $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ so defined all satisfy the cyclic conditions, and this is also easily verified directly. They occur in the integrals of motion of the Newton system

$$\begin{aligned}\ddot{q}_1 &= -10q_1^2+4q_2,\\ \ddot{q}_2 &= -16q_1q_2+10q_1^3+4q_3,\\ \ddot{q}_3 &= -20q_1q_3-8q_2^2+30q_1^2q_2-15q_1^4+d,\end{aligned}\tag{3.8}$$

which, under the substitution $q_1=u/4$, is equivalent to the integrated form

$$\frac{1}{64}(u_{6x}+14uu_{4x}+28u_xu_{xxx}+21u_{xx}^2+70uu_x^2+70u^2u_{xx}+35u^4)=d$$

of the seventh order stationary KdV flow [5]. Indeed, this system has three quadratic integrals of motion of the form $E^{(i)} = \dot{q}^T A^{(i)}\, \dot{q} + k^{(i)} - d\, D^{(i)}$, $i = 0, 1, 2$, where

$$
\begin{aligned}
k^{(0)} =& \, 24q_1^3q_2^2 - 8q_1q_2^3 - 10q_1^5q_2 - 16q_1q_3^2 - 10q_1^4q_3 \\
& + 8q_1^2q_2q_3 - 8q_2^2q_3, \\
k^{(1)} =& \, 8q_1^2q_2^2 + 10q_1^4q_2 - 5q_1^6 - 8q_2^3 + 4q_3^2 - 24q_1q_2q_3, \\
k^{(2)} =& - 20q_1^2q_3 + 8q_2q_3 - 16q_1q_2^2 + 20q_1^3q_2 - 6q_1^5,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
D^{(0)} &= - 2(q_1q_2 + q_3), \\
D^{(1)} &= - (q_1^2 + 2q_2), \\
D^{(2)} &= - 2q_1.
\end{aligned}
$$

The system can be reconstructed from any one of these integrals as

$$
\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla(k^{(i)} - d\, D^{(i)}).
$$

It was shown in [5] to be bi-Hamiltonian and completely integrable. The parameter d was used as an extra phase space variable in the bi-Hamiltonian formulation. Notice that the $D^{(i)}$ occur as coefficients in

$$
\det(G + \mu\widetilde{G}) = -2(q_1q_2 + q_3) - (q_1^2 + 2q_2)\mu - 2q_1\mu^2 - \mu^3.
$$

All this fits nicely into the general scheme to be developed in section 5, where we construct a large class of bi-Hamiltonian Newton systems containing this one as a special case. (In this particular example, the matrix $\widetilde{G}$ happens to be independent of q. This will not be the case in general.)

Remark 3.9. For $n = 2$ every matrix satisfying the cyclic conditions is the cofactor matrix of an elliptic coordinates matrix, as equation (3.1) shows. For $n > 2$ this is not the case, as we have already noticed in remark 3.7. As another example, a matrix with the block structure

$$
A = \begin{pmatrix} \operatorname{cof} G_1(q_1, \ldots, q_r) & 0 \\ 0 & \operatorname{cof} G_2(q_{r+1}, \ldots, q_n) \end{pmatrix},
$$

where G_1 and G_2 are elliptic coordinate matrices of smaller dimensions, satisfies the cyclic conditions but cannot in general be written as the cofactor matrix of a single elliptic coordinates matrix. Applying proposition 3.2 we can obtain matrices for which the same is true, but without the blocks of zeros immediately revealing them as "decomposable."

An interesting open problem is how to detect whether the reverse process is possible, i.e., if a given solution A of the cyclic conditions can be transformed, by changing variables according to proposition 3.2, into such a decomposable form with "cofactor blocks" along the diagonal and zeros elsewhere. If in that case $k(q) = k_1(q_1, \ldots, q_r) + k_2(q_{r+1}, \ldots, q_n)$ in the new variables, then the QLN system $0 = \delta^+(\dot{q}^T A(q)\, \dot{q} + k(q))$ splits into the direct sum of two smaller QLN systems, one for $q_1, \ldots, q_r$ and one for $q_{r+1}, \ldots, q_n$, to which the theory that we will develop for "cofactor systems" can be applied separately.

4 Hamiltonian formulation and cofactor systems

Now we turn to the question of integrability of QLN systems. The notion of complete integrability concerns *Hamiltonian systems*. If one has a Hamiltonian formulation for some system under study, then the task of showing the system's integrability is just a matter of finding sufficiently many Poisson commuting integrals of motion. In this section, we present a (nonstandard) Hamiltonian formulation for a certain class of QLN systems, the *cofactor systems*.

Recall that a *Poisson manifold* is a manifold endowed with *Poisson bracket*, i.e., a bilinear antisymmetric mapping $\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfies the Leibniz rule and the Jacobi identity. In coordinates $(x_1, \ldots, x_n)$ the bracket takes the form

$$\{f, g\}(x) = (\overline{\nabla} f(x))^T \Pi(x) \overline{\nabla} g(x)$$

for some antisymmetric *Poisson matrix* $\Pi(x)$, where $\overline{\nabla} = (\partial_{x_1}, \ldots, \partial_{x_n})^T$. A *Hamiltonian system* on $\mathcal{M}$ is a dynamical system of the form $\dot{x}_i = \{x_i, H\}$, or $\dot{x} = \Pi\overline{\nabla} H$, for some function $H(x)$.

Conservative Newton systems $\ddot{q} = -\nabla V(q)$ on R^n admit the standard Hamiltonian formulation

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{\nabla} H, \quad \text{where } H(q,p) = \frac{1}{2} p^T p + V(q),$$

on the phase space $\mathcal{M} = R^{2n}$ with coordinates (q, p). In this section, the manifold $\mathcal{M}$ will be R^{2n+1} with coordinates (q, p, d) and we will investigate the possibility of finding a nonstandard Hamiltonian formulation for some nonconservative Newton systems $\ddot{q} = M(q)$. The idea is that several known nonstandard Hamiltonian formulations of integrable Newton systems derived from soliton equations [5, 6] or bi-Hamiltonian formulations for systems with separable potentials [3, 7], involve Poisson matrices on R^{2n+1} with a certain block structure (see (4.1) below). We investigate what the most general form of a Poisson matrix with this structure is. The answer leads us to define *cofactor systems*, which are just the systems which admit this type of Hamiltonian formulation. The previously known systems are special instances of this class. Our results generalize the ones found using similar methods in [2, 3] for the cases $n = 2$ or $M = -\nabla V$.

Theorem 4.1 (Poisson matrix). *Let $\mathcal{M}$ denote the space R^{n+n+1} with coordinates (q, p, d). Let Π be an antisymmetric $(n+n+1) \times (n+n+1)$-matrix with the block structure*

$$\Pi = \begin{pmatrix} 0 & \frac{\lambda}{2} G(q) & p \\ * & \frac{\lambda}{2} F(q,p) & \widehat{M}(q,d) \\ * & * & 0 \end{pmatrix}, \tag{4.1}$$

where F and G are $n \times n$-matrices, p and $\widehat{M}$ column n-vectors, λ a nonzero real parameter (introduced for later convenience) and stars denote elements determined by antisymmetry. Then Π is a Poisson matrix if and only if:

1. *G is an elliptic coordinates matrix, i.e.,*

$$G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma \tag{4.2}$$

for some $\alpha \in R$, $\beta \in R^n$ *and* $\gamma = \gamma^T \in R^{n\times n}$.

2. F *is given by*

$$F(q,p) = Np^T - pN^T, \quad \text{where} \quad N(q) = \alpha q + \beta. \tag{4.3}$$

3. $\widehat{M}$ *has the structure*

$$\widehat{M}(q,d) = M(q) + \lambda d\, N(q), \tag{4.4}$$

where $M(q)$ *satisfies the equations*

$$0 = P_{ij} - P_{ji}, \quad \text{where } P_{ij} = 3N_i M_j + \sum_{k=1}^{n} G_{ki}\partial_k M_j, \tag{4.5}$$

for all $i, j = 1, \ldots, n$. *If* $\det G(q) \neq 0$, *this is equivalent to*

$$M = -\frac{1}{2}A^{-1}\nabla k, \quad \text{for some function } k(q), \tag{4.6}$$

where $A(q) = \operatorname{cof} G(q)$. *In other words,* $\ddot{q} = M(q)$ *is the QLN system generated by* $E = \dot{q}^T A(q)\,\dot{q} + k(q) = \dot{q}^T \operatorname{cof} G(q)\,\dot{q} + k(q)$.

Moreover, then the function

$$\widehat{E}(q,p,d) = p^T \operatorname{cof} G(q)\, p + k(q) - \lambda d \det G(q) \tag{4.7}$$

is a Casimir of Π, *i.e.,*

$$\Pi\,\overline{\nabla} E = 0,$$

where $\overline{\nabla} = (\partial_{q_1}, \ldots, \partial_{q_n}, \partial_{p_1}, \ldots, \partial_{p_n}, \partial_d)^T$.

Proof. First of all, F must be antisymmetric in order for Π to be so. Then we must determine what form F, G and $\widehat{M}$ must take in order for the Jacobi identity to be satisfied for all combinations of the coordinates q, p, d. Let us use the abbreviation $J(f,g,h) = \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}$. We find that $J(q_i,q_j,q_k)$ and $J(q_i,q_j,p_k)$ are both identically zero for all i, j, k, while $J(q_i,q_j,d) = \frac{\lambda}{2}(G_{ij} - G_{ji})$, which implies that the matrix $G(q)$ must be symmetric. Further, $J(q_i,p_j,d) = \frac{\lambda}{2}(p^T\nabla G_{ij} - F_{ij}) - p_i\partial_d\widehat{M}_j$, which shows that $\partial_d\widehat{M}(q,d)$ is independent of d, and thus $\widehat{M}(q,d) = M(q) + \lambda d N(q)$ for some $M(q)$ and $N(q)$. ($\nabla = (\partial_{q_1}, \ldots, \partial_{q_n})^T$, as usual, and λ is introduced here for convenience.) With this expression for $\widehat{M}$ we obtain $0 = p^T\nabla G_{ij} - F_{ij} - 2p_i N_j$. Adding and subtracting this expression and the corresponding one with i and j interchanged, and using $G = G^T$, $F = -F^T$, we obtain

$$F_{ij} = N_i p_j - N_j p_i,$$

which is (4.3) except that we do not know the form of N yet, and

$$p^T\nabla G_{ij} = N_i p_j + N_j p_i.$$

Let us now go back to writing ∂_k instead of ∂_{q_k}, since only derivatives with respect to the q variables remain. Taking $i = j$ we see that $N_i = \frac{1}{2}\partial_i G_{ii}$ and that G_{ii} and

N_i must depend on q_i only. For i, j and k different, we obtain $\partial_i G_{ij} = N_j(q_j)$, $\partial_j G_{ij} = N_i(q_i)$, and $\partial_k G_{ij} = 0$. Since mixed derivatives are equal, this gives $\partial_i N_i(q_i) = \partial_j N_j(q_j)$ for all i, j, and so $\partial_1 N_1 = \cdots = \partial_n N_n = \alpha$ for some constant α. This shows that $N_i = \alpha q_i + \beta_i$, from which it follows that $G_{ij} = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}$. We have now established (4.2) and (4.3). With F, G and N given by these formulas it is easy to check that $J(q_i, p_j, p_k)$ and $J(p_i, p_j, p_k)$ vanish identically. For the only remaining condition, we obtain $J(p_i, p_j, d) = \frac{\lambda}{2}(P_{ij} - P_{ji})$, from which (4.5) follows. When $\det G \neq 0$, the equations $0 = P_{ij} - P_{ji}$ are equivalent, through the forming of suitable linear combinations, to the equations $0 = \partial_i [AM]_j - \partial_j [AM]_i$, where $A(q) = \operatorname{cof} G(q)$ (the proof of this is slightly technical and has therefore been relegated to the appendix). It follows that there is a function $k(q)$ such that $AM = -\frac{1}{2}\nabla k$. We have now completely determined the structure of the Poisson matrix Π. Recall from theorem 3.4 that $A = \operatorname{cof} G$ satisfies the cyclic conditions, so that $M = -\frac{1}{2}A^{-1}\nabla k$ really is a QLN system.

It remains to verify that the function $\widehat{E}$ given by (4.7) is a Casimir of Π. One needs to use the facts that $GA = (\det G)I$ and $G\nabla(p^T A p) = 2FAp$. The latter equality is established as follows. By theorem 3.4, A satisfies the cyclic conditions, so that $\partial_k A_{ij} = -\partial_i A_{kj} - \partial_j A_{ki}$. Thus, using (3.3) and (3.5), we obtain

$$\begin{aligned}
[G\nabla(p^T A p)]_a &= \sum_k G_{ak}\, \partial_k \left(\sum_{i,j} A_{ij} p_i p_j \right) = -2 \sum_{i,j,k} G_{ak} (\partial_i A_{kj}) p_i p_j \\
&= -2 \sum_{i,j,k} \big(\partial_i (G_{ak} A_{kj}) - (\delta_{ia} N_k + \delta_{ik} N_a) A_{kj} \big) p_i p_j \\
&= -2 \left(\sum_{i,j} \partial_i (\delta_{aj} \det G) p_i p_j - \sum_{j,k} N_k A_{jk} p_a p_j - N_a \sum_{i,j} A_{ij} p_i p_j \right) \\
&= 2 \sum_{k,m} (N_a p_k - N_k p_a) A_{km} p_m = 2[FAp]_a.
\end{aligned}$$

Knowing this, the result $\Pi\overline{\nabla}\widehat{E} = 0$ follows from a relatively straightforward calculation which we omit here. □

Remark 4.2. If we assume from the outset that $M(q) = -\nabla V(q)$, as was done in [3], then (4.5) takes the form

$$0 = \sum_{r=1}^{n} \big(G_{ir}\, \partial_{rj} V - G_{jr}\, \partial_{ri} V \big) + 3 \big(N_i \partial_j V - N_j \partial_i V \big). \tag{4.8}$$

As pointed out in [3], this system of equations has been found before as a criterion for the separability of the potential V. We will return to this in section 7.

We need a name for the type of QLN systems occurring in theorem 4.1.

Definition 4.3 (Cofactor system). A QLN system $\delta^+ E = 0$ generated by $E = \dot{q}^T A\, \dot{q} + k$, where A is the cofactor matrix of a nonsingular elliptic coordinates matrix, i.e.,

$$A(q) = \operatorname{cof} G(q), \quad G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \det G(q) \neq 0,$$

will be called a *cofactor system*, and E an integral of motion of *cofactor type*.

In two dimensions any QLN system is a cofactor system, by remark 3.9. Theorem 4.1 leads immediately to a Hamiltonian formulation for cofactor systems:

Theorem 4.4 (Hamiltonian formulation). *Let $\ddot{q} = M(q)$ be a cofactor system, generated by $E = \dot{q}^T A(q)\,\dot{q} + k(q)$, with $A = \operatorname{cof} G$. Then, using the notation of theorem 4.1, there is on the extended phase space $\mathcal{M} = R^{2n+1}$ with coordinates (q, p, d) a related Hamiltonian system*

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \begin{pmatrix} p \\ M(q) + \lambda d\, N(q) \\ 0 \end{pmatrix} = \Pi\,\overline{\nabla} d, \tag{4.9}$$

whose motion on the hyperplane $d = 0$ coincides with the motion of the original system $\ddot{q} = M(q)$ in $(q, \dot{q} = p)$-space.

Proof. Since $M = -\frac{1}{2}A^{-1}\nabla k$ by theorem 2.1, all the conditions of theorem 4.1 are satisfied. Thus, Π is a Poisson matrix and the system is Hamiltonian. Trajectories with initial values in the hyperplane $d = 0$ remain there, since $\dot{d} = 0$. The motion

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \begin{pmatrix} p \\ M(q) \\ 0 \end{pmatrix}$$

in that hyperplane is clearly equivalent to $\ddot{q} = M(q)$. □

Remark 4.5. The restriction of the extended system (4.9) to any hyperplane of constant d (not necessarily $d = 0$) is equivalent to the Newton system $\ddot{q} = M(q) + \lambda d\, N(q)$, which is just the QLN system generated by $\widehat{E}(q, \dot{q}, d) = \dot{q}^T A\,\dot{q} + k(q) - \lambda d \det G(q)$, since $-\frac{1}{2}A^{-1}\nabla\big(k(q) - \lambda d \det G(q)\big) = M + \lambda d\, N$, by (3.5). Here we can view d just as a parameter in $\widehat{E}$, which is indeed how it first turns up in integrable Newton systems derived from soliton theory. In that context, d is typically an integration constant appearing when integrating the stationary flow of some soliton PDE. See for instance [5, 6] and example 3.8.

Remark 4.6. We have shown that one integral of motion of cofactor type is enough for a Newton system $\ddot{q} = M(q)$ to admit a certain type of Hamiltonian formulation, but it is of course not enough to guarantee integrability of any kind. If the extended system (4.9) admits $n - 1$ functionally independent Poisson commuting extra integrals of motion in addition to the Casimir $\widehat{E}$ and the Hamiltonian d, then it is completely integrable. Indeed, the restriction of the system to any level surface of $\widehat{E}$ is a Hamiltonian system [8, Prop. 6.19], which is symplectic, since Π obviously has rank $2n$ if $\det G \neq 0$, and has n commuting integrals of motion. Since the original Newton system $\ddot{q} = M(q)$ is obtained by restriction to the hyperplane $d = 0$, it can in this case be considered as completely integrable too. For instance, the system in example 3.8 falls into this category; setting $p = \dot{q}$ it is actually the "extended system," while what we have called here the "original system" corresponds to the case $d = 0$.

This, however, does not mean that any cofactor system with n integrals $E(q, \dot{q})$, $F_2(q, \dot{q})$, …, $F_n(q, \dot{q})$ must be integrable in this sense, because it may not be possible to incorporate d-dependence into the F_i to even make them integrals $\widehat{F}_i(q, p, d)$ of the extended system, not to mention that the $\widehat{F}_i$ have to Poisson commute. In the next section, we will see how it is possible to overcome this difficulty for the class of *cofactor pair* systems, i.e., systems with *two* integrals of motion of cofactor type, by using a slightly different (bi-Hamiltonian) extended system. The system in example 3.8 is in fact a cofactor pair system, but of a rather special kind ($\widetilde{G}$ is constant), which is why already the theory in this section is sufficient for proving its integrability. (Actually, one can get by with even less. That system has a Lagrangian with indefinite kinetic energy $\dot{q}_1\dot{q}_3 + \frac{1}{2}\dot{q}_2^2$, so when introducing momenta $s_1 = \dot{q}_3$, $s_2 = \dot{q}_2$, $s_3 = \dot{q}_1$ as was done in [5], one obtains a canonical Hamiltonian formulation.)

5 Bi-Hamiltonian formulation and cofactor pair systems

This section forms the central part of the paper. We show that *cofactor pair systems*, i.e., QLN systems with *two* independent integrals of motion of cofactor type, automatically must have n quadratic integrals of motion, and that they under some non-degeneracy assumptions can be considered as completely integrable via embedding into bi-Hamiltonian completely integrable systems in $(2n+1)$-dimensional phase space. The rest of the article is then devoted to the explicit construction of cofactor pair systems in large numbers, and to showing that many known integrable Newton systems from the literature, in particular conservative systems with separable potentials, fit into this framework as special cases. However, the main part of the class of cofactor pair systems seems not to have been considered before.

We now show how the results from the previous section lead naturally to the concept of a cofactor pair system. The matrix Π in theorem 4.1 depends linearly on the parameters α, β, γ in the G, N and F blocks. In order to construct a pencil of compatible Poisson matrices, let these parameters in turn depend linearly on a variable μ:

$$\begin{aligned} \alpha_\mu &= \alpha + \mu\widetilde{\alpha}, \\ \beta_\mu &= \beta + \mu\widetilde{\beta}, \\ \gamma_\mu &= \gamma + \mu\widetilde{\gamma}, \end{aligned} \tag{5.1}$$

where α, β, γ and $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are two separate sets of parameters. Then the corresponding G_μ, N_μ and F_μ also depend linearly on μ:

$$\begin{aligned} G_\mu &= \alpha_\mu qq^T + q\beta_\mu^T + \beta_\mu q^T + \gamma_\mu &&= G + \mu\widetilde{G}, \\ N_\mu &= \alpha_\mu q + \beta_\mu &&= N + \mu\widetilde{N}, \\ F_\mu &= N_\mu p^T - pN_\mu^T &&= F + \mu\widetilde{F}, \end{aligned} \tag{5.2}$$

where, for instance, $N = \alpha q + \beta$ and $\widetilde{N} = \widetilde{\alpha} q + \widetilde{\beta}$, and similarly for G and F. On the other hand, $A_\mu = \operatorname{cof} G_\mu$ is a polynomial in μ of degree $n-1$:

$$A_\mu = \operatorname{cof}(G + \mu\widetilde{G}) = \sum_{i=0}^{n-1} A^{(i)}\mu^i. \tag{5.3}$$

Note that $A^{(0)} = \operatorname{cof} G$ and $A^{(n-1)} = \operatorname{cof}\widetilde{G}$, that G and $\widetilde{G}$ are both elliptic coordinates matrices, and that the matrices $A^{(i)}$ so defined all satisfy the cyclic conditions (2.2) (cf. proposition 3.6 and remark 3.7).

If we now require the M block in the matrix Π *not* to depend on μ, we obtain the following result.

Theorem 5.1 (μ-dependent Poisson matrix). *Let G_μ, N_μ and F_μ be given by (5.2), and suppose that G and $\widetilde{G}$ are nonsingular and linearly independent. Then the matrix*

$$\Pi_\mu = \begin{pmatrix} 0 & \frac{\lambda}{2} G_\mu(q) & p \\ * & \frac{\lambda}{2} F_\mu(q,p) & M(q) + \lambda d\, N_\mu(q) \\ * & * & 0 \end{pmatrix} \tag{5.4}$$

is a Poisson matrix (for all μ) if and only if the Newton system $\ddot{q} = M(q)$ has n quadratic integrals of motion

$$E^{(i)} = \dot{q}^T A^{(i)}(q)\,\dot{q} + k^{(i)}(q), \quad i = 0, \dots, n-1, \tag{5.5}$$

where the matrices $A^{(i)}$ are defined by (5.3).

Moreover, then the function

$$\widehat{E}_\mu(q,p,d) = p^T A_\mu(q)\, p + k_\mu(q) - \lambda d \det G_\mu(q), \tag{5.6}$$

where

$$k_\mu(q) = \sum_{i=0}^{n-1} k^{(i)}(q)\mu^i, \tag{5.7}$$

is a Casimir of Π_μ.

Proof. Nearly all of the requirements of theorem 4.1 are automatically fulfilled. What remains is that we must have $A_\mu M = -\frac{1}{2}\nabla k_\mu$ for some function $k_\mu(q)$ in order for Π_μ to be Poisson. If this is to be an identity in μ, k_μ must have the form $k_\mu = \sum_{i=0}^{n-1} k^{(i)}\mu^i$, and $A^{(i)}M = -\frac{1}{2}\nabla k^{(i)}$ must hold for all i. The latter condition is, by theorem 2.1, equivalent to $E^{(i)} = \dot{q}^T A^{(i)}\,\dot{q} + k^{(i)}$ being an integral of motion of $\ddot{q} = M(q)$, which proves the first part of the theorem. The Casimir follows immediately from theorem 4.1. □

Remark 5.2 (Poisson pencil). The matrix Π_μ splits in the following way:

$$\begin{aligned} \Pi_\mu &= \Pi + \mu\Pi_0 \\ &= \begin{pmatrix} 0 & \frac{\lambda}{2} G(q) & p \\ * & \frac{\lambda}{2} F(q,p) & M(q) + \lambda d\, N(q) \\ * & * & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & \frac{\lambda}{2} \widetilde{G}(q) & 0 \\ * & \frac{\lambda}{2} \widetilde{F}(q,p) & \lambda d\, \widetilde{N}(q) \\ * & * & 0 \end{pmatrix}, \end{aligned} \tag{5.8}$$

where Π is Poisson by theorem 4.1, and likewise Π_0, by a similar (but simpler) calculation. (Alternatively, we could infer this from $\Pi_0 = \lim_{\mu\to\infty} \Pi_\mu/\mu$.) Thus, Π_μ is a *Poisson pencil* of compatible Poisson matrices Π and Π_0.

We have already, in example 3.8 (with $d = 0$), seen a Newton system of the type required in theorem 5.1. In the remainder of this article, we will show that such systems exist in large numbers, including for example systems with separable potentials, and we will also show that they are completely integrable (in a slightly generalized sense). To begin with, we have the following theorem, which says that the existence of n integrals of motion of the special form required in theorem 5.1 is guaranteed by the existence of just *two* integrals of motion of cofactor type. This is clearly a feature which cannot be seen until one considers more than two dimensions, and thus it has no counterpart in the two-dimensional theory [2].

Theorem 5.3 ("2 implies n"). *In the notation of theorem 5.1, if the Newton system $\ddot{q} = M(q)$ has integrals of motion $E^{(0)}$ and $E^{(n-1)}$ of cofactor type, then it also has integrals of motion of the form $E^{(2)}, \dots, E^{(n-2)}$.*

Proof. The question is whether each vector field $A^{(i)}M$ has a potential $-\frac{1}{2}k^{(i)}$, given that $A^{(0)}M$ and $A^{(n-1)}M$ do, where the matrices $A^{(i)}$ are defined by (5.3). We will show this in a rather indirect way. By theorem 4.1, applied first with $(\operatorname{cof} G)M = -\frac{1}{2}\nabla k^{(0)}$ and then with $(\operatorname{cof} \widetilde{G})M = -\frac{1}{2}\nabla k^{(n-1)}$, the matrices

$$\Pi' = \begin{pmatrix} 0 & \frac{\lambda}{2}G(q) & p \\ * & \frac{\lambda}{2}F(q,p) & M(q) + \lambda d\, N(q) \\ * & * & 0 \end{pmatrix}$$

and

$$\Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}\widetilde{G}(q) & p \\ * & \frac{\lambda}{2}\widetilde{F}(q,p) & M(q) + \lambda d\, \widetilde{N}(q) \\ * & * & 0 \end{pmatrix}$$

are both Poisson. At the same time,

$$\Pi' - \Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}(G - \widetilde{G}) & 0 \\ * & \frac{\lambda}{2}(F - \widetilde{F}) & \lambda d\,(N - \widetilde{N}) \\ * & * & 0 \end{pmatrix}$$

has the form of Π_0 in (5.8), so it is also Poisson. This implies [8, Lemma 7.20] that

$$\Pi' + \mu\Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}G_\mu & (1+\mu)p \\ * & \frac{\lambda}{2}F_\mu & (1+\mu)M + \lambda d\, N_\mu \\ * & * & 0 \end{pmatrix}$$

is Poisson for all μ, and also for all λ, since λ is just an arbitrary numerical parameter. Replacing λ with $\lambda(1+\mu)$, and dividing the matrix by $(1+\mu)$, we obtain precisely the matrix Π_μ in (5.4), which we thus have shown to be Poisson for all μ. Theorem 5.1 now implies that $\ddot{q} = M(q)$ has n integrals of motion $E^{(i)}$, as claimed. □

Remark 5.4. In [9] Newton systems on R^{2m} are constructed which have m quadratic and m quartic integrals of motion. This shows that the existence of two quadratic integrals of motion which are not of cofactor type is not sufficient for n quadratic integrals to exist.

