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A New Class of Integrable Newton Systems

Hans LUNDMARK

Dept. of Mathematics, Linköping University, SE–581 83 Linköping, Sweden
E-mail: halun@mai.liu.se

Abstract

A new class of integrable Newton systems in R^n is presented. They are characterized by the existence of two quadratic integrals of motion of so-called cofactor type, and are therefore called cofactor pair systems. This class includes as special cases conservative systems separable in elliptic or parabolic coordinates, as well as many Newton systems previously derived as reductions of soliton hierarchies.

1 Introduction

Throughout this note, elements of R^n are written as column vectors. The superscript T denotes transpose of a matrix. By a *Newton system* we shall mean a system of ODEs of the form

$$\ddot{q} = M(q), \qquad q = (q_1, \ldots, q_n)^T \in R^n, \tag{1}$$

which for example arises as the equations of motion of a unit mass particle moving in R^n under the influence of a (velocity-independent) force field $M(q)$. As a special case we have the conservative systems

$$\ddot{q} = -\nabla V(q), \tag{2}$$

which of course are very well known from classical mechanics. In fact, the powerful tools of Lagrangian and Hamiltonian mechanics are directly applicable to conservative systems, while less is known about general (nonconservative) Newton systems.

In this note we present a new class of completely integrable Newton systems which are, in general, not conservative in the sense of (2) (although they are of course conservative in the sense of having many integrals of motion).

2 Background

For a two-dimensional conservative system the energy $E = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + V(q)$ is conserved. If there exists a second integral of motion F as well, then E and F are in involution (if we switch to the Hamiltonian viewpoint by setting $p = \dot{q}$), and the system is completely integrable. If F depends quadratically on $\dot{q}$,

$$F = A_{11}(q)\,\dot{q}_1^2 + 2A_{12}(q)\,\dot{q}_1\dot{q}_2 + A_{22}(q)\,\dot{q}_2^2 + k(q), \tag{3}$$

then the following holds:

- The coefficients $A_{ij}(q)$ satisfy the cyclic equations

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0 \tag{4}$$

for all i, j, k (let $A_{12} = A_{21}$, so that the A_{ij} form the entries of a symmetric matrix A). The general solution to these equations is

$$\begin{aligned} A_{11} &= \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22}, \\ A_{12} &= -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}), \\ A_{22} &= \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11}. \end{aligned} \tag{5}$$

- The Newton system $\ddot{q} = -\nabla V(q)$ can be recovered not only from E (i.e., from the potential V), but also from the second integral of motion F (if $\det A \neq 0$):

$$\ddot{q} = -\frac{1}{2} A(q)^{-1} \nabla k(q). \tag{6}$$

This, incidentally, is equivalent to the "quasi-Lagrangian" equations

$$0 = \delta_i^+ F \equiv \frac{\partial F}{\partial q_i} + \frac{d}{dt}\frac{\partial F}{\partial \dot{q}_i}, \qquad i = 1, 2. \tag{7}$$

- V satisfies the *Bertrand–Darboux equation* [1, sec. 152] associated with the matrix A:

$$\begin{aligned} 0 = {} & (\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12})(\partial_{22} V - \partial_{11} V) \\ & + \left(\alpha\left(q_1^2 - q_2^2\right) + 2\beta_1 q_1 - 2\beta_2 q_2 + \gamma_{11} - \gamma_{22}\right) \partial_{12} V \\ & - 3(\alpha q_2 + \beta_2)\partial_1 V + 3(\alpha q_1 + \beta_1)\partial_2 V, \end{aligned} \tag{8}$$

the characteristic coordinates of which are either elliptic, parabolic or cartesian, depending on the parameters α, β_i, γ_{ij}. The potential V is separable in these coordinates (in the Hamilton–Jacobi sense).

The fact that the Newton system is completely determined by F makes it natural to dispense with E and consider in their own right (nonconservative) Newton systems generated through (6) by any quadratic integral of motion of the form given by (3) and (5). This was done in [2], where it was shown that two-dimensional Newton systems with *two* independent quadratic integrals of motion of this kind are in fact completely integrable (although in a slightly nonstandard sense; see below). This new class of integrable systems constitutes a very natural extension of the class of separable conservative systems, and is significantly larger.

We will not spend more time on the results in [2] here, since more general statements follow below (for arbitrary n). The details of these new results can be found in [3].

