

Quasi-Lagrangian systems of Newton equations

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Systems of Newton equations of the form $\ddot{q} = -\frac{1}{2}A^{-1}(q)\nabla k$ with an integral of motion quadratic in velocities are studied. These equations generalize the potential case (when $A=I$, the identity matrix) and they admit a curious quasi-Lagrangian formulation which differs from the standard Lagrange equations by the plus sign between terms. A theory of such quasi-Lagrangian Newton (qLN) systems having two functionally independent integrals of motion is developed with focus on two-dimensional systems. Such systems admit a bi-Hamiltonian formulation and are proved to be completely integrable by embedding into five-dimensional integrable systems. They are characterized by a linear, second-order partial differential equation PDE which we call the fundamental equation. Fundamental equations are classified through linear pencils of matrices associated with qLN systems. The theory is illustrated by two classes of systems: separable potential systems and driven systems. New separation variables for driven systems are found. These variables are based on sets of nonconfocal conics. An effective criterion for existence of a qLN formulation of a given system is formulated and applied to dynamical systems of the Hénon–Heiles type. © 1999 American Institute of Physics. [S0022-2488(99)00912-3]

I. INTRODUCTION

In this paper we introduce and study such systems of Newton equations $\ddot{q} = M(q)$ that can be generated as equations of the form

$$0 = \frac{d}{dx} \frac{\partial E}{\partial \dot{q}} + \frac{\partial E}{\partial q} \equiv \delta^+ E \tag{1.1}$$

by an energylike function quadratic in \dot{q} ,

$$E(q, \dot{q}) = \sum_{i,j=1}^n A_{ij}(q) \dot{q}_i \dot{q}_j + k(q) \equiv \dot{q}^t A \dot{q} + k(q), \tag{1.2}$$

where $A(q)$ is an $n \times n$ symmetric matrix with real entries $A_{ij}(q)$. Here and in what follows we use the standard mechanical notation $q = (q_1, \dots, q_n)^t$, $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)^t$, for position and velocity vectors (the superscript t denotes the transpose of a matrix), where $\dot{q}_k = (\partial/\partial x)q_k$, $k = 1, \dots, n$, with $x \in \mathbf{R}$ being the independent (time) variable. By Newton equations we mean second-order ordinary differential equations (ODEs) of the form: acceleration \ddot{q} is equal to the velocity independent force $M(q)$. The force M may be potential or not.

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The equations in (1.1) are called here *quasi-Lagrangian* (qL) equations since they differ from the Lagrange equations for $E(q, \dot{q})$ by sign between terms only. These equations are shortly denoted $0 = \delta^+ E = (\delta_1^+ E, \dots, \delta_n^+ E)^t$ where

$$\delta_k^+ E = \frac{d}{dx} \frac{\partial E}{\partial \dot{q}_k} + \frac{\partial E}{\partial q_k}.$$

The qL equations are not invariant with respect to arbitrary point transformation, but it can be easily shown that they remain invariant with respect to the affine change of variables $q = SQ + h$ where $Q = (Q_1, \dots, Q_n)^t$ are the new variables and $S \in GL(n), h \in \mathbf{R}^n$.

In the present article we shall mainly discuss quasi-Lagrangian sets of Newton equations (qLN) generated by a function E of the form (1.2) in the two-dimensional space of variables $q = (q_1, q_2) = (r, w)$. This class of equations (which seems to be completely new) is a very interesting class because of its rich differential-algebraic structure and also because it contains (as special cases) the well understood class of point-separable potential Newton equations $\ddot{q} = -\partial V(q)/\partial q$ and the class of nonpotential Newton equations of the triangular form $\ddot{r} = M_1(r, w), \ddot{w} = M_2(w)$ which we shall call *driven* systems. The qLN systems are not necessarily Lagrangian and thus they do not have any straightforward Hamiltonian formulation.

In this paper we develop a theory of completely integrable sets of qLN equations characterized by the existence of two functionally independent integrals of motion quadratic in velocities: E as above and $F = \dot{q}^t B(q) \dot{q} + l(q)$. The existence of a second integral of motion has far-reaching consequences; it eventually leads to wide classes of completely integrable qLN systems.

Example 1.1: The function $E = r\dot{r}\dot{w} - w\dot{r}^2 - \alpha w r^2 + \frac{1}{2}dr^2 + (w^2/2r^4)$ when inserted into (1.1) gives rise to

$$0 = \begin{bmatrix} \frac{d}{dx} \frac{\partial E}{\partial \dot{r}} + \frac{\partial E}{\partial r} \\ \frac{d}{dx} \frac{\partial E}{\partial \dot{w}} + \frac{\partial E}{\partial w} \end{bmatrix} = \begin{bmatrix} -2w \left(\ddot{r} - \alpha r + \frac{w}{r^5} \right) + r(\ddot{w} - 4\alpha w + d) \\ r \left(\ddot{r} - \alpha r + \frac{w}{r^5} \right) \end{bmatrix} = \begin{bmatrix} -2w & r \\ r & 0 \end{bmatrix} \begin{bmatrix} \ddot{r} - M_1(r, w) \\ \ddot{w} - M_2(w) \end{bmatrix}, \tag{1.3}$$

which is equivalent to a set of two Newton equations

$$\begin{aligned} \ddot{r} &= \alpha r - \frac{w}{r^5} \equiv M_1(r, w), \\ \ddot{w} &= 4\alpha w - d \equiv M_2(w), \end{aligned} \tag{1.4}$$

since the matrix

$$\begin{bmatrix} -2w & r \\ r & 0 \end{bmatrix}$$

is nonsingular. We see that the operation $0 = \delta^+ E$ generates *linear combinations* of the Newton equations (1.4).

Equations (1.4) were discovered accidentally as a Newton parametrization of the second stationary flow of the Harry Dym hierarchy:¹

$$0 = (\frac{1}{4}\partial^3 - \alpha\partial)(\alpha u^{-3/2} - \frac{5}{16}\dot{u}^2 u^{-7/2} + \frac{1}{4}\ddot{u} u^{-5/2}) = (\frac{1}{4}\partial^3 - \alpha\partial)(-r^5\ddot{r} - \alpha r^6)$$

(here $\partial = \partial/\partial x$), where we substituted $u = r^{-4}$. The substitution $w = -r^5\dot{r} + \alpha r^6$ gives the system (1.4). The particular feature of (1.4) is that it is a driven system: the equation for w can be solved independently and then the solution $w(x)$ drives the equation for r .

II. GENERAL PROPERTIES OF QUASI-LAGRANGIAN NEWTON SYSTEMS

Let us consider an n -dimensional qL system $0 = \delta^+ E$ with (quadratic in velocities) energylike function

$$E(q, \dot{q}) = \sum_{i,j=1}^n A_{ij}(q) \dot{q}_i \dot{q}_j + k(q) \tag{2.1}$$

with a symmetric (which can be assumed without loss of generality) matrix $A(q) = A^t(q)$. We shall formulate the necessary and sufficient condition for the matrix $A(q)$ to make the equations $0 = \delta^+ E$ equivalent to the set of equations

$$0 = \ddot{q} - M(q) \tag{2.2}$$

with a velocity independent force $M(q) = (M_1(q), \dots, M_n(q))^t$.

Theorem 2.1: *For the function E given by (2.1) with a nonsingular matrix $A(q)$ the following conditions are equivalent:*

- (1) *The equations $0 = \delta^+ E$ are equivalent to the set of Newton equations $\ddot{q} = M(q)$ with velocity independent forces $M = -\frac{1}{2}A^{-1}(q)\nabla k(q)$.*
- (2) *The function E is an integral of motion for the qL system $0 = \delta^+ E$.*
- (3) *The matrix elements $A_{ij}(q)$ satisfy the following set of ‘‘cyclic’’ differential equations:*

$$0 = \partial_i A_{jk}(q) + \partial_j A_{ki}(q) + \partial_k A_{ij}(q) \quad \text{for all } i, j, k = 1, \dots, n. \tag{2.3}$$

Throughout the whole article the symbol ∇ denotes the gradient operator and $\partial_i = \partial/\partial q_i$. Later on we will also use the notation $\partial_{ij} = \partial^2/\partial q_i \partial q_j$.

Statement (2) of the above theorem explains the name ‘‘energylike’’ for the function E .

Proof: Let us calculate the i th equation in $0 = \delta^+ E$:

$$\begin{aligned} 0 = \delta_i^+ E &= \frac{d}{dx} \frac{\partial E}{\partial \dot{q}_i} + \frac{\partial E}{\partial q_i} = \frac{d}{dx} \left(2 \sum_j A_{ij}(q) \dot{q}_j \right) + \sum_{j,k} \partial_i A_{jk}(q) \dot{q}_j \dot{q}_k + \partial_i k \\ &= 2 \sum_j A_{ij}(q) \ddot{q}_j + \partial_i k + \sum_{j,k} (\partial_i A_{jk}(q) + \partial_j A_{ki}(q) + \partial_k A_{ij}(q)) \dot{q}_j \dot{q}_k. \end{aligned} \tag{2.4}$$

The last equality in (2.4) is due to the symmetry of $A(q)$. Thus, clearly, $2A\ddot{q} + \nabla k = 0$ if and only if the equations (2.3) are satisfied and the equivalence of (1) and (3) is established.

Let us now calculate the total derivative of E with respect to x :

$$\begin{aligned} \dot{E} &= \sum_i \left(2 \sum_j A_{ij} \ddot{q}_j + \partial_i k \right) \dot{q}_i + \sum_{i,j,k} \partial_k A_{ij} \dot{q}_i \dot{q}_j \dot{q}_k \\ &= \sum_i \left(2 \sum_j A_{ij} \ddot{q}_j + \partial_i k \right) \dot{q}_i + \frac{1}{3} \sum_{i,j,k} (\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) \dot{q}_i \dot{q}_j \dot{q}_k. \end{aligned} \tag{2.5}$$

The second term on the right-hand side of the above equation has been rewritten by renaming indices. It contains precisely the cyclic conditions (2.3). So, if one (and thus both) of the statements (1) and (3) are satisfied, then both terms in (2.5) vanish. On the other hand, if $\dot{E} = 0$, then terms at different powers of \dot{q}_i in (2.5) must be equal to zero, which implies both 1 and 3.

Remark 2.2: For $n=2$ the general solution of equations (2.3) can easily be found. It is

$$\begin{aligned}
 A_{11}(w) &= aw^2 + bw + \alpha, \\
 2A_{12}(r, w) &= -2arw - br - cw + \beta, \\
 A_{22}(r) &= ar^2 + cr + \gamma,
 \end{aligned}
 \tag{2.6}$$

with some real constants $a, b, c, \alpha, \beta, \gamma$. The corresponding qLN equations read explicitly as

$$0 = \begin{bmatrix} \frac{d}{dx} \frac{\partial E}{\partial \dot{r}} + \frac{\partial E}{\partial r} \\ \frac{d}{dx} \frac{\partial E}{\partial \dot{w}} + \frac{\partial E}{\partial w} \end{bmatrix} = 2 \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \ddot{r} - M_1(r, w) \\ \ddot{w} - M_2(r, w) \end{bmatrix},$$

where

$$\begin{aligned}
 M_1(r, w) &= \frac{1}{2 \det(A)} \left(A_{12} \frac{\partial k}{\partial w} - A_{22} \frac{\partial k}{\partial r} \right), \\
 M_2(r, w) &= \frac{1}{2 \det(A)} \left(A_{12} \frac{\partial k}{\partial r} - A_{11} \frac{\partial k}{\partial w} \right).
 \end{aligned}$$

The remaining part of this work is mostly devoted to the case when qLN equations $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k$ generated by E admit a *second* (quadratic in velocities) integral of motion $F(q, \dot{q}) = \sum_{i,j=1}^n B_{ij}(q)\dot{q}_i\dot{q}_j + l(q) \equiv \dot{q}^t B(q)\dot{q} + l(q)$ which is linearly, and therefore functionally, independent of E .

Theorem 2.3 (qLN systems with two integrals): *Let the qLN system of Newton equations,*

$$0 = \delta^+ E = 2A(\ddot{q} + \frac{1}{2}A^{-1}\nabla k),
 \tag{2.7}$$

generated by the function $E(q, \dot{q}) = \dot{q}^t A(q)\dot{q} + k(q)$, admit a second, functionally independent quadratic integral of motion $F(q, \dot{q}) = \dot{q}^t B(q)\dot{q} + l(q)$. Then we have the following.

(1) *The matrix $B(q)$ has the same structure as the matrix $A(q)$ in the sense that the coefficients $B_{ij}(q)$ of $B(q)$ satisfy the set of cyclic differential equations (2.3).*

(2) *If $\det(B) \neq 0$, then*

$$A^{-1}\nabla k = B^{-1}\nabla l,
 \tag{2.8}$$

and so the qLN system $0 = \delta^+ F = 2B(\ddot{q} + \frac{1}{2}B^{-1}\nabla l)$ generates the same Newton equations as E .

(3) *Any differentiable function $f(E, F)$ generates the same system of Newton equations [by $0 = \delta^+ f(E, F)$] as E does. In particular, any linear combination, $\lambda E + \mu F$ generates the same system of Newton equations.*

The statement (2) shows one of the peculiar features of qLN systems: all quadratic (in velocities) integrals of motion of a qLN system generate the same system (see also Sec. 151 in Ref. 2).