Theorem 5.3 motivates the following definition:

Definition 5.5 (Cofactor pair system). An n-dimensional QLN system with two independent quadratic integrals of motion $E = \dot{q}^T A\, \dot{q} + k$ and $\widetilde{E} = \dot{q}^T \widetilde{A}\, \dot{q} + \widetilde{k}$, where A and $\widetilde{A}$ both are cofactor matrices of linearly independent nonsingular elliptic coordinates matrices, i.e.,

$$\begin{aligned} A(q) &= \operatorname{cof} G(q), \quad G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \det G(q) \neq 0, \\ \widetilde{A}(q) &= \operatorname{cof} \widetilde{G}(q), \quad \widetilde{G}(q) = \widetilde{\alpha} q q^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}, \quad \det \widetilde{G}(q) \neq 0, \end{aligned}$$

will be called a *cofactor pair system.*

Note that A and $\widetilde{A}$ are the same as $A^{(0)}$ and $A^{(n-1)}$ in theorems 5.1 and 5.3, and similarly for k, $\widetilde{k}$ and E, $\widetilde{E}$. By theorem 5.3, a cofactor pair system always has n quadratic integrals of motion $E^{(i)} = \dot{q}^T A^{(i)}\, \dot{q} + k^{(i)}$ which can be found by solving the equations $-2A^{(i)}M = \nabla k^{(i)}$ for $k^{(i)}$. Theorem 5.1 leads immediately to the following theorem, which is the key to explaining in what sense cofactor pair systems can be considered to be integrable.

Theorem 5.6 (Bi-Hamiltonian formulation). *Let $\ddot{q} = M(q)$ be a cofactor pair system. Then there is on the extended phase space $\mathcal{M} = R^{2n+1}$ with coordinates (q, p, d) a related bi-Hamiltonian system*

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \Pi\, \overline{\nabla}(\lambda d \det \widetilde{G}) = \Pi_0\, \overline{\nabla}(\widetilde{E} - \lambda d D^{(n-1)}), \tag{5.9}$$

where Π and Π_0 are given by (5.8)*, and $D^{(n-1)}$ is defined by*

$$D_\mu = \det G_\mu = \det(G + \mu \widetilde{G}) = \sum_{i=0}^{n} D^{(i)} \mu^i. \tag{5.10}$$

The trajectories of this system on the hyperplane $d = 0$ coincide with the trajectories of the original system in $(q, \dot{q} = p)$-space, but are traversed with $\lambda \det \widetilde{G}(q)$ times the velocity at each point.

Proof. From theorem 5.1 we know that

$$\widehat{E}_\mu(q, p, d) = p^T A_\mu(q)\, p + k_\mu(q) - \lambda d D_\mu(q) = \sum_{i=0}^{n} \widehat{E}^{(i)} \mu^i,$$

is a Casimir of the Poisson pencil $\Pi_\mu = \Pi + \mu \Pi_0$. Collecting powers of μ, we obtain the following bi-Hamiltonian chain:

$$\begin{aligned} 0 =& \Pi_\mu \overline{\nabla} \widehat{E}_\mu = (\Pi + \mu \Pi_0) \overline{\nabla} \left(\sum_{i=0}^{n} \widehat{E}^{(i)} \mu^i \right) \\ =& \Pi \overline{\nabla} \widehat{E}^{(0)} + \mu \left[\Pi \overline{\nabla} \widehat{E}^{(1)} + \Pi_0 \overline{\nabla} \widehat{E}^{(0)} \right] \\ &+ \cdots + \mu^n \left[\Pi \overline{\nabla} \widehat{E}^{(n)} + \Pi_0 \overline{\nabla} \widehat{E}^{(n-1)} \right] \\ &+ \mu^{n+1} \Pi_0 \overline{\nabla} \widehat{E}^{(n)}. \end{aligned} \tag{5.11}$$

Since $\widehat{E}^{(n)} = -\lambda d D^{(n)} = -\lambda d \det \widetilde{G}$ and $\widehat{E}^{(n-1)} = E^{(n-1)} - \lambda d D^{(n-1)} = \widetilde{E} - \lambda d D^{(n-1)}$, we identify at μ^n the bi-Hamiltonian system (5.9). Computing the right-hand side of the system explicitly yields

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \Pi\,\overline{\nabla}(\lambda d \det \widetilde{G}) = \lambda \begin{pmatrix} (\det \widetilde{G})p \\ -\frac{\lambda}{2} dG(\nabla \det \widetilde{G}) + (\det \widetilde{G})(M + \lambda d\, N) \\ -dp^T \nabla \det \widetilde{G} \end{pmatrix},$$

which for $d = 0$ reduces to

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \lambda \det \widetilde{G} \begin{pmatrix} p \\ M \\ 0 \end{pmatrix}.$$

The last claim follows. □

Corollary 5.7. *If the functions $\widehat{E}^{(i)}(q,p,d)$, $i = 0,\dots,n$, are functionally independent, then the bi-Hamiltonian system* (5.9) *is completely integrable.*

Proof. This follows by similar reasoning as in remark 4.6, since the functions $\widehat{E}^{(i)}(q,p,d)$ Poisson commute with respect to Π and Π_0 by Magri's theorem [10]. □

Remark 5.8. Since a completely integrable Hamiltonian system can, in principle, be solved by quadrature, the same is true for cofactor pair systems satisfying the assumptions of corollary 5.7. The final step, from the solution of the extended bi-Hamiltonian system back to the original Newton system, is just a matter of reparameterizing the trajectories to obtain the correct velocity at each point. This can be done with one further quadrature.

Remark 5.9. The assumption about functional independence of the functions $\widehat{E}^{(i)}$ seems to be fulfilled for most cofactor pair systems, like for instance in example 3.8. As an example of a degenerate case when it is not, consider $G = qq^T$ and $\widetilde{G} = I$ for $n = 3$. The system $\ddot{q} = -q$ is a cofactor pair system with these matrices. It is just a harmonic oscillator, so it is integrable. The integrals of motion in the cofactor chain (for the extended system) are $\widehat{E}^{(0)} = 0$, $\widehat{E}^{(1)} = l_{12}^2 + l_{13}^2 + l_{23}^2$, where $l_{ij} = q_i p_j - q_j p_i$, $\widehat{E}^{(2)} = p^T p + q^T q + \lambda d q^T q$, and $\widehat{E}^{(3)} = \lambda d$. Since $A^{(0)} = \operatorname{cof} G = 0$, the cofactor chain does not provide us with all the integrals of motion of this system. (G is singular in this example, which simplifies the formulas a little, but it could be replaced with the nonsingular matrix $G = qq^T + I$ with essentially the same results; $\widehat{E}^{(0)}$ would not be zero, but the functions $\widehat{E}^{(i)}$ would be dependent.)

The harmonic oscillator above is integrable, so there are integrals of motion which do not appear in the degenerate cofactor chain. It might also be possible that there exist non-integrable cofactor pair systems, with dependent $\widehat{E}^{(i)}$ and no other integrals of motion.

The standard test for functional independence is the following: the functions $\widehat{E}^{(i)}(q,p,d)$ are functionally dependent in an open set U if and only if their gradients $\overline{\nabla}\widehat{E}^{(i)}(q,p,d)$ are linearly dependent everywhere in U [8]. This shows that a sufficient conditions for the functions $E^{(i)}$ to be functionally independent is that the vectors $A^{(i)}p$, $i = 0,\dots,n-1$ are linearly independent. It would be nice to have some simple criterion, expressed directly in terms of G and $\widetilde{G}$, which would guarantee this, but we have not been able to find any such. As the above example shows, it is not enough that G and $\widetilde{G}$ are nonsingular.

Remark 5.10. Theorems 4.1 and 5.6 generalize the corresponding results obtained in [2] for $n = 2$. When $\widetilde{G} = I$ they reproduce, respectively, theorem 4.1 and corollary 4.2 of [3] (bi-Hamiltonian formulation for separable potentials; see also section 7).

6 The fundamental equations and recursive construction of cofactor pair systems

Considering the results of the previous section, which show that cofactor pair systems can be considered as completely integrable, it is natural to ask how large the class of such systems is, and how to find or identify them in practice. In this section, we show that cofactor pair systems are closely related to a system of $\binom{n}{2}$ second order linear PDEs, which we call the *fundamental equations*. This yields an extremely simple method of constructing infinite families of cofactor pair systems.

Definition 6.1 (Fundamental equations). Let $G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma$ and $\widetilde{G}(q) = \widetilde{\alpha} q q^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}$ be elliptic coordinates matrices. Let, as usual, $N = \alpha q + \beta$ and $\widetilde{N} = \widetilde{\alpha} q + \widetilde{\beta}$. The *fundamental equations* associated with the pair $(G, \widetilde{G})$ are, for $i, j = 1, \ldots, n$,

$$
\begin{aligned}
0 = &\sum_{r,s=1}^{n} \big(G_{ir}\widetilde{G}_{js} - G_{jr}\widetilde{G}_{is}\big)\partial_{rs}K \\
&+ 3\sum_{r=1}^{n} \big(G_{ir}\widetilde{N}_j + \widetilde{G}_{jr}N_i - G_{jr}\widetilde{N}_i - \widetilde{G}_{ir}N_j\big)\partial_r K \\
&+ 6\big(N_i\widetilde{N}_j - N_j\widetilde{N}_i\big)K.
\end{aligned} \tag{6.1}
$$

The number of independent equations is (at most) $\binom{n}{2}$ since the equations are antisymmetric in i and j.

The coefficients in this system are polynomials in q. The highest powers of q cancel in each coefficient, so that the coefficient at $\partial_{rs}K$ is in general of degree three, at $\partial_r K$ of degree two, and at K of degree one.

The fundamental equations are antisymmetric not only with respect to i and j, but also under swapping of corresponding parameters with and without tilde. This means that the fundamental equations associated with the pair $(G, \widetilde{G})$ are the same as the fundamental equations for $(\widetilde{G}, G)$, or even for any linear combination $(\lambda_1 G + \lambda_2 \widetilde{G}, \mu_1 G + \mu_2 \widetilde{G})$. Consequently, we might say that the fundamental equations are associated with the linear span of the matrices G and $\widetilde{G}$. The following theorem shows the intimate connection between cofactor pair systems and the corresponding fundamental equations.

Theorem 6.2 (Fundamental equations). *For a cofactor pair system with integrals of motion* $E = \dot{q}^T A\,\dot{q} + k$ *and* $\widetilde{E} = \dot{q}^T \widetilde{A}\,\dot{q} + \widetilde{k}$, *where* $A = \operatorname{cof} G$ *and* $\widetilde{A} = \operatorname{cof} \widetilde{G}$, *the functions*

$$
K'(q) = \frac{k(q)}{\det G(q)} \quad \textit{and} \quad K''(q) = \frac{\widetilde{k}(q)}{\det \widetilde{G}(q)},
$$

although in general different, both satisfy the fundamental equations associated with the pair of matrices $(G, \widetilde{G})$.

Conversely, for each solution K *of the fundamental equations the two different QLN systems*

$$0 = \delta^+ \widetilde{E}, \quad \textit{where} \quad \widetilde{E} = \dot{q}^T \widetilde{A}\, \dot{q} + \widetilde{k}, \quad \widetilde{k} = K \det \widetilde{G},$$
$$0 = \delta^+ F, \quad \textit{where} \quad F = \dot{q}^T A\, \dot{q} + l, \quad l = K \det G$$

are both cofactor pair systems. Explicitly, there exist extra integrals of motion

$$E = \dot{q}^T A\, \dot{q} + k \quad \textit{and} \quad \widetilde{F} = \dot{q}^T \widetilde{A}\, \dot{q} + \widetilde{l}$$

for the first and second system respectively.

Proof. The cofactor pair system can be written $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k = -\frac{1}{2} \widetilde{A}^{-1} \nabla \widetilde{k} = M$. This means that the vector field $AM = -\frac{1}{2} \nabla k$ satisfies the integrability conditions

$$\partial_a [AM]_b - \partial_b [AM]_a = 0$$

for all a, b. As shown in the proof of theorem 4.1, these conditions are equivalent to the equations

$$0 = P_{ij} - P_{ji}, \text{where } P_{ij} = 3 N_i M_j + \sum_{k=1}^{n} G_{ik} \partial_k M_j. \tag{6.2}$$

Expressing $M = -\frac{1}{2} \widetilde{A}^{-1} \nabla \widetilde{k}$ in terms of $K = K'' = \widetilde{k} / \det \widetilde{G}$ yields

$$-2M = \widetilde{A}^{-1} \nabla \widetilde{k} = \widetilde{A}^{-1} (\nabla K \det \widetilde{G} + K \nabla \det \widetilde{G}) = \widetilde{G} \nabla K + 2K\, \widetilde{N},$$

where we have used equation (3.5) (with tildes attached) and the relation $\widetilde{G}\widetilde{A} = (\det \widetilde{G}) I$. Substituting this into (6.2) we obtain after a short calculation the fundamental equations (6.1), which thus are satisfied by $K = K''$. Exchanging the roles of E and $\widetilde{E}$, we find that $K = K' = k / \det G$ satisfies the corresponding equations with coefficients with and without tilde interchanged. But this is in fact the same system, since (6.1) is completely antisymmetric under that operation.

The second part of the theorem follows easily by doing the same calculations backwards. Indeed, if K satisfies the fundamental equations and we let $\widetilde{k} = K \det \widetilde{G}$, then the vector field $-2AM = A(\widetilde{G} \nabla K + 2K\, \widetilde{N}) = A \widetilde{A}^{-1} \nabla \widetilde{k}$ satisfies the integrability conditions, so there exists a function k such that $\nabla k = A \widetilde{A}^{-1} \nabla \widetilde{k}$. Thus, $\ddot{q} = \frac{1}{2} A^{-1} \nabla k = \frac{1}{2} \widetilde{A}^{-1} \nabla \widetilde{k}$ is the cofactor pair system $\delta^+ E = 0 = \delta^+ \widetilde{E}$. Similarly, if we let $l = K \det G$, then, because of the antisymmetry, the fundamental equations are also equivalent to the integrability conditions for the vector field $\widetilde{A} A^{-1} \nabla l$, so that we obtain the cofactor pair system $\delta^+ \widetilde{F} = 0 = \delta^+ F$. □

Corollary 6.3. $K(q) = 1 / \det G(q)$ *and* $K(q) = 1 / \det \widetilde{G}(q)$ *are solutions of the fundamental equations* (6.1).

Proof. The Newton system $\ddot{q} = 0$ is trivially a cofactor pair systems for any pair $(G, \widetilde{G})$ and any constant k and $\widetilde{k}$, so we just take $k = \widetilde{k} = 1$ in the preceding theorem. □

Remark 6.4. Since $A^{-1}\nabla k = -2\ddot{q} = \widetilde{A}^{-1}\nabla\widetilde{k}$, the first part of the theorem can be expressed by saying that the equation

$$A^{-1}\nabla(K' \det G) = \widetilde{A}^{-1}\nabla(K'' \det\widetilde{G})$$

is an auto-Bäcklund transformation between solutions K' and K'' of the fundamental equations. For example, when $G = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\widetilde{G} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ this reproduces the Cauchy–Riemann equations, a well-known auto-Bäcklund transformation for the Laplace equation, which is the fundamental equation in this case (when $n = 2$, there is just one fundamental equation).

Theorem 6.2 opens up the possibility of recursively constructing families of solutions to the fundamental equations, or, equivalently, constructing families of cofactor pair systems whose integrals of motion all have the same "kinetic" parts $\dot{q}^T A^{(i)}\dot{q}$, determined by $\operatorname{cof}(G + \mu\widetilde{G}) = \sum_0^{n-1} A^{(i)}\mu^i$, but different "potential" parts $k^{(i)}$. We can combine the two statements of the theorem as the following diagram illustrates:

$$\boxed{\text{Cofactor pair system } \delta^+\widetilde{E} = 0.} \xrightarrow{K = \frac{\widetilde{k}}{\det\widetilde{G}}} \boxed{\text{Solution } K} \xrightarrow{l = K \det G} \boxed{\text{Cofactor pair system } \delta^+ F = 0.}$$

I.e., starting with a cofactor pair system $\delta^+ E = 0 = \delta^+\widetilde{E}$, we obtain another cofactor pair system $\delta^+ F = 0 = \delta^+\widetilde{F}$ by defining $l = K \det G = (\widetilde{k}/\det\widetilde{G}) \det G$ and determining $\widetilde{l}$ from

$$\nabla\widetilde{l} = \widetilde{A}A^{-1}\nabla l. \tag{6.3}$$

That this integration is possible is precisely what theorem 6.2 says. Then we can repeat the procedure to find yet another cofactor pair system $\delta^+ G = 0 = \delta^+\widetilde{G}$, and so on. We can also go to the left, thereby producing a bi-infinite sequence of cofactor pair systems

$$\cdots \longleftrightarrow \delta^+ E = 0 \longleftrightarrow \delta^+ F = 0 \longleftrightarrow \delta^+ G = 0 \longleftrightarrow \cdots$$

The next theorem shows that there is a purely algebraic relation between the integrals of motion of adjacent systems in this sequence. This means that we can get from one system to the next without having to integrate (6.3), but instead we need to keep track of all n integrals of motion of each system.

Theorem 6.5 (Recursion formula). *Let $0 = \delta^+ E$ and $\delta^+ F = 0$ be cofactor pair systems related as in the second part of theorem 6.2. Let, as usual, $E^{(i)} = \dot{q}^T A^{(i)}\dot{q} + k^{(i)}$ and $F^{(i)} = \dot{q}^T A^{(i)}\dot{q} + l^{(i)}$, $i = 0, \ldots, n-1$ denote their integrals of motion, where $\widetilde{k} = k^{(n-1)}$ and $l = l^{(0)}$, and let $k_\mu = \sum_{i=0}^{n-1} k^{(i)}\mu^i$ and $l_\mu = \sum_{i=0}^{n-1} l^{(i)}\mu^i$. Then, up to an arbitrary additive constant in each $l^{(i)}$,*

$$l_\mu = \frac{\det(G + \mu\widetilde{G})}{\det\widetilde{G}}\,\widetilde{k} - \mu k_\mu, \tag{6.4}$$

with the inverse relationship

$$k_\mu = \frac{1}{\mu}\left(\frac{\det(G + \mu\widetilde{G})}{\det G}\, l - l_\mu\right). \tag{6.5}$$

Proof. This proof is quite technical and has therefore been put in the appendix. □

Remark 6.6. Setting

$$\frac{\det(G+\mu\widetilde{G})}{\det\widetilde{G}} = \sum_{i=0}^{n} X^{(i)}\mu^i$$

we can write (6.4) as

$$\begin{pmatrix} l^{(0)} \\ l^{(1)} \\ \vdots \\ l^{(n-2)} \\ l^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & & & & X^{(0)} \\ -1 & 0 & & & X^{(1)} \\ & & \ddots & & \vdots \\ & & -1 & 0 & X^{(n-2)} \\ & & & -1 & X^{(n-1)} \end{pmatrix} \begin{pmatrix} k^{(0)} \\ k^{(1)} \\ \vdots \\ k^{(n-2)} \\ k^{(n-1)} \end{pmatrix}, \tag{6.6}$$

which is sometimes convenient. We note that the matrix is (minus) what is known as the companion matrix of the polynomial $\mu^n + X^{(n-1)}\mu^{n-1} + \cdots + X^{(0)}$.

Remark 6.7 (Families of cofactor pair systems). With the help of the recursion theorem we can easily construct a bi-infinite family of cofactor pair systems for any given pair $(G, \widetilde{G})$. Namely, we observe that any k_μ which is independent of q gives rise to the trivial cofactor pair system $\ddot{q} = 0$, which can be used as a starting point for the recursion. For example, we can take $k_\mu = \mu^{n-1}$ and iterate (6.4) to obtain the "upwards" part of the family, or start with $l_\mu = 1$ and iterate (6.5) to obtain the "downwards" part. (Starting with other choices of constant k_μ or l_μ will only lead to systems which are linear combinations of the systems in this family.) For systems $\ddot{q} = M(q)$ obtained in this way, $M_1(q), \ldots, M_n(q)$ will always be a rational functions. However, if we find some cofactor pair system which does not depend rationally on q, then we can use the recursion formula in both directions to obtain another bi-infinite family, associated with this system, whose members will all be non-rational. This is illustrated in example 7.7.

Example 6.8. To illustrate the procedure in the case $n = 3$, define elliptic coordinates matrices G and $\widetilde{G}$ by

$$\alpha = 1,\ \beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \quad \widetilde{\alpha} = 0,\ \widetilde{\beta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \widetilde{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$
\begin{aligned}
&\operatorname{cof}(G+\mu\widetilde{G}) = \\
&= \operatorname{cof}\left[\begin{pmatrix} q_1^2+1 & q_1q_2 & q_1q_3 \\ q_1q_2 & q_2^2+2 & q_2q_3 \\ q_1q_3 & q_2q_3 & q_3^2+3 \end{pmatrix} + \mu\begin{pmatrix} 1 & 0 & q_1 \\ 0 & 1 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}\right] \\
&= \begin{pmatrix} 3q_2^2+2q_3^2+6 & -3q_1q_2 & -2q_1q_3 \\ -3q_1q_2 & 3q_1^2+q_3^2+3 & -q_2q_3 \\ -2q_1q_3 & -q_2q_3 & 2q_1^2+q_2^2+2 \end{pmatrix} \\
&+ \mu\begin{pmatrix} q_3^2+4q_3+3 & 0 & -q_1q_3-2q_1 \\ 0 & q_3^2+2q_3+3 & -q_2q_3-q_2 \\ -q_1q_3-2q_1 & -q_2q_3-q_2 & q_1^2+q_2^2+3 \end{pmatrix} \\
&+ \mu^2\begin{pmatrix} -q_2^2+2q_3 & q_1q_2 & -q_1 \\ q_1q_2 & -q_1^2+2q_3 & -q_2 \\ -q_1 & -q_2 & 1 \end{pmatrix} \\
&= A^{(0)} + \mu A^{(1)} + \mu^2 A^{(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
\det(G+\mu\widetilde{G}) = {} & (6q_1^2+3q_2^2+2q_3^2+6) + (3q_1^2+3q_2^2+3q_3^2+4q_3+9)\mu \\
& + (-2q_1^2-q_2^2+q_3^2+6q_3+3)\mu^2 + (-q_1^2-q_2^2+2q_3)\mu^3.
\end{aligned}
$$

An application of the "upwards" recursion formula (6.4) with $k_\mu = \mu$ gives $l_\mu = l^{(0)} + l^{(1)}\mu + l^{(2)}\mu^2$, where

$$
\begin{aligned}
l^{(0)} &= \frac{6q_1^2+3q_2^2+2q_3^2+6}{-q_1^2-q_2^2+2q_3}, \\
l^{(1)} &= \frac{3q_1^2+3q_2^2+3q_3^2+4q_3+9}{-q_1^2-q_2^2+2q_3}, \\
l^{(2)} &= \frac{-2q_1^2-q_2^2+q_3^2+6q_3+3}{-q_1^2-q_2^2+2q_3}.
\end{aligned}
$$

This corresponds to the nontrivial Newton system

$$
\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla l^{(i)} = \frac{-1}{(-q_1^2-q_2^2+2q_3)^2}\begin{pmatrix} q_1q_3+q_1 \\ q_2q_3+2q_2 \\ q_3^2-3 \end{pmatrix} \tag{6.7}
$$

with integrals of motion $\dot{q}^T A^{(i)}\dot{q} + l^{(i)}$ $(i = 0, 1, 2)$.

Applying the "downwards" recursion formula (6.5) with $l_\mu = 1$ gives $k_\mu = k^{(0)} + k^{(1)}\mu + k^{(2)}\mu^2$, where

$$
\begin{aligned}
k^{(0)} &= \frac{3q_1^2+3q_2^2+3q_3^2+4q_3+9}{6q_1^2+3q_2^2+2q_3^2+6}, \\
k^{(1)} &= \frac{-2q_1^2-q_2^2+q_3^2+6q_3+3}{6q_1^2+3q_2^2+2q_3^2+6}, \\
k^{(2)} &= \frac{-q_1^2-q_2^2+2q_3}{6q_1^2+3q_2^2+2q_3^2+6},
\end{aligned}
$$

corresponding to the Newton system

$$\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla k^{(i)} = \frac{1}{(6q_1^2+3q_2^2+2q_3^2+6)^2}\begin{pmatrix} 2q_1q_3+6q_1 \\ 2q_2q_3+3q_2 \\ 2q_3^2-6 \end{pmatrix} \tag{6.8}$$

with integrals of motion $\dot{q}^T A^{(i)} \dot{q} + k^{(i)}$ $(i = 0, 1, 2)$.

The systems (6.7) and (6.8) are integrable in the sense described in the previous section, but it is not known if they admit, for example, any kind of variable separation. Further systems in the recursive sequence are easily computed with the help of symbolic algebra software, but the expressions quickly become rather long.

7 Identifying cofactor pair system

There is a straightforward way of testing if a given Newton system $\ddot{q} = M(q)$ is a cofactor pair system.

Theorem 7.1. *The Newton system $\ddot{q} = M(q)$ admits an integral of motion $E = \dot{q}^T \operatorname{cof} G(q)\, \dot{q} + k(q)$ of cofactor type if and only if the equations*

$$0 = P_{ij} - P_{ji}, \quad \text{where } P_{ij} = 3N_i M_j + \sum_{k=1}^{n} G_{ki}\partial_k M_j, \tag{7.1}$$

viewed as a linear system for the parameters α, β, γ in $G = \alpha q q^T + q\beta^T + \beta q^T + \gamma$ and $N = \alpha q + \beta$, has a nontrivial solution with G nonsingular. It is a cofactor pair system if and only if there is a two-parameter family of solutions $G = sG' + tG''$, from which it is possible to choose $G = s_1G' + t_1G''$ and $\widetilde{G} = s_2G' + t_2G''$ nonsingular and linearly independent.

Proof. These equations occurred previously as equations (4.5). The claim follows immediately from the statement in theorem 4.1 connecting equations (4.5) and (4.6). □

Example 7.2 (Harry Dym stationary flow). As a simple example, let us apply this test to the system (1.4) from example 1.1. Inserting $M(q)$ from (1.4) into (7.1) (with $i = 1$, $j = 2$) yields $0 = -\alpha\, q_1^{-5} q_2^2 - 5\beta_1\, q_1^{-6} q_2^2 + \gamma_{22}\, q_1^{-5} - 5\gamma_{12}\, q_1^{-6} q_2 +$ (polynomial terms). Since different powers are linearly independent we must have $\alpha = \beta_1 = \gamma_{12} = \gamma_{22} = 0$, and with these values the polynomial terms cancel as well, leaving β_2 and γ_{11} free to attain any values. Thus

$$G = s\begin{pmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the general solution of (7.1) in this case. The matrices at s and t correspond to the two known quadratic integrals of motion E and F from example 1.1. If we want both G and $\widetilde{G}$ to be nonsingular, we can take for example $\left(\begin{smallmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & q_1 \\ q_1 & 2q_2 \end{smallmatrix}\right)$.