3 Cofactor pair systems

Now let $n \geq 2$ be arbitrary. Consider a function E quadratic in $\dot{q}$:

$$E(q, \dot{q}) = \dot{q}^T A(q)\, \dot{q} + k(q) = \sum_{i,j=1}^{n} A_{ij}(q)\, \dot{q}_i \dot{q}_j + k(q), \tag{9}$$

where $A(q)$ is a symmetric $n \times n$ matrix. If E is an integral of motion for the Newton system $\ddot{q} = M(q)$, then it is easy to show that (4) and (6) hold, just like in the case $n = 2$. However, the general solution of the cyclic equations (4) is quite intricate if $n > 2$. The following concept provides a useful subclass of solutions.

Definition 3.1 (Elliptic coordinates matrix). *A symmetric $n \times n$-matrix $G(q)$ whose entries are quadratic polynomials in q of the form*

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij} \tag{10}$$

will be called an elliptic coordinates matrix. Using matrix multiplication, $G(q)$ can be written

$$G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \qquad \text{where} \quad \alpha \in R, \quad \beta \in R^n, \quad \gamma = \gamma^T \in R^{n\times n}. \tag{11}$$

The reason for the terminology is that the eigenvalues $u_1(q), \ldots, u_n(q)$ of an elliptic coordinates matrix (under some assumptions) determine a change of variables from Cartesian coordinates q to generalized elliptic coordinates u.

Theorem 3.2 (Cofactor matrix). *If $G(q)$ is an elliptic coordinates matrix, then its cofactor matrix (also called the adjoint matrix) $A(q) = \operatorname{cof} G(q)$ satisfies the cyclic conditions* (4)*.*

Notice that for $n = 2$, this gives the general solution (5). For $n > 2$, however, not all solutions to (4) are of this form.

If $\det G \neq 0$, then $E = \dot{q}^T(\operatorname{cof} G)\dot{q} + k$ generates a Newton system $\ddot{q} = -\frac{1}{2}(\operatorname{cof} G)^{-1}\nabla k$, which has E as an integral of motion "of cofactor type". Consider now two nonsingular elliptic coordinates matrices, $G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma$ and $\widetilde{G}(q) = \widetilde{\alpha} q q^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}$, and let $E = \dot{q}^T(\operatorname{cof} G)\dot{q} + k$ and $\widetilde{E} = \dot{q}^T(\operatorname{cof} \widetilde{G})\dot{q} + \widetilde{k}$.

Definition 3.3. *If a Newton system is generated by both E and $\widetilde{E}$,*

$$\ddot{q} = -\frac{1}{2}(\operatorname{cof} G)^{-1}\nabla k = -\frac{1}{2}(\operatorname{cof} \widetilde{G})^{-1}\nabla \widetilde{k}, \tag{12}$$

then it is called a cofactor pair system.

For a given pair $(G, \widetilde{G})$ it is not obvious that there exist functions k and $\widetilde{k}$ such that (12) holds. We will see in the next section that there in fact are many such functions.

A cofactor pair system has, by definition, two integrals of motion E and $\widetilde{E}$ of cofactor type. The following theorem says that such a system must in fact have at least n integrals of motion.

Theorem 3.4 ("2 implies n"). *The cofactor pair system* (12) *has n quadratic integrals of motion*

$$E^{(i)} = \dot{q}^T A^{(i)}\, \dot{q} + k^{(i)}, \qquad i = 0, \ldots, n-1, \tag{13}$$

where the matrices $A^{(0)}, \ldots, A^{(n-1)}$ are defined by

$$A_\mu = \operatorname{cof}\,(G + \mu\widetilde{G}) = \sum_{i=0}^{n-1} A^{(i)} \mu^i. \tag{14}$$

Note that $E^{(0)} = E$ and $E^{(n-1)} = \widetilde{E}$, so the two integrals of motion of cofactor type sit at either end of this “cofactor chain” of integrals of motion. This theorem indicates that cofactor pair systems might have interesting integrability properties. Indeed, in [3] it is shown that for a given cofactor pair system there is a corresponding completely integrable bi-Hamiltonian system, in $(n+n+1)$-dimensional phase space with coordinates (q,p,d), whose trajectories in the hyperplane $(q,p,0)$ agree up to reparametrization with the trajectories of the cofactor pair system in the $(q,\dot{q})$ plane. In this sense, cofactor pair systems can be considered completely integrable. Lack of space prevents us from entering into details here.