Proof: The requirement $\dot{B} = 0$ yields [cf. (2.5)]

$$\begin{aligned}
 0 &= \sum_i \left(2 \sum_j B_{ij}(q)\ddot{q}_j + \partial_i l \right) \dot{q}_i + \sum_{i,j,k} \partial_k B_{ij} \dot{q}_i \dot{q}_j \dot{q}_k \\
 &= \sum_i \dot{q}_i \left(2B \left(-\frac{1}{2}A^{-1}\nabla k \right) + \nabla l \right)_i + \sum_{i,j,k} \partial_k B_{ij} \dot{q}_i \dot{q}_j \dot{q}_k,
 \end{aligned}
 \tag{2.9}$$

where the index i at the vector expression containing matrices B and A^{-1} denotes its i th component. The equality is satisfied identically with respect to \dot{q} and so both sums must be separately equal to zero. It follows that B_{ij} satisfy the cyclic conditions $\partial_i B_{jk} + \text{cycl} = 0$ and that $2B(-\frac{1}{2}A^{-1}\nabla k) + \nabla l = 0$. The latter yields precisely the equation (2.8) since we assumed $\det(B) \neq 0$. So the statements (1) and (2) are proved.

The operator δ^+ acts as differentiation on the algebra of constants of motion, so that

$$0 = \delta^+ f(E, F) = \frac{\partial f}{\partial E} \delta^+ E + \frac{\partial f}{\partial F} \delta^+ F = 2 \left(\frac{\partial f}{\partial E} A + \frac{\partial f}{\partial F} B \right) (\ddot{q} - M)$$

(where $M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$), which proves the statement (3) of the theorem. Q.E.D.

It is important to stress that the equation (2.8) is the necessary and sufficient condition for the equivalence of the qLN system (2.7) and the qLN system generated by $F = \dot{q}^t B(q) \dot{q} + l(q)$. This condition will be used later.

III. qLN EQUATIONS IN TWO DIMENSIONS

We shall from now on restrict our considerations to the case $n = 2$. We will use the notation $q = (q_1, q_2)^t = (r, w)^t$. The case of arbitrary n is studied in a separate paper.³

For $n = 2$ Theorem 2.3 contains two special cases which explain the connection of our theory with classical results² about separable potential Newton equations and with the class of driven systems where one of the Newton equations depends only on a single variable r or w and can be solved on its own.

Corollary 3.1: Assume that the Newton equations

$$\ddot{r} = M_1(r, w), \quad \ddot{w} = M_2(r, w) \tag{3.1}$$

generated by the integral $E = \dot{q}^t A(q) \dot{q} + k(q)$ [with the matrix A given by (2.6)] as $0 = \delta^+ E$ have a potential force: $M_1 = -\partial V / \partial r$, $M_2 = -\partial V / \partial w$. Then the potential $V(r, w)$ satisfies the Bertrand–Darboux equation²

$$0 = (V_{ww} - V_{rr})(-2arw - br - cw + \beta) + 2V_{rw}(aw^2 - ar^2 + bw - cr + \alpha - \gamma) + 3V_r(2aw + b) - 3V_w(2ar + c) \tag{3.2}$$

(where the indices at V denote partial derivatives with respect to r and w) with the coefficients $a, b, c, \alpha, \beta, \gamma$ being exactly the coefficients of the polynomials in entries of the matrix A as given by (2.6). This means that the Newton system (3.1) can be solved by separating variables in the related Hamilton–Jacobi equation (see Ref. 2).

Proof: If M is potential, then, according to Theorem 2.3, $M = -\frac{1}{2}A^{-1}\nabla k = -\nabla V$ and so $\nabla k = 2A\nabla V$. The potential V exists provided that $\partial^2 k / \partial r \partial w = \partial^2 k / \partial w \partial r$. This yields exactly the Bertrand–Darboux equation (3.2) for V . Q.E.D.

Remark 3.2: The quantity $k(r, w) / \det(A)$ satisfies the same Bertrand–Darboux equation as the potential V . This result can be verified directly but it also follows from Theorem 4.1 in the next section.

Remark 3.3: Let us emphasize that the Hamiltonian system

$$\dot{r} = s, \quad \dot{w} = z, \quad \dot{s} = -\frac{\partial V}{\partial r}, \quad \dot{z} = -\frac{\partial V}{\partial w}$$

generated by a separable natural Hamiltonian $H = \frac{1}{2}(s^2 + z^2) + V(r, w)$ can be reconstructed as the qLN system $0 = \delta^+ E = 2A(\ddot{q} + \frac{1}{2}A^{-1}\nabla k)$ from its second integral of motion E . This is easy to see, since the above Hamilton equations are equivalent to $\ddot{r} = -\partial V / \partial r, \ddot{w} = -\partial V / \partial w$.

The second class of equations satisfying the assumptions of Theorem 2.3 is the class of qLN systems of the form

$$\ddot{r} = M_1(r, w), \quad \ddot{w} = M_2(w), \tag{3.3}$$

which naturally generalizes the system in Example 1.1. Such systems are called *driven* since the equation for w can be solved independently and then $w(x)$ can be substituted into the equation for r . Observe that the second equation (and thus the whole system) admits an extra integral of motion of the form $F(w, \dot{w}) = \dot{w}^2/2 - \int M_2(w) dw$. The qLN system $0 = \delta^+ E$ attains the form (3.3) if and only if the second component M_2 of the force $-\frac{1}{2}A^{-1}\nabla k$ does not depend on r :

$$\frac{\partial}{\partial r} (A^{-1}\nabla k)_2 = 0. \tag{3.4}$$

Example 3.4: The qLN equations generated by the function

$$E = r\dot{r}\dot{w} - w\dot{r}^2 + k(r, w)$$

are driven [i.e., have the form (3.3)] provided that $k(r, w)$ satisfies the following second-order PDE:

$$0 = \frac{\partial}{\partial r} \left(\frac{1}{r} k_r + \frac{2w}{r^2} k_w \right),$$

which is a specialization of (3.4). The general solution of the above equation is

$$k(r, w) = f\left(\frac{r^2}{w}\right) + r^2 g(w)$$

with arbitrary twice differentiable functions f and g . The corresponding qLN system attains the form

$$\ddot{r} = -rg'(w) + \frac{r}{w^2} f'\left(\frac{r^2}{w}\right), \quad \ddot{w} = -2 \frac{d}{dw} (wg(w))$$

and can be solved by quadratures (see Sec. VII). The second integral of motion of our system, $F = \dot{w}^2/2 - \int M_2(w) dw = \dot{w}^2/2 + 2wg(w)$, yields the matrix B

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

which is singular so F does not generate our system. However, any linear combination $\lambda E + \mu F$ of E and F (with both λ and $\mu \neq 0$) is another integral of motion with a nonsingular matrix $B' = \lambda A + \mu B$ and thus it generates the same driven system as E .

Existence of two functionally independent constants of motion does not automatically imply Liouville integrability since we also need a Hamiltonian formulation for our equations of motion. Our systems usually do not have a Lagrangian formulation and so they do not have the standard Hamiltonian formulation. On the other hand, the special system discussed in Example 1.1, being a stationary flow of the Harry Dym hierarchy, is expected to be integrable. The question thus arises if/when our qLN systems possess a nonstandard Hamiltonian formulation. In Sec. VI we shall demonstrate the existence of new Poisson structures for qLN systems and their close relationship with Poisson pencils for separable potentials. We shall also explain there when and in what sense our qLN systems are integrable.

IV. FUNDAMENTAL EQUATION

We shall now characterize those two-dimensional qLN systems which admit two (quadratic in velocities) functionally independent integrals of motion E and F , with the force $M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$. We remind the reader that for $n=2$ we use the notation $q = (q_1, q_2)^t = (r, w)^t$.

Let us consider two symmetric 2×2 matrices $A(r, w)$ and $B(r, w)$ both satisfying the cyclic conditions (2.3). According to Remark 2.2 they must have the following structure,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \tag{4.1}$$

with the polynomial entries given by [cf. (2.6)]

$$\begin{aligned} A_{11}(w) &= a_1 w^2 + b_1 w + \alpha_1, \\ 2A_{12}(r, w) &= -2a_1 r w - b_1 r - c_1 w + \beta_1, \\ A_{22}(r) &= a_1 r^2 + c_1 r + \gamma_1, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} B_{11}(w) &= a_2 w^2 + b_2 w + \alpha_2, \\ 2B_{12}(r, w) &= -2a_2 r w - b_2 r - c_2 w + \beta_2, \\ B_{22}(r) &= a_2 r^2 + c_2 r + \gamma_2, \end{aligned} \tag{4.3}$$

with some arbitrary real constants a_1, \dots, γ_2 .

Theorem 4.1 (fundamental equation): *Let*

$$\begin{bmatrix} \ddot{r} \\ \ddot{w} \end{bmatrix} = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l, \tag{4.4}$$

with nonsingular 2×2 matrices A, B given by (4.1), (4.2), and (4.3), be a set of qLN equations. Then the functions $K_1 = k/\det(A)$ and $K_2 = l/\det(B)$ both satisfy the same linear, second-order, partial differential equation

$$\begin{aligned} 0 &= 2(A_{12}B_{22} - A_{22}B_{12})K_{rr} - 2(A_{11}B_{22} - A_{22}B_{11})K_{rw} + 2(A_{11}B_{12} - A_{12}B_{11})K_{ww} \\ &\quad + 3(A_{12}\partial_r B_{22} - B_{12}\partial_r A_{22} + A_{22}\partial_w B_{11} - B_{22}\partial_w A_{11})K_r - 3(A_{11}\partial_r B_{22} - B_{11}\partial_r A_{22} \\ &\quad + A_{12}\partial_w B_{11} - B_{12}\partial_w A_{11})K_w + 3(\partial_r A_{22}\partial_w B_{11} - \partial_r B_{22}\partial_w A_{11})K, \end{aligned} \tag{4.5}$$

which explicitly reads

$$\begin{aligned} 0 &= 2K_{rr}[\gamma_2\beta_1 - \gamma_1\beta_2 + (b_2\gamma_1 - \gamma_2b_1 + \beta_1c_2 - c_1\beta_2)r + (\gamma_1c_2 - \gamma_2c_1)w + (b_2c_1 - c_2b_1 + a_2\beta_1 \\ &\quad - a_1\beta_2)r^2 + 2(\gamma_1a_2 - \gamma_2a_1)wr + (a_1b_2 - a_2b_1)r^3 + (a_2c_1 - c_2a_1)wr^2] + 4K_{rw}[\alpha_2\gamma_1 - \alpha_1\gamma_2 \\ &\quad + (\alpha_2c_1 - \alpha_1c_2)r + (b_2\gamma_1 - \gamma_2b_1)w + (\alpha_2a_1 - \alpha_1a_2)r^2 + (\gamma_1a_2 - \gamma_2a_1)w^2 + (b_2c_1 - c_2b_1)rw \\ &\quad + (a_1b_2 - a_2b_1)wr^2 + (a_2c_1 - c_2a_1)rw^2] + 2K_{ww}[\alpha_1\beta_2 - \alpha_2\beta_1 + (\alpha_2b_1 - \alpha_1b_2)r \\ &\quad + (\alpha_2c_1 - \alpha_1c_2 + b_1\beta_2 - b_2\beta_1)w + (a_1\beta_2 - a_2\beta_1 + b_2c_1 - c_2b_1)w^2 + 2(\alpha_2a_1 - \alpha_1a_2)wr \\ &\quad + (a_2c_1 - c_2a_1)w^3 + (a_1b_2 - a_2b_1)rw^2] + 3K_r[2b_2\gamma_1 - 2\gamma_2b_1 + \beta_1c_2 - c_1\beta_2 \\ &\quad + (3b_2c_1 - 3c_2b_1 + 2a_2\beta_1 - 2a_1\beta_2)r + 4(\gamma_1a_2 - \gamma_2a_1)w + 4(a_1b_2 - a_2b_1)r^2 \end{aligned}$$

$$\begin{aligned}
 &+4(a_2c_1 - c_2a_1)rw] + 3K_w[2\alpha_2c_1 - 2\alpha_1c_2 + b_1\beta_2 - b_2\beta_1 + (2a_1\beta_2 - 2a_2\beta_1 \\
 &+ 3b_2c_1 - 3c_2b_1)w + 4(\alpha_2a_1 - \alpha_1a_2)r + 4(a_2c_1 - c_2a_1)w^2 + 4(a_1b_2 - a_2b_1)rw] \\
 &+ 6K[b_2c_1 - c_2b_1 + 2(a_1b_2 - a_2b_1)r + 2(a_2c_1 - c_2a_1)w] \tag{4.6}
 \end{aligned}$$

with K denoting either K_1 or K_2 and $K_r = \partial K / \partial r$, $K_{rr} = \partial^2 K / \partial r^2$ and so on.

Conversely, any solution $K_2(q)$ of the equation (4.5) generates two different systems of qLN equations $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k_1 = -\frac{1}{2}B^{-1}\nabla l_1$ and $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k_2 = -\frac{1}{2}B^{-1}\nabla l_2$, where the functions k_1, k_2, l_1, l_2 are determined by the equations

$$\begin{aligned}
 l_1 &= K_2 \det(B), \quad \nabla k_1 = AB^{-1}\nabla(K_2 \det(B)), \\
 k_2 &= K_2 \det(A), \quad \nabla l_2 = BA^{-1}\nabla(K_2 \det(A)). \tag{4.7}
 \end{aligned}$$

We will call the equation (4.5) the *fundamental equation* associated with the matrices A and B .

The fundamental equation plays a crucial role in our theory of qLN systems. Observe that it is invariant with respect to the transformation $A \mapsto \lambda A + \mu B$, $B \mapsto \lambda' A + \mu' B$, $(\lambda, \lambda', \mu, \mu' \in \mathbf{R})$ since the coefficients at every monomial in this equation are skew-symmetric in A and B . This is consistent with statement (3) of Theorem 2.3, which asserts that if any pair E, F of functions generates a qLN system, then the linear combinations $\lambda E + \mu F$ and $\lambda' E + \mu' F$ also generate the same system. This explains that the assumption of nonsingularity for both A and B is nonessential since if $\det(A) \neq 0$, a singular matrix B can always be substituted by an invertible matrix $B' = \lambda A + \mu B$. We shall investigate further properties of the fundamental equation in the next theorem.