Example 7.3 (KdV stationary flow). For a three-dimensional example, consider the Newton system (3.8) from example 3.8. If we apply theorem 7.1 to this system, we obtain first $0 = P_{12} - P_{21} = 60\alpha\, q_1^4 + 90\beta_1\, q_1^3 +$ (lower order terms), from which it follows that $\alpha = \beta_1 = 0$. This simplifies the expressions considerably. What remains is $0 = P_{12} - P_{21} = (30\gamma_{11} + 34\beta_2)q_1^2 + 4(\beta_3 + \gamma_{12})q_1 - (20\beta_2 + 16\gamma_{11})q_2 + 4(\gamma_{13} - \gamma_{22})$, which forces $\beta_2 = \gamma_{11} = 0$, $\beta_3 = -\gamma_{12}$ and $\gamma_{13} = \gamma_{22}$. Taking this into account, we find $P_{13} - P_{31} = -4\gamma_{23}$ and $P_{23} - P_{32} = -4\gamma_{23}\, q_1 - 4\gamma_{33}$, which gives $\gamma_{23} = \gamma_{33} = 0$. Consequently, the most general matrix G for which the system has an integral of motion of the form $\dot q^T(\operatorname{cof} G)\dot q + k(q)$ is

$$G = s\begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix} + t\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this way we recover the matrices G and $\widetilde{G}$ from example 3.8.

The system (3.8) has an indefinite Lagrangian $\dot q_1 \dot q_3 + \dot q_2^2/2 - V(q) + d\, q_1$, where $-2V(q) = k^{(2)}$ using our notation from (3.9). This gives a canonical Hamiltonian formulation via the Legendre transformation to momenta $s_1 = \dot q_3$, $s_2 = \dot q_2$, $s_3 = \dot q_1$, and there is also a second, non-canonical, Hamiltonian formulation given in [5]. Except for naming the momenta in the reverse order, this bi-Hamiltonian formulation is just a special case of the one in theorem 5.6. The system was shown in [11] to be separable in the Hamilton–Jacobi sense, using results about so-called quasi-bi-Hamiltonian systems [12]. The same can be shown to hold for any cofactor pair system where one of the matrices G or $\widetilde{G}$ (say $\widetilde{G}$, as in this case) is independent of q. Briefly, when changing to momenta $s = \widetilde{G}^{-1}p$ instead of p, our bi-Hamiltonian formulation of theorem 5.6 takes the form required for the methods used in [11] to apply. However, it is not known if general cofactor pair system, with both G and $\widetilde{G}$ depending on q, can be solved through separation of variables. A separation procedure not using the Hamilton–Jacobi equation was given in [2] for a special class of two-dimensional cofactor pair systems, the so-called *driven systems*. Similar results have been found also for $n > 2$ and will be published in a separate paper.

Finally, the fundamental equations (6.1) for $K = k^{(2)}/\det \widetilde{G} = -k^{(2)} = 2V$ associated with the pair $(G, \widetilde{G})$ reduce to precisely the system (4.20) for V in [5], found there as the conditions for the Jacobi identity of the non-canonical Poisson matrix to be fulfilled. The authors note that any V satisfying these equations gives rise to a completely integrable bi-Hamiltonian system, but do not address the question of finding such V. Our recursion formula (6.4), which in this case can be written

$$\begin{pmatrix} k^{(0)} \\ k^{(1)} \\ k^{(2)} \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 2q_1q_2 + 2q_3 \\ -1 & 0 & q_1^2 + 2q_2 \\ 0 & -1 & 2q_1 \end{pmatrix} \begin{pmatrix} k^{(0)} \\ k^{(1)} \\ k^{(2)} \end{pmatrix},$$

immediately provides us with an infinite family of solutions, one of which corresponds to the Newton system (3.8). In fact, starting with $k^{(0)} = k^{(1)} = 0$, $k^{(2)} = -1$, and iterating, we obtain the $k^{(i)}$ of (3.9) after five steps.

Separable potentials

There is an interesting special case of theorem 7.1 that deserves mentioning. The Newton system $\ddot{q} = M(q)$ is conservative when $G = I$ (identity matrix) is a solution of (7.1). A two-parameter solution $sG + tI$ with G non-constant indicates a special kind of cofactor pair system, namely a conservative system with separable potential (in the Hamilton–Jacobi sense). Indeed, I being a solution implies that $M = -\nabla V$ for some potential V, and inserting this into (7.1) shows that G and V satisfy the equations (cf. remark 4.2)

$$\begin{aligned} 0 &= \sum_{r=1}^{n} \big(G_{ir}\,\partial_{rj}V - G_{jr}\,\partial_{ri}V\big) + 3\big(N_i\partial_j V - N_j\partial_i V\big) \\ &= \sum_{r=1}^{n} \Big((\alpha q_i q_r + \beta_i q_r + \beta_r q_i + \gamma_{ir})\,\partial_{rj}V - (\alpha q_j q_r + \beta_j q_r + \beta_r q_j + \gamma_{jr})\,\partial_{ri}V \Big) \\ &\quad + 3\big((\alpha q_i + \beta_i)\partial_j V - (\alpha q_j + \beta_j)\partial_i V\big), \end{aligned} \tag{7.2}$$

which have been found before in various forms [13, 14, 15, 16, 3] as a criterion for the potential V to be separable in generalized elliptic coordinates or some degeneration thereof. The matrix G determines in which coordinates the separation takes place, in a way which we will now describe briefly. The proofs of the following three propositions, which finally justify the terminology "elliptic coordinates matrix," can be found in the appendix.

Proposition 7.4 (Standard form). *Let $G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma$ be an elliptic coordinates matrix with α and β not both zero. Any (G, I) cofactor pair system can be transformed by an orthogonal change of reference frame $q \to Sq + v$, $S \in SO(n)$, $v \in R^n$, to an equivalent system where G has the standard form*

$$G(q) = -qq^T + \operatorname{diag}(\lambda_1, \dots, \lambda_n), \tag{7.3}$$

if $\alpha \neq 0$, or

$$G(q) = e_n q^T + q e_n^T + \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1}, 0), \tag{7.4}$$

where $e_n = (0, \dots, 0, 1)^T$, if $\alpha = 0$, $\beta \neq 0$.

Proposition 7.5 (Elliptic coordinates). *If*

$$G(q) = -qq^T + \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

then the eigenvalues $u_1(q), \dots, u_n(q)$ of G satisfy

$$\prod_{i=1}^{n}(z - u_i) \Bigg/ \prod_{j=1}^{n}(z - \lambda_j) = 1 + \sum_{m=1}^{n} \frac{q_m^2}{z - \lambda_m}, \tag{7.5}$$

which, when all λ_i are distinct, is the defining equation for generalized elliptic coordinates u with parameters $(\lambda_1, \dots, \lambda_n)$.

Proposition 7.6 (Parabolic coordinates). *Let $e_n = (0, \dots, 0, 1)^T$. If*

$$G(q) = e_n q^T + q e_n^T + \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1}, 0),$$

then the eigenvalues $u_1(q), \dots, u_n(q)$ *of* G *satisfy*

$$-\prod_{i=1}^{n}(z-u_i) \Bigg/ \prod_{j=1}^{n-1}(z-\lambda_j) = \sum_{m=1}^{n-1} \frac{q_m^2}{z-\lambda_m} + (2q_n - z), \tag{7.6}$$

which, when all λ_i *are distinct, is the defining equation for generalized parabolic coordinates* u *with parameters* $(\lambda_1, \dots, \lambda_{n-1})$.

Now, to find the separation coordinates, first change Euclidean reference frame so as to transform G to standard form. Then change to the elliptic or parabolic coordinates defined by the eigenvalues of G, and the Hamilton–Jacobi equation separates. See for example [16] for a nice summary of the theory. (There are some technicalities concerning degenerate cases; see [13].)

The equations (7.2) are precisely the fundamental equations (6.1) for the special case $\widetilde{G} = I$ (and with V instead of K). Thus, the recursion theorem 6.5 provides an easy way of producing separable potentials for any elliptic coordinates matrix G. In fact, when G takes one of the standard forms above, the recursion formula reduces to known recursion formulas [15] for elliptic and parabolic separable potentials respectively.

Example 7.7. When $n = 2$ and $\widetilde{G} = I$, the matrix form (6.6) of the recursion formula reduces to

$$\begin{pmatrix} l \\ \widetilde{l} \end{pmatrix} = \begin{pmatrix} 0 & \det G \\ -1 & \operatorname{tr} G \end{pmatrix} \begin{pmatrix} k \\ \widetilde{k} \end{pmatrix},$$

with $\widetilde{k}$ corresponding to the potential V and k occurring in the second quadratic integral of motion. We can solve the recursion explicitly by computing the powers of the matrix. For example, with

$$G = \begin{pmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{pmatrix} \tag{7.7}$$

it is not hard to show that we recover the combinatorial potentials

$$V_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} q_1^{2k} (2q_2)^{m-2k}, \quad m \geq 0, \tag{7.8}$$

found in [17], as well as the accompanying downwards family

$$V_{-m} = \frac{(-1)^m}{q_1^{2m}} V_{m-1}, \quad m \geq 1. \tag{7.9}$$

These potentials are all separable in the parabolic coordinates $u_{1,2} = q_2 \pm \sqrt{q_1^2 + q_2^2}$ defined by the eigenvalues of G, and in fact constitute the two-dimensional case of a more general family of parabolic separable potentials in n dimensions [15].

The two-dimensional Kepler potential

$$W_0(q) = -(q_1^2 + q_2^2)^{-1/2} \tag{7.10}$$

(which is not rational in q) is also separable in these same parabolic coordinates, with second integral $F_0 = \dot{q}^T (\operatorname{cof} G) \dot{q} + 2q_2 W_0(q) = 2q_2 \dot{q}_1^2 - 2q_1 \dot{q}_1 \dot{q}_2 + 2q_2 W_0(q)$.

Starting the recursion with $k_\mu = (q_2 + \mu)W_0(q)$, we obtain what might be called the Kepler family of parabolic separable potentials:

$$W_m = (-q_2 V_{m-1} + V_m)W_0, \quad m \geq 0, \tag{7.11}$$

and

$$W_{-m} = (q_2 V_{-m} - V_{-(m-1)})W_0,, \quad m \geq 1, \tag{7.12}$$

where the V_i are given by (7.8) and (7.9).

8 Conclusions

We have introduced the class of cofactor pair Newton systems in n dimensions, and explained their integrability properties through embedding into bi-Hamiltonian systems in extended phase space. As well as providing many new integrable systems, this gives a framework into which several previously known systems fit, such as separable potentials and some integrable Newton systems derived from soliton theory. Perhaps the most remarkable feature of cofactor pair systems is the algebraic structure of their integrals of motion; namely, that a Newton system with two integrals of motion of cofactor type must have an entire "cofactor chain" consisting of n quadratic integrals of motion. We have shown how to construct infinite families of cofactor pair systems, and how to determine if a given Newton system is a cofactor pair system. Whether all cofactor pair systems can be integrated through some kind of variable separation is an interesting open question, but only partial results are known yet.

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A Appendix

This appendix contains the missing part of the proof of the Poisson matrix theorem 4.1 as well as the proofs of the recursion theorem 6.5 and propositions 7.4, 7.5, and 7.6.

Proof of theorem 4.1. It remains to show that the $\binom{n}{2}$ equations ($1 \leq i < j \leq n$)

$$0 = P_{ij} - P_{ji}, \quad \text{where } P_{ij} = 3N_i M_j + \sum_{k=1}^{n} G_{ik} \partial_k M_j, \tag{A.1}$$

are equivalent to the $\binom{n}{2}$ equations ($1 \leq a < b \leq n$)

$$0 = \partial_a [AM]_b - \partial_b [AM]_a, \quad \text{where } A(q) = \operatorname{cof} G(q), \det G(q) \neq 0. \tag{A.2}$$

Consider the following linear combination of equations (A.2):

$$
\begin{aligned}
&\sum_{a<b}(G_{ia}G_{jb} - G_{ja}G_{ib})(\partial_a[AM]_b - \partial_b[AM]_a) \\
&= \frac{1}{2}\sum_{a,b}(G_{ia}G_{jb} - G_{ja}G_{ib})(\partial_a[AM]_b - \partial_b[AM]_a) \\
&= \sum_{a,b}(G_{ia}G_{jb} - G_{ja}G_{ib})\partial_a[AM]_b \\
&= Q_{ij} - Q_{ji},
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{ij} =& \sum_{a,b} G_{ia}G_{jb}\partial_a[AM]_b \\
=& \sum_{a,b} \partial_a(G_{ia}G_{jb}[AM]_b) - \sum_{a,b} \partial_a(G_{ia}G_{jb})[AM]_b \\
=& \sum_{a} \partial_a(G_{ia}[GAM]_j) - \sum_{a}(\partial_a G_{ia})[GAM]_j \\
& - \sum_{a,b} G_{ia}(\delta_{aj}N_b + \delta_{ab}N_j)[AM]_b \\
=& \sum_{a} G_{ia}\partial_a(M_j \det G) - G_{ij}N^T AM - N_j[GAM]_i \\
=& (\det G)\left(\sum_{a} G_{ia}\partial_a M_j - N_j M_i\right) + M_j[G\,2AN]_i - G_{ij}N^T AM \\
=& (\det G)\left(\sum_{a} G_{ia}\partial_a M_j - N_j M_i + 2M_j N_i\right) - G_{ij}N^T AM.
\end{aligned}
$$

Here we have made use of (3.3) and (3.5), as well as the fact $GA = (\det G)I$. It follows that

$$
Q_{ij} - Q_{ji} = (\det G)(P_{ij} - P_{ji})
$$

so that (A.2) implies (A.1). The opposite implication follows from the fact that equations (A.1) can be linearly combined to yield (A.2). Explicitly, the inverse transformation of the linear combination above is obtained by multiplying (A.1) by the algebraic complement of $G_{ia}G_{jb} - G_{ja}G_{ib}$ in $\det G$ and summing over $i < j$. This completes the proof. (It can be noted, for completeness, that the implication (A.1) $\Longrightarrow$ (A.2) does not require $\det G \neq 0$.) □

Before turning to the proof of theorem 6.5 we need some preliminaries. From (3.5) we know that $\nabla \det G = 2AN$, or, since $2N = \nabla \operatorname{tr} G$ and $GA = (\det G)I$,

$$
G\nabla \det G = (\det G)\nabla \operatorname{tr} G. \tag{A.3}
$$

This can be generalized in the following way.

Lemma A.1. *If $X = \widetilde{G}^{-1}G$, where G and $\widetilde{G}$ are elliptic coordinates matrices, then*

$$
X\nabla \det X = (\det X)\nabla \operatorname{tr} X. \tag{A.4}
$$

Moreover,

$$\frac{X\nabla \det X}{\det X} = \frac{(X+\mu I)\nabla \det(X+\mu I)}{\det(X+\mu I)}. \tag{A.5}$$

Proof. Multiplying (A.4) by $\widetilde{G}$, we see that it is equivalent to

$$G\nabla\left(\frac{\det G}{\det \widetilde{G}}\right) = \frac{\det G}{\det \widetilde{G}}\,\widetilde{G}\,\nabla\operatorname{tr}(\widetilde{G}^{-1}G).$$

Using (A.3) we find that the left-hand side equals

$$G\left(\frac{\nabla \det G}{\det \widetilde{G}} - \det G\frac{\nabla \det \widetilde{G}}{(\det \widetilde{G})^2}\right) = \frac{\det G}{\det \widetilde{G}}\left(\nabla \operatorname{tr} G - G\widetilde{G}^{-1}\nabla \operatorname{tr} \widetilde{G}\right).$$

We are done if we can show that the expression in parentheses equals $\widetilde{G}\,\nabla\operatorname{tr}(\widetilde{G}^{-1}G)$. Let us temporarily use the notation $\widetilde{H} = \widetilde{G}^{-1}$. Then the general formula for the derivative of an inverse matrix, together with (3.3), yields

$$\partial_k \widetilde{H}_{rs} = -[\widetilde{H}(\partial_k\widetilde{G})\widetilde{H}]_{rs} = -\widetilde{H}_{kr}[\widetilde{H}\widetilde{N}]_s - \widetilde{H}_{ks}[\widetilde{H}\widetilde{N}]_r.$$

Now we can compute

$$\begin{aligned}
[\widetilde{G}\nabla\operatorname{tr}(\widetilde{G}^{-1}G)]_m &= \sum_{k,r,s} \widetilde{G}_{mk}\partial_k(\widetilde{H}_{rs}G_{sr}) \\
&= \sum_{k,r,s} \widetilde{G}_{mk}\big(-\widetilde{H}_{kr}[\widetilde{H}\widetilde{N}]_s - \widetilde{H}_{ks}[\widetilde{H}\widetilde{N}]_r\big)G_{sr} \\
&\quad + \sum_{k,r,s} \widetilde{G}_{mk}\widetilde{H}_{rs}(\delta_{ks}N_r + \delta_{kr}N_s) \\
&= -2\sum_{r,s}[\widetilde{G}\widetilde{H}]_{mr}[\widetilde{H}\widetilde{N}]_s G_{sr} + 2[\widetilde{G}\widetilde{H}N]_m \\
&= -2[G\widetilde{H}\widetilde{N}]_m + 2N_m \\
&= [\nabla\operatorname{tr} G - G\widetilde{G}^{-1}\nabla\operatorname{tr}\widetilde{G}]_m.
\end{aligned}$$

This establishes (A.4).

To prove (A.5), observe that (A.4) can be applied with $X+\mu I = \widetilde{G}^{-1}(G+\mu\widetilde{G})$ instead of X, since $G+\mu\widetilde{G}$ is an elliptic coordinates matrix. This shows that (A.5) is just a restatement of the identity $\nabla\operatorname{tr}(X+\mu I) = \nabla\operatorname{tr} X$. □

Proof of theorem 6.5. We remind the reader that $l = l^{(0)}$, $\widetilde{k} = k^{(n-1)}$, $A = A^{(0)}$, and $\widetilde{A} = A^{(n-1)}$. If $\ddot{q} = M(q)$ is the cofactor pair system $\delta^+ E = 0 = \delta^+\widetilde{E}$, then we know that it is generated by any of its integrals of motions, so that $-2A_\mu M = \nabla k_\mu$. In particular, $M = -\frac{1}{2}\widetilde{A}^{-1}\nabla\widetilde{k}$, which shows that

$$\nabla k_\mu = A_\mu \widetilde{A}^{-1}\nabla\widetilde{k}. \tag{A.6}$$

Similarly, l_μ is determined up to integration constant by

$$\nabla l_\mu = A_\mu A^{-1}\nabla l. \tag{A.7}$$

The relationship between $\widetilde{k}$ and l is by construction given by $K = \widetilde{k}/\det\widetilde{G} = l/\det G$, where K is some solution of the fundamental equations. This is in agreement with the recursion formula (6.4) that we are trying to prove. What needs to be verified is consequently that the expression (6.4) for l_μ as a function of k_μ satisfies (A.7), given that k_μ satisfies (A.6). Rewriting this in terms of $X = \widetilde{G}^{-1}G$, we have to verify that

$$\nabla\left[\det(X+\mu I)\widetilde{k} - \mu k_\mu\right] = \frac{\det(X+\mu I)}{\det X}(X+\mu I)^{-1}X\,\nabla\left[(\det X)\widetilde{k}\right] \tag{A.8}$$

when

$$\nabla k_\mu = \det(X+\mu I)(X+\mu I)^{-1}\nabla\widetilde{k}.$$

With the help of lemma A.1, we find

$$\begin{aligned}
&(X+\mu I)\times[\text{RHS of (A.8)}] = \frac{\det(X+\mu I)}{\det X}X\,\nabla\left[(\det X)\widetilde{k}\right]\\
&= \det(X+\mu I)\frac{X\nabla\det X}{\det X}\widetilde{k} + \det(X+\mu I)(X+\mu I-\mu I)\nabla\widetilde{k}\\
&= (X+\mu I)\nabla\det(X+\mu I)\widetilde{k} + (X+\mu I)\det(X+\mu I)\nabla\widetilde{k} - \mu(X+\mu I)\nabla k_\mu\\
&= (X+\mu I)\times[\text{LHS of (A.8)}].
\end{aligned}$$

This completes the proof of (6.4). The inverse formula (6.5) follows immediately, since $k^{(n-1)} = l^{(0)}\det\widetilde{G}/\det G$. □

Proof of proposition 7.4. We need to study how the velocity-dependent parts of the integrals of motion transform under the stated change of variables. Clearly, $\dot{q}^T I\,\dot{q}$ in $\widetilde{E}$ does not change, while $\dot{q}^T(\operatorname{cof}G(q))\,\dot{q}$ in E goes to $(S\dot{q})^T(\operatorname{cof}G(Sq+v))(S\dot{q}) = \dot{q}^T\operatorname{cof}(S^TG(Sq+v)\,S)\dot{q}$, since $S^T = \operatorname{cof}S$ if $S\in SO(n)$. Thus, we must show that we can choose S and v such that $S^TG(Sq+v)\,S$ takes the stated standard form.

Consider first the case $\alpha \neq 0$. Dividing G by $-\alpha$ and adjusting the cofactor chain accordingly, we can assume $\alpha = -1$ without loss of generality. If we then take $v = \beta$ and choose S so as to diagonalize the symmetric matrix $\gamma + \beta\beta^T$, i.e., $S^T(\gamma+\beta\beta^T)S = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$, it is easily verified that $S^TG(Sq+v)\,S = -qq^T + \operatorname{diag}(\lambda_1,\dots,\lambda_n)$.

Similarly, in the case $\alpha = 0$ we can assume that the vector β is normalized. Direct calculation shows that $S^TG(Sq+v)\,S = (S^T\beta)q^T + q(S^T\beta)^T + S^T(\gamma + \beta v^T + v\beta^T)S$, which, if the last column in the orthogonal matrix S equals β, equals $e_nq^T + qe_n^T + S^T\gamma S + e_nv^T + ve_n^T$. Now, to choose the remaining columns of S, let R be any orthogonal matrix with last column β, and let P be an orthogonal $(n-1)\times(n-1)$ matrix which diagonalizes the upper left $(n-1)\times(n-1)$ block Q in $R^T\gamma R$, i.e., $P^TQP = \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1})$. Setting

$$S = R\begin{pmatrix} P & 0\\ 0 & 1\end{pmatrix},$$

we find $S^T\gamma S = \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1},0) + e_nc^T + ce_n^T$ for some vector c. Finally we complete the proof by taking $v = -c$, which gives $S^TG(Sq+v)\,S = e_nq^T + qe_n^T + \operatorname{diag}(\lambda_1,\dots,\lambda_{n-1},0)$. □

Proof of proposition 7.5. We have $\prod_{i=1}^n (z - u_i) = \det(zI - G) = \det(qq^T + \operatorname{diag}(z - \lambda_1, \dots, z - \lambda_n))$, so the statement follows from taking $\mu_i = z - \lambda_i$ in the identity

$$\begin{aligned}\det(qq^T + \operatorname{diag}(\mu_1, \dots, \mu_n)) &= \prod_{i=1}^n \mu_i + \sum_{m=1}^n q_m^2 \left(\prod_{\substack{i=1\\ i\neq m}}^n \mu_i \right) \\ &= \left(1 + \sum_{m=1}^n \frac{q_m^2}{\mu_m} \right) \prod_{i=1}^n \mu_i,\end{aligned} \tag{A.9}$$

which can be proved by induction on the dimension n, as follows. It is obviously true for $n = 1$. Let $A(q) = \operatorname{cof}(qq^T + \operatorname{diag}(\mu_1, \dots, \mu_n))$. The diagonal entries A_{aa} are determinants of the same form as the one we are computing, so they are $A_{aa} = \prod_{i\neq a} \mu_i + \sum_{m\neq a} q_m^2 (\prod_{i\neq m,a} \mu_i)$ by the induction hypothesis. From them the off-diagonal entries are found, using the cyclic conditions $\partial_a A_{ab} = -\frac{1}{2}\partial_b A_{aa}$ (theorem 3.4), to be $A_{ab} = -q_a q_b \prod_{i\neq a,b} \mu_i$ (there can be no constant term since all entries from row a in G that occur in the determinant A_{ab} contain the factor q_a). A cofactor expansion along any row or column now yields (A.9). □

Proof of proposition 7.6. This is similar to the elliptic case, but easier. The proposition follows quickly once we prove

$$\begin{aligned}&\det(e_n q^T + q e_n^T + \operatorname{diag}(\mu_1, \dots, \mu_n)) \\ &\quad = \left(\prod_{i=1}^{n-1} \mu_i \right) (2q_n + \mu_n) - \sum_{m=1}^{n-1} q_m^2 \left(\prod_{\substack{i=1\\ i\neq m}}^{n-1} \mu_i \right) \\ &\quad = \left(\prod_{i=1}^{n-1} \mu_i \right) \left(2q_n + \mu_n - \sum_{m=1}^{n-1} \frac{q_m^2}{\mu_m} \right).\end{aligned} \tag{A.10}$$

The elements in the cofactor matrix which correspond to nonzero elements in the first column are A_{11}, which by induction is given by (A.10) with sum and product indices starting from 2 instead of 1, and $A_{n1} = -q_1 \prod_{i=2}^{n-1} \mu_i$. Cofactor expansion along the first column completes the proof. □

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Paper 2

Driven Newton equations and separable time-dependent potentials

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Abstract

We present a class of time-dependent potentials in R^n that can be integrated by separation of variables: by embedding them into so-called cofactor pair systems of higher dimension, we are led to a time-dependent change of coordinates that allows the time variable to be separated off, leaving the remaining part in separable Stäckel form.

1 Introduction

Newton's law of force in mechanics leads to second order ordinary differential equations $\ddot{q} = M(q, \dot{q}, t)$, where $q = (q^1, \dots, q^n)$ are coordinates on some manifold Q, the configuration space of the system. Often the force M is derived from a potential $V(q,t)$ and the equations can be written in Lagrangian form

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0, \qquad L(q, \dot{q}, t) = \frac{1}{2} g_{ij}\dot{q}^i \dot{q}^j - V(q,t),$$

or, via the Legendre transformation, in Hamiltonian form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad H(q,p,t) = \frac{1}{2} g^{ij} p_i p_j + V(q,t).$$

Here g_{ij} is the metric tensor on Q, with inverse g^{ij}, and (q^i, p_j) are (adapted) coordinates on the cotangent bundle T^*Q.

Powerful techniques have been developed for solving such equations; in particular the well-known Hamilton–Jacobi method, where one tries to find new coordinates $u = u(q)$ on Q, in terms of which the Hamilton–Jacobi equation corresponding to H can be solved by separation of variables. If this succeeds, the mechanical system can be integrated by quadratures.

We will restrict ourselves to Euclidean n-space, i.e., $Q = R^n$ and $g_{ij} = \delta_{ij}$. The coordinates will be written with lower indices in this case, and regarded as a column vector $q = (q_1, \dots, q_n)^T$, the T denoting matrix transposition.

Consider a Newton system which does not contain time t or velocity $\dot{q}$ explicitly:

$$\ddot{q} = M(q).$$

If there is a potential, the system takes the form

$$\ddot{q} = -\nabla V(q), \qquad \nabla = \frac{\partial}{\partial q} = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n}\right)^T,$$

and then the energy $E = \frac{1}{2}\dot{q}^T\dot{q} + V(q)$ is conserved ($\dot{E} = 0$). The separability theory for such time-independent potentials in Euclidean space is highly developed. It is known that separation of the corresponding Hamilton–Jacobi equation can only take place in so-called generalized elliptic coordinates or some degeneration thereof [1]. There even exists an effective algorithm for determining whether or not a given potential $V(q)$, expressed in Cartesian coordinates, is separable, and if so, in which coordinate system [2].