4 The fundamental equations

Let $G(q)$ and $\widetilde{G}(q)$ as above be given, and take any function $k(q)$. We wish to find another function $\widetilde{k}(q)$ such that (12) holds, i.e.,

$$\nabla\widetilde{k} = (\operatorname{cof}\widetilde{G})(\operatorname{cof}G)^{-1}\nabla k. \tag{15}$$

This is possible provided that the integrability conditions $\partial_i\partial_j\widetilde{k} = \partial_j\partial_i\widetilde{k}$ are satisfied. This yields a system of second order PDEs for k, which turns out to take a much simpler form after the substitution $k(q) = K(q)\det G(q)$. The system for K obtained in this way is called the *fundamental equations* associated with the pair $(G,\widetilde{G})$. Setting $N = \alpha q+\beta$ and $\widetilde{N} = \widetilde{\alpha}q + \widetilde{\beta}$, it reads

$$\begin{aligned} 0 = & \sum_{r,s=1}^{n} \left(G_{ir}\widetilde{G}_{js} - G_{jr}\widetilde{G}_{is}\right)\partial_{rs}K \\ & + 3\sum_{r=1}^{n}\left(G_{ir}\widetilde{N}_j + \widetilde{G}_{jr}N_i - G_{jr}\widetilde{N}_i - \widetilde{G}_{ir}N_j\right)\partial_r K \\ & + 6\left(N_i\widetilde{N}_j - N_j\widetilde{N}_i\right)K, \qquad i,j = 1,\ldots,n. \end{aligned} \tag{16}$$

The number of independent equations is (at most) $\binom{n}{2}$ since the equations (by construction) are antisymmetric in i and j.

What is remarkable is that the fundamental equations are also antisymmetric with respect to the interchange of coefficients with and without tilde. This means that if we have a cofactor pair system, then the fundamental equations are not only satisfied by $K = k/\det G$, but also by $K = \widetilde{k}/\det\widetilde{G}$ (these two solutions are in general different). Conversely, given a solution K it is possible to construct two different cofactor pair systems. Combining these two facts, it is easy to recursively construct infinite families of cofactor pair systems for any given pair $(G,\widetilde{G})$, since the trivial cofactor pair system $\ddot{q} = 0$ (k and $\widetilde{k}$ constant) provides a system with which to start. The following theorem gives the precise relationship between adjacent systems in the recursion.

Theorem 4.1 (Recursion formula). *Consider a cofactor pair system with integrals of motion $E^{(i)} = \dot{q}^T A^{(i)}\dot{q} + k^{(i)}$ as in Theorem 3.4. Then we obtain another cofactor pair system with integrals of motion $F^{(i)} = \dot{q}^T A^{(i)}\dot{q} + l^{(i)}$ through the formula*

$$l_\mu = \frac{\det(G+\mu\widetilde{G})}{\det\widetilde{G}}\widetilde{k} - \mu k_\mu, \tag{17}$$

where $k_\mu = \sum_{i=0}^{n-1} k^{(i)}\mu^i$ *and* $l_\mu = \sum_{i=0}^{n-1} l^{(i)}\mu^i$. *The inverse relationship is*

$$k_\mu = \frac{1}{\mu}\left(\frac{\det(G+\mu\widetilde{G})}{\det G}\, l - l_\mu\right). \tag{18}$$

5 Special cases

A cofactor pair system with $\widetilde{G} = I$ (identity matrix) is the same as an ordinary conservative Newton system, with an extra integral of motion of cofactor type. In this case, the fundamental equations (with V instead of K) reduce to a known criterion (if $n = 2$, the Bertrand–Darboux equation (8)) for V to be separable in generalized elliptic coordinates or some degeneration thereof [4, 5, 6]. In fact, the separation coordinates are given by the eigenvalues of the elliptic coordinates matrix $G(q)$. The recursion formula reduces to known recursion formulas for separable potentials [7].

In the more general (but far from most general) case that $\widetilde{G}$ is a constant matrix, the cofactor pair systems can be shown to be *Pfaffian quasi-bi-Hamiltonian*, and thus solvable by variable separation in a suitable Hamilton–Jacobi equation [8, 9]. Several known integrable Newton systems derived as reductions of soliton equations belong to this category.

The question of separability for general cofactor pair systems is still open. It is believed that this can lead to new interesting results in the theory of separability (see [2] for some developments in this direction).

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