Notice that in the second part of Theorem 4.1 one has to reconstruct l_2 and k_1 by integrating the expressions for ∇l_2 and for ∇k_1 . This can always be done, as the above theorem implicitly states. Also, notice that in the fundamental equation (4.6) all terms of degree 4 and higher cancel so that the polynomial degree of coefficients in this equation is less than or equal to 3.

Proof (of Theorem 4.1): Our qLN system (4.4) is generated by either of the two functions $E(q, \dot{q}) = \dot{q}^t A \dot{q} + k$ and $F(q, \dot{q}) = \dot{q}^t B \dot{q} + l$ and so the condition (2.8), i.e., $A^{-1}\nabla k = B^{-1}\nabla l$, must be satisfied. This implies that $\nabla l = BA^{-1}\nabla k$. This equation for the function l has solutions if and only if its compatibility condition $l_{rw} = l_{wr}$ is satisfied. This yields a PDE for the function k which, after the substitution $k = K_1 \det(A)$ and with use of the cyclic conditions (2.3), yields that K_1 satisfies equation (4.5). By inserting into this equation the explicit form of the polynomials A_{11}, \dots, B_{22} we obtain (4.6). On the other hand, the condition (2.8) implies also $\nabla k = AB^{-1}\nabla l$, and its compatibility condition $k_{rw} = k_{wr}$ gives a PDE which in terms of $K_2 = l / \det(B)$ must attain the form (4.5) with interchanged entries of A and B (since the equation $\nabla k = AB^{-1}\nabla l$ becomes $\nabla l = BA^{-1}\nabla k$ when one exchanges A, k and B, l). Due to the skew-symmetry of coefficients of the equation for K_1 with respect to the entries of matrices A, B [clearly seen from the form of (4.5)], the obtained equation for K_2 differs from the equation for K_1 by a minus sign on the right-hand side only. This proves that K_1 and K_2 both satisfy (4.5) (notice, however, that this does *not* imply $K_1 = K_2$).

The existence of k_1 [i.e., the possibility of integrating the equations (4.7) in order to obtain k_1] follows from the fact that the condition $\partial^2 k_1 / \partial r \partial w = \partial^2 k_1 / \partial w \partial l$ together with $\nabla k_1 = AB^{-1}\nabla(K_2 \det(B))$ yields precisely the fundamental equation for K_2 which is satisfied due to assumptions. One can similarly prove the existence of l_2 . The second statement of the theorem can now be proved by checking that both pairs k_1, l_1 and k_2, l_2 given by (4.7) satisfy the condition (2.8) and thus give rise to two systems of qLN equations. Q.E.D.

Remark 4.2: For $B(q) = \frac{1}{2}I$ (a 2×2 identity matrix) the equation (4.5) becomes the Bertrand–Darboux equation (3.2) characterizing all separable potentials since in this case $\ddot{q} = -\frac{1}{2}B^{-1}\nabla l = -\nabla l$ is a potential equation.

The next theorem shows that there exists a recursive relation between two different qLN systems constructed from a given solution $K_2(q)$ of the fundamental equation (4.5). This makes it possible to construct a doubly infinite sequence of qLN systems corresponding to a given fundamental equation.

Theorem 4.3 (recursion theorem): *Let k_1, l_1 and k_2, l_2 be two pairs of functions determined by a given solution K_2 of the fundamental equation (4.5) as in (4.7). Then these functions are related by the following linear algebraic equations:*

$$k_2 = l_1 \det(AB^{-1}), \quad l_2 = l_1 \operatorname{Tr}(AB^{-1}) - k_1 \tag{4.8}$$

(where Tr denotes trace of matrix). Moreover, in the infinite sequence

$$\begin{array}{ccccccc}
 & k_0 & & k_1 & & k_2 & & \\
 & & \nearrow & & \nearrow & & \nearrow & \\
 \cdots & \downarrow & & K_1 & \downarrow & K_2 & \downarrow & K_3 & \cdots \\
 & & \nearrow & & \nearrow & & \nearrow & & \\
 & l_0 & & l_1 & & l_2 & & &
 \end{array} \tag{4.9}$$

of triples $(K_m, k_m, l_m), m \in \mathbf{Z}$, defined recursively by

$$k_m = l_{m-1} \det(AB^{-1}), \quad l_m = l_{m-1} \operatorname{Tr}(AB^{-1}) - k_{m-1} \tag{4.10}$$

and by

$$K_m = k_m / \det(A) = l_{m-1} / \det(B),$$

the functions k_m and l_m satisfy $A^{-1} \nabla k_m = B^{-1} \nabla l_m$ and thus they both determine the same (for a given m) qLN system $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k_m = -\frac{1}{2} B^{-1} \nabla l_m$. All functions K_m satisfy the fundamental equation (4.5) and are related through the following two-step recursion:

$$K_{m+1} = K_m \operatorname{Tr}(AB^{-1}) - K_{m-1} \det(AB^{-1}). \tag{4.11}$$

The above recursion is reversible. The solution K_m placed between l_{m-1} and k_m determines both l_{m-1} and k_m . The recursion (4.11) is soluble. Namely, if we denote the eigenvalues of the matrix AB^{-1} by λ_1 and λ_2 , then it can be proved that for the case $\lambda_1 \neq \lambda_2$ the solution of (4.11) is

$$K_m = \frac{1}{\lambda_1 - \lambda_2} (K_1 - \lambda_2 K_0) \lambda_1^m + \frac{1}{\lambda_1 - \lambda_2} (K_0 \lambda_1 - K_1) \lambda_2^m,$$

while in the case $\lambda_1 = \lambda_2$ the solution of (4.11) becomes

$$K_m = K_0 \lambda_1^m + \left(\frac{K_1}{\lambda_1} - K_0 \right) m \lambda_1^m.$$

In both cases K_0 and K_1 are two subsequent solutions of the fundamental equation in the sequence (4.9) which are related by

$$\nabla(K_1 \det(B)) = BA^{-1} \nabla(K_0 \det(A)).$$

In order to prove the recursion theorem we need the following lemma.

Lemma 4.4: *Let $X = AB^{-1}$ with matrices A, B as above. Then*

$$X^{-1} \nabla(\det(X)) = \nabla(\operatorname{Tr}(X)).$$

This lemma follows from the cyclic properties (2.3) of matrices A and B by a lengthy but straightforward calculation.

Proof (of the recursion theorem): Consider a solution K_2 of the fundamental equation and the functions $k_1, l_1; k_2, l_2$ defined by (4.7). Then obviously $k_2/\det(A)=l_1/\det(B)$, which immediately implies $k_2=l_1 \det(AB^{-1})$. Let $X=AB^{-1}$. Then

$$\begin{aligned} \nabla l_2 - \nabla(l_1 \operatorname{Tr}(X) - k_1) &= X^{-1} \nabla(K_2 \det(A)) - \nabla(K_2 \det(B) \operatorname{Tr}(X)) + X \nabla(K_2 \det(B)) \\ &= X^{-1} \nabla(K_2 \det(A)) - (\operatorname{Tr}(X)I - X) \nabla(K_2 \det(B)) - K_2 \det(B) \nabla(\operatorname{Tr}(X)) \\ &= X^{-1} \nabla(K_2 \det(A)) - X^{-1} \det(X) \nabla(K_2 \det(B)) - K_2 \det(B) \nabla(\operatorname{Tr}(X)) \\ &= K_2 \det(B) (X^{-1} \nabla(\det(X)) - \nabla(\operatorname{Tr}(X))) = 0, \end{aligned}$$

where we used that $X^2 - \operatorname{Tr}(X)X + \det(X)I = 0$ as follows from the Cayley–Hamilton theorem. The last equality is due to Lemma 4.4 above. Thus $l_2 = l_1 \operatorname{Tr}(X) - k_1$ up to a nonessential additive constant. This proves the first assertion of the theorem.

If we now define the sequence $\{(k_m, l_m)\}$ via the recursive procedure (4.10), then a simple induction argument shows that each pair (k_m, l_m) satisfies the condition (2.8) and thus both k_m and l_m determine the same qLN system. Moreover, each $K_m = k_m/\det(A) = l_{m-1}/\det(B)$ is a solution of the fundamental equation as theorem (4.1) states. Finally, to obtain (4.11) it is enough to insert the formula $K_m = k_m/\det(A) = l_{m-1}/\det(B)$ into the second equation in (4.10). Q.E.D.

Example 4.5 (cf. Remark 4.2): For $B = \frac{1}{2}I$ (the potential case) the recursion (4.8) takes the form

$$k_2 = 4V_1 \det(A), \quad V_2 = 2 \operatorname{Tr}(A) V_1 - k_1$$

with $V_1 = l_1$. This is the separable case when (4.5) reduces to the Bertrand–Darboux equation. In the generic case, i.e., when $a \neq 0$ in (2.6), the matrix $A(q)$ can be reduced [with the use of affine transformations $q = SQ + h$ with $S \in GL(2, \mathbf{R}), h \in \mathbf{R}^2$, see also Sec. V] to the form

$$A(q) = \begin{bmatrix} -q_2^2 + \lambda_2 & q_1 q_2 \\ q_1 q_2 & -q_1^2 + \lambda_1 \end{bmatrix}.$$

If we now start with the harmonic oscillator potential $V_1 = \frac{1}{2}(q_1^2 + q_2^2)$, then the condition $\nabla V_1 = \frac{1}{2}A^{-1} \nabla k_1$ gives $k_1 = \lambda_2 q_1^2 + \lambda_1 q_2^2$ and the recursion formulas specify to

$$k_2 = 2(q_1^2 + q_2^2)(\lambda_1 \lambda_2 - \lambda_2 q_1^2 - \lambda_1 q_2^2),$$

$$V_2 = \lambda_1 q_1^2 + \lambda_2 q_2^2 - (q_1^2 + q_2^2)^2,$$

thus reproducing the potential of the Garnier system.⁴ It can be shown that the above formulas prolongate to the $n=2$ case of the recursion for the Jacobi family of elliptic separable potentials.⁵

In order to explain the character of the recursion (4.9) more completely, let us consider instead of the pair (A, B) of cyclic matrices another pair $(A + \mu B, B)$ with $\mu \in \mathbf{R}$. As it can be shown (see below), this pair determines the same fundamental equation as the pair (A, B) does. By choosing a solution K_2 of the fundamental equation and the pair $(A + \mu B, B)$ we arrive at a different qLN system $\ddot{q} = M_\mu(q) = -\frac{1}{2}(A + \mu B)^{-1} \nabla(K_2 \det(A + \mu B))$. It turns out that the force $M_\mu(q)$ is a linear combination of two neighboring forces in the sequence (4.9) generated by K_2 .

Lemma 4.6: Let A and B be two 2×2 matrices satisfying the cyclic conditions (2.3) and let K be a solution of the fundamental equation associated with A and B . Let also $\mu \in \mathbf{R}$. Then

$$(A + \mu B)^{-1} \nabla(K \det(A + \mu B)) = A^{-1} \nabla(K \det(A)) + \mu B^{-1} \nabla(K \det(B)).$$

This lemma is a consequence of Lemma 4.4. It says that a solution of a given fundamental equation determines the force M (and so the system of qLN equations) up to linear combinations of two consecutive systems in the recursion (4.9).

As we have mentioned, the matrices A and B uniquely determine the fundamental equation. The choice of A, B which generate a given fundamental equation is, however, not unique since the pair $A' = \alpha A + \beta B, B' = \gamma A + \delta B$ determines the same equation. One can also ask to what extent a given fundamental equation determines the pair (A, B) . The precise relationship between pairs (A, B) and the fundamental equation is explained in the following theorem.

Theorem 4.7: *Let (A, B) be a pair of linearly independent matrices A, B satisfying the cyclic conditions (2.3). Then there is a 1-1 relationship between the linear span $\{\lambda A + \mu B: \lambda, \mu \in \mathbf{R}\}$ of A and B and the fundamental equation (4.5), i.e.,*

- (1) *any two linearly independent matrices $A' = \alpha A + \beta B, B' = \gamma A + \delta B$ determine the same fundamental equation as (A, B) does.*
- (2) *If the pair (A', B') determines the same fundamental equation as (A, B) does, then the matrices A' and B' belong to the linear span $\{\lambda A + \mu B\}$ of A and B .*

Proof: An easy calculation shows that the fundamental equation associated with the matrices $A' = \alpha A + \beta B$ and $B' = \gamma A + \delta B$ differs from the fundamental equation associated with the matrices A and B by the multiplicative factor $\alpha\delta - \beta\gamma$ on the right-hand side, i.e., by the nonzero determinant of the transformation between (A, B) and (A', B') , and so it is, in fact, the same equation. This shows assertion (1) of the theorem.

Assume now that the equation (4.5) is associated with a pair (A, B) . Consider the vector $\vec{X} = (X_1, X_2, X_3)^t \in \mathbf{R}^3$ of the coefficients of (4.5) at the highest derivatives K_{rr}, K_{rw}, K_{ww} , respectively. Then

$$\begin{aligned} X_1 &= A_{12}B_{22} - A_{22}B_{12}, \\ X_2 &= A_{22}B_{11} - A_{11}B_{22}, \\ X_3 &= A_{11}B_{12} - A_{12}B_{11}, \end{aligned} \tag{4.12}$$

or

$$\vec{X} = \vec{A} \times \vec{B}, \tag{4.13}$$

where $\vec{A} = (A_{11}, A_{12}, A_{22})^t$ and $\vec{B} = (B_{11}, B_{12}, B_{22})^t$ are three-dimensional vectors depending on r and w . Hence, for a fixed (r, w) both vectors \vec{A} and \vec{B} are orthogonal to \vec{X} . The coefficients at $K_r, K_w,$ and K yield equations which are differential consequences of (4.12) and so they do not impose any additional restrictions on (A, B) . Suppose now that there exist matrices A' and B' satisfying the cyclic condition (2.3) and associated with the same fundamental equation. This means that the equation (4.13) has another solution, i.e., that $\vec{X} = \vec{A}' \times \vec{B}'$, so that the vectors \vec{A}' and \vec{B}' are orthogonal to \vec{X} and, in consequence, they are linear combinations of \vec{A} and \vec{B} : $\vec{A}' = \alpha\vec{A} + \beta\vec{B}, \vec{B}' = \gamma\vec{A} + \delta\vec{B}$ with some coefficients that may depend on r and w . For the corresponding matrices it immediately follows that

$$A' = \alpha A + \beta B, \quad B' = \gamma A + \delta B.$$

It remains to show that the coefficients $\alpha, \beta, \gamma, \delta$ in fact do not depend on r nor w . This can be shown by inserting the explicit form (4.2) and (4.3) of entries of matrices A, B, A' and B' into (4.12). This shows assertion (2) of the theorem. Q.E.D.