Less is known in the time-dependent case. One of the aims of this paper is to show how certain Newton systems in R^n with time-dependent potential can be integrated by viewing them as driven systems in R^N, with $N > n$, as the following example illustrates.

Example 1.1. Consider the time-dependent potential

$$V(x_1, x_2, t) = \frac{1}{x_1 x_2 - t} \tag{1.1}$$

and the corresponding Newton system in R^2:

$$\begin{aligned} \ddot{x}_1 &= -\frac{\partial V}{\partial x_1} = \frac{x_2}{(x_1 x_2 - t)^2}, \\ \ddot{x}_2 &= -\frac{\partial V}{\partial x_2} = \frac{x_1}{(x_1 x_2 - t)^2}. \end{aligned} \tag{1.2}$$

In order to integrate this system, we introduce the following auxiliary Newton system in R^3, where the first equation drives the other two:

$$\begin{aligned} \ddot{q}_1 &= 0, \\ \ddot{q}_2 &= \frac{q_3}{(q_2 q_3 - q_1)^2}, \\ \ddot{q}_3 &= \frac{q_2}{(q_2 q_3 - q_1)^2}. \end{aligned} \tag{1.3}$$

We think of the q coordinates as partitioned into *driving* coordinates y and *driven* coordinates x:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \equiv \begin{pmatrix} y \\ x_1 \\ x_2 \end{pmatrix}.$$

The particular solution $y(t) \equiv q_1(t) = t$ clearly gives rise to the system (1.2) under the identification $x_1 = q_2$, $x_2 = q_3$. The Newton system (1.3) in R^3 is what we call a *cofactor system* (see section 2), which means that it has the form

$$\ddot{q} = -A(q)^{-1}\nabla W(q) = -\frac{1}{\det G(q)} G(q)\nabla W(q),$$

where $A = \operatorname{cof} G = (\det G)G^{-1}$ is the cofactor matrix of a symmetric matrix $G(q)$ of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}.$$

Equivalently, $\frac{1}{2}\dot{q}^T A(q)\dot{q} + W(q)$ is an integral of motion (of *cofactor type*) for the system.

In this specific case, as is easily verified, the system (1.3) can be written as $\ddot{q} = -G\nabla W/(\det G)$ with

$$G(q) = \begin{pmatrix} 2q_1 & q_2 & q_3 \\ q_2 & 0 & 1 \\ q_3 & 1 & 0 \end{pmatrix}, \qquad W(q) = -\frac{q_2^2 + q_3^2}{q_2 q_3 - q_1}.$$

According to the general theory to be developed in this paper, such a driven cofactor system can be integrated using a time-dependent change of coordinates

$$u_1 = \lambda_1(t, x_1, x_2),$$
$$u_2 = \lambda_2(t, x_1, x_2),$$

where $\lambda_1(q)$ and $\lambda_2(q)$ are the roots of the equation $\det(G(q) - \lambda\widetilde{G}) = 0$, with $\widetilde{G} = \mathrm{diag}(0, 1, 1)$.

It turns out that by defining corresponding momenta s_1 and s_2 appropriately, the equations of motion for (u_1, u_2) can be put in Hamiltonian form with a time-dependent *separable* Hamiltonian. Consequently, $u_1(t)$ and $u_2(t)$ can be found using a variant of the Hamilton–Jacobi method. Changing back to old coordinates, we find $x_1(t)$ and $x_2(t)$, and the problem is solved.

We will fill in the details of this example after explaining the method in general.

2 Quasi-potential Newton systems of cofactor type

The general framework in which we are working was developed in [3] and [4]. We will now quote the definitions and results needed here, some of which have already been hinted at above.

We use the shorthand $\partial_i = \partial/\partial q_i$. The notation $\mathrm{cof}\, X$ means the cofactor matrix of a square matrix X. If X is nonsingular, then $\mathrm{cof}\, X = (\det X)X^{-1}$.

Proposition 2.1. *The "energy-like" function*

$$E(q, \dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(q)\dot{q}_i\dot{q}_j + W(q) = \frac{1}{2}\dot{q}^T A(q)\,\dot{q} + W(q), \tag{2.1}$$

with $A(q)$ a symmetric $n \times n$ matrix, is an integral of motion of the Newton system $\ddot{q} = M(q)$ in R^n if and only if

1. *The matrix entries $A_{ij}(q)$ satisfy the cyclic conditions*

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0, \quad i, j, k = 1, \dots, n. \tag{2.2}$$

(The general solution of these equations is a subspace, of dimension $\frac{1}{12}n(n+1)^2(n+2)$, of the vector space of symmetric matrices whose entries are polynomials of degree at most two in $q_1, \dots, q_n$.)

2. *The force $M(q)$ satisfies $A(q)M(q) + \nabla W(q) = 0$.*

Definition 2.2 (Quasi-potential system). A Newton system of the form

$$\ddot{q} = -A(q)^{-1}\nabla W(q),$$

where the matrix A satisfies the cyclic conditions (2.2), is called a *quasi-potential* system. By the proposition above, $E = \frac{1}{2}\dot{q}^T A\,\dot{q} + W$ is an integral of motion for the system, and it is said to generate the system, since the system is completely determined by $A(q)$ and $W(q)$, and hence by E. (Special case: if $A = I$ is the identity matrix, then W is a potential for the system and E is the usual energy.)

Definition 2.3 (Elliptic coordinates matrix G). A symmetric matrix of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}, \qquad i, j = 1, \ldots, n, \tag{2.3}$$

is called an *elliptic coordinates matrix*. Using matrix multiplication, $G(q)$ can be written

$$G(q) = \alpha q q^T + \beta q^T + q \beta^T + \gamma, \tag{2.4}$$

with α a scalar, q and β column vectors, and γ a symmetric matrix.

Put briefly, the eigenvalues $u_1(q), \ldots, u_n(q)$ of $G(q)$ give the change of coordinates from Cartesian coordinates q to elliptic coordinates $u = u(q)$. See [4] for a more detailed explanation.

Definition 2.4 (Associated vector N). To a given elliptic coordinates matrix G we associate the column vector $N = \alpha q + \beta = \frac{1}{2}\nabla \operatorname{tr} G$.

Proposition 2.5. *If G is an elliptic coordinates matrix, N the associated vector, and $A = \operatorname{cof} G$, then*

$$\nabla \det G = 2\, AN. \tag{2.5}$$

The preceding proposition is frequently useful. It implies, for example, that $A = \operatorname{cof} G$ satisfies

$$(\det G)\partial_k A_{ij} = 2[AN]_k A_{ij} - [AN]_i A_{kj} - [AN]_j A_{ik}, \tag{2.6}$$

from which the following remarkable property of elliptic coordinates matrices follows.

Proposition 2.6. *If $G(q)$ is an elliptic coordinates matrix, then $A(q) = \operatorname{cof} G(q)$ satisfies the cyclic conditions* (2.2)*.*

Corollary 2.7. *If $G(q)$ and $\widetilde{G}(q)$ are elliptic coordinates matrices, then the matrices $A^{(0)}(q), \ldots, A^{(n-1)}(q)$ defined by the generating function*

$$\operatorname{cof}(G + \mu\widetilde{G}) = \sum_{k=0}^{n-1} A^{(k)} \mu^i \tag{2.7}$$

all satisfy the cyclic conditions (2.2)*.*

Remark 2.8. Note that $A^{(0)} = \operatorname{cof} G$ and $A^{(n-1)} = \operatorname{cof} \widetilde{G}$.

We will also need a proposition that does not occur in [4].

Proposition 2.9. *With G, N and $A = \operatorname{cof} G$ as above,*

$$\nabla(N^T AN) = 2\,\alpha\, AN. \tag{2.8}$$

Proof. Equation (2.6) implies that $\sum_{i,j}(\partial_k A_{ij})N_i N_j = 0$, from which the statement follows easily. □

Definition 2.10 (Cofactor system). A *cofactor system* is a quasi-potential Newton system of the special form

$$\ddot{q} = -A(q)^{-1}\nabla W(q) = -\frac{1}{\det G(q)}\, G(q)^{-1}\nabla W(q),$$

where $A = \operatorname{cof} G$, and G is a nonsingular elliptic coordinates matrix. The integral of motion $E = \frac{1}{2}\dot{q}^T A\,\dot{q} + W = \frac{1}{2}\dot{q}^T(\operatorname{cof} G)\,\dot{q} + W$ is said to be of cofactor type.

Definition 2.11 (Cofactor pair system). A *cofactor pair system* is a Newton system which has two independent integrals of motion of cofactor type,

$$E = \frac{1}{2}\dot{q}^T(\operatorname{cof} G)\,\dot{q} + W \qquad \text{and} \qquad \widetilde{E} = \frac{1}{2}\dot{q}^T(\operatorname{cof} \widetilde{G})\,\dot{q} + \widetilde{W}.$$

Equivalently, it is a system which can be written as

$$\ddot{q} = -A^{-1}\nabla W = -\widetilde{A}^{-1}\nabla\widetilde{W}, \tag{2.9}$$

where $A = \operatorname{cof} G$ and $\widetilde{A} = \operatorname{cof}\widetilde{G}$.

Theorem 2.12 (Two implies n). *A cofactor pair system $\ddot{q} = M(q)$ in R^n has n integrals of motion*

$$E^{(k)} = \frac{1}{2}\dot{q}^T A^{(k)}\,\dot{q} + W^{(k)}, \qquad k = 0, \ldots, n-1, \tag{2.10}$$

where the matrices $A^{(k)}$ are given by (2.7) *and the quasi-potentials $W^{(k)}$ are determined (up to irrelevant additive constants) by $\nabla W^{(k)} = -A^{(k)}M$.*

Remark 2.13. Note that the original integrals of motion $E = E^{(0)}$ and $\widetilde{E} = E^{(n-1)}$ of cofactor type sit at either end of this "cofactor chain" of integrals.

Remark 2.14. It is sometimes convenient to handle the integrals of motion using a generating function

$$E_\mu = \sum_{k=0}^{n-1} E^{(k)}\mu^k = \frac{1}{2}\dot{q}^T \operatorname{cof}(G + \mu\widetilde{G})\dot{q} + W_\mu, \tag{2.11}$$

where $W_\mu = \sum_{k=0}^{n-1} W^{(k)}\mu^k$.

Remark 2.15. For W to be well defined by $\nabla W = -AM$, the compatibility conditions $\partial_i[AM]_j = \partial_j[AM]_i$ have to be satisfied for all i and j. This, of course, is the reason that not every Newton system $\ddot{q} = M(q)$ has a potential V, and also that not every Newton system has a quasi-potential W, even though by allowing $A(q) \neq I$ we enlarge the class of systems under consideration.

Now, for $\ddot{q} = M(q)$ to be a cofactor pair system, two sets of compatibility conditions need to be satisfied simultaneously; $\partial_i[AM]_j = \partial_j[AM]_i$ and $\partial_i[\widetilde{A}M]_j = \partial_j[\widetilde{A}M]_i$. For given G and $\widetilde{G}$, this is a rather strong restriction on M. In fact, according to the theorem, it is so strong that if $\partial_i[A^{(k)}M]_j = \partial_j[A^{(k)}M]_i$ holds for $A^{(0)} = A = \operatorname{cof} G$ and $A^{(n-1)} = \widetilde{A} = \operatorname{cof}\widetilde{G}$, then it must hold for *all* the matrices $A^{(k)}$.

Definition 2.16 (Fundamental equations). The *fundamental equations* associated to a pair $(G,\widetilde{G})$ of elliptic coordinates matrices is the following set of $\binom{n}{2}$ second order linear PDEs:

$$\begin{aligned} 0 &= \sum_{r,s=1}^{n} \big(G_{ir}\widetilde{G}_{js} - G_{jr}\widetilde{G}_{is}\big)\partial_{rs}K \\ &\quad + 3\sum_{r=1}^{n}\big(G_{ir}\widetilde{N}_j + \widetilde{G}_{jr}N_i - G_{jr}\widetilde{N}_i - \widetilde{G}_{ir}N_j\big)\partial_r K \\ &\quad + 6\big(N_i\widetilde{N}_j - N_j\widetilde{N}_i\big)K, \qquad i,j=1,\dots,n. \end{aligned} \tag{2.12}$$

Here $N = \alpha q + \beta$ is the vector associated to G, with the same parameters α and β as in $G = \alpha q q^T + \beta q^T + q\beta^T + \gamma$, and similarly for $\widetilde{N}$.

Theorem 2.17. *Let*

$$\ddot{q} = -(\operatorname{cof} G)^{-1}\nabla W = -(\operatorname{cof}\widetilde{G})^{-1}\nabla\widetilde{W} \tag{2.13}$$

be a cofactor pair system. Then the functions $K_1 = W/\det G$ and $K_2 = \widetilde{W}/\det\widetilde{G}$, while in general different, both satisfy the fundamental equations (2.12) *associated to the pair $(G,\widetilde{G})$.*

Conversely, if K satisfies (2.12) *and we set $W = K\det G$, then there is a function $\widetilde{W}$ such that* (2.13) *holds. And if we set $\widetilde{W} = K\det\widetilde{G}$, then there is a function W such that* (2.13) *holds (but these W and $\widetilde{W}$ are in general not the same as those in the previous sentence!).*

Remark 2.18. Once again, this is all about compatibility conditions. If G, $\widetilde{G}$, and W are given, then $\widetilde{W}$ is well defined by (2.13) if and only if

$$\partial_i\big[(\operatorname{cof}\widetilde{G})(\operatorname{cof} G)^{-1}\nabla W\big]_j = \partial_j\big[(\operatorname{cof}\widetilde{G})(\operatorname{cof} G)^{-1}\nabla W\big]_i$$

for all i and j. This is a system of $\binom{n}{2}$ second order linear PDEs for W, with coefficients depending in a complicated way on G and $\widetilde{G}$. Substituting $K = W/\det G$ and forming suitable linear combinations of the equations simplifies this system to precisely the fundamental equations (2.12). These being completely antisymmetric with respect to coefficients with and without tilde, the result is the same if we go the other way around, interchanging the roles of W and $\widetilde{W}$.

Remark 2.19. This theorem leads to a recursive procedure for explicitly constructing infinite families of cofactor pair systems. See [4] for details.

In [4] it was shown, using the theory of bi-Hamiltonian systems, that cofactor pair systems generically are completely integrable, but it was not clear if they admit some kind of separation of variables. The special case $\widetilde{G} = I$ corresponds to conservative systems with an extra integral of motion of cofactor type. Such systems are precisely those with potentials separable in the elliptic (or parabolic) coordinates given by the eigenvalues of $G(q)$, so in that case we have a concrete method of integration. A recent preprint [5], which appeared during the work on this paper, deals with separation of variables for generic cofactor pair systems, with both G and $\widetilde{G}$ nonsingular (and nonconstant, in general). Here, we study the very degenerate case of cofactor pair systems with $\widetilde{G} = \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1)$. As we will see in the next section, these systems admit a somewhat nonstandard integration by separation of variables, and there is a surprising connection with time-dependent potentials.

3 Driven systems

From now on we fix positive integers m and n, and let $N = m + n$. (Hopefully there is no risk of confusing this N with the vector N associated to an elliptic coordinates matrix G.) Let us begin by defining some notation.

Definition 3.1 (Block notation). If X is an $N \times N$ matrix, with $N = m + n$, then we use arrow subscripts to denote blocks in X, as follows:

$$X = \begin{pmatrix} X_{\nwarrow} & X_{\nearrow} \\ X_{\swarrow} & X_{\searrow} \end{pmatrix} \quad \text{with sizes} \quad \begin{bmatrix} m \times m & m \times n \\ n \times m & n \times n \end{bmatrix}. \tag{3.1}$$

Similarly, if Y is a column vector in R^N, then

$$Y = \begin{pmatrix} Y_{\uparrow} \\ Y_{\downarrow} \end{pmatrix} \quad \text{with sizes} \quad \begin{bmatrix} m \\ n \end{bmatrix}. \tag{3.2}$$

So, for instance, $[X_{\nearrow}]_{ij} = X_{i,m+j}$.

We will consider *driven* Newton systems in R^N, where $N = m + n$. By this we mean that the first m equations depend only on the first m variables, so that they form a Newton system in R^m on their own:

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1, \ldots, q_m), \\ &\vdots \\ \ddot{q}_m &= M_m(q_1, \ldots, q_m), \\ \ddot{q}_{m+1} &= M_{m+1}(q_1, \ldots, q_m; q_{m+1}, \ldots, q_{m+n}), \\ &\vdots \\ \ddot{q}_{m+n} &= M_{m+n}(q_1, \ldots, q_m; q_{m+1}, \ldots, q_{m+n}). \end{aligned} \tag{3.3}$$

Definition 3.2 (Vectors x and y). Since we will consider the time evolution of $q_{\uparrow}$ and $q_{\downarrow}$ separately, we write $y = q_{\uparrow}$ and $x = q_{\downarrow}$ to simplify the notation.

With this definition, the system (3.3) can be written as

$$\begin{aligned} \ddot{y} &= M_{\uparrow}(y), \\ \ddot{x} &= M_{\downarrow}(y, x). \end{aligned} \tag{3.4}$$

As in example 1.1, $(y_1, \ldots, y_m)$ are called *driving* variables and $(x_1, \ldots, x_n)$ are called *driven* variables. The system $\ddot{y} = M_{\uparrow}(y)$ is called the *driving* system, since its solution $y = y(t)$, when fed into $\ddot{x} = M_{\downarrow}(y(t), x)$, drives the evolution of the x variables.

An important observation is that if

$$G = \alpha q q^T + \beta q^T + q \beta^T + \gamma$$

is an $N \times N$ elliptic coordinates matrix, then

$$G_{\nwarrow} = \alpha y y^T + \beta_{\uparrow} y^T + y(\beta_{\uparrow})^T + \gamma_{\nwarrow},$$

so that $G_{\nwarrow}(y)$ is an $m \times m$ elliptic coordinates matrix in the y variables. (Similarly for $G_{\searrow}(x)$, but we will not use that here.)

The major part of this paper is devoted to proving the following theorem.

Theorem 3.3 (Driven cofactor systems). *Suppose that a driven Newton system in R^{m+n} is of cofactor type:*

$$\ddot{q} = \begin{pmatrix} M_{\uparrow}(y) \\ M_{\downarrow}(y,x) \end{pmatrix} = -\big(\operatorname{cof} G(q)\big)^{-1} \frac{\partial W}{\partial q}(q). \tag{3.5a}$$

Suppose also that G is not constant (i.e., that α and β are not both zero), that $\det G_{\nwarrow} \neq 0$, *and that there is a potential $V(y,x)$, with y occuring parametrically, such that*

$$M_{\downarrow}(y,x) = -\frac{\partial V}{\partial x}(y,x). \tag{3.5b}$$

Then the driving system is a cofactor system in R^m. Namely, there is a function $w(y)$ such that

$$\ddot{y} = -\big(\operatorname{cof} G_{\nwarrow}(y)\big)^{-1} \frac{\partial w}{\partial y}(y). \tag{3.6}$$

Moreover, for any given solution $y = y(t)$ of the driving system $\ddot{y} = M_{\uparrow}(y)$, the system

$$\ddot{x} = M_{\downarrow}(y(t),x) = -\frac{\partial V}{\partial x}(y(t),x), \tag{3.7}$$

given by the time-dependent potential $V(y(t),x)$, has n (time-dependent) integrals of motion. Under some technical assumptions, stated in definition 3.8, its solution $x(t)$ can be found by quadratures.

The main idea is to recognize systems of the form (3.5) as a degenerate form of cofactor pair systems, with $\widetilde{G} = \operatorname{diag}(0,\dots,0,1,\dots,1)$. Separation coordinates are provided by the roots of the equation $\det(G(q) - u\widetilde{G}) = 0$. In these new coordinates, the system (3.7) takes Hamiltonian form (although not with V as the potential), so the Hamilton–Jacobi method can be applied. The time variable can be separated off, after which the Stäckel conditions are satisfied, so that the space variables separate as well.

We now proceed stepwise with the proof, in sections 3.1 through 3.6.

3.1 Driven cofactor systems as cofactor pair systems

Definition 3.4 (Matrix J). Let J denote the $N \times N$ diagonal matrix

$$J = \operatorname{diag}(0,\dots,0,1,\dots,1), \tag{3.8}$$

with m zeros and n ones along the diagonal ($N = m + n$).

Proposition 3.5. *A system of the form* (3.5) *is a cofactor pair system with*

$$\widetilde{G}(q) = \lambda\, G(q) + J =: \widetilde{G}_\lambda(q), \tag{3.9}$$

for any λ *such that* $\det \widetilde{G}_\lambda \neq 0$*. Conversely, any such cofactor pair system has the form* (3.5)*.*

We note that since G is assumed nonsingular by the definition of cofactor system, $\det(\lambda\, G(q) + J)$ cannot vanish identically, so there are λ such that $\det \widetilde{G}_\lambda \neq 0$. The reason for taking $\widetilde{G} = \widetilde{G}_\lambda$ instead of just $\widetilde{G} = J$ is that the theorems we use about cofactor pair systems require both G and $\widetilde{G}$ to be nonsingular. However, many of the results will be the same as if applying the theorems formally with $\widetilde{G} = J$ directly, so we will regard such systems as cofactor pair systems associated with the pair (G, J).

The proof of proposition 3.5 uses the following lemma [4], which follows from the algebraic properties of an elliptic coordinates matrix G.

Lemma 3.6. *If* $M = -(\det G)^{-1} G \nabla W$*, then*

$$-\partial_j M_i = \sum_{r=1}^{N} G_{ir}\, \partial_{rj} K + 3 N_i\, \partial_j K,$$

where $K(q) = W(q)/\det G(q)$*.*

Proof of proposition 3.5. By construction, the given cofactor system

$$\ddot{q} = M(q) = -(\operatorname{cof} G)^{-1} \nabla W = -(\det G)^{-1} G \nabla W,$$

has an integral of motion of cofactor type $E = \frac{1}{2} \dot{q}^T (\operatorname{cof} G) \dot{q} + W$. Now fix some constant λ such that $\det \widetilde{G}_\lambda \neq 0$. Theorem 2.17 says that the system is a cofactor *pair* system with $\widetilde{G} = \widetilde{G}_\lambda$, i.e., admits an additional integral of motion of cofactor type

$$\widetilde{E}_\lambda = \frac{1}{2} \dot{q}^T (\operatorname{cof} \widetilde{G}_\lambda) \dot{q} + \widetilde{W}_\lambda,$$

if and only if $K = W/\det G$ satisfies the fundamental equations (2.12) associated to the pair $(G, \widetilde{G}_\lambda)$.

The antisymmetry of the fundamental equations shows that any pair $(G, \lambda\, G + J)$ gives rise to the same fundamental equations as the pair (G, J), so we simply plug $\widetilde{G} = J$ into the fundamental equations (2.12) (with n replaced by $m + n$). To begin with, since J is diagonal and constant (so that $\widetilde{N} = 0$), we obtain

$$\begin{aligned} 0 = \sum_{r=1}^{m+n} G_{ir} J_{jj} \partial_{rj} K - \sum_{r=1}^{m+n} G_{jr} J_{ii} \partial_{ri} K \\ + 3\big(J_{jj} N_i \partial_j K - J_{ii} N_j \partial_i K\big), \qquad i, j = 1, \dots, m+n. \end{aligned} \tag{3.10}$$

Now $J_{ii} = 0$ or 1 as $i \leq m$ and $i > m$, respectively. From this it is immediate that (3.10) is identically satisfied if $i, j \leq m$. Using lemma 3.6 to express the remaining equations (3.10) for K in terms of $M = -(\det G)^{-1} G \nabla (K \det G)$ gives $0 = \partial_j M_i$

for $i \leq m < j$, and $0 = \partial_i M_j - \partial_j M_i$ for $m < i, j$. Clearly, these equations are equivalent to M having the block structure

$$M(q) = \begin{pmatrix} M_\uparrow(y) \\ M_\downarrow(y, x) \end{pmatrix}$$

and (at least locally) a "partial potential" V such that $M_\downarrow = -\partial V / \partial x$. □

3.2 Integrals of motion

Proposition 3.7. *The system* (3.5) *has* $n+1$ *integrals of motion* $E^{(0)}, \dots, E^{(n)}$ *given by the generating function*

$$\begin{aligned} E_\mu &= \sum_{k=0}^{n} E^{(k)} \mu^k \\ &= \sum_{k=0}^{n} \left(\frac{1}{2} \dot{q}^T A^{(k)} \dot{q} + W^{(k)} \right) \mu^k \\ &= \frac{1}{2} \dot{q}^T \operatorname{cof}(G + \mu J)\, \dot{q} + W_\mu \end{aligned} \tag{3.11}$$

for some functions $W^{(k)}$. *The integral* $E^{(n)}$ *has the form*

$$E^{(n)}(y, \dot{y}) = \frac{1}{2} \dot{y}^T \operatorname{cof} G_\nwarrow(y)\, \dot{y} + w(y), \tag{3.12}$$

and is an integral of motion of the driving system $\ddot{y} = M_\uparrow(y)$, *of cofactor type in the* y *variables.*

Proof. According to theorem 2.12, our cofactor pair system should have a chain of $N = m + n$ integrals of motion. Here, however, that number is reduced since some of them will be linearly dependent. More specifically, for arbitrary λ such that $\det \widetilde{G}_\lambda \neq 0$, theorem 2.12 gives us integrals $E_\lambda^{(0)}, \dots, E_\lambda^{(N-1)}$ which we write using a generating function

$$E_{\lambda,\mu} = \sum_{k=0}^{m+n-1} E_\lambda^{(k)} \mu^k = \frac{1}{2} \dot{q}^T \operatorname{cof}(G + \mu \widetilde{G}_\lambda) \dot{q} + W_{\lambda,\mu} \tag{3.13}$$

as in (2.11). By construction, $\dot{E}_{\lambda,\mu} = 0$ for all values of μ and all λ such that $\det \widetilde{G}_\lambda \neq 0$. But $E_{\lambda,\mu}$ depends polynomially on λ and μ, since $\operatorname{cof}(G + \mu \widetilde{G}_\lambda) = \operatorname{cof}(G + \mu(\lambda G + J)) = \operatorname{cof}((1 + \mu\lambda) G + \mu J))$ does. Hence, $\dot{E}_{\lambda,\mu} = 0$ identically. In particular, if we set $\lambda = 0$ we extract the constant term with respect to λ, which is just the E_μ of (3.11), a polynomial in μ whose coefficients are integrals of motion.