V. AFFINE INEQUIVALENT FORMS OF FUNDAMENTAL EQUATION

In this section we are interested in characterizing all different types of two-dimensional qLN systems admitting two functionally independent integrals of motion E and F which are quadratic in velocities, i.e., systems of the form

$$\ddot{q} = M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l, \tag{5.1}$$

where M is the force of the system. Every such system is described by a pair of matrices $A(q), B(q)$ satisfying the cyclic conditions (2.3) and by a pair of functions $k(q), l(q)$ satisfying $A^{-1}\nabla k = B^{-1}\nabla l$. We remind the reader that the functions $k/\det(A)$ and $l/\det(B)$ satisfy the same fundamental equation with the coefficients completely determined by the matrix elements of A and B .

Let us first consider how the qLN system $0 = \delta^+ E = 2A(\ddot{q} + \frac{1}{2}A^{-1}\nabla k)$ transforms under the affine transformation of coordinates

$$q = SQ + h, \quad S \in \text{GL}(2, \mathbf{R}), \quad h \in \mathbf{R}^2, \tag{5.2}$$

where $Q = (Q_1, \dots, Q_n)^t$. It is easy to see that, under the affine transformation (5.2), the generating function E transforms as

$$E(q(Q), \dot{q}(\dot{Q})) = \dot{Q}^t S^t A(q(Q)) SQ + k(q(Q)), \tag{5.3}$$

where $q(Q) = SQ + h$ and so $\dot{q}(\dot{Q}) = S\dot{Q}$. It can be shown by a direct verification that the transformed matrix

$$A_Q(Q) = S^t A(q(Q)) S \tag{5.4}$$

in (5.3) also satisfies the cyclic conditions (2.3) and therefore (5.3) generates a qLN system. This means that the qLN system $0 = 2A(\ddot{q} + \frac{1}{2}A^{-1}\nabla k)$ is indeed *invariant* with respect to the affine change of coordinates (5.2).

Let us now consider the system (5.1). Using (5.4) one can prove that the fundamental equation associated with the pair (A, B) of matrices is also invariant with respect to the affine transformations (5.2). This means that we can simplify this fundamental equation by performing an appropriate affine change of coordinates. However, Theorem 4.7 makes it possible to classify fundamental equations, and therefore the corresponding qLN systems, by classifying *pairs* of matrices (A, B) . Instead of working with the coefficients of the fundamental equation we can thus work with linear spans $\{\lambda A + \mu B\}$ of A and B . Since the affine transformations do not change the polynomial degree of matrices A, B , the set of all linear spans of A and B can be divided into affine inequivalent classes corresponding to different polynomial degree of A and B . Each equivalence class will be represented by the algebraically simplest pair of matrices obtained by the use of affine transformations and linear combining of matrices (since the latter leave the fundamental equation unchanged, see above).

In order to be more precise we shall introduce some notation. By $A^{(i)}$ ($i=0,1,2$) we will denote all matrices A which satisfy the cyclic conditions (2.3) and have the highest degree of polynomial entries equal to i . So, for example, the general form of matrices in the class $A^{(1)}$ is

$$\begin{bmatrix} bq_1 + \alpha & -\frac{1}{2}bq_1 - \frac{1}{2}cq_2 + \frac{1}{2}\beta \\ * & cq_1 + \gamma \end{bmatrix}$$

with arbitrary constants (parameters) $b, c, \alpha, \beta, \gamma$. We will use the symbol $*$ to denote matrix elements determined by the symmetry of a given matrix. Moreover, by $[A^{(i)}, B^{(j)}]$ ($i, j=0,1,2$) we will denote the class of (nonordered) pairs (A, B) of linearly independent matrices A, B such that one of the matrices belongs to $A^{(i)}$ and the other to $B^{(j)}$. We have, of course, $[A^{(i)}, B^{(j)}] = [A^{(j)}, B^{(i)}]$ and so we have precisely six such classes. Obviously, all classes $[A^{(i)}, B^{(j)}]$ are invariant with respect to the affine transformations (5.2). Notice that if $(A, B) \in [A^{(2)}, B^{(2)}]$, we can kill the coefficient a_2 at the second degree monomials in B by subtraction $B \mapsto B - (a_2/a_1)A$ so every element of this class can be reduced to an element in $[A^{(2)}, B^{(1)}]$. Thus we have to consider only five classes. It is easy to realize that the five classes $[A^{(i)}, B^{(j)}]$ with $i \geq j, j < 2$ are invariant with respect to the affine transformations (5.2) and with respect to taking

linear combinations of pairs (A,B) . These two operations can now be used to find for every class a simple representing pair (A,B) which has a minimal number of free parameters (a simple representative). Consider, for example, the class $[A^{(2)}, B^{(0)}]$. The general form of matrices belonging to this class is

$$A = \begin{bmatrix} a_1q_2^2 + b_1b_2 + \alpha_1 & -a_1q_1q_2 - \frac{1}{2}b_1q_1 - \frac{1}{2}c_1q_2 + \beta_1/2 \\ * & a_1q_1^2 + c_1q_1 + \gamma_1 \end{bmatrix}, \tag{5.5}$$

$$B = \begin{bmatrix} \alpha_2 & \frac{\beta_2}{2} \\ \frac{\beta_2}{2} & \gamma_2 \end{bmatrix}.$$

Translation by the vector $h = -(1/2a_1)(c_1, b_1)^t$ kills b_1 and c_1 in the matrix A . Since translations obviously preserve the form of B the above pair of matrices attains the form

$$A = \begin{bmatrix} a_1q_2^2 + \alpha_1 & -a_1q_1q_2 + \frac{\beta_1}{2} \\ * & a_1q_1^2 + \gamma_1 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_2 & \frac{\beta_2}{2} \\ \frac{\beta_2}{2} & \gamma_2 \end{bmatrix}$$

with some new constants denoted by the same letters as in (5.5). Further, the transformation $A \mapsto A - (\beta_1/\beta_2)B$ kills the coefficient β_1 in A . In the case when $\beta_2 = 0$ we can still kill β_1 in A by the linear transformation $q = SQ$ with

$$S = \begin{bmatrix} \frac{-t\gamma_1 - \beta_1/2}{\alpha_1 + t\beta_1/2} & 1 \\ 1 & t \end{bmatrix},$$

where $t \in \mathbf{R}$ must be chosen so that $\alpha_1 + t\beta_1/2 \neq 0$ and $\det(S) \neq 0$, which can be always done. Finally, we can divide both matrices by a_1 and $2\alpha_2$, respectively. So, a simple representative of the class $[A^{(2)}, B^{(0)}]$ has the form

$$A = \begin{bmatrix} q_2^2 + \alpha_1 & -q_1q_2 \\ * & q_1^2 + \gamma_1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{\beta_2}{2} \\ \frac{\beta_2}{2} & \gamma_2 \end{bmatrix} \tag{5.6}$$

with four essential parameters. Both separable and driven systems belong to this class since $B = \frac{1}{2}I$ for separable systems and $B = \text{diag}(\frac{1}{2}, 0)$ for driven systems.

We perform a similar reduction for each class $[A^{(i)}, B^{(j)}], i \geq j, j < 2$. The results are presented below.

It is also easy to see that one can pass from one invariant class $[A^{(i)}, B^{(j)}]$ to another by specifying values of free parameters. For example, by setting $a_1 = 0$ we obtain $[A^{(1)}, B^{(1)}]$ from $[A^{(2)}, B^{(1)}]$; by setting $b_2 = c_2 = 0$ we get $[A^{(1)}, B^{(0)}]$ and so on as shown in Fig. 1. This figure presents—for all classes $[A^{(i)}, B^{(j)}]$ —complete results of simplification of a generic pair (A,B) belonging to each class with the use of linear combinations and affine transformations.

Below we list the form of the fundamental equation (4.6) corresponding to the simple representative pair (A,B) of each class as given in Fig. 1. We use the notation $K_i = \partial K / \partial q_i, K_{ij} = \partial^2 K / \partial q_j \partial q_i$.

- (a) For $[A^{(2)}, B^{(1)}]$,

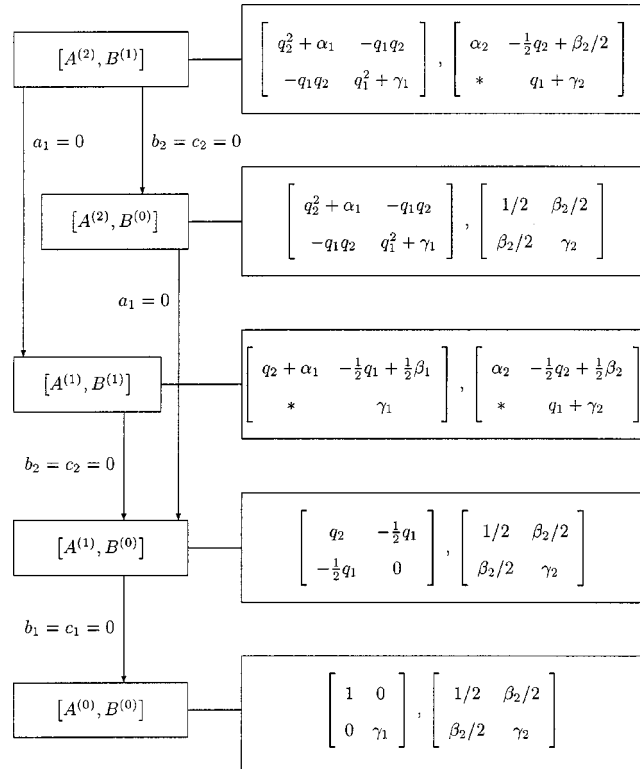


FIG. 1. Classification diagram.

$$0 = K_{11}(-\gamma_1\beta_2 + \gamma_1q_2 - 2\gamma_2q_1q_2 - \beta_2q_1^2 - q_1^2q_2) + 2K_{12}(\alpha_2\gamma_1 - \alpha_1\gamma_2 - \alpha_1q_1 - \gamma_2q_2^2 + \alpha_2q_1^2 - q_1q_2^2) + K_{22}(\alpha_1\beta_2 - \alpha_1q_2 + 2\alpha_2q_1q_2 + \beta_2q_2^2 - q_2^3) + 3K_1(-\beta_2q_1 - 2\gamma_2q_2 - 2q_1q_2) + 3K_2(-\alpha_1 + 2\alpha_2q_1 + \beta_2q_2 - 2q_2^2) - 6q_2K.$$

(b) For $[A^{(2)}, B^{(0)}]$,

$$0 = K_{11}(-\gamma_1\beta_2 - 2\gamma_2q_1q_2 - \beta_2q_1^2) + 2K_{12}(\gamma_1 - \alpha_1\gamma_2 - \gamma_2q_2^2 + q_1^2) + K_{22}(\alpha_1\beta_2 + 2q_1q_2 + \beta_2q_2^2) + 3K_1(-\beta_2q_1 - 2\gamma_2q_2) + 3K_2(2q_1 + \beta_2w). \tag{5.7}$$

(c) For $[A^{(1)}, B^{(1)}]$,

$$0 = 2K_{11}(\gamma_2\beta_1 - \gamma_1\beta_2 + (-\gamma_2 + \beta_1)q_1 + q_2 - q_1^2) + 4K_{12}(\alpha_2\gamma_1 - \alpha_1\gamma_2 - \alpha_1q_1 - \gamma_2q_2 - q_1q_2) + 2K_{22}(\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2q_1 + (\beta_2 - \alpha_1)q_2 - q_2^2) + 3K_1(-2\gamma_2 + \beta_1 - 3q_1) + 3K_2(-2\alpha_1 + \beta_2 - 3q_2) - 6K.$$

(d) For $[A^{(1)}, B^{(0)}]$,

$$0 = 2\gamma_2q_1K_{11} + 4\gamma_2q_2K_{12} - 2(q_1/2 + \beta_2q_2)K_{22} + 6\gamma_2K_1 - 3\beta_2K_2. \tag{5.8}$$

(e) For $[A^{(0)}, B^{(0)}]$,

$$0 = \gamma_1\beta_2K_{11} + 2(\alpha_1\gamma_2 - \alpha_2\gamma_1)K_{12} - \alpha_1\beta_2K_{22}. \tag{5.9}$$

What we present here is an illustrative characterization of different types of fundamental equations in terms of matrix pairs (A, B) . This provides a good intuitive description of the world of qLN equations and helps to specify where two particular classes—separable potentials and driven systems—belong. An alternative way of classifying qLN equations with two quadratic integrals of motion is to simplify the fundamental equation (4.5) with the use of affine transformations as has been done for the Bertrand–Darboux equation² (see Example 5.1 below). This may amount to a similar picture as we have presented above, but the principles of simplification of the third-order polynomials at K_{rr} , K_{rw} , and K_{ww} are more difficult to discern. This is yet to be done.

Example 5.1: The classification of types of the Bertrand–Darboux equation with respect to Euclidean transformations leads to three forms of this equation which are separable in either elliptic, parabolic, or Cartesian coordinates. According to Corollary 3.1, if a potential two-dimensional Newton system

$$\ddot{q}_1 = -\frac{\partial V}{\partial q_1}, \quad \ddot{q}_2 = -\frac{\partial V}{\partial q_2}$$

with $V = V(q_1, q_2)$ possesses a second integral of motion of the form $E = \dot{q}^t A \dot{q} + k(q)$ with $A \in A^{(2)}$, then it has the qLN form $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla V$ with $B = \frac{1}{2}I$ where I is 2×2 identity matrix. Moreover, the potential V must satisfy the Bertrand–Darboux equation (3.2). This system belongs to the class $[A^{(2)}, B^{(0)}]$ with $\beta_2 = 0$ and with $\alpha_2 = \gamma_2 = \frac{1}{2}$. The corresponding fundamental equation is exactly the Bertrand–Darboux equation since in this case $K = V/\det(B) = 4V$. The simplification procedure described above does not alter the form of the matrix $B = \frac{1}{2}I$ and so the corresponding fundamental equation (5.7) attains the form

$$0 = (V_{22} - V_{11})q_1q_2 + V_{12}(q_1^2 - q_2^2 + \gamma_1 - \alpha_1) - 6q_2V_1 + 6q_1V_2, \tag{5.10}$$

which has only one essential parameter $\gamma_1 - \alpha_1$. This form of the Bertrand–Darboux equation separates in the elliptic coordinates.² The specification $a_1 = 0$ reduces the class $[A^{(2)}, B^{(0)}]$ to $[A^{(1)}, B^{(0)}]$. The corresponding fundamental equation after the simplification procedure attains the form (5.8). In the case when $B = \frac{1}{2}I$, the final form of B (after simplification) will be exactly the same (i.e., with $\beta_2 = 0$, $\alpha_2 = \gamma_2 = \frac{1}{2}$), and so the fundamental equation (5.8) reads

$$0 = q_1(V_{11} - V_{22}) + 2q_2V_{12} + 3V_1.$$

It does not contain any parameters now. This equation separates in the parabolic coordinates.² Further specification $b_1 = c_1 = 0$ leads to the class $[A^{(0)}, B^{(0)}]$ of constant symmetric matrices. The corresponding fundamental equation in the course of simplification attains the form (5.9) which in the case $B = \frac{1}{2}I$ (again, this form of B survives the simplification procedure—in this case just the diagonalization of A by a rotation) yields $V_{\xi\eta} = 0$, where ξ, η are coordinates which originate by a rotation of the Cartesian coordinates x, y . This is the case of the Bertrand–Darboux equation separable in (rotated) Cartesian coordinates.

The above example indicates that the fundamental equation plays the same role in the theory of qLN equations as the Bertrand–Darboux equation does in the theory of separable potential forces $M = -\partial V/\partial q$. For separable potentials the characteristic coordinates of the Bertrand–Darboux equation determine the coordinates of separation which makes it possible to solve the corresponding Newton equations by quadratures. In Sec. VII we prove a similar result for the class of two-dimensional driven systems by showing that the characteristic coordinates of the fundamental equation associated with a given driven system separate this system, i.e., that in these coordinates it is possible to integrate the system by quadratures. The question whether the characteristic coordinates of the fundamental equation separate general qLN systems admitting two integrals of motion remains to be investigated. We have here to do with a much broader theory depending on five essential parameters while the Bertrand–Darboux equation depends on one parameter only.

VI. HAMILTONIAN STRUCTURES AND COMPLETE INTEGRABILITY

In this section we will establish a Hamiltonian formulation of two-dimensional qLN systems and discuss their complete integrability. Let us consider first the qLN system $0 = \delta^+ E = 2A(\ddot{q} + \frac{1}{2}A^{-1}\nabla k(q))$ generated by the function $E = \dot{q}^t A(q)\dot{q} + k(q)$, $q = (q_1, q_2)^t$ with the 2×2 matrix $A(q)$ satisfying the cyclic conditions (2.3). This system usually does not have any Lagrangian formulation and thus it does not have the standard Hamiltonian formulation. However, we can always embed this system in a Hamiltonian qLN system in the five-dimensional phase space of variables (q_1, q_2, p_1, p_2, d) as the following theorem states.

Theorem 6.1 (Hamiltonian form of qLN systems): *Let*

$$0 = \ddot{q} + \frac{1}{2}A^{-1}(q)\nabla(k(q) + d\lambda \det(A(q))) \tag{6.1}$$

with $q = (q_1, q_2)^t$ be the qLN system generated by

$$\hat{E} = \dot{q}^t A(q)\dot{q} + k(q) + d\lambda \det(A(q)) \equiv E + d\lambda \det(A)$$

with some constant λ and with $d \in \mathbf{R}$. Let also \mathcal{M} be the extended five-dimensional phase space of variables (q_1, q_2, p_1, p_2, d) with $p_i = \dot{q}_i$, $i = 1, 2$. Then the system (6.1) is equivalent to

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{d} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -(\lambda/2)G(q) & p \\ (\lambda/2)G^t(q) & -(\lambda/2)F(q,p) & \hat{M}(q,d) \\ -p^t & -\hat{M}^t(q,d) & 0 \end{array} \right] \nabla_{\mathcal{M}} d \equiv \Pi_A \nabla_{\mathcal{M}} d, \tag{6.2}$$

where $\nabla_{\mathcal{M}} = (\partial/\partial q_1, \partial/\partial q_2, \partial/\partial p_1, \partial/\partial p_2, \partial/\partial d)^t$ is the gradient operator in \mathcal{M} and where the 2×2 matrices G and F and the vector \hat{M} are given by

$$G(q) = \det(A)A^{-1} = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{12} & A_{11} \end{bmatrix},$$

$$F_{12}(q,p) = \frac{1}{2} \left(\frac{\partial A_{22}}{\partial q_1} p_2 - \frac{\partial A_{11}}{\partial q_2} p_1 \right), \quad F = -F^t,$$

$$\hat{M}(q,d) = M(q) - \frac{1}{2}d\lambda A^{-1}\nabla(\det(A))$$

with $M(q) = -\frac{1}{2}A^{-1}\nabla k$ being the force of the qLN system $0 = \delta^+ E$. Moreover, the antisymmetric matrix Π_A is Poisson and so (6.2) is the Hamiltonian formulation of (6.1).

Notice that the matrix G obtained above is symmetric due to the symmetry of A .

Proof: Since $\nabla_{\mathcal{M}} d = (0, 0, 0, 0, 1)^t$, the equation (6.2) yields $\dot{q} = p$, $\dot{p} = \hat{M} = -\frac{1}{2}A^{-1}\nabla(k + d\lambda \det(A))$, $\dot{d} = 0$, i.e., it reproduces (6.1). The matrix Π_A is antisymmetric and it is straightforward to verify that it satisfies the Jacobi identity in the phase space \mathcal{M} . Q.E.D.

We remind the reader that the operator $\Pi: T^*\mathcal{M} \rightarrow T\mathcal{M}$ mapping fiberwise the cotangent bundle $T^*\mathcal{M}$ of \mathcal{M} into the tangent bundle $T\mathcal{M}$ is Poisson if the bilinear mapping $\{\cdot, \cdot\}_{\Pi}: C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ defined for any pair of functions $f, g: \mathcal{M} \rightarrow \mathbf{R}$ by

$$\{f, g\}_{\Pi} = \langle \nabla_{\mathcal{M}} f, \Pi \nabla_{\mathcal{M}} g \rangle$$

(where $\langle \cdot, \cdot \rangle$ is the dual map between cotangent and tangent spaces of \mathcal{M}) is a Poisson bracket.

Remark 6.2: In the hyperplane $d = 0$ the solutions of (6.2) coincide with the solutions of $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k$. Thus our original qLN system $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k$ is in a natural way embedded in the Hamiltonian system (6.2).

Proposition 6.3: The function

$$\hat{E} = p^t A(q) p + k(q) + d \lambda \det(A(q))$$

is a Casimir function for the Poisson operator Π_A in (6.2), that is, $\Pi_A \nabla_{\mathcal{M}} \hat{E} = 0$.

One can check this proposition by a direct verification.

A statement converse to the second statement of Theorem 6.1 also holds.

Theorem 6.4: *Let the antisymmetric matrix*

$$\Pi = \left[\begin{array}{cc|c} 0 & -(\lambda/2)G(q) & p \\ (\lambda/2)G^t(q) & -(\lambda/2)F(q,p) & \hat{M}(q,d) \\ \hline -p^t & -\hat{M}^t(q,d) & 0 \end{array} \right] \tag{6.3}$$

be a Poisson operator in the space of variables (q,p,d) . Then

(1) $G(q)$ must have the form

$$G(q) = \begin{bmatrix} aq_1^2 + cq_1 + \gamma & aq_1q_2 + \frac{b}{2}q_1 + \frac{c}{2}q_2 - \frac{\beta}{2} \\ * & aq_2^2 + bq_2 + \alpha \end{bmatrix} \tag{6.4}$$

(thus it is symmetric) with some constants $a, b, c, \alpha, \beta, \gamma$ and so

$$G = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{6.5}$$

for some symmetric matrix $A(q)$ satisfying the cyclic conditions (2.3). In other words, $\Pi = \Pi_A$ with Π_A defined in (6.2) and with

$$A = \begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix}.$$

(2) $F(q,p)$ must have the form

$$F_{12}(q,p) = \frac{1}{2} \left(\frac{\partial A_{22}}{\partial q_1} p_2 - \frac{\partial A_{11}}{\partial q_2} p_1 \right), \quad F = -F^t. \tag{6.6}$$

(3) $\hat{M}(q,d)$ must have the form

$$\hat{M}(q,d) = M(q) + d \lambda N(q),$$

where $-2AM(q) = \nabla k$ for some function $k(q)$, so if $\det(G) \neq 0$, then $M(q) = -\frac{1}{2}A^{-1} \nabla k$, and where $N(q) = -\frac{1}{2}A^{-1} \nabla(\det(A))$.

Proof: The conditions $\{\{q_i, q_j\}_\Pi, q_k\}_\Pi + \text{cycl} = 0$ and $\{\{q_i, q_j\}_\Pi, p_k\}_\Pi + \text{cycl} = 0$ (where ‘cycl’ means the cyclic permutation of expressions) hold identically due to the block structure of Π . The condition $\{\{q_i, q_j\}_\Pi, d\}_\Pi + \text{cycl} = 0$ yields the symmetry of G : $G = G^t$. Further,

$$0 = \{\{q_i, p_j\}_\Pi, d\}_\Pi + \text{cycl} = -\frac{\lambda}{2} \left(p_1 \frac{\partial G_{ij}}{\partial q_1} + p_2 \frac{\partial G_{ij}}{\partial q_2} \right) + \frac{\lambda}{2} F_{ij} - p_i \frac{\partial \hat{M}_j}{\partial d}.$$

Let us denote the right-hand side of the above equality by $-(\lambda/2)R_{ij}$. Notice that $\partial\hat{M}_j/\partial d$ cannot depend on d , and so we have $\partial\hat{M}_j/\partial d = \lambda N_j(q)$ for some vector $N(q) = (N_1(q), N_2(q))^t$ which yields $\hat{M}(q, d) = M(q) + d\lambda N(q)$ for some vector $M(q)$. By taking linear combinations of the conditions $R_{ij} = 0$ and using the symmetry of G and the antisymmetry of F we get the following sets of equations:

$$\frac{\partial G_{11}}{\partial q_2} = \frac{\partial G_{22}}{\partial q_1} = 0, \quad \frac{\partial G_{11}}{\partial q_1} = 2N_1, \quad \frac{\partial G_{22}}{\partial q_2} = 2N_2, \tag{6.7}$$

$$\frac{\partial G_{12}}{\partial q_1} = N_2, \quad \frac{\partial G_{12}}{\partial q_2} = N_1, \tag{6.8}$$

$$F_{12} = p_2 N_1 - p_1 N_2. \tag{6.9}$$

The equations (6.7) show that G_{11} and N_1 depend only on q_1 and that G_{22} and N_2 depend only on q_2 . The equations (6.8) give $\partial N_1/\partial q_1 = \partial^2 G_{12}/\partial q_1 \partial q_2 = \partial N_2/\partial q_2$ and so all terms in this expression must be equal to a constant a . Integration yields

$$N_1 = a q_1 + c/2, \quad N_2 = a q_2 + b/2, \tag{6.10}$$

where b and c are integration constants. Substituting (6.10) into (6.7) and (6.8) and integrating we get (6.4). If we now introduce the symmetric matrix A by the equality (6.5) and use (6.7) then (6.9) will attain the form (6.6).

It is straightforward to check that with the above forms of F and G the conditions $\{\{p_i, p_j\}_\Pi, p_k\}_\Pi + \text{cycl} = 0$ and $\{\{q_i, p_j\}_\Pi, p_k\}_\Pi + \text{cycl} = 0$ are satisfied identically.

Further, the condition $\{\{p_1, p_2\}_\Pi, d\}_\Pi + \text{cycl} = 0$ after some calculations attains the form

$$0 = \frac{\partial}{\partial q_1} (G_{11}M_2 - G_{21}M_1) - \frac{\partial}{\partial q_2} (G_{22}M_1 - G_{12}M_2),$$

which means that in the vector

$$\begin{bmatrix} G_{22}M_1 - G_{12}M_2 \\ -G_{21}M_1 + G_{11}M_2 \end{bmatrix} = \begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = AM,$$

the mixed derivatives of its components are equal and so this vector is equal to the gradient of some function $-\frac{1}{2}k(q)$, that is, $AM = -\frac{1}{2}\nabla k$ or $M = -\frac{1}{2}A^{-1}\nabla k$.

Finally, by direct calculation we verify that $N = -\frac{1}{2}A^{-1}\nabla(\det(A))$ and so statement (3) of the theorem is proved. Q.E.D.

Remark 6.5: This theorem generalizes the result of Ref. 6. In particular, if we assume $M = -\nabla V(q)$, then we recover the known second Poisson operator for separable potential systems.⁷

Notice that \hat{M} is the force of the two-dimensional qLN system (6.1). This means that every Poisson operator of the form (6.3) is a Poisson operator for some qLN system of the form (6.1).