The reason why E_μ is only of degree n (instead of $m+n-1$) is that the matrix J has so few nonzero elements that the expansion of $\operatorname{cof}(G + \mu J)$ in powers of μ terminates "prematurely" (the details in this expansion are explained below, after the proof):

$$
\begin{aligned}
\operatorname{cof}(G+\mu J) &= \operatorname{cof} G + \dots + \\
&+ \begin{pmatrix} A_{\nwarrow}^{(n-1)} & -(\operatorname{cof} G_{\nwarrow})G_{\nearrow} \\ -\big((\operatorname{cof} G_{\nwarrow})G_{\nearrow}\big)^T & (\det G_{\nwarrow}) I_{n\times n} \end{pmatrix} \mu^{n-1} \\
&+ \begin{pmatrix} \operatorname{cof} G_{\nwarrow} & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times n} \end{pmatrix} \mu^n \\
&=: \sum_{k=0}^{n} A^{(k)} \mu^k.
\end{aligned}
\tag{3.14}
$$

All the coefficients in the generating function $E_{\lambda,\mu}$ in (3.13) are linear combinations of these $n+1$ basic integrals $E^{(0)}, \dots, E^{(n)}$, so even though one can obtain a seemingly longer chain (with $N = m+n$ integrals) by taking $\lambda \neq 0$, it would not contain any essentially new integrals of motion. (Note also that the polynomial E_μ is what we would have obtained by applying theorem 2.12 formally with the singular matrix $\widetilde{G} = J$ instead of $\widetilde{G}_\lambda$.)

The integral $E^{(n)}$ has the form

$$
\begin{aligned}
E^{(n)} &= \frac{1}{2} \begin{pmatrix} \dot{y}^T & \dot{x}^T \end{pmatrix} \begin{pmatrix} \operatorname{cof} G_{\nwarrow}(y) & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times n} \end{pmatrix} \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} + W^{(n)}(y,x) \\
&= \frac{1}{2} \dot{y}^T \operatorname{cof} G_{\nwarrow}(y)\, \dot{y} + w(y),
\end{aligned}
\tag{3.15}
$$

where clearly $W^{(n)} = w(y)$ cannot depend on x if $E^{(n)}$ is to be an integral of motion. Consequently, $E^{(n)}(y,\dot{y})$ must be an integral of motion of the driving system $\ddot{y} = M_{\uparrow}(y)$, and it is of cofactor type in the y variables. □

In (3.14) we have written out some blocks in the matrices $A^{(n-1)}$ and $A^{(n)}$ for future reference (in the proof of proposition 3.16). These can be found either by analyzing the cofactor expansion directly or by writing the identity

$$
(G+\mu J) \operatorname{cof}(G+\mu J) = \det(G+\mu J)\, I_{N\times N}
$$

as

$$
\begin{aligned}
JA^{(n)}\mu^{n+1} + \big(JA^{(n-1)} + GA^{(n)}\big)\mu^n + \dots \\
= \big(0\, \mu^{n+1} + (\det G_{\nwarrow})\mu^n + \dots\big)\, I_{N\times N}
\end{aligned}
$$

and identifying coefficients block-wise at μ^{n+1} and μ^n, using that the matrices $A^{(i)}$ are symmetric. The block $A_{\nwarrow}^{(n-1)}$ does not enter into this identity until at the power μ^{n-1}, and depends on G in a more complicated way. Fortunately, the only information about $A_{\nwarrow}^{(n-1)}$ that we will need is that $A^{(n-1)}$ satisfies the cyclic conditions (2.2) which connect derivatives of $A_{\nwarrow}^{(n-1)}$ to derivatives of the other blocks, which are known explicitly.

We have now completed the proof of the first statement of theorem 3.3, namely, that the driving system is a cofactor system in the y variables.

Moreover, for any given solution $y = y(t)$ of the driving system, we can consider $E^{(0)}, \dots, E^{(n-1)}$ as functions of $(x, \dot{x}, t)$, and these constitute n time-dependent integrals of motion of the driven system (3.7) given by the time-dependent potential $V(y(t), x)$. These are the integrals referred to at the end of theorem 3.3.

3.3 Separation coordinates

Our remaining task (which is much more complicated) is to show how to integrate the driven system $\ddot{x} = -\frac{\partial V}{\partial x}(y(t), x)$, given a solution $y(t)$ of the driving system $\ddot{y} = M_{\uparrow}(y)$. This will be accomplished using a change of variables $(y, x) \mapsto (v, u)$ on R^{m+n} defined as follows:

Definition 3.8 (Variables u and v, roots λ). Let $v_i = y_i$ for $i = 1, \ldots, m$. Let $u_j = \lambda_j(y, x)$ for $j = 1, \ldots, n$, where $\lambda_1, \ldots, \lambda_n$ are the roots of the n-th degree polynomial equation

$$\det\bigl(G(y, x) - \lambda J\bigr) = 0. \tag{3.16}$$

(We assume that this really defines a coordinate system. This requires, to begin with, that all the roots λ_j are non-constant as functions of q. Moreover, the gradients of the v_i and u_j must be linearly independent. Because of lemma 3.11 below, this holds at least in a neighbourhood of any point where all $\lambda_j(q)$ are distinct.)

Definition 3.9 (Polynomial $U(\mu)$). Let

$$U(\mu) = (u_1 - \mu)(u_2 - \mu) \ldots (u_n - \mu). \tag{3.17}$$

It follows from the definition of the u_k as roots of the polynomial $\det(G - \mu J)$, which has the leading term $(-\mu)^n \det G_{\nwarrow}$, that

$$\det(G - \mu J) = U(\mu) \det G_{\nwarrow}. \tag{3.18}$$

Our aim is to express the integrals of motion $E^{(0)}, \ldots, E^{(n)}$ in terms of the new coordinates v and u, and likewise for the equations of motion for the system (although for that purpose we view $x \mapsto u = \lambda(y(t), x)$, where $y(t)$ is a given solution of the driving system, as a time-dependent change of variables in R^n; more about that later). The remainder of this subsection contains technical preparations for these tasks.

Definition 3.10 (Matrix Ψ). Let Ψ denote the $N \times N$ matrix of partial derivatives of v and u with respect to y and x, arranged so that the columns of Ψ are the gradients of v and u with respect to $q = \left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right)$:

$$\begin{aligned} \Psi &= \begin{pmatrix} \nabla v_1 & \ldots & \nabla v_m & \nabla u_1 & \ldots & \nabla u_n \end{pmatrix} \\ &= \begin{pmatrix} e_1 & \ldots & e_m & \nabla \lambda_1 & \ldots & \nabla \lambda_n \end{pmatrix}, \end{aligned} \tag{3.19}$$

where e_i is the column vector with 1 in position i and 0 elsewhere. (In the block notation of (3.1), $\Psi_{\nwarrow} = I_{m \times m}$ and $\Psi_{\swarrow} = 0_{n \times m}$.)

With this definition we have

$$\begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix} = \Psi^T \dot{q}, \qquad \dot{q} = \frac{1}{\det \Psi} (\operatorname{cof} \Psi^T) \begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix}, \tag{3.20}$$

and also

$$\nabla = \begin{pmatrix} \partial/\partial y \\ \partial/\partial x \end{pmatrix} = \Psi \begin{pmatrix} \partial/\partial v \\ \partial/\partial u \end{pmatrix}. \tag{3.21}$$

(Note that $\partial/\partial y \neq \partial/\partial v$ even though $y = v$, hence the need for the different names.)

The following lemma will give us information about the last n columns in the matrix Ψ (or, equivalently, about the blocks $\Psi_{\nearrow}$ and $\Psi_{\searrow}$).

Lemma 3.11 (Eigenvalues and eigenvectors). *Let $G(q)$ and $\widetilde{G}(q)$ be elliptic coordinates matrices. If $\lambda = \lambda(q)$ is a simple root of* $\det(G - \lambda\,\widetilde{G}) = 0$*, then $\nabla\lambda(q)$ is the corresponding "eigenvector":*

$$\Big(G(q) - \lambda(q)\,\widetilde{G}(q)\Big)\nabla\lambda(q) = 0. \tag{3.22}$$

If λ_1 and λ_2 are two different such roots, then

$$(\nabla\lambda_1)^T \widetilde{G}\,\nabla\lambda_2 = 0. \tag{3.23}$$

Proof. Let $G_r = G - r\,\widetilde{G}$ and $p(r) = \det G_r$. For each r, G_r is an elliptic coordinates matrix, with associated vector $N_r = N - r\,\widetilde{N}$, where $N = \alpha q + \beta$ and $\widetilde{N} = \widetilde{\alpha} q + \widetilde{\beta}$. If we apply proposition 2.5 to G_r we get $\nabla p(r) = 2(\operatorname{cof} G_r)N_r$. Now compute the gradient of $p(\lambda(q)) \equiv 0$:

$$\begin{aligned} 0 &= (\nabla p)(\lambda(q)) + p'(\lambda(q))\,\nabla\lambda(q) \\ &= 2\operatorname{cof}\big(G - \lambda(q)\,\widetilde{G}\big)\big(N - \lambda(q)\,\widetilde{N}\big) + p'(\lambda(q))\,\nabla\lambda(q) \end{aligned} \tag{3.24}$$

Multiplying this by $G - \lambda(q)\,\widetilde{G}$ yields, since $\det\big(G - \lambda(q)\,\widetilde{G}\big) = 0$ by definition of λ,

$$0 = p'(\lambda(q))\,\big(G - \lambda(q)\,\widetilde{G}\big)\nabla\lambda(q).$$

But $p'(\lambda(q)) \neq 0$ since $\lambda(q)$ is assumed to be a simple root of p. The first statement follows.

The second statement comes from the simple observation that if $GX_1 = \lambda_1\,\widetilde{G}X_1$ and $GX_2 = \lambda_2\,\widetilde{G}X_2$, then, since G and $\widetilde{G}$ are symmetric,

$$0 = (GX_1)^T X_2 - X_1^T(GX_2) = (\lambda_1 - \lambda_2)X_1^T\widetilde{G}X_2.$$

□

Lemma 3.11, with $\widetilde{G} = J$, says that

$$G\nabla u_k = u_k\,J\nabla u_k, \tag{3.25}$$

and that $\nabla u_1, \ldots, \nabla u_n$ (which are the last n columns of Ψ) are "J-orthogonal,"

$$(\nabla u_j)^T J\,(\nabla u_k) = 0, \qquad \text{if } j \neq k. \tag{3.26}$$

Thus, the columns $(\nabla u_j)_\downarrow$ of the lower right $n \times n$ block $\Psi_\searrow$ in Ψ are orthogonal in R^n in the ordinary Euclidean sense, with squared lengths $\Delta_1, \dots, \Delta_n$, where

$$\Delta_k = \big((\nabla u_k)_\downarrow\big)^T (\nabla u_k)_\downarrow = \sum_{i=1}^{n} (\Psi_{n+i,n+k})^2. \tag{3.27}$$

Consequently, since the first m columns in Ψ are just $e_1, \dots, e_m$, the interpretation of an $n \times n$ determinant as a volume in R^n shows that

$$(\det \Psi)^2 = \Delta_1 \Delta_2 \cdots \Delta_n. \tag{3.28}$$

It also follows that, with $\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_n)$ and $\mathcal{U} = \operatorname{diag}(u_1, \dots, u_n)$,

$$\Psi^T J \, \Psi = \begin{pmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & \Delta \end{pmatrix} \tag{3.29}$$

and

$$\Psi^T G \, \Psi = \begin{pmatrix} G_\nwarrow & 0_{m\times n} \\ 0_{n\times m} & \mathcal{U}\Delta \end{pmatrix}. \tag{3.30}$$

3.4 Integrals of motion in separation coordinates

Now we will transform the integrals of motion $E^{(0)}, \dots, E^{(n)}$ given by (3.11) to the new coordinates (v, u).

Kinetic part

We begin with the "kinetic" part $\dot q^T \operatorname{cof}(G + \mu J)\, \dot q$. Write $G_\mu = G + \mu J$ for simplicity. Equation (3.20) gives

$$\dot q^T (\operatorname{cof} G_\mu)\, \dot q = \frac{1}{(\det \Psi)^2} \begin{pmatrix} \dot v^T & \dot u^T \end{pmatrix} \operatorname{cof}\big(\Psi^T G_\mu \Psi\big) \begin{pmatrix} \dot v \\ \dot u \end{pmatrix}.$$

Equations (3.29) and (3.30) show that

$$\Psi^T G_\mu \Psi = \begin{pmatrix} G_\nwarrow & 0_{m\times n} \\ 0_{n\times m} & \mathcal{U}_\mu \Delta \end{pmatrix},$$

where

$$\mathcal{U}_\mu = \mathcal{U} + \mu I_{n\times n} = \operatorname{diag}(u_1 + \mu, \dots, u_n + \mu). \tag{3.31}$$

This, together with (3.28), gives

$$\frac{1}{(\det \Psi)^2} \operatorname{cof}\big(\Psi^T G_\mu \Psi\big) = \begin{pmatrix} \det \mathcal{U}_\mu \, \operatorname{cof} G_\nwarrow & 0_{m\times n} \\ 0_{n\times m} & (\det G_\nwarrow)\Delta^{-1} \operatorname{cof} \mathcal{U}_\mu \end{pmatrix}.$$

Sandwiching this between $\begin{pmatrix} \dot v^T & \dot u^T \end{pmatrix}$ and $\begin{pmatrix} \dot v \\ \dot u \end{pmatrix}$, we finally obtain

$$\begin{aligned} \dot q^T (\operatorname{cof} G_\mu)\, \dot q &= (\det \mathcal{U}_\mu)\; \dot v^T (\operatorname{cof} G_\nwarrow)\, \dot v \\ &\quad + (\det G_\nwarrow)\; \dot u^T \left(\Delta^{-1} \operatorname{cof} \mathcal{U}_\mu\right) \dot u. \end{aligned} \tag{3.32}$$

Note that $\det \mathcal{U}_\mu = \prod_1^n (u_i + \mu)$ is the generating function for the elementary symmetric polynomials in the n variables $\{u_1, \dots, u_n\}$, while the kth entry in the diagonal matrix $\operatorname{cof} \mathcal{U}_\mu$ generates the elementary symmetric polynomials in the $n - 1$ variables $\{u_1, \dots, u_n\} \setminus \{u_k\}$.

Structure of Δ_k

Next we prove a statement about how Δ_k, defined by (3.27), depends on u and v. This result is important for showing separability later.

Proposition 3.12. *The quantities $\Delta_1, \dots, \Delta_n$ satisfy*

$$\Delta_k(u,v)\, U'(u_k)\, \det G_{\nwarrow}(v) = f_k(u_k), \qquad k = 1, \dots, n, \tag{3.33}$$

where each of the functions $f_1, \dots, f_n$ depends on one variable only, as indicated. (But $U'(u_k)$, which is just the derivative of $U(\mu) = \prod(u_i - \mu)$ evaluated at $\mu = u_k$, depends on all the variables u_i.)

Proof. Recall that $\Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_n) = (\Psi_{\searrow})^T \Psi_{\searrow}$, by (3.29). Since the columns ∇u_k make up the blocks $\Psi_{\nearrow}$ and $\Psi_{\searrow}$, the "upper part" of (3.25) shows that

$$G_{\nwarrow} \Psi_{\nearrow} + G_{\nearrow} \Psi_{\searrow} = 0_{m \times n}. \tag{3.34}$$

Recall from (3.18) that

$$\begin{aligned} \det(G - \mu J) &= U(\mu) \det G_{\nwarrow} \\ &= \det G_{\nwarrow} \Big((-\mu)^n + (-\mu)^{n-1} (\textstyle\sum u_i) + \dots \Big). \end{aligned}$$

By proposition 2.5,

$$\begin{aligned} \nabla \det(G - \mu J) &= 2 \operatorname{cof}(G - \mu J)\, N \\ &= 2 \Big((-\mu)^n A^{(n)} + (-\mu)^{n-1} A^{(n-1)} + \dots \Big) N. \end{aligned}$$

(Note that N is the vector associated to $G - \mu J$ as well as to G, since J is constant.) Hence, in particular,

$$2\, A^{(n-1)} N = \nabla \big((\det G_{\nwarrow}) \textstyle\sum u_i \big).$$

Now, $(\nabla \det G_{\nwarrow})_{\downarrow} = 0$ since $G_{\nwarrow}$ depends only on the y variables, and consequently

$$2\, (A^{(n-1)} N)_{\downarrow} = (\det G_{\nwarrow}) \textstyle\sum (\nabla u_i)_{\downarrow} = (\det G_{\nwarrow})\, \Psi_{\searrow} 1_n,$$

where $1_n \in R^n$ is the column vector with all ones. If we use what we know from (3.14) about the block structure of $A^{(n-1)}$ and divide by $\det G_{\nwarrow}$, this takes the form

$$2 \begin{pmatrix} -G_{\nwarrow}^{-1} G_{\nearrow} \\ I_{n \times n} \end{pmatrix}^T N = \Psi_{\searrow} 1_n. \tag{3.35}$$

Combining (3.34) and (the transpose of) (3.35), we find

$$\begin{aligned} 2\, N^T \begin{pmatrix} \Psi_{\nearrow} \\ \Psi_{\searrow} \end{pmatrix} &= 2\, N^T \begin{pmatrix} -G_{\nwarrow}^{-1} G_{\nearrow} \Psi_{\searrow} \\ \Psi_{\searrow} \end{pmatrix} \\ &= (\Psi_{\searrow} 1_n)^T \Psi_{\searrow} \\ &= 1_n^T \Delta \\ &= \begin{pmatrix} \Delta_1 & \Delta_2 & \dots & \Delta_n \end{pmatrix}. \end{aligned}$$

In other words,

$$\Delta_k = 2\,N^T \nabla u_k, \qquad k = 1, \dots, n. \tag{3.36}$$

As a special case of (3.24), with $\widetilde{G} = J$, $\widetilde{N} = 0$, $p(\mu) = \det(G - \mu J) = U(\mu) \det G_{\nwarrow}$, and $\lambda = u_k$, we have

$$U'(u_k)\,(\det G_{\nwarrow})\,\nabla u_k = -2\,\operatorname{cof}(G - u_k J)\,N, \tag{3.37}$$

which, because of (3.36), when multiplied from the left by $2\,N^T$ yields

$$U'(u_k)\,(\det G_{\nwarrow})\,\Delta_k = -4\,N^T \operatorname{cof}(G - u_k J)\,N, \tag{3.38}$$

The left hand side here is what we claim depends on u_k only, and we will prove this by showing that the gradient of the right hand side is proportional to ∇u_k. (Clearly, a function $f(v, u)$ depends on u_k alone iff $\frac{\partial f}{\partial u_k} \nabla u_k$ is the only contribution when computing ∇f with the chain rule.)

Proposition 2.9, applied to $G - \mu J$ (which has the same α and N as G), shows that

$$\nabla\big(N^T \operatorname{cof}(G - \mu J) N\big) = 2\,\alpha\,\operatorname{cof}(G - \mu J)\,N.$$

Hence, by the chain rule,

$$\begin{aligned} &\nabla\big(N^T \operatorname{cof}(G - u_k J) N\big) \\ &\qquad = 2\,\alpha\,\operatorname{cof}(G - u_k J)\,N + \frac{d}{d\mu}\Big[N^T \operatorname{cof}(G - \mu J) N\Big]_{\mu = u_k} \nabla u_k. \end{aligned}$$

It is manifest that the second term is proportional to ∇u_k, and so is in fact also the first term, because of (3.37). This finishes the proof of proposition 3.12. $\square$

Remark 3.13. In all the examples we have computed, it turns out that $f_i(q_i) = f(q_i)$ for a single function f, but we have no proof that this is always true. In any case, it is not needed for proving separability here.

Solution of the fundamental equations

We previously (in the proof of proposition 3.5) investigated the fundamental equations associated to the pair (G, J):

$$0 = \frac{\partial M_i}{\partial q_j} \qquad \text{for } i \le m < j, \tag{3.39}$$

$$0 = \frac{\partial M_i}{\partial q_j} - \frac{\partial M_j}{\partial q_i} \qquad \text{for } m < i, j, \tag{3.40}$$

where

$$M = -\frac{G\,\nabla(K \det G)}{\det G}$$

is the right-hand side in the cofactor pair system $\ddot{q} = M(q)$ generated by $E^{(0)} = \frac{1}{2}\dot{q}^T (\operatorname{cof} G)\dot{q} + K \det G$.

Proposition 3.14. *In terms of the separations coordinates* (v, u)*, the general solution of the fundamental equations* (3.39) *and* (3.40) *is*

$$K(v,u) = \frac{1}{\det G_{\nwarrow}(v)} \left(w(v) + \sum_{k=1}^{n} \frac{g_k(u_k)/u_k}{U'(u_k)} \right), \tag{3.41}$$

where $g_1(u_1), \ldots, g_n(u_n)$ *are arbitrary functions of one variable, and* $U'(u_k)$ *is as in proposition 3.12.*

Proof. Recall from (3.21) that

$$\nabla \equiv \begin{pmatrix} \partial_y \\ \partial_x \end{pmatrix} = \Psi \begin{pmatrix} \partial_v \\ \partial_u \end{pmatrix},$$

while (3.18) shows that $\det G = u_1 \ldots u_n \det G_{\nwarrow}(v)$. Hence,

$$\begin{aligned} -M &= G\Psi \begin{pmatrix} \partial_v (K \det G) \\ \partial_u (K \det G) \end{pmatrix} \Big/ \det G \\ &= \frac{\begin{pmatrix} G_{\nwarrow} & 0 \\ G_{\swarrow} & \Psi_{\searrow} \mathcal{U} \end{pmatrix} \begin{pmatrix} u_1 \ldots u_n \, \partial_v (K \det G_{\nwarrow}) \\ (\det G_{\nwarrow}) \partial_u (u_1 \ldots u_n K) \end{pmatrix}}{u_1 \ldots u_n \det G_{\nwarrow}} \\ &= \begin{pmatrix} \dfrac{G_{\nwarrow} \partial_v (K \det G_{\nwarrow})}{\det G_{\nwarrow}} \\ \dfrac{G_{\swarrow} \partial_v (K \det G_{\nwarrow})}{\det G_{\nwarrow}} + \Psi_{\searrow} \begin{pmatrix} \partial_{u_1}(u_1 K) \\ \vdots \\ \partial_{u_n}(u_n K) \end{pmatrix} \end{pmatrix}, \end{aligned}$$

where $G\Psi$ was computed using (3.25). Equation (3.39) says that the upper part

$$M_{\uparrow} = -\frac{G_{\nwarrow} \partial_v (K \det G_{\nwarrow})}{\det G_{\nwarrow}}$$

depends only on the y (or v) variables, which happens if and only if

$$K \det G_{\nwarrow} = w(v) + F(u).$$

The function $w(y)$ here is the same as in theorem 3.3, since the driving system $\ddot{y} = M_{\uparrow}$ is generated by $E^{(n)} = \frac{1}{2} \dot{y}^T (\operatorname{cof} G_{\nwarrow}) \dot{y} + w(y)$.

The function $F(u)$ is then determined by (3.40), which obviously is only interesting if $i \neq j$. In this case, if we set $i = m + k$ and $j = m + l$, the first term in

$$M_i = M_{m+k} = -\frac{[G_{\swarrow}]_{\text{row } k} \, \partial_v (K \det G_{\nwarrow})}{\det G_{\nwarrow}} - [\Psi_{\searrow}]_{\text{row } k} \begin{pmatrix} \partial_{u_1}(u_1 K) \\ \vdots \\ \partial_{u_n}(u_n K) \end{pmatrix}$$

does not depend on $q_j = x_l$, since row k of $G_{\swarrow}$ depends on x_k and y only. Then, since by the definition of Ψ

$$[\Psi_{\searrow}]_{\text{row } k} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_k} & \dfrac{\partial u_2}{\partial x_k} & \cdots & \dfrac{\partial u_n}{\partial x_k} \end{pmatrix},$$

we find

$$\begin{aligned}\frac{\partial M_i}{\partial q_j} &= -\frac{\partial}{\partial x_l}\sum_{s=1}^{n}\frac{\partial u_s}{\partial x_k}\partial_{u_s}(u_s K)\\ &= -\sum_{s=1}^{n}\frac{\partial^2 u_s}{\partial x_l \partial x_k}\,\partial_{u_s}(u_s K) - [\Psi_{\searrow}]_{\text{row } k}\,\frac{\partial}{\partial x_l}\begin{pmatrix}\partial_{u_1}(u_1 K)\\ \vdots \\ \partial_{u_n}(u_n K)\end{pmatrix}.\end{aligned}$$

In the second term we substitute $K = \big(w(v) + F(u)\big)/\det G_{\nwarrow}(v)$ and plug what we have into (3.40). The first term cancels out in the subtraction, leaving

$$\begin{aligned}0 &= \frac{\partial M_i}{\partial q_j} - \frac{\partial M_j}{\partial q_i}\\ &= -\frac{1}{\det G_{\nwarrow}}\left([\Psi_{\searrow}]_{\text{row } k}\,\frac{\partial}{\partial x_l}\begin{pmatrix}\partial_{u_1}(u_1 F)\\ \vdots \\ \partial_{u_n}(u_n F)\end{pmatrix} - [\Psi_{\searrow}]_{\text{row } l}\,\frac{\partial}{\partial x_k}\begin{pmatrix}\partial_{u_1}(u_1 F)\\ \vdots \\ \partial_{u_n}(u_n F)\end{pmatrix}\right).\end{aligned}$$

Now, since $\partial_x = \Psi_{\searrow}\partial_u$, this shows that

$$0 = [\Psi_{\searrow}]_{\text{row } l}\,\Omega\,[\Psi_{\searrow}^T]_{\text{column } k} - [\Psi_{\searrow}]_{\text{row } k}\,\Omega\,[\Psi_{\searrow}^T]_{\text{column } l},$$

where Ω (temporarily) denotes the $n \times n$ matrix with entries $\Omega_{ab} = \partial_{u_a}\partial_{u_b}(u_b F)$. In other words, $0 = \Psi_{\searrow}(\Omega - \Omega^T)\Psi_{\searrow}^T$, or, finally,

$$\frac{\partial^2}{\partial u_a \partial u_b}\Big((u_a - u_b)F(u)\Big) = 0, \quad a, b = 1, \ldots, n. \tag{3.42}$$

This equation occurs in classical separability theory in connection with separation in elliptic and parabolic coordinates. It is known to have the general solution

$$F(u) = \sum_{k=1}^{n}\frac{F_k(u_k)}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(u_k - u_j)},$$

with arbitrary functions $F_1(u_1), \ldots, F_n(u_n)$ depending on one variable each (see Lemma 1 and Lemma 2 in [6]). Hence, we have the general solution

$$K(v, u) = \frac{1}{\det G_{\nwarrow}(v)}\left(w(v) + \sum_k \frac{F_k(u_k)}{\prod\limits_{j\neq k}(u_k - u_j)}\right). \tag{3.43}$$

For our purposes, it turns out to be most convenient to write this in the form (3.41). □

Potential part

It remains to investigate the form of the "potential" parts $W^{(0)}, \ldots, W^{(n)}$ in the (v, u) coordinates.

Proposition 3.15. *The functions $W^{(0)}, \dots, W^{(n-1)}$ take the following form when expressed in the (v,u) coordinates:*

$$W^{(a)}(v,u) = \sigma_{n-a}(u)\, w(v) + \sum_{k=1}^{n} \frac{\sigma_{n-a-1}(\breve{u}_k)\, g_k(u_k)}{U'(u_k)}, \tag{3.44}$$

where $\sigma_b(u)$ denotes the elementary symmetric polynomial of degree b in the n variables $\{u_1,\dots,u_n\}$, and $\sigma_b(\breve{u}_k)$ denotes the elementary symmetric polynomial of degree b in the $n-1$ variables $\{u_1,\dots,u_n\}\setminus\{u_k\}$. As above, $g_1(u_1),\dots,g_n(u_n)$ are functions of one variable, and $U'(u_k)$ is as in proposition 3.12.