We are now in position to investigate complete integrability of qLN systems admitting two quadratic, functionally independent integrals of motion. Notice first that Theorem 6.4 provides us with an alternative way of characterizing qLN systems generated by a quadratic integral of motion E : by starting with a Poisson operator of the form (6.3) we arrive at qLN systems generated by the Hamiltonian $H(q, p, d) = d$ which admit a quadratic integral E . In a similar way the following theorem characterizes all qLN systems admitting two independent quadratic integrals E, F .

Theorem 6.6 (Poisson pencil): Consider the antisymmetric operator

$$\Pi_\mu = \left[\begin{array}{cc|c} 0 & -(\lambda/2)G_\mu(q) & p \\ (\lambda/2)G_\mu^t(q) & -(\lambda/2)F_\mu(q,p) & M(q) + d\lambda N_\mu(q) \\ \hline -p^t & -M^t(q) - d\lambda N_\mu^t(q) & 0 \end{array} \right], \tag{6.11}$$

where

$$G_\mu = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} - \mu \begin{bmatrix} B_{22} & -B_{12} \\ -B_{21} & B_{11} \end{bmatrix} \equiv G_A - \mu G_B,$$

with both matrices A and B satisfying the cyclic conditions (2.3),

$$[F_\mu]_{12} = \frac{1}{2} \left(\frac{\partial(A_{22} - \mu B_{22})}{\partial q_1} p_2 - \frac{\partial(A_{11} - \mu B_{11})}{\partial q_2} p_1 \right) \equiv [F_A]_{12} - \mu [F_B]_{12}$$

(with $F = -F^t$, $F_A = -F_A^t$, $F_B = -F_B^t$) and

$$N_\mu = -\frac{1}{2}A^{-1}\nabla(\det(A)) + \frac{1}{2}\mu B^{-1}\nabla(\det(B)) \equiv N_A - \mu N_B.$$

Then Π_μ is Poisson if and only if

$$M(q) = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l \tag{6.12}$$

for some functions $k(q)$ and $l(q)$. Moreover, if we let

$$\Pi_\mu = \Pi_1 - \mu \Pi_2 \equiv \left[\begin{array}{cc|c} 0 & -(\lambda/2)G_A & p \\ (\lambda/2)G_A^t & -(\lambda/2)F_A & M + d\lambda N_A \\ \hline -p^t & -M^t - d\lambda N_A^t & 0 \end{array} \right] - \mu \left[\begin{array}{cc|c} 0 & -(\lambda/2)G_B & 0 \\ (\lambda/2)G_B^t & -(\lambda/2)F_B & d\lambda N_B \\ \hline 0 & -d\lambda N_B^t & 0 \end{array} \right],$$

then both operators Π_1 and Π_2 are Poisson and so $\Pi_\mu = \Pi_1 - \mu \Pi_2$ is a Poisson pencil.

Proof: According to the proof of Theorem 6.4 the matrix Π_μ satisfies all the Jacobi identities except possibly for $\{\{p_1, p_2\}_{\Pi_\mu}, d\}_{\Pi_\mu} + \text{cycl} = 0$, since Π_μ differs from $\Pi_{A-\mu B} = \Pi_A - \mu \Pi_B$ by the form of $M(q)$ only. Like in the proof of Theorem 6.4 we find that $\{\{p_1, p_2\}_{\Pi_\mu}, d\}_{\Pi_\mu} + \text{cycl} = 0$ yields that the mixed derivatives of the components of the vector $-2(A - \mu B)M$ are equal and so $-2(A - \mu B)M = \nabla(k - \mu l)$ for some functions $k(q)$ and $l(q)$. By comparing coefficients at different powers of μ we get $-2AM = \nabla k$ and $-2BM = \nabla l$ and thus $M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$.

Further, $\Pi_1 = \Pi_A$ in the notation of Theorem 6.4 so it is Poisson. Easy calculation shows that Π_2 is Poisson, too. Q.E.D.

The above theorem states that if $M(q)$ is the force of a qLN system admitting two functionally independent integrals of motion, then the matrix Π_μ is a Poisson pencil. We will establish its Casimir function, which will be a polynomial in μ . This will lead to a bi-Hamiltonian chain containing the qLN system (6.1). We will prove that this chain is completely integrable. In this way we will show that our original qLN system $\dot{q} = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$ can be naturally embedded in a completely integrable bi-Hamiltonian system.

Proposition 6.7: Suppose that Π_μ is Poisson, i.e., that (6.12) is satisfied. Then the function

$$H_\mu = p^t(A - \mu B)p + k - \mu l + d\lambda \det(A - \mu B) \tag{6.13}$$

is a Casimir function for Π_μ , i.e., $\Pi_\mu \nabla H_\mu = 0$.

Proof: This proposition is a consequence of Proposition 6.3. If we modify the matrix Π_A by substituting the matrix A by $A - \mu B$ and substituting k with $k - \mu l$ we obtain the matrix

$$\left[\begin{array}{cc|c} 0 & -\frac{\lambda}{2}G_\mu & p \\ \frac{\lambda}{2}G_\mu^t & -\frac{\lambda}{2}F_\mu & \tilde{M}_\mu \\ \hline -p^t & -\tilde{M}_\mu^t & 0 \end{array} \right], \tag{6.14}$$

where

$$\tilde{M}_\mu = -\frac{1}{2}(A - \mu B)^{-1}\nabla(k - \mu l) - \frac{1}{2}d\lambda(A - \mu B)^{-1}\nabla(\det(A - \mu B)).$$

Due to Proposition 6.3 the function (6.13) is the Casimir of (6.14). However, (6.14) is, in fact, equal to Π_μ since it can be verified that $-\frac{1}{2}(A - \mu B)^{-1}\nabla(k - \mu l) = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$ and that $(A - \mu B)^{-1}\nabla(\det(A - \mu B)) = A^{-1}\nabla(\det(A)) - \mu B^{-1}\nabla(\det(B))$. Q.E.D.

Let us collect terms in H_μ at different powers of μ :

$$H_\mu = p^t A p + k + d\lambda \det(A) + \mu(-p^t B p - l - d\lambda Y) + \mu^2(d\lambda \det(B)) \equiv \hat{E} + \mu \hat{F} + \mu^2 \hat{H}$$

with $Y = B_{11}A_{22} + B_{22}A_{11} - 2B_{12}A_{12}$. Then the above proposition gives

$$\begin{aligned} 0 &= \Pi_\mu \nabla H_\mu = (\Pi_1 - \mu \Pi_2) \nabla(\hat{E} + \mu \hat{F} + \mu^2 \hat{H}) \\ &= \Pi_1 \nabla \hat{E} + \mu(\Pi_1 \nabla \hat{F} - \Pi_2 \nabla \hat{E}) + \mu^2(\Pi_1 \nabla \hat{H} - \Pi_2 \nabla \hat{F}) - \mu^3 \Pi_2 \nabla \hat{H}. \end{aligned}$$

By equating to zero the coefficients at different powers of μ we obtain the following bi-Hamiltonian chain:

$$\begin{aligned} \Pi_1 \nabla \hat{E} &= 0, \\ \Pi_1 \nabla \hat{F} &= \Pi_2 \nabla \hat{E}, \\ \Pi_1 \nabla \hat{H} &= \Pi_2 \nabla \hat{F}, \\ 0 &= \Pi_2 \nabla \hat{H}. \end{aligned} \tag{6.15}$$

Theorem 6.8: *The bi-Hamiltonian chain (6.15) is completely integrable, i.e., both nontrivial bi-Hamiltonian vector fields,*

$$V_1 = \Pi_1 \nabla \hat{F} = \Pi_2 \nabla \hat{E}, \quad V_2 = \Pi_1 \nabla \hat{H} = \Pi_2 \nabla \hat{F},$$

in (6.15) are completely integrable.

Proof (modification of the proof of Liouville–Arnold theorem⁸): Consider the two-dimensional manifold $\mathcal{N} = \{x \in \mathcal{M} : \hat{E}(x) = E_0, \hat{F}(x) = F_0, \hat{H}(x) = H_0\}$ in \mathcal{M} . Poisson brackets of all pairs of $\hat{E}, \hat{F}, \hat{H}$ induced by both structures Π_0 and Π_1 are equal to zero, since the functions $\hat{E}, \hat{F}, \hat{H}$ all belong to the same bi-Hamiltonian chain. For instance, $\{\hat{F}, \hat{H}\}_{\Pi_1} = \langle \nabla \hat{F}, \Pi_1 \nabla \hat{H} \rangle = \langle \nabla \hat{F}, \Pi_2 \nabla \hat{F} \rangle = \{\hat{F}, \hat{F}\}_{\Pi_2} = 0$ with the second equality being a consequence of the bi-Hamiltonian structure of V_2 . It follows that the Lie bracket $[V_1, V_2]$ of both vector fields V_1 and V_2 is equal to zero, $[V_1, V_2] = [\Pi_1 \nabla \hat{F}, \Pi_1 \nabla \hat{H}] = 0$, since the mappings $\Pi_i \nabla$ ($i=1,2$) are Lie algebra homomorphisms between the Lie algebra of vector fields on \mathcal{M} and the Lie algebra of all smooth functions on \mathcal{M} with the Lie bracket defined by $[f_1, f_2] = \{f_1, f_2\}_{\Pi_i}$. Moreover, $\langle \nabla \hat{E}, V_1 \rangle = \langle \nabla \hat{E}, \Pi_2 \nabla \hat{E} \rangle$

$=\{\hat{E}, \hat{E}\}_{\Pi_2}=0$ and similarly $\langle \nabla \hat{F}, V_1 \rangle = \langle \nabla \hat{H}, V_2 \rangle = 0$, which proves that V_1 is tangent to \mathcal{N} . In the same way one can show that V_2 is also tangent to \mathcal{N} . Direct verification shows that V_1 and V_2 are linearly independent. We thus have a two-dimensional submanifold \mathcal{N} in \mathcal{M} equipped with a pair of linearly independent, commuting vector fields V_1 and V_2 . We can now apply the construction of Liouville-Arnold⁸ and conclude that both V_1 and V_2 are completely integrable. Q.E.D.

Corollary 6.9: The qLN system $\dot{q} = M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l$ with two linearly independent matrices A and B satisfying the cyclic conditions (2.3) is completely integrable in the sense that the trajectories of the system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p \\ M \end{bmatrix} \tag{6.16}$$

coincide on the hyperplane $d=0$ with the trajectories of the completely integrable five-dimensional system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{d} \end{bmatrix} = V_2 = \Pi_1 \nabla \hat{H} = \Pi_2 \nabla \hat{F}. \tag{6.17}$$

Proof: Consider the vector field V_2 from (6.15). Obviously

$$V_2 = \Pi_1 \nabla \hat{H} = \lambda \det(B) \Pi_1 \nabla d + \lambda d \Pi_1 \nabla (\det(B))$$

and so in the hyperplane $d=0$ we have

$$V_2|_{d=0} = \lambda \det(B) \begin{bmatrix} p \\ M \\ 0 \end{bmatrix}, \tag{6.18}$$

which means that the hyperplane $d=0$ is invariant with respect to the action of the vector field V_2 . The formula (6.18) also shows that in the hyperplane $d=0$ the vector field of the system (6.17) is parallel to the vector field of the system (6.16) and so their trajectories must coincide. Q.E.D.

Thus we have shown that the system (6.16) is embedded in the completely integrable bi-Hamiltonian system (6.17). The trajectories of (6.16) stay on the intersection of invariant manifolds for (6.17) with the hyperplane $d=0$. Also, since we can now solve the system (6.17) by quadratures the time evolution of the coefficient $\lambda \det(B)$ in (6.18) can be calculated which makes it possible to solve the system (6.16) by quadratures too.

VII. NEW TYPES OF SEPARATION VARIABLES FOR DRIVEN qLN SYSTEMS

In this section we study an important class of two-dimensional qLN equations called driven systems. We find for all such systems their separation variables and prove their integrability by quadratures. The variables of separation are of a completely new type: they consist of families of conics which are non-confocal in contrast with the classical separability theory for potential systems.

We remind the reader that we call a two-dimensional Newton system *driven* if one of the two differential equations depends on one variable only. By renaming the variables if necessary, we can always arrange for such a system to take the form

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1, q_2), \\ \ddot{q}_2 &= M_2(q_2). \end{aligned} \tag{7.1}$$

The second equation can be solved on its own and its solution $q_2(x)$ then determines the equation for q_1 , which explains the name ‘‘driven.’’ A driven system always has one integral of motion $F = \dot{q}_2^2/2 - \int M_2 dq_2$, obtained by integrating the second equation once, but, in general, there need not exist any others.

Here we shall consider driven systems that admit a quasi-Lagrangian formulation $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k(q)$. Here, as usual, $A(q)$ is a nondegenerate 2×2 matrix satisfying the cyclic conditions (2.3), i.e., a matrix of the form

$$A = \begin{bmatrix} aq_2^2 + bq_2 + \alpha & -aq_1q_2 - \frac{b}{2}q_1 - \frac{c}{2}q_2 + \frac{\beta}{2} \\ -aq_1q_2 - \frac{b}{2}q_1 - \frac{c}{2}q_2 + \frac{\beta}{2} & aq_1^2 + cq_1 + \gamma \end{bmatrix}. \tag{7.2}$$

Such a system always has *two* functionally independent integrals of motion $E = \dot{q}^t A \dot{q} + k(q)$ and $F = \dot{q}_2^2/2 - \int M_2 dq_2$.

By examining the second component of the equation $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k(q)$, we immediately see that a qLN system is driven iff

$$A_{12}\partial_1 k - A_{11}\partial_2 k = 2 \det(A)M_2(q_2), \tag{7.3}$$

for some function $M_2(q_2)$ depending on q_2 only. We can produce driven qLN systems with any given $M_2(q_2)$ and $A(q)$ by solving for $k(q)$ in this equation. The case $A_{11} = 0$ is degenerate and will be treated separately later (see Remark 7.6), so we assume from now on that $A_{11} \neq 0$.

We start by introducing separation variables for (7.1) as characteristic coordinates for (7.3).

Definition 7.1: Define curvilinear coordinates $(u, v) = (u(q), v(q))$ as follows. Let u be a parameter indexing the family of characteristic curves of (7.3) given by

$$\dot{q}(x) = \begin{bmatrix} A_{12}(q(x)) \\ -A_{11}(q(x)) \end{bmatrix}, \tag{7.4}$$

and let $v = q_2$.

In other words, the curves given by (7.4) are the coordinate curves of constant u . For a given matrix A these curves can be explicitly calculated. In Theorem 7.7 we will describe these curves more explicitly. Let us just note for the moment that they are not parallel to the curves of constant v , because of the assumption $A_{11} \neq 0$. Thus the above description really defines a coordinate system (at least locally). There is some freedom in the choice of u , but this will not affect our results. By abuse of notation we will write $f(q_1, q_2)$ and $f(u, v)$ for the same function f expressed in different coordinate systems.

Lemma 7.2: The general solution of (7.3) is

$$k(u, v) = f(u) + D(u, v)g(v), \tag{7.5}$$

where f is an arbitrary function, $D = \det(A)$, and

$$g(q_2) = \frac{-2}{A_{11}(q_2)} \int M_2(q_2) dq_2.$$

Proof: Along each characteristic curve $q(x)$ given by (7.4) we can consider (7.3) as an ODE

$$\frac{d}{dx} k(q(x)) = 2D(q(x))M_2(q_2(x)),$$

with general solution

$$k(q(x)) = D(q(x))g(q_2(x)) + f,$$

where f is a constant of integration. This can be verified by direct differentiation; the cyclic conditions imply that

$$\frac{d}{dx}D(q(x)) = \partial_1 D\dot{q}_1 + \partial_2 D\dot{q}_2 = \partial_1 DA_{12} + \partial_2 D(-A_{11}) = -D\partial_2 A_{11},$$

and thus

$$\begin{aligned} \frac{d}{dx}k(q) &= \frac{d}{dx}(D(q)g(q_2) + f) = \frac{dD}{dx}g + D\partial_2 g\dot{q}_2 \\ &= -D\partial_2 A_{11}g + D\partial_2 g(-A_{11}) \\ &= -D\partial_2(A_{11}g) = 2DM_2. \end{aligned}$$

The constant of integration f can be different for different characteristic curves, so when we express the result in terms of u and v , f will depend on u (but not on v). Q.E.D.

Lemma 7.3: Equation (7.3) is equivalent, under the substitution $k = K \det(A)$, to the equation

$$A_{12}\partial_{11}K - A_{11}\partial_{12}K - \frac{3}{2}\partial_2 A_{11}\partial_1 K = 0, \tag{7.6}$$

which is the fundamental equation (4.5) associated with the matrices A and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Proof: Equation (7.3) implies

$$\partial_1 \left(\frac{-A_{12}\partial_1 k + A_{11}\partial_2 k}{\det(A)} \right) = 0.$$

Conversely, this expression can be integrated to give (7.3), where $M_2(q_2)$ is an arbitrary function of integration. By substituting $k = K \det(A)$ and simplifying the resulting expression using the cyclic conditions one obtains (7.6). Comparison with the general expression for the fundamental equation in Theorem 4.1 proves the second statement of the lemma. Q.E.D.

Remarks 7.4: The fundamental equation (7.6) is hyperbolic. Its characteristic coordinates are precisely the coordinates (u, v) of Definition (7.1). The general solution is $K(u, v) = f(u)/D(u, v) + g(v)$, as can be seen by combining the above lemmas.

Let us turn to the question of how to integrate a driven qLN system. The solution $q_2(x)$ of the second equation can be found by quadrature from $F = \dot{q}_2^2/2 - \int M_2 dq_2$:

$$\int \frac{dq_2}{\sqrt{2F + 2\int M_2 dq_2}} = \pm \int dx. \tag{7.7}$$

Inserting $q_2(x)$ and $\dot{q}_2(x)$ into

$$E = A_{11}(q_2)\dot{q}_1^2 + 2A_{12}(q_1, q_2)\dot{q}_1\dot{q}_2 + A_{22}(q_1)\dot{q}_2^2 + k(q_1, q_2)$$

would give a first-order ODE for $q_1(x)$, but there is no obvious way to solve this equation since the variables q_1 and x do not separate. We will now show how to proceed instead.

Theorem 7.5: Every driven qLN system can be integrated by quadratures using the characteristic coordinates (u, v) of the fundamental equation (7.6) as separation variables.

Proof: We use the notation of Lemma 7.2. Let the system be generated by $E = \dot{q}^t A \dot{q} + k(q)$ with $k(u, v) = f(u) + D(u, v)g(v)$. Since $v = q_2$, we can express F as $F = \frac{1}{2}(\dot{v}^2 + A_{11}(v)g(v))$ and calculate $v(x)$ by quadrature, as above. Now note that since the curves of constant u by definition have tangent $\dot{q} = (A_{12}, -A_{11})^t$, we must have $\nabla u = \rho(q)(A_{11}, A_{12})^t$ for some function $\rho(q)$, whose exact form depends on the choice of u . This gives $\dot{u} = \partial_1 u \dot{q}_1 + \partial_2 u \dot{q}_2 = \rho(q)(A_{11} \dot{q}_1 + A_{12} \dot{q}_2)$, and thus

$$\begin{aligned} \dot{u}^2 &= \rho^2 A_{11} \left(A_{11} \dot{q}_1^2 + 2A_{12} \dot{q}_1 \dot{q}_2 + \frac{A_{12}^2}{A_{11}} \dot{q}_2^2 \right) = \rho^2 A_{11} \left(E - A_{22} \dot{q}_2^2 - k(q) + \frac{A_{12}^2}{A_{11}} \dot{q}_2^2 \right) \\ &= \rho^2 A_{11} \left(E - \left(A_{22} - \frac{A_{12}^2}{A_{11}} \right) \dot{v}^2 - f(u) - Dg(v) \right) \\ &= \rho^2 A_{11} \left(E - \frac{D}{A_{11}} (2F - A_{11}g(v)) - f(u) - Dg(v) \right) \\ &= \rho^2 A_{11} \left(E - \frac{2D}{A_{11}} F - f(u) \right). \end{aligned}$$

In order to complete the proof, we will show that $\rho(u, v) = \phi(u)|A_{11}(v)|^{-3/2}$ and $D(u, v)/A_{11}(v) = \psi(u)$ for some functions ϕ and ψ , since this implies that the variables u and x separate. Explicitly, we can then find $u(x)$ from the quadrature

$$\int \frac{du}{\phi(u)\sqrt{E - 2\psi(u)F - f(u)}} = \pm \int \frac{dx}{A_{11}(v(x))}, \tag{7.8}$$

after which the inverse coordinate transformation gives us $q_1(x)$. Notice that for a given matrix A , the characteristic coordinates (u, v) can be calculated explicitly so that the function ρ and thus ϕ and ψ can be easily calculated and used in the quadrature (7.8) above. The theorem covers, however, all the cases at once without any need of calculating ρ explicitly.

To see that $\rho(u, v) = \phi(u)|A_{11}(v)|^{-3/2}$, note that $\partial_{12}u = \partial_{21}u$ implies that $\rho(q)$ satisfies the PDE

$$0 = \partial_1(\rho A_{12}) - \partial_2(\rho A_{11}) = A_{12}\partial_1\rho - A_{11}\partial_2\rho - \frac{3}{2}\partial_2 A_{11}\rho,$$

which has the same characteristic curves (7.4) as Eq. (7.3). Along such a curve we determine ρ by integrating

$$\frac{d}{dx} \rho(q(x)) = \frac{3}{2} \partial_2 A_{11}(q_2(x)) \rho(q(x)),$$

which, taking into account $\dot{q}_2(x) = -A_{11}(q_2(x))$, gives

$$\rho(q(x)) = \phi|A_{11}(q_2(x))|^{-3/2}.$$

The integration constant ϕ can be different on different characteristic curves, so changing to (u, v) coordinates we obtain

$$\rho(u, v) = \phi(u)|A_{11}(v)|^{-3/2},$$

as desired.

Finally, we calculate the total derivative of the function $\psi(q) = D(q)/A_{11}(q_2)$ along a characteristic curve:

$$\frac{d}{dx} \psi(q(x)) = \partial_1 \left(A_{22} - \frac{A_{12}^2}{A_{11}} \right) A_{12} - \partial_2 \left(A_{22} - \frac{A_{12}^2}{A_{11}} \right) A_{11}.$$

Using the cyclic conditions, we find that this expression is identically zero. This implies that ψ is constant along the coordinate curves of constant u , i.e., $\psi = \psi(u)$. This completes the proof.

Q.E.D.

Remark 7.6: The degenerate case $A_{11} = 0$ can be treated as follows. Since $a = b = \alpha = 0$, the expression (7.2) for A reduces to

$$A = \begin{bmatrix} 0 & -\frac{c}{2}q_2 + \frac{\beta}{2} \\ -\frac{c}{2}q_2 + \frac{\beta}{2} & cq_1 + \gamma \end{bmatrix} \tag{7.9}$$

and Eq. (7.3) reduces to

$$A_{12} \partial_1 k = 2(-A_{12}^2) M_2(q_2),$$

with the general solution

$$k(q) = -2A_{12}(q_2) M_2(q_2) q_1 + k_2(q_2).$$

We calculate $q_2(x)$ by quadrature as before. Inserting $q_2(x)$ and $\dot{q}_2(x)$ into $E = 2A_{12} \dot{q}_1 \dot{q}_2 + A_{22} \dot{q}_2^2 + k(q_1, q_2)$ yields in this case an equation of the form $\dot{q}_1(x) + \xi(x) q_1(x) = \eta(x)$, from which we can find $q_1(x)$ by quadrature.

Theorem 7.7: *The separation coordinates for driven qLN systems, i.e., the characteristic coordinates (u, v) of the fundamental equation (7.6), are of one of the following types, determined by the coefficients in the matrix A :*

- (1) fanlike hyperbolic, if $a \neq 0$ and $b^2/4 - a\alpha = 0$;
- (2) axial hyperbolic, if $a \neq 0$ and $b^2/4 - a\alpha < 0$;
- (3) two-point elliptic-hyperbolic, if $a \neq 0$ and $b^2/4 - a\alpha > 0$;
- (4) one-point parabolic, if $a = 0$ and $b \neq 0$; and
- (5) parallel parabolic, if $a = 0$ and $b = 0$.

Proof: We will compute explicitly the curves given by (7.4), which constitute the curves of constant u . (The curves of constant v are just horizontal lines, since $v = q_2$.) Inserting the explicit expression (7.2) for the matrix A into (7.4), we obtain

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = - \begin{bmatrix} aq_1q_2 + \frac{b}{2}q_1 + \frac{c}{2}q_2 - \frac{\beta}{2} \\ aq_2^2 + bq_2 + \alpha \end{bmatrix}. \tag{7.10}$$

When solving these equations, we distinguish four different cases, depending on the values of the parameters in A .

The case $a \neq 0$. By setting $r_1 = aq_1 + c/2$ and $r_2 = aq_2 + b/2$, which is just rescaling of the axes and translation of the origin, we transform (7.10) into

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} -r_1 r_2 + C_1 \\ -r_2^2 + C_2 \end{bmatrix}, \quad \text{where} \begin{cases} C_1 = bc/4 + a\beta/2, \\ C_2 = b^2/4 - a\alpha. \end{cases} \quad (7.11)$$

Subcase $C_2 = 0$ (type 1). Either $r_2 = 0$, or $r_2 = (x + D_1)^{-1}$ and $r_1 = C_1(x + D_1)/2 + D_2(x + D_1)^{-1}$, where D_1 and D_2 are constants of integration. Eliminating x and writing u instead of D_2 , we obtain

$$r_1 = \frac{C_1}{2r_2} + ur_2, \quad (7.12)$$

which represents a family of hyperbolas, each with asymptotes $r_2 = 0$ and $r_2 = r_1/u$. The solution $r_2 = 0$ found above corresponds to the limiting cases $u \rightarrow \pm \infty$ (see Fig. 2).

Subcase $C_2 \neq 0$ (type 2 and 3). The substitution $s_1 = r_1 - C_1 r_2 / C_2$, $s_2 = r_2$ yields

$$\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{bmatrix} = \begin{bmatrix} -s_1 s_2 \\ -s_2^2 + C_2 \end{bmatrix}, \quad (7.13)$$

and thus

$$\frac{ds_1}{ds_2} = \frac{\dot{s}_1}{\dot{s}_2} = \frac{s_2}{s_2^2 - C_2} s_1,$$

resulting in

$$s_1^2 = u^2 |s_2^2 - C_2|. \quad (7.14)$$

If $C_2 < 0$ (type 2), this represents in the s plane a family of hyperbolas centered around the s_1 axis, with asymptotes $s_2 = \pm s_1/u$ and vertices $(\pm u\sqrt{-C_2}, 0)$ [Fig. 3(a)].

If $C_2 > 0$ (type 3), we obtain in the region $|s_2| > \sqrt{C_2}$ a family of hyperbolas with asymptotes $s_2 = \pm s_1/u$ and vertices $(0, \pm \sqrt{C_2})$, and in the region $|s_2| < \sqrt{C_2}$ a family of ellipses with vertices

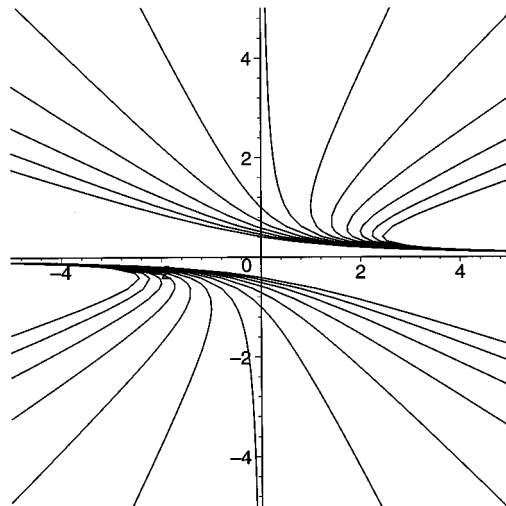


FIG. 2. Fan-like hyperbolic.