In particular, the function $W^{(n)}$ depends on the v coordinates only:

$$W^{(n)} = w(v). \tag{3.45}$$

Proof. We have seen that $W^{(n)} = w(y)$ depends only on y in the original coordinates, hence also $W^{(n)} = w(v)$. We also know that $K = W^{(0)}/\det G$ is a solution of the fundamental equations, so according to (3.41)

$$\begin{aligned} W^{(0)}(v,u) &= \frac{\det G(v,u)}{\det G_{\nwarrow}(v)} \left(w(v) + \sum_{k=1}^{n} \frac{g_k(u_k)/u_k}{U'(u_k)} \right) \\ &= u_1 \dots u_n\, w(v) + \sum_{k=1}^{n} \frac{\sigma_{n-1}(\breve{u}_k)\, g_k(u_k)}{U'(u_k)}. \end{aligned} \tag{3.46}$$

With M determined by $W^{(0)}$, the remaining $W^{(a)}$ are determined (up to irrelevant additive constants) by the relation $\nabla W^{(a)} = -A^{(a)} M$, or

$$\nabla W_\mu = \sum_{a=0}^{n} \nabla W^{(a)} \mu^a = -\left(\sum_{a=0}^{n} A^{(a)} \mu^a \right) M = \operatorname{cof}(G+\mu J) \frac{G}{\det G} \nabla W^{(0)}.$$

We multiply by $(\det G)\Psi^T(G+\mu J)$ from the left and use (3.21), (3.29) and (3.30) to obtain the equivalent condition

$$(\det G) \begin{pmatrix} G_{\nwarrow} & 0 \\ 0 & (\mathcal{U}+\mu I)\Delta \end{pmatrix} \begin{pmatrix} \partial_v W_\mu \\ \partial_u W_\mu \end{pmatrix} = \det(G+\mu J) \begin{pmatrix} G_{\nwarrow} & 0 \\ 0 & \mathcal{U}\Delta \end{pmatrix} \begin{pmatrix} \partial_v W^{(0)} \\ \partial_u W^{(0)} \end{pmatrix}.$$

It is a tedious but fairly straightforward calculation, which we omit, to verify that this is satisfied by

$$W_\mu = \left(\prod_{i=1}^{n} (u_i + \mu) \right) w(v) + \sum_{k=1}^{n} \left(\prod_{\substack{j=1 \\ j\neq k}}^{n} (u_j+\mu) \right) \frac{g_k(u_k)}{U'(u_k)},$$

from which $W^{(a)}$ can be read off as the coefficient of μ^a. □

Summary

We have now determined the form of the integrals of motion in separation coordinates (v,u). We have seen that

$$E^{(n)} = \frac{1}{2} \dot{v}^T \big(\operatorname{cof} G_{\nwarrow}(v)\big) \dot{v} + w(v) \tag{3.47}$$

depends only on v, while the form of $E^{(0)}, \ldots, E^{(n-1)}$ is obtained from (3.32) and (3.44):

$$E^{(a)} = \sigma_{n-a}(u)\, E^{(n)} + \sum_{k=1}^{n} \sigma_{n-a-1}(\breve{u}_k) \left(\frac{1}{2} (\det G_{\nwarrow}) \frac{\dot{u}_k^2}{\Delta_k} + \frac{g_k(u_k)}{U'(u_k)} \right). \tag{3.48}$$

If we let $s_k = \dot{u}_k / \Delta_k$ and use proposition 3.12, we can write this as

$$E^{(a)} = \sigma_{n-a}(u)\, E^{(n)} + \sum_{k=1}^{n} \frac{\sigma_{n-a-1}(\breve{u}_k)}{U'(u_k)} \left(\frac{1}{2} f_k(u_k)\, s_k^2 + g_k(u_k) \right). \tag{3.49}$$

Note in particular that

$$E^{(n-1)} = \left(\sum_{k=1}^{n} u_k \right) E^{(n)} + \sum_{k=1}^{n} \left(\frac{f_k(u_k)\, s_k^2 + g_k(u_k)}{U'(u_k)} \right). \tag{3.50}$$

3.5 The equations of motion are Hamiltonian

Given some solution $y = y(t)$ (or $v = v(t)$) of the driving system, we now consider $u = u(y(t), x)$ as a time-dependent change of variables in R^n. We want to express the driven system $\ddot{x} = -\frac{\partial V}{\partial x}(y(t), x)$ in terms of the u variables. Note that since $E^{(n)}$ is an integral of motion for the driving system, it can from now on be treated as simply a constant, the value of which is determined by which solution $y(t)$ is taken.

Proposition 3.16. *The equations of motion for the u variables can be put into canonical Hamiltonian form*

$$\dot{u} = \frac{\partial h}{\partial s}(u, s, t),$$
$$\dot{s} = -\frac{\partial h}{\partial u}(u, s, t),$$

with momenta $s_1, \ldots, s_n$ defined by

$$s_k = \frac{\dot{u}_k}{\Delta_k} \tag{3.51}$$

(Δ_i as in proposition 3.12), and with the time-dependent Hamiltonian

$$h(u, s, t) = \frac{1}{\det G_{\nwarrow}(y(t))} \left(\left(\sum_{k=1}^{n} u_k \right) E^{(n)} + \sum_{k=1}^{n} \left(\frac{f_k(u_k)\, s_k^2 + g_k(u_k)}{U'(u_k)} \right) \right). \tag{3.52}$$

Proof. First we see from (3.50) that h is simply $E^{(n-1)} / \det G_{\nwarrow}$, expressed in terms of u, s, and t. Now, with $p = \dot{x}$ the system $\ddot{x} = -\frac{\partial V}{\partial x}(y(t), x)$ has a canonical Hamiltonian formulation

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, t),$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, t),$$

where $H(x,p,t) = \frac{1}{2}p^T p + V(y(t),x)$. Consider the extended phase space R^{2n+1} with coordinates (x,p,t). With $T = t$, the variables (u,s,T) constitute a different coordinate system on this space. The vector field in extended phase space that corresponds to the canonical phase flow is encoded in the 1-form $p^T dx - H\,dt$ (by spanning the kernel of its exterior derivative). It follows that the equations of motion are canonical in the new coordinates, with Hamiltonian h, if the two 1-forms

$$p^T dx - H\,dt \quad \text{and} \quad s^T du - h\,dT$$

have the same exterior derivative [7, sec. 45]. (Here we view dx and du as column vectors of 1-forms dx_i and du_i, in order to be consistent with our previous matrix notation.) Since here we have $dT = dt$, the proof amounts to showing that

$$d\big(p^T dx - s^T du + (h - H)\,dt\big) = 0. \tag{3.53}$$

The computations will be performed in the (x,p,t) coordinates, and whenever we write y we mean the given function $y(t)$. Note also that since $G_{\nwarrow}$ depends only on the y variables, it too will be a function of t only. In particular, $\det G_{\nwarrow}$ is a function of t only.

We need to express $s^T du$ and h in terms of the (x,p,t) coordinates. Recall that by the definition 3.10 of the matrix Ψ we have

$$\Big(\nabla u_1, \ldots, \nabla u_n\Big) = \begin{pmatrix} \Psi_{\nearrow} \\ \Psi_{\searrow} \end{pmatrix}.$$

Since

$$du_i = \sum_{k=1}^{m} \frac{\partial u}{\partial y_k} \dot{y}_k\,dt + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} dx_k$$

we obtain

$$du = (\Psi_{\nearrow})^T \dot{y}\,dt + (\Psi_{\searrow})^T dx,$$

that is,

$$\dot{u} = (\Psi_{\nearrow})^T \dot{y} + (\Psi_{\searrow})^T \dot{x}.$$

If we transpose and multiply from the right by $\Delta^{-1} = \operatorname{diag}(\Delta_k^{-1})$, we get

$$s^T = \dot{y}^T \Psi_{\nearrow} \Delta^{-1} + p^T \Psi_{\searrow} \Delta^{-1}.$$

Now we define an $m \times n$ matrix Ξ by

$$\Xi = \Psi_{\nearrow} (\Psi_{\searrow})^{-1}. \tag{3.54}$$

Since $\Delta = (\Psi_{\searrow})^T \Psi_{\searrow}$, it follows that

$$\Psi_{\searrow} \Delta^{-1} (\Psi_{\nearrow})^T = \Xi^T,$$
$$\Psi_{\nearrow} \Delta^{-1} (\Psi_{\nearrow})^T = \Xi\,\Xi^T.$$

Consequently,

$$s^T du = p^T dx + (\dot{y}^T \Xi\, \Xi^T \dot{y} + \dot{y}^T \Xi p) dt + \dot{y}^T \Xi\, dx. \tag{3.55}$$

Furthermore, (3.34) shows that $G_{\nearrow} = -G_{\nwarrow}\Xi$, so that the expression for the block $A_{\nearrow}^{(n-1)}$ from (3.14) can be written as

$$A_{\nearrow}^{(n-1)} = -(\operatorname{cof} G_{\nwarrow}) G_{\nearrow} = (\det G_{\nwarrow})\Xi. \tag{3.56}$$

Hence, since from (3.14) we also have $A_{\searrow}^{(n-1)} = (\det G_{\nwarrow}) I$, we find the following expression for h:

$$\begin{aligned} h &= \frac{E^{(n-1)}}{\det G_{\nwarrow}} \\ &= \frac{1}{\det G_{\nwarrow}} \left(\frac{1}{2} \begin{pmatrix} \dot{y}^T & p^T \end{pmatrix} A^{(n-1)} \begin{pmatrix} \dot{y} \\ p \end{pmatrix} + W^{(n-1)} \right) \\ &= \frac{1}{2} p^T p + \dot{y}^T \Xi p + \frac{\frac{1}{2} \dot{y}^T A_{\nwarrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\nwarrow}}. \end{aligned} \tag{3.57}$$

So far we have

$$\begin{aligned} & p^T dx - s^T du + (h - H)\, dt = \\ &\qquad = \left(\frac{\frac{1}{2} \dot{y}^T A_{\nwarrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\nwarrow}} - V - \dot{y}^T \Xi\, \Xi^T \dot{y} \right) dt - \dot{y}^T \Xi\, dx, \end{aligned}$$

and the exterior derivative of this is zero iff

$$\frac{\partial}{\partial x} \left(\frac{\frac{1}{2} \dot{y}^T A_{\nwarrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\nwarrow}} - V - \dot{y}^T \Xi\, \Xi^T \dot{y} \right) + \frac{\partial}{\partial t} \left(\Xi^T \dot{y} \right) = 0.$$

Now $\frac{\partial}{\partial t}(\Xi^T \dot{y}) = \frac{\partial \Xi^T}{\partial t} \dot{y} + \Xi^T \ddot{y}$, and from $\ddot{q} = -[A^{(n-1)}]^{-1} \nabla W^{(n-1)}$ it follows that

$$\begin{aligned} \frac{\partial W^{(n-1)}}{\partial x} &= -[A^{(n-1)} \ddot{q}]_{\downarrow} = -(A_{\nearrow}^{(n-1)})^T \ddot{y} + A_{\searrow}^{(n-1)} \frac{\partial V}{\partial x} = \\ &\qquad\qquad = (\det G_{\nwarrow}) \left(-\Xi^T \ddot{y} + \frac{\partial V}{\partial x} \right), \end{aligned}$$

so it remains to show that

$$\frac{\partial}{\partial x} \left(\frac{\frac{1}{2} \dot{y}^T A_{\nwarrow}^{(n-1)} \dot{y}}{\det G_{\nwarrow}} - \dot{y}^T \Xi\, \Xi^T \dot{y} \right) + \frac{\partial \Xi^T}{\partial t} \dot{y} = 0.$$

To simplify the notation for this final computation, write

$$\det G_{\nwarrow} = D, \quad A_{\nwarrow}^{(n-1)} = (a_{ij}), \quad \text{and} \quad A_{\nearrow}^{(n-1)} = (b_{ij}).$$

Then $(b_{ij}) = -(\operatorname{cof} G_{\nwarrow}) G_{\nearrow} = D\Xi$, by (3.56). In this notation, what we must show is

$$\frac{1}{2D} \sum_{i,j=1}^{m} \frac{\partial a_{ij}}{\partial x_k} \dot{y}_i \dot{y}_j - \frac{1}{D^2} \sum_{i,j=1}^{m} \sum_{l=1}^{n} \frac{\partial (b_{il} b_{jl})}{\partial x_k} \dot{y}_i \dot{y}_j + \sum_{i=1}^{m} \frac{\partial}{\partial t} \left(\frac{b_{ik}}{D} \right) \dot{y}_i = 0. \tag{3.58}$$

To begin with, since $G_\nwarrow$ is independent of x and $G_\nearrow$ is linear in x, we see that b_{ij} is linear in x. More precisely, since

$$\frac{\partial [G_\nearrow]_{rj}}{\partial x_k} = \frac{\partial G_{r,m+j}}{\partial q_{m+k}} = \delta_{jk} N_r,$$

applying proposition 2.5 with y instead of q gives

$$\frac{\partial b_{ij}}{\partial x_k} = -\sum_{r=1}^{m} [\operatorname{cof} G_\nwarrow]_{ir} (\delta_{jk} N_r) = -\delta_{jk} [(\operatorname{cof} G_\nwarrow) N_\uparrow]_i = -\frac{\delta_{jk}}{2} \frac{\partial D}{\partial y_i}.$$

Furthermore,

$$\sum_{i=1}^{m} \frac{\partial}{\partial t} \left(\frac{b_{ik}}{D} \right) \dot{y}_i = \sum_{i,j=1}^{m} \frac{\partial}{\partial y_j} \left(\frac{b_{ik}}{D} \right) \dot{y}_i \dot{y}_j.$$

Finally, since $A^{(n-1)}$ satisfies the cyclic conditions,

$$\frac{\partial a_{ij}}{\partial x_k} = \frac{\partial A^{(n-1)}_{ij}}{\partial q_{m+k}} = -\frac{\partial A^{(n-1)}_{j,m+k}}{\partial q_i} - \frac{\partial A^{(n-1)}_{m+k,i}}{\partial q_j} = -\frac{\partial b_{jk}}{\partial y_i} - \frac{\partial b_{ik}}{\partial y_j}.$$

Plugging all this into (3.58), it is easy to verify that everything cancels out, which completes the proof. □

3.6 Separation of the time-dependent Hamilton–Jacobi equation

The time-dependent Hamilton–Jacobi equation corresponding to the Hamiltonian $h(u, s, t)$ of proposition 3.16 is

$$h(u, \frac{\partial F}{\partial u}, t) + \frac{\partial F}{\partial t} = 0. \tag{3.59}$$

A complete solution $F(u, \alpha, t)$ can be obtained by separation of variables, as we will now show. We number the parameters $\alpha_0, \dots, \alpha_{n-1}$ since they will in fact be just the values of the integrals of motion $E^{(0)}, \dots, E^{(n-1)}$, as will be clear from by comparing (3.62) below with (3.49).

To begin with, since the time variable t appears in K only in the overall multiplicative factor $1/(\det G_\nwarrow)$, it can be separated off by assuming a solution for F of the form

$$F(u, \alpha, t) = S(u, \alpha) - \alpha_{n-1} \int \frac{1}{\det G_\nwarrow(y(t))} \, dt. \tag{3.60}$$

With the explicit expression for h from proposition 3.16 we get the following equation for $S(u, \alpha)$:

$$\left(\sum_{k=1}^{n} u_k \right) E^{(n)} + \sum_{k=1}^{n} \left(\frac{\frac{1}{2} f_k(u_k) \left(\frac{\partial S}{\partial u_k} \right)^2 + g_k(u_k)}{U'(u_k)} \right) = \alpha_{n-1}. \tag{3.61}$$

In order to find a complete solution, depending on all the parameters α_i, we will use Stäckel's method. Consider the n equations

$$\sum_{k=1}^{n} \frac{\sigma_{n-a-1}(\check{u}_k)}{U'(u_k)} \left(\frac{1}{2} f_k(u_k) \left(\frac{\partial S}{\partial u_k} \right)^2 + g_k(u_k) \right) = \alpha_a - \sigma_{n-a}(u)\, E^{(n)}, \qquad (3.62)$$

where $a = 0, \ldots, n-1$. If we can find a solution of this system, it will be a complete solution of (3.61), since it will depend on all α_i. (Of course it will solve (3.61) which is just the last equation of the system, corresponding to $a = n-1$).

Now (3.62) is a linear system of equations for the expression in parentheses, and the matrix of coefficients is the inverse of a Stäckel matrix (similar to the one occuring when separating in elliptic or parabolic coordinates). In fact, the matrix can be inverted using known properties of symmetric polynomials, resulting in

$$\frac{1}{2} f_k(u_k) \left(\frac{\partial S}{\partial u_k} \right)^2 + g_k(u_k) = -P(-u_k), \qquad k = 1, \ldots, n, \qquad (3.63)$$

where the polynomial P is given by

$$P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1} + E^{(n)} z^n. \qquad (3.64)$$

It is now clear that the additive Ansatz

$$S(u, \alpha) = S_1(u_1, \alpha) + \cdots + S_n(u_n, \alpha)$$

yields a separated solution, provided that each function S_k satisfies the separation ODE

$$\frac{1}{2} \left(\frac{dS_k}{du_k} \right)^2 = \frac{-g_k(u_k) - P(-u_k)}{f_k(u_k)}. \qquad (3.65)$$

Consequently,

$$F(u, \alpha, t) = \sum_{k=1}^{n} \int \sqrt{-2\, \frac{g_k(u_k) + P(-u_k)}{f_k(u_k)}}\, du_k - \alpha_{n-1} \int \frac{1}{\det G_{\nwarrow}(y(t))}\, dt \qquad (3.66)$$

is a complete solution, and in the usual way it generates a canonical transformation to variables (β, α), where $\beta_i = \frac{\partial F}{\partial \alpha_i}$. These new variables will be constant during the motion, with values determined by the initial condition. One can then (at least in principle) solve for $u = u(\beta, \alpha, t)$, and hence $x = x(\beta, \alpha, t)$. This finishes the proof of theorem 3.3.

4 The case of one driven equation

The case when only the last equation is driven by the other ones is easier to handle, since it does not require the Hamilton–Jacobi method, as we shall soon see. Specializing our previous results to this case by setting $n = 1$, we find the following. If a system of the form

$$\begin{aligned} \ddot{y}_1 &= M_1(y_1, \ldots, y_m), \\ &\vdots \\ \ddot{y}_m &= M_m(y_1, \ldots, y_m), \\ \ddot{x} &= -\frac{\partial V}{\partial x}(y_1, \ldots, y_m; x) \end{aligned} \qquad (4.1)$$

has an integral of motion $E^{(0)}$ of cofactor type, then it must have an extra integral of motion $E^{(1)} = \frac{1}{2}\dot{y}^T \operatorname{cof} G_{\nwarrow}(y)\,\dot{y} + w(y)$ depending only on the variables y. We change to new coordinates $(v_1, \dots, v_m, u)$, where $v = y$ and u is the zero of the first degree polynomial $\det(G - \lambda J)$. Here $J = \operatorname{diag}(0, \dots, 0, 1)$, so $\det(G - \lambda J) = \det G - \lambda \det G_{\nwarrow}$, hence

$$u = \frac{\det G(y, x)}{\det G_{\nwarrow}(y)}.$$

In the new variables, $E^{(1)}$ remains unchanged (with v instead of y), while $E^{(0)}$ takes the form given by (3.48),

$$E^{(0)} = uE^{(1)} + \frac{1}{2}\frac{\det G_{\nwarrow}(v)}{\Delta}\,\dot{u}^2 + g(u),$$

where, according to (3.27) and proposition 3.12,

$$\Delta = \left(\frac{\partial u}{\partial x}\right)^2 = \frac{f(u)}{\det G_{\nwarrow}(v)}$$

for some function $f(u)$. Hence,

$$E^{(0)} = uE^{(1)} + \frac{1}{2}\frac{\big(\det G_{\nwarrow}(v)\big)^2}{f(u)}\,\dot{u}^2 + g(u). \tag{4.2}$$

Now, for a given solution $v(t) = y(t)$ of the driving system, we write this as

$$\left(\det G_{\nwarrow}(v(t))\,\frac{du}{dt}\right)^2 = 2\,f(u)\big(E^{(0)} - uE^{(1)} - g(u)\big),$$

or

$$\frac{du}{\sqrt{2\,f(u)\big(E^{(0)} - uE^{(1)} - g(u)\big)}} = \frac{dt}{\det G_{\nwarrow}(v(t))},$$

which can be integrated by quadrature, since u and t are separated.

This procedure can be applied recursively to "triangular" systems, as in the following proposition. Note that for an arbitrary triangular system all we can do in general is to solve the first equation for $q_1(t)$. It is quite surprising that the existence of an integral of motion of cofactor type is enough to allow us to solve the system completely.

Proposition 4.1 (Triangular cofactor systems). *Suppose that the "triangular" Newton system*

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1), \\ \ddot{q}_2 &= M_2(q_1, q_2), \\ \ddot{q}_3 &= M_3(q_1, q_2, q_3), \\ &\;\vdots \\ \ddot{q}_N &= M_N(q_1, q_2, q_3, \dots, q_N), \end{aligned} \tag{4.3}$$

is of cofactor type. Suppose also that no upper left $k \times k$ block in G is constant or singular ($k = 1, \dots, N-1$). Then the system can be integrated by quadratures.

Proof. The whole system is of the type considered above (driven, with $n = 1$), so it can be integrated provided that the driving system, consisting of the $N-1$ first equations, can be integrated. By what we said above, the driving system must have an integral of motion of cofactor type, so it is itself a triangular cofactor system, of one dimension less. Since the first equation can be integrated (being one-dimensional), the statement follows by induction. □

In each step of the integration procedure one new variable $u = u_k$ is introduced. Denoting the determinant of the upper left $k \times k$ block in G by $D_k(q_1, \dots, q_k)$, we can write the separation variables $(u_1, \dots, u_N)$ as

$$u_1 = q_1 \qquad \text{and} \qquad u_i = \frac{D_i}{D_{i-1}}, \quad i = 2, \dots, N.$$

5 Examples

Example 5.1 (Example 1.1 continued). We can now fill in the missing details in our first example. We had

$$M(q) = \frac{1}{(q_2 q_3 - q_1)^2} \begin{pmatrix} 0 \\ q_3 \\ q_2 \end{pmatrix}.$$

With

$$G(q) = \begin{pmatrix} 2q_1 & q_2 & q_3 \\ q_2 & 0 & 1 \\ q_3 & 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find from $A_\mu = \operatorname{cof}(G + \mu J)$ that

$$A^{(0)} = \operatorname{cof} G = \begin{pmatrix} -1 & q_3 & q_2 \\ q_3 & -q_3^2 & q_2 q_3 - 2q_1 \\ q_2 & q_2 q_3 - 2q_1 & -q_2^2 \end{pmatrix},$$

$$A^{(1)} = \begin{pmatrix} 0 & -q_2 & -q_3 \\ -q_2 & 2q_1 & 0 \\ -q_3 & 0 & 2q_1 \end{pmatrix}, \qquad A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The relation $\nabla W^{(k)} = -A^{(k)} M$ then yields

$$W^{(0)} = -\frac{q_2^2 + q_3^2}{q_2 q_3 - q_1}, \qquad W^{(1)} = \frac{2q_1}{q_2 q_3 - q_1}, \qquad W^{(2)} = 0.$$

We introduce new variables (v, u_1, u_2), where $v = q_1$ and $u_{1,2}$ are the roots of

$$0 = \det(G - uJ) = 2(q_2 q_3 - q_1) + (q_2^2 + q_3^2)u + (2q_1)u^2.$$

With (y, x_1, x_2) instead of (q_1, q_2, q_3), we see that $u_1 + u_2 = -(x_1^2 + x_2^2)/2y$ and $u_1 u_2 = 2(x_1 x_2 - y)/2y$, so that

$$\begin{aligned} \left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 &= v(1 - u_1)(1 - u_2), \\ \left(\frac{x_1 - x_2}{\sqrt{2}}\right)^2 &= -v(1 + u_1)(1 + u_2). \end{aligned} \tag{5.1}$$

Except for the factor v, the new variables (u_1, u_2) are elliptic coordinates aligned along axes that are rotated $\pi/4$ relative to the Cartesian coordinates (x_1, x_2). With $u_1 < -1 < u_2 < 1$, the coordinate curves are ellipses (for u_1) and hyperbolas (for u_2). The example (1.2) is obtained by taking the particular solution $y(t) = v(t) = q_1(t) = t$ of the driving equation $\ddot{q}_1 = 0$, and in this case we get a factor t with the effect of expanding the entire coordinate web as time increases, so these coordinates might be called "expanding elliptic coordinates."

We can express (u_1, u_2) in terms of (y, x_1, x_2) as

$$u_{1,2} = \frac{-1}{4y}\left(x_1^2 + x_2^2 \pm \sqrt{(x_1^2 + x_2^2)^2 - 16y(x_1x_2 - y)}\right),$$

and then a straightforward computation gives the quantities

$$\Delta_{1,2} = \left(\frac{\partial u_{1,2}}{\partial x_1}\right)^2 + \left(\frac{\partial u_{1,2}}{\partial x_2}\right)^2 = \frac{1}{2y^2}\left(x_1^2 + x_2^2 \pm \frac{(x_1^2 + x_2^2)^2 - 8yx_1x_2}{\sqrt{(x_1^2 + x_2^2)^2 - 16y(x_1x_2 - y)}}\right).$$

With $U(\mu) = (u_1 - \mu)(u_2 - \mu)$ we find that $(\det G_{\nwarrow})U'(u_1)\Delta_1 = 2y(u_1 - u_2)\Delta_1 = 4(1 - u_1^2)$ and $(\det G_{\nwarrow})U'(u_2)\Delta_2 = 2y(u_2 - u_1)\Delta_1 = 4(1 - u_2^2)$ depend only on one variable, as predicted by Proposition 3.12. So in this case we have $f_1 = f_2 = f$, where $f(u) = 4(1 - u^2)$.

The functions $W^{(k)}$, expressed in the new variables, take the form

$$\begin{aligned} W^{(0)} &= 2\,\frac{u_1 + u_2}{u_1u_2} = u_2\,\frac{-2/u_1}{U'(u_1)} + u_1\,\frac{-2/u_2}{U'(u_2)}, \\ W^{(1)} &= \frac{2}{u_1u_2} = \frac{-2/u_1}{U'(u_1)} + \frac{-2/u_2}{U'(u_2)}, \\ W^{(2)} &= 0, \end{aligned}$$

in accordance with Proposition 3.15.

We can now write down the integrals of motion $E^{(k)} = \frac{1}{2}\dot{q}^T A^{(k)}\,\dot{q} + W^{(k)}$ in terms of the variables (v, u_1, u_2). With $s_i = \dot{u}_i/\Delta_i$, we find

$$\begin{aligned} E^{(2)} &= \frac{\dot{v}^2}{2}, \\ E^{(1)} &= (u_1 + u_2)E^{(2)} + \frac{4(1 - u_1^2)\frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + \frac{4(1 - u_2^2)\frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)}, \\ E^{(0)} &= u_1u_2\,E^{(2)} + u_2\,\frac{4(1 - u_1^2)\frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + u_1\,\frac{4(1 - u_2^2)\frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)}. \end{aligned}$$

The new Hamiltonian is $h = E^{(1)}/\det G_{\nwarrow}(v(t))$, or, with $v(t) = t$,

$$h(u, s, t) = \frac{1}{2t}\left(\frac{u_1 + u_2}{2} + \frac{4(1 - u_1^2)\frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + \frac{4(1 - u_2^2)\frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)}\right).$$

The time-dependent Hamilton–Jacobi equation $h(u, \partial F/\partial u, t) + \partial F/\partial t = 0$ admits a separated complete solution of the form

$$F(u_1, u_2, \alpha_1, \alpha_2, t) = S_1(u_1, \alpha_0, \alpha_1) + S_2(u_2, \alpha_0, \alpha_1) - \frac{\alpha_1}{2}\ln|t|,$$

where S_1 and S_2 satisfy the separation equations

$$\frac{1}{2}\left(\frac{dS_1}{du_1}\right)^2 = \frac{\frac{2}{u_1} - \alpha_0 + \alpha_1 u_1 - \frac{u_1^2}{2}}{4(1-u_1^2)},$$
$$\frac{1}{2}\left(\frac{dS_2}{du_2}\right)^2 = \frac{\frac{2}{u_2} - \alpha_0 + \alpha_1 u_2 - \frac{u_2^2}{2}}{4(1-u_2^2)}.$$

From $\beta_k = \partial F/\partial \alpha_k$ we finally obtain

$$\beta_1(u_1,u_2,t,\alpha_0,\alpha_1) = \int^{u_1} \frac{x}{2R} dx + \int^{u_2} \frac{x}{2R} dx - \frac{1}{2}\ln|t|,$$
$$\beta_0(u_1,u_2,t,\alpha_0,\alpha_1) = \int^{u_1} \frac{-1}{2R} dx + \int^{u_2} \frac{-1}{2R} dx,$$

where

$$R(x,\alpha_1,\alpha_2) = \sqrt{2(1-x^2)\left(\frac{2}{x} - \alpha_0 + \alpha_1 x - \frac{x^2}{2}\right)}.$$

This gives the solution $u(\beta,\alpha,t)$ in implicit form.

Example 5.2 (A triangular system). An interesting example of a triangular cofactor system appears when applying the recursive method for constructing cofactor pair systems given in [4] to the matrices

$$G = \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}, \qquad \widetilde{G} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Starting with $W^{(0)} = W^{(1)} = 0$ and $W^{(2)} = -1$, one obtains after four steps the system

$$\begin{aligned} \ddot{q}_1 &= -4q_1, \\ \ddot{q}_2 &= 6q_1^2 - 4q_2, \\ \ddot{q}_3 &= -10q_1^3 + 12q_1q_2 - 4q_3, \end{aligned} \tag{5.2}$$

which is a cofactor pair system with respect to the given matrices G and $\widetilde{G}$. Since the third equation is driven by the first two, the system is also a cofactor pair system with respect to G and $J = \mathrm{diag}(0,0,1)$. In fact, the most general matrix G for which the system has an integral of motion of the form $\frac{1}{2}\dot{q}^T(\mathrm{cof}\, G)\dot{q} + W(q)$ is

$$c_1 \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so it might be called a "cofactor quadruple system." (The third matrix comes from the fact that there is a function $U(q)$ such that $M_2 = \partial_3 U$ and $M_3 = \partial_2 U$.) Anyway, we know from section 4 that the driving system is a cofactor system with respect to

$$G_{\nwarrow} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Since this matrix is constant, we cannot use it for integrating the driving system, but it so happens that the driving system is a cofactor system with respect to any matrix of the form

$$c_1 \begin{pmatrix} -1 & q_1 \\ q_1 & 2q_2 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, forgetting about (5.2) for the moment, we consider the two-dimensional driving system

$$\ddot{q} = \begin{pmatrix} -4q_1 \\ 6q_1^2 - 4q_2 \end{pmatrix} = -\frac{g\nabla w}{\det g}, \tag{5.3}$$

where now

$$g = \begin{pmatrix} -1 & q_1 \\ q_1 & 2q_2 \end{pmatrix}, \qquad w = \frac{3}{2}q_1^4 + 2q_1^2 q_2 - 2q_2^2.$$

(In this example, we use lowercase letters for quantities referring to the two-dimensional system (5.3).) In the new variables $v = q_1$ and $u = \det g / \det g_{\nwarrow} = q_1^2 + 2q_2$, we have the integrals of motion

$$e^{(1)} = \frac{1}{2}\dot{q}_1^2 + 2q_1^2 = \frac{1}{2}\dot{v}^2 + 2v^2$$

from the first equation, and (after a short calculation)

$$e^{(0)} = \frac{1}{2}\dot{q}^T(\operatorname{cof} g)\dot{q} + w(q) = ue^{(1)} - \frac{\dot{u}^2}{8} - \frac{u^2}{2}.$$

The function $v(t) = q_1(t)$ is just a harmonic oscillation, whose amplitude determines the numerical value of $e^{(1)}$ (or the other way around):

$$q_1(t) = \sqrt{\frac{e^{(1)}}{2}} \sin 2(t - t_1). \tag{5.4}$$

The value of $e^{(0)}$ is determined by the initial conditions for q_1 and q_2. Then $u(t)$, and hence $q_2(t) = (u(t) - v(t)^2)/2$, can be found from the separable ODE

$$\frac{du}{dt} = \sqrt{8\left(ue^{(1)} - \frac{u^2}{2} - e^{(0)}\right)}.$$

This gives

$$u(t) = \sqrt{(e^{(1)})^2 - 2e^{(0)}} \sin 2(t - t_2) + e^{(1)},$$

so that

$$q_2(t) = \frac{1}{2}\left(\sqrt{(e^{(1)})^2 - 2e^{(0)}} \sin 2(t - t_2) + e^{(1)}\left(1 - \frac{1}{2}\sin^2 2(t - t_1)\right)\right). \tag{5.5}$$

Having found $q_1(t)$ and $q_2(t)$, we return to the three-dimensional system (5.2):

$$\ddot{q} = \begin{pmatrix} -4q_1 \\ 6q_1^2 - 4q_2 \\ -10q_1^3 + 12q_1q_2 - 4q_3 \end{pmatrix} = -\frac{G\nabla W}{\det G},$$

where

$$G = \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}, \qquad W = 6q_1^2 q_2^2 - 4q_1^4 q_2 + 4q_1 q_2 q_3 - 2q_3^2 - 4q_1^3 q_3.$$

Here we take new variables $v_1 = q_1$, $v_2 = q_2$, and $u = \det G / \det G_{\nwarrow} = 2(q_1 q_2 + q_3)$. The integrals of motion turn out to be

$$E^{(1)} = \frac{1}{2}\dot{v}^T \operatorname{cof} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \dot{v} + 4v_1 v_2 - 2v_1^3$$

and

$$E^{(0)} = \frac{1}{2}\dot{q}^T (\operatorname{cof} G)\dot{q} + W(q) = uE^{(1)} - \frac{\dot{u}^2}{8} - \frac{u^2}{2},$$

so the equation for $u(t)$ can again be separated (in exactly the same way as above). After finding $u(t)$, we finally obtain $q_3(t) = u(t)/2 - q_1(t)q_2(t)$, that is

$$q_3(t) = \frac{1}{2}\left(\sqrt{E^{(1)} - 2E^{(0)}} \sin 2(t - t_3) + E^{(1)}\right) - q_1(t)q_2(t). \tag{5.6}$$

By inserting the expressions for $q_1(t)$ and $q_2(t)$ into the expression for $E^{(1)}$ we find that it depends on the previous integration constants $e^{(0)}$, $e^{(1)}$, t_1, t_2 through the equation

$$E^{(1)} = \sqrt{2e^{(1)}}\sqrt{e^{(1)} - 2e^{(0)}} \cos 2(t_2 - t_1).$$

On the other hand, $E^{(0)}$ and t_3 are independend of the previous integration constants.

Example 5.3 (Construction of driven systems). Given a cofactor system

$$\ddot{y} = M_{\uparrow}(y) = -\big(\operatorname{cof} g(y)\big)^{-1} \frac{\partial w}{\partial y}(y),$$

how can it be extended to a driven system

$$\ddot{q} = \begin{pmatrix} M_{\uparrow}(y) \\ M_{\downarrow}(y, x) \end{pmatrix} = -\big(\operatorname{cof} G(q)\big)^{-1} \frac{\partial W}{\partial q}(q)$$

of the type considered in this paper? First of all, the restriction that the elliptic coordinates matrix $G(q)$ must have $g(y) = G_{\nwarrow}(y)$ as its upper left block fixes α, $\beta_{\uparrow}$ and $\gamma_{\nwarrow}$. The remaining entries of β and γ can be chosen at will (as long as G is nonsingular). Then we want to find some extension $M_{\downarrow}$ of the right-hand side which is compatible with the chosen matrix G (i.e., so that $W(q)$ exists). In separation coordinates, this amounts to specifying the functions $g_k(u_k)$ in the corresponding solution of the fundamental equations (proposition 3.14), the function $w(v) = w(y)$ already being determined by the driving system. One can find a family of possible $M_{\downarrow}$ in Cartesian coordinates directly by using the recursion formula from [4]. As it stands, this formula requires $\widetilde{G}$ to be nonsingular, but taking $\widetilde{G} = J$ can be justified like in the proof of proposition 3.7 (however, it only makes sense in the "downwards" recursion formula). We then find that if a driven system has integrals of motion given by the generating function $E_\mu = \frac{1}{2}\dot{q}^T A_\mu \dot{q} + W_\mu$ as in (3.11), then we obtain another driven system with integrals of motion $\frac{1}{2}\dot{q}^T A_\mu \dot{q} + U_\mu$ by setting

$$U_\mu = \frac{1}{\mu}\left(\frac{\det(G+\mu J)}{\det G} W^{(0)} - W_\mu\right). \tag{5.7}$$

It is clear that U_μ is a polynomial in μ of degree $n-1$, not n, which means that the new system (and any system obtained by iterating this process) is driven in the trivial way ($\ddot{y} = 0$). They correspond to solutions (3.41) of the fundamental equations with $w(y) = 0$. Adding

$$\frac{\det(G+\mu J)}{\det G_\nwarrow} w(y)$$

to U_μ gives a system with any $w(y)$ desired.

As an example, consider the two-dimensional Garnier potential $V = (q_1^2 + q_2^2)^2 - (\lambda_1 q_1^2 + \lambda_2 q_2^2)$. We will demonstrate how to find G and M_3 such that the system

$$\begin{aligned} \ddot{q}_1 &= -\partial_1 V(q_1, q_2), \\ \ddot{q}_2 &= -\partial_2 V(q_1, q_2), \\ \ddot{q}_3 &= M_3(q_1, q_2, q_3) \end{aligned} \tag{5.8}$$

is of cofactor type $\ddot{q} = -\frac{G}{\det G}\nabla W$. With $G_\nwarrow = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, corresponding to $w = V$, we have $\alpha = \beta_1 = \beta_2 = 0$, so we choose for example $\beta_3 = 1$ and extend γ with zeros to get

$$G = \begin{pmatrix} 1 & 0 & q_1 \\ 0 & 1 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}.$$

Applying (5.7) with $W_\mu = 1 + 0\mu$ and $J = \operatorname{diag}(0,0,1)$ gives $U_\mu = (\det G)^{-1}$, corresponding to the trivially driven system

$$\ddot{q} = -\frac{G}{\det G}\nabla\left(\frac{1}{\det G}\right) = \frac{2}{(2q_3 - q_1^2 - q_2^2)^2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To keep things simple we stop the recursion after this first step, and let

$$E_\mu = \frac{1}{2}\dot{q}^T \operatorname{cof}(G+\mu J)\dot{q} + (\det G)^{-1} + \frac{\det(G+\mu J)}{\det G_\nwarrow} V(q_1, q_2),$$

which then generates an extended system of the desired form,

$$\ddot{q} = -\frac{G}{\det G}\nabla\left((\det G)^{-1} + (\det G)V\right) = \begin{pmatrix} -\partial_1 V \\ -\partial_2 V \\ 2(2q_3 - q_1^2 - q_2^2)^{-2} - 2V \end{pmatrix}. \tag{5.9}$$

Since the Garnier potential is separable in elliptic coordinates it admits an extra integral of motion of cofactor type. This gives us the possibility to instead take

$$G_\nwarrow = \begin{pmatrix} \lambda_1 - q_1^2 & -q_1 q_2 \\ -q_1 q_2 & \lambda_2 - q_2^2 \end{pmatrix},$$

corresponding to

$$w = \lambda_2 q_1^4 + \lambda_1 q_2^4 + (\lambda_1 + \lambda_2) q_1^2 q_2^2 - \lambda_1 \lambda_2 (q_1^2 + q_2^2).$$

Here $\alpha = -1$ and $\beta_{\uparrow} = 0$, and we can for example extend $G_{\nwarrow}$ to

$$G = \begin{pmatrix} \lambda_1 - q_1^2 & -q_1 q_2 & -q_1 q_3 \\ -q_1 q_2 & \lambda_2 - q_2^2 & -q_2 q_3 \\ -q_1 q_3 & -q_2 q_3 & \lambda_3 - q_3^2 \end{pmatrix}.$$

In a similar way as above we get in this case (after some computation) the extended system

$$\begin{aligned} \ddot{q} &= -\frac{G}{\det G} \nabla \left(\frac{\det G_{\nwarrow}}{\det G} + \frac{\det G}{\det G_{\nwarrow}} w \right) \\ &= \begin{pmatrix} -\partial_1 V \\ -\partial_2 V \\ \lambda_1 \lambda_2 \, q_3 \left(\frac{2w(q_1, q_2)}{(\det G_{\nwarrow})^2} - \frac{2}{(\det G)^2} + \frac{q_1 (\partial_1 V)/\lambda_1 + q_2 (\partial_2 V)/\lambda_2}{\det G_{\nwarrow}} \right) \end{pmatrix}, \end{aligned} \tag{5.10}$$

where $\det G = \lambda_1 \lambda_2 \lambda_3 (1 - q_1^2/\lambda_1 - q_2^2/\lambda_2 - q_3^2/\lambda_3)$ and $\det G_{\nwarrow} = \lambda_1 \lambda_2 (1 - q_1^2/\lambda_1 - q_2^2/\lambda_2)$.

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Paper 3

Multiplicative structure of cofactor pair systems in Riemannian spaces

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Abstract

This paper deals with the explicit construction of cofactor pair system on Riemannian manifolds, as introduced by Crampin and Sarlet. We show that the recursion formula known in the Euclidean case holds in this more general setting as well. This is then generalized to a "multiplication" formula mapping two cofactor pair systems to a third one. As a special case, this formula contains the fact that the product of two holomorphic functions is again holomorphic.

1 Introduction

Cofactor systems and cofactor pair systems were introduced in Euclidean space R^n by the author [1], extending previous work in the two-dimensional case [2]. They were generalized to Riemannian manifolds Q by Crampin and Sarlet [3], who at the same time contributed a better geometric understanding of these systems, which are mechanical systems given by the "non-conservative" form of Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = M_i(q), \tag{1.1}$$

with force $M = M_i(q)\,dq^i$ of a particular form; see definitions 2.10 and 2.11 below. (Summation over repeated indices is understood, as usual.) Here

$$T(q, \dot{q}) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j \tag{1.2}$$

is the kinetic energy, with g_{ij} denoting the metric tensor on Q. Recall that in the "conservative" case $M_i = -\partial V(q)/\partial q^i$ the equations take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \tag{1.3}$$

with $L = T - V$, and the total energy $E = T + V$ is conserved. In the case considered here, the M_i have a different form, which nevertheless guarantees the existence of an "energy-like" integral of motion (quadratic in the velocity $\dot{q}$). Consequently, the systems are in fact conservative in a sense, despite the terminology.

The system (1.1) can be written in the equivalent form $\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = M^i$, where $M^i = g^{ij}M_j$, and Γ^i_{jk} is the the Christoffel symbol. In the Euclidean case this reduces to the vector equation $\ddot{q} = M(q)$, which was considered in [2, 1].

2 Preliminaries

This section contains all the background material needed to state and prove our new results. We define cofactor systems and cofactor pair systems, and quote without proof some known facts [1, 3]. We will mostly follow the notation of [3]. Throughout, Q denotes an n-dimensional Riemannian manifold with coordinates q^i and metric g_{ij}. We raise and lower indices freely using the metric, so that we consider for example A_{ij}, A^i_j and A^{ij} as different version of the same tensor A.

2.1 Quadratic integrals of motion

We are interested in integrals of motion of the system (1.1) which are quadratic in the velocity components q^i. The basic fact is the following.

Proposition 2.1. *A function E on the tangent bundle TQ, of the form*

$$E(q, \dot{q}) = \frac{1}{2} A_{ij}(q) \dot{q}^i \dot{q}^j + W(q), \tag{2.1}$$

with $A_{ij} = A_{ji}$, is an integral of motion of the system (1.1) *if and only if*

1. *The tensor A is a Killing tensor.*
2. *The force components M_i and the "quasi-potential" W are connected by*

$$A^i_j M_i + \frac{\partial W}{\partial q^j} = 0. \tag{2.2}$$

(Note that if A is nonsingular, then the integral of motion E determines the force M uniquely.)

We remind the reader that a (rank two) *Killing tensor* (with respect to the metric g) is a symmetric tensor A_{ij} such that

$$\nabla_i A_{jk} + \nabla_k A_{ij} + \nabla_j A_{ki} = 0 \tag{2.3}$$

for all i, j, k. The symbol ∇ denotes the covariant derivative associated to the Levi–Civita connection of g. Equivalently, A is a Killing tensor iff $\{H_A, H_g\} = 0$, where $H_A = \frac{1}{2} A^{ij} p_i p_j$, $H_g = \frac{1}{2} g^{ij} p_i p_j$, and $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket between functions on the cotangent bundle T^*Q. (That is, H_A is an integral of motion for the geodesic equations, which are the canonical equations generated by the Hamiltonian H_g.)

2.2 SCK tensors and associated operators

Of special importance to us are Killing tensors "of cofactor type," which are constructed using so-called SCK tensors [3, 4].

Definition 2.2 (Special conformal Killing (SCK) tensor). A symmetric tensor J_{ij} satisfying

$$\nabla_k J_{ij} = \frac{1}{2} (\alpha_i g_{jk} + \alpha_j g_{ik}) \tag{2.4}$$

for some 1-form $\alpha = \alpha_k \, dq^k$ is called a *special conformal Killing tensor* (*SCK tensor* for short). We say that α is the 1-form associated to the SCK tensor J.

Recall that a symmetric tensor B_{ij} on Q is called a *conformal Killing tensor* if

$$\nabla_i B_{jk} + \nabla_k B_{ij} + \nabla_j B_{ki} = c_i g_{jk} + c_k g_{ij} + c_j g_{ki} \tag{2.5}$$

for some 1-form $c = c_k \, dq^k$. If $c = df$ for some function f, then B is called a conformal Killing tensor of *gradient type*. It is easily verified that an SCK tensor J is indeed a conformal Killing tensor, and that it is of gradient type with $c = \alpha = d(\operatorname{tr} J) = d(J^i_i)$.

Proposition 2.3. *Let J be an SCK tensor. Then the identity $J^* d(\det J) = (\det J)\alpha$ holds. The Nijenhuis torsion N_J is zero. If J is nonsingular, its cofactor tensor $A = \operatorname{cof} J$ is a Killing tensor. (Such a Killing tensor is said to be of* cofactor type.*)*

Remark 2.4. Proposition 2.3 involves the following concepts, which are well defined for $(1,1)$ tensors J^i_j on any manifold:

1. The determinant $\det J$; it is just the determinant of the matrix of components J^i_j in any coordinate system.
2. The cofactor tensor $A = \operatorname{cof} J$; it is defined by letting A^i_j be the usual cofactor matrix of J^i_j, such that $A^i_k J^k_j = (\det J)\delta^i_j$.
3. The adjoint (or transpose) J^*. While J acts on vectors so that JX is the vector with components $(JX)^i = J^i_j X^j$, the adjoint J^* acts on 1-forms so that $J^*\alpha$ is the 1-form with components $(J^*\alpha)_j = J^i_j \alpha_i$.
4. The Nijenhuis torsion N_J; this is the $(1,2)$ tensor defined by

   $$N_J(X,Y) = [JX, JY] - J\big([JX, Y] + [X, JY] - J[X,Y]\big),$$

   for vectors X and Y (the bracket is the usual Lie bracket of vector fields on Q).

Since we are on a Riemannian manifold, we can compute the determinant, cofactor tensor, adjoint, and Nijenhuis torsion of any rank 2 tensor by first raising or lowering an index if necessary. In the case of the cofactor tensor, we often lower or raise that index back afterwards. If J_{ij} is symmetric, then the tensor A_{ij} obtained in this way (raising an index on J_{ij}, computing $A^i_j = \operatorname{cof} J^i_j$, and lowering an index on A^i_j) is also symmetric.

Remark 2.5. Any Riemannian manifold admits a trivial SCK tensor, namely the metric $J = g$. A cofactor system with $J = g$ is nothing but a conservative system (1.3) with W coinciding with the usual potential V. Further SCK tensors may or may not exist, depending on the metric g. In the Euclidean case (with Cartesian coordinates, so that we need not bother about upper indices), an SCK tensor has the form

$$J_{ij} = a\, q_i q_j + b_i q_j + b_j q_i + c_{ij}$$

for arbitrary constants a, b_i, $c_{ij} = c_{ji}$, so at least R^n admits plenty of SCK tensors. (In [1] such a tensor J_{ij} was called an *elliptic coordinates matrix* and denoted by G_{ij}. The associated 1-form $\alpha = 2(aq_i + b_i)dq^i$, was denoted by $2N$ and written as a column vector.)

Next, following Crampin and Sarlet, we define two useful linear operators associated to an SCK tensor: d_J, which is a "derivation of type d_* in the sense of Frölicher–Nijenhuis theory," and a related operator D_J. Both of these act on the exterior algebra of differential forms on Q, although here we will only need to apply them to functions and 1-forms.

Definition 2.6 (Operator d_J). Let J be a nonsingular SCK tensor. The operator d_J is defined by the following properties:

1. It is a graded derivation of degree 1, i.e., it takes k-forms to $(k+1)$-forms, and

$$d_J(\alpha \wedge \beta) = (d_J\alpha) \wedge \beta + (-1)^k \alpha \wedge (d_J\beta) \tag{2.6}$$

   when α is a k-form.

2. It anti-commutes with the exterior derivative d,

$$d_J d + d d_J = 0. \tag{2.7}$$

3. It acts on functions (0-forms) according to

$$d_J f = J^* df = J^i_j \frac{\partial f}{\partial q^i} dq^j. \tag{2.8}$$

Using these properties one easily derives the action of d_J on a 1-form β, which we write down here for the reader's convenience:

$$d_J(\beta_a dq^a) = \left(J^i_a \frac{\partial \beta_b}{\partial q^i} - \frac{\partial J^i_b}{\partial q^a} \beta_i \right) dq^a \wedge dq^b.$$

Proposition 2.7. *The operator d_J satisfies $d_J^2 = 0$ (since $N_J = 0$). The condition $d_J\theta = 0$ for a k-form θ is necessary and sufficient for the local existence of a $(k-1)$-form ϕ such that $\theta = d_J\phi$ (in other words, "d_J-closed" is equivalent to "locally d_J-exact").*

Definition 2.8 (Operator D_J). Let J be a nonsingular SCK tensor. The operator D_J is defined by

$$D_J\theta := \frac{d_J\big((\det J)\theta\big)}{\det J} = d_J\theta + \alpha \wedge \theta. \tag{2.9}$$

Here α is the 1-form associated with J, and the equality of the two expressions follows from the property $J^* d(\det J) = (\det J)\alpha$ in proposition 2.3.

Proposition 2.9. *The operator D_J is not a derivation, but $D_J^2 = 0$ and "D_J-closed" is equivalent to "locally D_J-exact."*

2.3 Cofactor systems

Let A be a Killing tensor. Recall from proposition 2.1 that if $A^i_j M_i + \frac{\partial W}{\partial q^j} = 0$, or in other words, if $A^* M + dW = 0$, then $E = \frac{1}{2} A_{ij} \dot{q}^i \dot{q}^j + W$ is an integral of motion of the system (1.1). Restricting attention to Killing tensors of cofactor type, where $A = \operatorname{cof} J$ for a nonsingular SCK tensor J, this condition takes the form

$$M = -(A^*)^{-1} dW = -\frac{J^* dW}{\det J} = -\frac{d_J W}{\det J} = -D_J \left(\frac{W}{\det J} \right). \tag{2.10}$$

Given M, the condition $D_J M = 0$ is sufficient for a function $W' = W/\det J$ to exist (locally), such that $M = -D_J W'$. This leads us to the following definition.

Definition 2.10 (Cofactor system). A *cofactor system* is a system of the form (1.1) where the 1-form $M = M_k \, dq^k$ satisfies $D_J M = 0$ for some nonsingular SCK tensor J.

So a cofactor system has a force M of the form (2.10), and it always admits an integral of motion of cofactor type, i.e., of the following form:

$$E(q, \dot{q}) = \frac{1}{2} \big(\operatorname{cof} J(q) \big)_{ij} \dot{q}^i \dot{q}^j + W(q). \tag{2.11}$$

The force M is uniquely determined by E according to (2.10).

2.4 Cofactor pair systems

A cofactor pair system is a system which is a cofactor system in two different ways, or, equivalently, which admits two independent integrals of motion of cofactor type. Somewhat surprisingly, this implies the existence of n integrals of motion, as we will see below. As one may suspect from this, cofactor pair systems are indeed completely integrable (but in a slightly nonstandard sense, see [1, 3]).

Definition 2.11 (Cofactor pair system). A *cofactor pair system* is a system of the form (1.1) where the 1-form $M = M_k \, dq^k$ satisfies $D_J M = D_{\widetilde{J}} M = 0$ for two independent nonsingular SCK tensors J and $\widetilde{J}$.

Remark 2.12. Cofactor pair systems with either J or $\widetilde{J}$ equal to the metric, say $\widetilde{J} = g$, have a potential $\widetilde{W} = V$ (cf. remark 2.5) which is *separable* in the Hamilton–Jacobi sense. The separation coordinates are given by the eigenvalues of the nontrivial SCK tensor J, at least if these are all distinct. Coinciding eigenvalues in J indicates partial separability of V. This is discussed in detail in the Euclidean case in [5].

Cofactor pair systems are best analyzed using properties of the differential operator D_J, as in [3]. This powerful formalism gives much simpler proofs of many of the statements in [1]. To begin with, $J_\mu = J + \mu \widetilde{J}$ is again an SCK tensor for any constant μ, with corresponding 1-form $\alpha_\mu = \alpha + \mu \widetilde{\alpha}$. Moreover, since $D_J = (d_J + \alpha \wedge)$ and $D_J^2 = D_{\widetilde{J}}^2 = D_{J+\mu\widetilde{J}}^2 = 0$, it follows that

$$D_{J_\mu} = D_J + \mu D_{\widetilde{J}} \tag{2.12}$$

and

$$D_J D_{\widetilde{J}} + D_{\widetilde{J}} D_J = 0. \tag{2.13}$$

Since $D_J M = D_{\widetilde{J}} M = 0$, there are functions W and $\widetilde{W}$ such that

$$M = -D_J \left(\frac{W}{\det J} \right) = -D_{\widetilde{J}} \left(\frac{\widetilde{W}}{\det \widetilde{J}} \right), \tag{2.14}$$

so a cofactor system admits two integrals of motion of cofactor type,

$$E = \frac{1}{2} (\operatorname{cof} J)_{ij} \dot{q}^i \dot{q}^j + W(q) \qquad \text{and} \qquad \widetilde{E} = \frac{1}{2} (\operatorname{cof} \widetilde{J})_{ij} \dot{q}^i \dot{q}^j + \widetilde{W}(q). \tag{2.15}$$

Theorem 2.13 (Fundamental equation). *With the notation above, the two functions $W/\det J$ and $\widetilde{W}/\det \widetilde{J}$ both satisfy the* fundamental equation *associated to the pair $(J, \widetilde{J})$,*

$$D_J D_{\widetilde{J}} \phi = 0. \tag{2.16}$$

Conversely, to a given solution ϕ of the fundamental equation there corresponds two cofactor pair systems, one with $M = -D_J \phi$ and another with $M = -D_{\widetilde{J}} \phi$.

Proof. This follows immediately from (2.14), (2.13) and the property $D_J^2 = D_{\widetilde{J}}^2 = 0$. □

Theorem 2.14 (Two implies n). *A cofactor pair system admits n integrals of motion*

$$E^{(k)} = \frac{1}{2} A^{(k)}_{ij} \dot{q}^i \dot{q}^j + W^{(k)}(q), \qquad k = 0, 1, \ldots, n-1, \tag{2.17}$$

where the Killing tensors $A^{(0)}, \ldots, A^{(n-1)}$ are defined by the generating function

$$A_\mu = \sum_{k=0}^{n-1} A^{(k)} \mu^k = \operatorname{cof}(J + \mu \widetilde{J}) = \operatorname{cof} J_\mu. \tag{2.18}$$

Proof. For any μ such that J_μ is nonsingular, the following holds. Since $D_{J_\mu} M = D_J M + \mu D_{\widetilde{J}} M = 0$, there exists W_μ such that

$$M = -D_{J_\mu} \left(\frac{W_\mu}{\det J_\mu} \right), \tag{2.19}$$

or (cf. (2.10))

$$dW_\mu = -A_\mu^* M. \tag{2.20}$$

Since $A_\mu = \operatorname{cof} J_\mu$ depends polynomially on μ, so does W_μ. Moreover, $E_\mu = \frac{1}{2} (A_\mu)_{ij} \dot{q}^i \dot{q}^j + W_\mu$ is an integral of motion. It follows that the coefficients $E^{(k)}$ at different powers μ^k all are integrals of motion. □

This proof, which closely follows [3], is reproduced here since the polynomial W_μ, which is the generating function of the "quasi-potentials" $W^{(k)}$, will be important in what follows.

It is worth pointing out that $E^{(0)} = E$ and $E^{(n-1)} = \widetilde{E}$, at the ends of the "cofactor chain" E_μ of integrals, are the two integrals of motion of cofactor type whose existence are immediately implied by the definition of a cofactor pair system.

3 Multiplicative structure

This section contains our new results.

Crampin and Sarlet do not discuss the construction, or even existence, of cofactor pair systems. However, they hint at the fact that the operators D_J and $D_{\widetilde{J}}$ constitute a "gauged bi-differential calculus" (in the terminology of [6]), which makes it possible to recursively construct families of cofactor pair systems. This was actually done in the Euclidean case in [1], although with a less transparent notation and not using that terminology.

3.1 Recursive construction of cofactor pair systems

Suppose the manifold Q admits two independent SCK tensors J and $\widetilde{J}$. We will construct a bi-infinite family of cofactor pair systems, all with the same SCK tensors J and $\widetilde{J}$, but with different forces M_m ($m = 0, \pm 1, \pm 2, \dots$). (In special cases the sequence may be periodic, as we will see below, but generically all M_m are different.)

For readers familiar with the bi-differential calculi introduced by Dimakis and Müller-Hoissen [6], it might be helpful to note that our M_m and ϕ_m correspond to their $\chi^{(m)}$ and $J^{(m)}$, respectively. A curious feature here is that we are using the bi-differential calculus for jumping between different integrable systems, whereas in [6] it is used for jumping between different integrals of motion (or conservation laws) of one single integrable system.

We want to find sequences of functions W_m and $\widetilde{W}_m$ such that

$$M_m = -D_J\left(\frac{W_m}{\det J}\right) = -D_{\widetilde{J}}\left(\frac{\widetilde{W}_m}{\det \widetilde{J}}\right) \qquad (m = 0, \pm 1, \pm 2, \dots). \tag{3.1}$$

To begin with, there exists a trivial cofactor pair system, namely the system of geodesic equations, obtained by taking $M = 0$ in (1.1). So we let $M_0 = 0$, which is accomplished by taking W_0 and $\widetilde{W}_0$ constant, for instance

$$W_0 = \widetilde{W}_0 = 1.$$

Now we define W_1 so that the relation

$$\frac{W_1}{\det J} = \frac{\widetilde{W}_0}{\det \widetilde{J}} \tag{3.2}$$

holds, i.e., we set $W_1 = \frac{\det J}{\det \widetilde{J}}\widetilde{W}_0$. Then

$$D_{\widetilde{J}}\left[D_J\left(\frac{W_1}{\det J}\right)\right] = -D_J D_{\widetilde{J}}\left(\frac{\widetilde{W}_0}{\det \widetilde{J}}\right) = -D_J D_J\left(\frac{W_0}{\det J}\right) = 0,$$

so there exists a function $\widetilde{W}_1'$ such that $D_J\left(\frac{W_1}{\det J}\right) = D_{\widetilde{J}}(\widetilde{W}_1')$. Setting $\widetilde{W}_1 = \widetilde{W}_1' \det \widetilde{J}$ we obtain

$$D_J\left(\frac{W_1}{\det J}\right) = D_{\widetilde{J}}\left(\frac{\widetilde{W}_1}{\det \widetilde{J}}\right) =: -M_1.$$

Similarly, we define $W_2 = \frac{\det J}{\det \widetilde{J}} \widetilde{W}_1$ and so on, and obtain the upper part $(M_m)_{m\geq 0}$ of the sequence. Simply by reversing the roles of W and $\widetilde{W}$ we can go backwards as well, obtaining the lower part $(M_m)_{m\leq 0}$ of the sequence. Explicitly, we define $\widetilde{W}_{-1}$ through the relation

$$\frac{\widetilde{W}_{-1}}{\det \widetilde{J}} = \frac{W_0}{\det J}, \tag{3.3}$$

or $\widetilde{W}_{-1} = \frac{\det \widetilde{J}}{\det J} W_0$, and determine W_{-1} from the relation

$$D_J\left(\frac{W_{-1}}{\det J}\right) = D_{\widetilde{J}}\left(\frac{\widetilde{W}_{-1}}{\det \widetilde{J}}\right) =: -M_{-1}.$$

Then we set $\widetilde{W}_{-2} = \frac{\det \widetilde{J}}{\det J} W_{-1}$, etc.

We can also think of this procedure as producing an infinite sequence of solutions $\phi_m = \frac{W_m}{\det J}$ of the fundamental equation. Adjacent solutions are related through

$$D_J \phi_m = D_{\widetilde{J}} \phi_{m+1} \quad (= -M_m). \tag{3.4}$$

3.2 The recursion formula

In the procedure described above, W_{k+1} was obtained algebraically from $\widetilde{W}_k$, but it was necessary to perform an integration to obtain $\widetilde{W}_{k+1}$. In [1] it was shown, in the Euclidean case, how this integration can be avoided by in each step keeping track of all the "quasi-potentials" $W^{(0)}, \ldots, W^{(n-1)}$ from theorem 2.14, and not just the outermost ones $W = W^{(0)}$ and $\widetilde{W} = W^{(n-1)}$. As we will show here, exactly the same formula (3.5) is still valid in the more general Riemannian setting. Actually, it is just a special case of theorem 3.4 below, but we give a separate proof for comparison with the proof in [1]. It is remarkable how much simpler the proof becomes with the use of the operator D_J.

From now on, we think of a cofactor pair system as given by its integrals of motion (encoded in the polynomial $W_\mu = \sum_{i=0}^{n-1} W^{(i)} \mu^i$) rather than by the force M. The SCK tensors J and $\widetilde{J}$ are fixed throughout. There is a purely algebraic relation between polynomials $(W_\mu)_k$ and $(W_\mu)_{k+1}$ corresponding to adjacent cofactor pair systems M_k and M_{k+1} in the above sequence, as the following theorem shows.

Theorem 3.1 (Recursion formula). *Adjacent cofactor pair systems in the sequence defined above are related through*

$$(W_\mu)_{k+1} = -\mu\,(W_\mu)_k + \frac{\det(J + \mu\widetilde{J})}{\det \widetilde{J}} \widetilde{W}_k. \tag{3.5}$$

Since $\widetilde{W}_k = W_k^{(n-1)}$ is the highest coeffiecient in $(W_\mu)_k$, the right hand side of (3.5) can – and should, as we will see below – be viewed as multiplication by $-\mu$, followed by reduction modulo the nth degree polynomial $\det(J + \mu\widetilde{J})/\det \widetilde{J}$ in order to obtain a polynomial of degree $n-1$ in μ.

For the proof, we need the following lemma.

Lemma 3.2. *The functions $W^{(0)}, \dots, W^{(n-1)}$ appear in the integrals of motion of a given $(J, \widetilde{J})$ cofactor pair system (as in theorem 2.14) if and only if the polynomial $W_\mu = \sum_{i=0}^{n-1} W^{(i)} \mu^i$ satisfies*

$$M = -D_{J_\mu}\left(\frac{W_\mu}{\det J_\mu}\right), \tag{3.6}$$

where $J_\mu = J + \mu \widetilde{J}$.

Proof. This is clear from the proof of theorem 2.14; see in particular (2.19). □

Proof of theorem 3.1. To begin with, identifying the coefficient at the lowest power μ^0 in (3.5) we see that $W_{k+1} = \frac{\det J}{\det \widetilde{J}} \widetilde{W}_k$, which agrees with the recursive definition of the sequence $(W_m, \widetilde{W}_m)$; cf. (3.2). Moreover, applying lemma 3.2 to M_k we obtain

$$\begin{aligned}
-D_{J_\mu}\left(\frac{(W_\mu)_{k+1}}{\det J_\mu}\right) &= D_{J_\mu}\left(\mu \frac{(W_\mu)_k}{\det J_\mu}\right) - D_{J_\mu}\left(\frac{1}{\det J_\mu} \frac{\det J_\mu}{\det \widetilde{J}} \widetilde{W}_k\right) \\
&= -\mu M_k - D_J\left(\frac{\widetilde{W}_k}{\det \widetilde{J}}\right) - \mu D_{\widetilde{J}}\left(\frac{\widetilde{W}_k}{\det \widetilde{J}}\right) \\
&= -\mu M_k - D_J\left(\frac{W_{k+1}}{\det J}\right) - \mu(-M_k) \\
&= M_{k+1}.
\end{aligned}$$

Using lemma 3.2 in the opposite direction, we conclude that the remaining coefficients in $(W_\mu)_{k+1}$ are correct as well. □

3.3 The multiplication formula

The following example was given in [2], where the present theory was first developed in R^2.

Example 3.3. Let $Q = R^2$ with coordinates (x, y), and define the SCK tensors

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding cofactor pair systems take the usual form (2.14), which here translates into the following condition for the functions W and $\widetilde{W}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial W/\partial x \\ \partial W/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial \widetilde{W}/\partial x \\ \partial \widetilde{W}/\partial y \end{pmatrix}.$$

This is nothing but the Cauchy–Riemann equations, so $W + i\widetilde{W}$ is a holomorphic function. In other words, cofactor pair systems with this choice of J and $\widetilde{J}$ are in one-to-one correspondence with holomorphic functions. The recursion formula (3.5) takes $W + i\widetilde{W}$ to $\widetilde{W} - iW$, so it corresponds to multiplying a holomorphic function by i. Consequently, the recursively constructed sequence of cofactor pair systems has period 4 in this case.

Inspired by this example, one might wonder if it is possible to "multiply" two different cofactor pair systems, just as the product of two holomorphic functions is a holomorphic function. It turns out that this is the case, and the formula is very simple. We need only multiply the corresponding polynomials and reduce modulo a certain nth degree polynomial so that we obtain a new polynomial of degree $n-1$.

Theorem 3.4 (Multiplication formula). *If U_μ and V_μ are polynomials associated (as in lemma 3.2) to cofactor pair systems with the same pair $(J, \widetilde{J})$, then so is*

$$W_\mu = U_\mu V_\mu \bmod P_\mu, \tag{3.7}$$

where $P_\mu = \det J_\mu / \det \widetilde{J}$ is a polynomial in μ of degree n, and $F_\mu \bmod P_\mu$ denotes the remainder in the polynomial division F_μ / P_μ.

Proof. The condition in lemma 3.2 for W_μ to correspond to a cofactor pair system can be reformulated as $M = -\frac{1}{\det J_\mu} d_{J_\mu} W_\mu$, or

$$d_{J_\mu} W_\mu = -(\det \widetilde{J}) P_\mu \, M, \quad \text{for some } M.$$

This, in turn, is equivalent to

$$d_{J_\mu} W_\mu \equiv 0 \pmod{P_\mu}, \tag{3.8}$$

since $d_{J_\mu} W_\mu$ and P_μ are both of degree n in μ.

So it suffices to show that W_μ, defined by (3.7), satisfies (3.8) whenever U_μ and V_μ do. We have $W_\mu = U_\mu V_\mu - Q_\mu P_\mu$, where Q_μ is the quotient in the polynomial division $(U_\mu V_\mu)/P_\mu$. Hence, assuming that U_μ and V_μ satisfy (3.8), and applying the formula $d_J(\det J) = (\det J)\alpha$ (from proposition 2.3) to J_μ, we find

$$\begin{aligned}
d_{J_\mu} W_\mu &= (d_{J_\mu} U_\mu) V_\mu + U_\mu (d_{J_\mu} V_\mu) - (d_{J_\mu} Q_\mu) P_\mu - Q_\mu (d_{J_\mu} P_\mu) \\
&\equiv 0\, V_\mu + U_\mu\, 0 - 0 - Q_\mu\, d_{J_\mu} \left(\frac{\det J_\mu}{\det \widetilde{J}} \right) \\
&= -Q_\mu \left(\frac{(\det J_\mu)(\alpha + \mu \widetilde{\alpha})}{\det \widetilde{J}} - (\det J_\mu) \frac{d_{J_\mu}(\det \widetilde{J})}{\det^2 \widetilde{J}} \right) \\
&= -Q_\mu P_\mu \left(\alpha + \mu \widetilde{\alpha} - \frac{d_{J_\mu}(\det \widetilde{J})}{\det \widetilde{J}} \right) \\
&\equiv 0 \pmod{P_\mu},
\end{aligned}$$

as was to be shown. □

4 Examples

Example 4.1. In example 3.3 we have $P_\mu = -\mu^2 - 1$, which means that multiplication of first degree polynomials modulo P_μ is equivalent to multiplication of complex numbers. Multiplying two cofactor pair systems using (3.7) is in this case equivalent to multiplying two holomorphic functions.

Example 4.2 (Parabolic separable potentials). Consider $Q = R^3$ with coordinates (x, y, z), and choose the SCK tensors

$$J = \begin{pmatrix} \lambda_1 & 0 & x \\ 0 & \lambda_2 & y \\ x & y & 2z \end{pmatrix} \qquad \text{and} \qquad \widetilde{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding Killing tensors are given by $A_\mu = \operatorname{cof}(J + \mu\widetilde{J})$, i.e.,

$$A^{(0)} = \begin{pmatrix} 2\lambda_2 z - y^2 & xy & -\lambda_2 x \\ xy & 2\lambda_1 z - x^2 & -y \\ -\lambda_2 x & -y & \lambda_1\lambda_2 \end{pmatrix} = \operatorname{cof} J,$$
$$A^{(1)} = \begin{pmatrix} \lambda_2 + 2z & 0 & -x \\ 0 & \lambda_1 + 2z & -y \\ -x & -y & \lambda_1 + \lambda_2 \end{pmatrix},$$
$$A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{cof} \widetilde{J}.$$

Since $\widetilde{J}$ is the identity matrix, $E^{(2)}$ is just the usual energy and $W^{(2)}$ is a potential, which is separable in the coordinates given by the eigenvalues of J (which can be shown to be parabolic coordinates on R^3 with parameters λ_1 and λ_2). The recursion and multiplication formulas make it trivial to produce such separable potentials. First we compute

$$P_\mu = \frac{\det(J + \mu\widetilde{J})}{\det \widetilde{J}}$$
$$= (2\lambda_1\lambda_2 z - \lambda_2 x^2 - \lambda_1 y^2) + \mu(\lambda_1\lambda_2 + 2(\lambda_1 + \lambda_2)z - x^2 - y^2) + \mu^2(\lambda_1 + \lambda_2 + 2z) + \mu^3.$$

With $W_\mu = (\mu + \lambda_1)(\mu + \lambda_2)$ as starting point, we multiply modulo P_μ by μ for going upwards, and by its inverse

$$\mu^{-1} = \frac{(\lambda_1\lambda_2 + 2(\lambda_1 + \lambda_2)z - x^2 - y^2) + \mu(\lambda_1 + \lambda_2 + 2z) + \mu^2}{2\lambda_1\lambda_2 z - \lambda_2 x^2 - \lambda_1 y^2}$$

for going downwards. In this way we reconstruct the family of parabolic separable potentials V_m found in [7], whose lowest-order members are

$$V_{-1} = \left(2z - \frac{x^2}{\lambda_1} - \frac{y^2}{\lambda_2}\right)^{-1},$$
$$V_0 = 1,$$
$$V_1 = -2z,$$
$$V_2 = 4z^2 + (x^2 + y^2),$$
$$V_3 = -8z^3 - 4z(x^2 + y^2) + (\lambda_1 x^2 + \lambda_2 y^2).$$

The potential V_m is the coefficient at the highest power of μ in the polynomial $\mu^m W_\mu \bmod P_\mu$. This example extends immediately to an arbitary number of dimensions.

Example 4.3 (Kepler potential). Consider the two-dimensional case of the previous example. The Kepler potential $V_K = -(x^2+y^2)^{-1/2}$ is separable in parabolic coordinates, with second integral of motion $\dot{x}(y\dot{x} - x\dot{y}) + yV_K$, corresponding to

$$J = \begin{pmatrix} 0 & x \\ x & 2y \end{pmatrix} \qquad \text{and} \qquad \widetilde{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $W_\mu = \mu V_K + yV_K$ for the Kepler system, and $P_\mu = \mu^2 + 2y\mu + x^2$. "Squaring" W_μ with the multiplication formula yields

$$(\mu + y)^2 V_K^2 \bmod P_\mu = \big(-(2y\,\mu - x^2) + 2y\,\mu + y^2\big)\frac{1}{x^2+y^2} = 1.$$

In other words: the Kepler system is its own inverse under this multiplication! This means that no new interesting potentials arise by multiplying it by itself. On the other hand, we can of course multiply it by powers of μ, resulting in a "Kepler family" of separable systems given by the polynomials $\mu^m(\mu + y)V_K \bmod P_\mu$ (an equivalent construction was done already in [1] using the recursion formula).

Example 4.4 (Potentials separable on the sphere). For an example in curved space, consider the unit sphere S^2 in R^3: $x^2 + y^2 + z^2 = 1$. The potential

$$V(x, y, z) = \frac{ax^2 + by^2 + cz^2}{(x^2 + y^2 + z^2)^2} \tag{4.1}$$

on R^3 (where a, b, c are distinct) is one in a "Neumann family" of potentials [7], all separable in the spherical-conical coordinates (r, u_1, u_2) defined by $r^2 = x^2+y^2+z^2$ and

$$\frac{x^2}{u-a} + \frac{y^2}{u-b} + \frac{z^2}{u-c} = r^2\,\frac{(u-u_1)(u-u_2)}{(u-a)(u-b)(u-c)}.$$

In terms of these coordinates we have

$$V(r, u_1, u_2) = \frac{a+b+c-u_1-u_2}{r^2} = \frac{a+b+c}{r^2} - \frac{1}{r^2}\frac{u_1^2 - u_2^2}{u_1 - u_2},$$

which agrees with the general form of a potential separable in spherical-conical coordinates: $V = f(r) + g(u_1, u_2)r^{-2}$, where $g = \frac{g_1(u_1) - g_2(u_2)}{u_1 - u_2}$.

What is usually called the Neumann potential is the restriction of V to S^2,

$$V_{\text{Neu}} = (ax^2 + by^2 + cz^2)\big|_{S^2} = a + b + c - \frac{u_1^2 - u_2^2}{u_1 - u_2},$$

which is also integrable since it is separable in (u_1, u_2) (elliptic coordinates on the sphere).

The system $\ddot{q} = -\nabla V$ in R^3 has three quadratic integrals of motion $E^{(k)} = \frac{1}{2}A_{ij}^{(k)}\dot{q}^i\dot{q}^j + W^{(k)}(q)$, where the Killing tensors $A^{(k)}$, associated with separability in spherical-conical coordinates [8], are

$$
\begin{aligned}
A^{(0)} &= \begin{pmatrix} cy^2 + bz^2 & -cxy & -bxz \\ -cxy & cx^2 + az^2 & -ayz \\ -bxz & -ayz & bx^2 + ay^2 \end{pmatrix}, \\
A^{(1)} &= \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}, \\
A^{(2)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

and the quasi-potentials $W^{(k)}$ are

$$
\begin{aligned}
W^{(0)} &= -\frac{bcx^2 + acy^2 + abz^2}{x^2 + y^2 + z^2}, \\
W^{(1)} &= \frac{ax^2 + by^2 + cz^2}{x^2 + y^2 + z^2}, \\
W^{(2)} &= V = \frac{ax^2 + by^2 + cz^2}{(x^2 + y^2 + z^2)^2}.
\end{aligned}
$$

We now consider the Neumann system on S^2, and use coordinates $X = x^2$, $Y = y^2$ (on a patch $x > 0$, $y > 0$, $z = \sqrt{1 - X - Y} > 0$ for example). This is inspired by [9], where also the tensor J below was given (as a block in a Poisson operator). The metric in these coordinates is

$$
g^{ij} = 4 \begin{pmatrix} X(1-X) & -XY \\ -XY & Y(1-Y) \end{pmatrix}.
$$

If we view $A^{(0)}$ and $A^{(1)}$ as $(1,1)$-tensors, i.e., as linear operators on tangent vectors, it is easy to see that the restriction of $A^{(1)}$ to S^2 is the identity mapping, while the restriction of $A^{(0)}$ to S^2 (call it B) is found by computing:

$$
\begin{aligned}
B\frac{\partial}{\partial X} &= A^{(0)} \begin{pmatrix} 1/2x \\ 0 \\ -1/2z \end{pmatrix} \\
&= \big((c-b)y^2 + b\big) \begin{pmatrix} 1/2x \\ 0 \\ -1/2z \end{pmatrix} + (a-c)y^2 \begin{pmatrix} 0 \\ 1/2y \\ -1/2z \end{pmatrix} \\
&= \big((c-b)Y + b\big)\frac{\partial}{\partial X} + (a-c)Y\frac{\partial}{\partial Y},
\end{aligned}
$$

and similarly for $B\frac{\partial}{\partial Y}$. We get

$$
B^i_j = \begin{pmatrix} (c-b)Y + b & (b-c)X \\ (a-c)Y & (c-a)X + a \end{pmatrix},
$$

and consequently $B = \operatorname{cof} J$ with the SCK tensor

$$
J^i_j = \begin{pmatrix} (c-a)X + a & (c-b)X \\ (c-a)Y & (c-b)Y + b \end{pmatrix}.
$$

The two quadratic integrals of motion of the Neumann system on S^2 can now easily be written down by restricting the functions $W^{(0)}$ and $W^{(1)}$ to S^2 and expressing them in terms of (X, Y):

$$
\begin{aligned}
F^{(0)} &= \frac{1}{2}(B_{11}\dot{X}^2 + 2B_{12}\dot{X}\dot{Y} + B_{22}\dot{Y}^2) - \bigl(ab + b(c-a)X + a(c-b)Y\bigr), \\
F^{(1)} &= \frac{1}{2}(g_{11}\dot{X}^2 + 2g_{12}\dot{X}\dot{Y} + g_{22}\dot{Y}^2) + \bigl(c + (a-c)X + (b-c)Y\bigr).
\end{aligned}
\tag{4.2}
$$

So far nothing new, but we can now use the recursion and multiplication formulas to produce new separable potentials, expressed directly in the coordinates (X, Y). We have, since g^i_j is the identity and hence $\det g = 1$,

$$
P_\mu = \frac{\det(J + \mu g)}{\det g} = D + T\mu + \mu^2,
\tag{4.3}
$$

where

$$
\begin{aligned}
D &= \det J = ab + b(c-a)X + a(c-b)Y, \\
T &= \operatorname{tr} J = a + b + (c-a)X + (c-b)Y.
\end{aligned}
\tag{4.4}
$$

Starting with $W_\mu = 1$ for instance, we find a family of separable potentials $V_n(X, Y)$ as the coefficient of μ in the first degree polynomial $\mu^n \bmod P_\mu$. The first nontrivial one is $V_2 = -T$, which is the Neumann potential (minus the irrelevant constant $a + b + c$). In fact, the family obtained in this way is essentially the Neumann family mentioned above (but restricted to S^2 and expressed in terms of X and Y).

We can also find a "self-inverse" system like the Kepler system in the previous example, namely $W_\mu = (T + 2\mu)/\sqrt{T^2 - 4D}$, since it is immediate that $(W_\mu)^2 \bmod (D + T\mu + \mu^2) = 1$. The potential in this case is

$$
V = \frac{2}{\sqrt{T^2 - 4D}} = \frac{2}{u_1 - u_2},
$$

since the separation coordinates $u_{1,2}$ are the roots of the polynomial $D + T\mu + \mu^2$. We can of course construct a family $\mu^n W_\mu \bmod P_\mu$ based on this potential as well. The potential V can also be extended to a potential on R^3 separable in spherical-conical coordinates, namely

$$
\begin{aligned}
V = \frac{2}{r^2\,(u_1 - u_2)} &= 2\bigl[(b-c)^2x^4 + (a-c)^2y^4 + (a-b)^2z^4 + \\
&\quad + 2(a-c)(b-c)x^2y^2 + 2(a-b)(c-b)x^2z^2 + 2(a-b)(a-c)y^2z^2\bigr]^{-1/2}.
\end{aligned}
$$

5 Addendum

After this work was finished another preprint [10] by Crampin and Sarlet appeared, which (among other things) contained material more or less equivalent to our sections 3.1 and 3.2. These results are all straightforward generalizations of results previously known for the Euclidean case. However, the main new result of this paper (the multiplication theorem 3.4) appears only here.

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