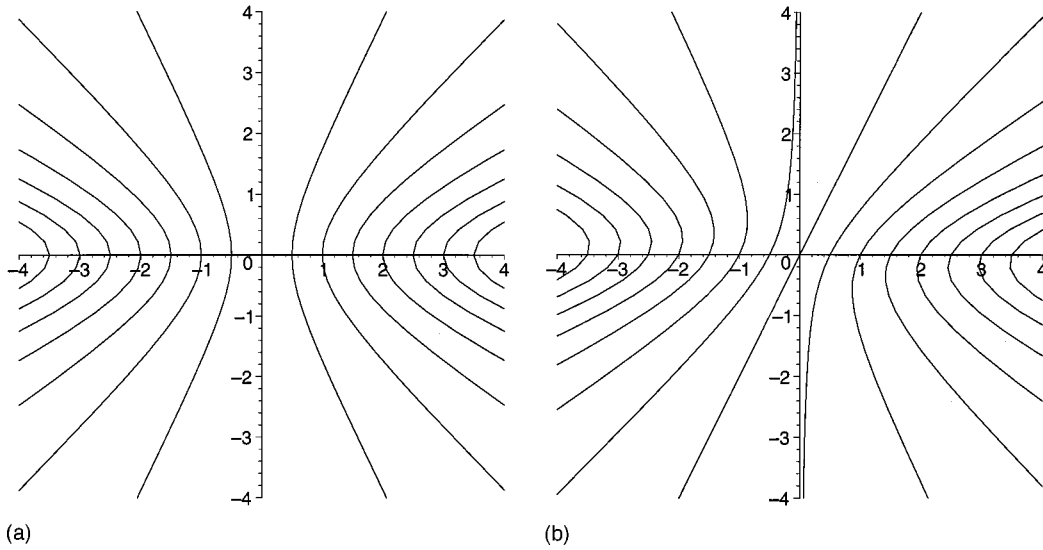


FIG. 3. (a) Axial-hyperbolic in the s -plane. (b) Axial-hyperbolic in the r -plane.

$(0, \pm\sqrt{C_2})$ and $(0, \pm u\sqrt{C_2})$. The corresponding curves in the r plane are obtained by a shear in the s_1 direction with factor C_1/C_2 [Figs. 3(b) and 4(b)]. They are still hyperbolas and ellipses, but not aligned parallel with the r axes.

The case $a=0$. Subcase $b \neq 0$ (type 4). Translating the origin by $r_1 = q_1 - (\alpha c/b^2 + \beta/b)$ and $r_2 = q_2 + \alpha/b$, we obtain

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b}{2}r_1 - \frac{c}{2}r_2 \\ -br_2 \end{bmatrix}, \tag{7.15}$$

which yields

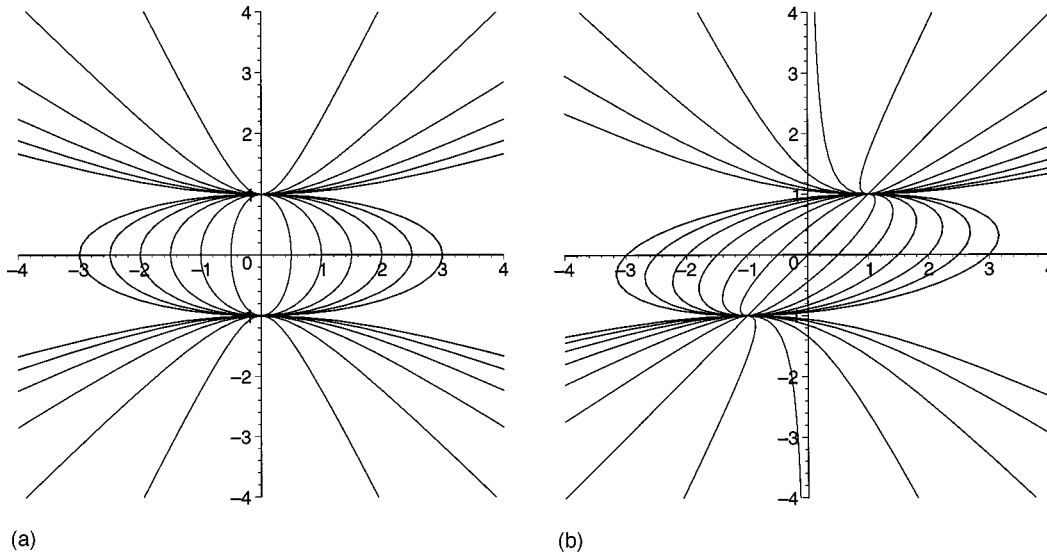


FIG. 4. (a) Two-point elliptic-hyperbolic in the s -plane. (b) Two-point elliptic-hyperbolic in the r -plane.

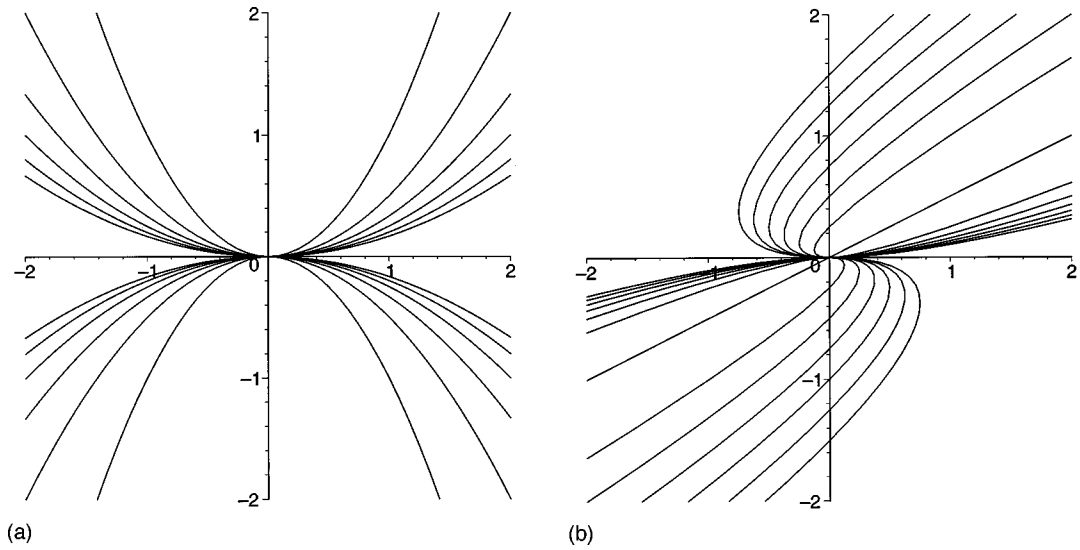


FIG. 5. (a) One-point parabolic in the s -plane. (b) One-point parabolic in the r plane.

$$\left(r_1 - \frac{c}{b}r_2\right)^2 = ur_2. \tag{7.16}$$

With $s_1 = r_1 - cr_2/b$, $s_2 = r_2$, we obtain in the s plane a family of parabolas $s_2 = s_1^2/u$ [Fig. 5(a)]. The corresponding curves in the r plane are parabolas obtained by a shear in the s_1 direction with factor c/b [Fig. 5(b)].

Subcase $b=0$ (type 5). Here we can assume that $\alpha \neq 0$, or else we get the degenerate case $A_{11} = 0$. A simple calculation shows that

$$q_1 = -\frac{c}{4\alpha}q_2^2 + \frac{\beta}{2\alpha}q_2 + u, \tag{7.17}$$

which is a family of translated parabolas seen in Fig. 6 (or straight lines, if $c=0$). Q.E.D.

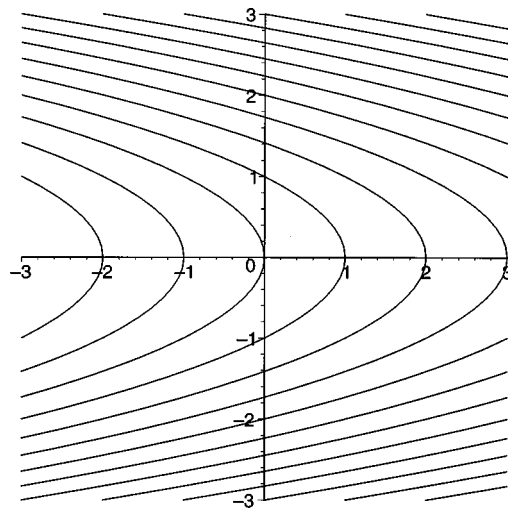


FIG. 6. Parallel-parabolic.

VIII. EXAMPLES AND APPLICATIONS

The notion of a qLN force $M(q) = -\frac{1}{2}A^{-1}(q)\nabla k(q)$ naturally generalizes the concept of a potential force $M(q) = -\nabla k(q)$, which is a special case. The qLN forces admit an integral of motion quadratic in velocities, which in the potential case becomes the energy integral. The function $k(q)$ may be called a ‘‘quasi-potential’’ of the force $M(q)$.

A given force is easy to test for the existence of a qLN formulation, provided that one knows the general form of the matrix $A(q)$ solving the cyclic condition (2.3). In two dimensions $A(q)$, given by

$$A(r, w) = \begin{bmatrix} aw^2 + bw + \alpha & -arw - \frac{b}{2}r - \frac{c}{2}w + \frac{\beta}{2} \\ -arw - \frac{b}{2}r - \frac{c}{2}w + \frac{\beta}{2} & ar^2 + cr + \gamma \end{bmatrix},$$

depends on six arbitrary parameters, and a qLN formulation exists provided that the mixed derivatives of $\nabla k(r, w) = -2A(r, w)M(r, w)$ are equal for some nonzero values of the parameters $a, b, c, \alpha, \beta, \gamma$. We thus have the following lemma:

Lemma 8.1: A given force $M(r, w) = (M_1(r, w), M_2(r, w))^t$ admits a qLN formulation $M(r, w) = -\frac{1}{2}A^{-1}(r, w)\nabla k(r, w)$ if and only if there is a nontrivial solution A , with $\det(A) \neq 0$, of the equation

$$0 = \frac{\partial}{\partial w} \left((aw^2 + bw + \alpha)M_1 + \left(-arw - \frac{b}{2}r - \frac{c}{2}w + \frac{\beta}{2} \right)M_2 \right) - \frac{\partial}{\partial r} \left(\left(-arw - \frac{b}{2}r - \frac{c}{2}w + \frac{\beta}{2} \right)M_1 + (ar^2 + cr + \gamma)M_2 \right). \tag{8.1}$$

Lemma 8.2 (Criterion of integrability, $n=2$): Equation (8.1) has a two-parameter family of solutions for $A(r, w)$ if and only if $\ddot{q} = M(q)$ admits two functionally independent integrals of motion E and F quadratic in velocities.

Proof: If such $E = \dot{q}^t A \dot{q} + k$ and $F = \dot{q}^t B \dot{q} + l$ exist, then $\lambda E + \mu F = \dot{q}^t (\lambda A + \mu B) \dot{q} + (\lambda k + \mu l)$ is an integral of motion for all λ and μ , and thus $\lambda A + \mu B$ is a two-parameter solution of (8.1).

Conversely, if there is a two-parameter solution $D(\lambda, \mu)$ of (8.1), then there are linearly independent integrals E and F with $A = D(1, 0)$ and $B = D(0, 1)$. Q.E.D.

These two lemmas make it simple to test a given two-dimensional force for the existence of a qLN formulation, and to show integrability if a two-parameter family of solutions for A exists.

Example 8.3 (gH-H system): The generalized Hénon–Heiles (gH-H) system⁹ is defined by the potential

$$V(q_1, q_2) = c_1 q_1 q_2^2 - \frac{c_2 q_1^3}{3} + \frac{c_0}{2q_2^2}, \quad c_1, c_2 \neq 0.$$

It is known to be integrable in three cases: the Korteweg–de Vries (KdV) case $6c_1 + c_2 = 0$, the Sawada–Kotera (S-K) case $c_1 + c_2 = 0$, and the Kaup–Kupershmidt (K-K) case $16c_1 + c_2 = 0$. In the KdV case, and also in the S-K case, if $c_0 = 0$, the second integral of motion is quadratic in velocities, in the other cases it is quartic. The system appears naturally when integrating the equation $0 = (\frac{1}{4}\partial^3 + 2c_1 u \partial + c_1 u_x)(\frac{1}{4}u_{xx} - \frac{1}{4}c_2 u^2)$, which for the above cases corresponds to the stationary flow of the fifth-order KdV, S-K, and K-K soliton equations. This observation explained¹⁰ the remarkable connection of the integrable cases of the gH-H system with soliton hierarchies.

We shall apply our criterion for existence of a qLN formulation to two Newton representations of the gH-H system; the original system in q variables

$$\begin{aligned} \ddot{q}_1 &= -\frac{\partial V}{\partial q_1} = -c_1 q_2^2 + c_2 q_1^2, \\ \ddot{q}_2 &= -\frac{\partial V}{\partial q_2} = -2c_1 q_1 q_2 + \frac{c_0}{q_2^3}, \end{aligned} \tag{8.2}$$

and another system in r variables

$$\begin{aligned} \ddot{r}_1 &= r_2 + (c_1 + c_2)r_1^2, \\ \ddot{r}_2 &= c_3 - 10c_1 r_1 r_2 - 10c_1 \left(c_1 + \frac{c_2}{3} \right) r_1^3, \end{aligned} \tag{8.3}$$

which is equivalent¹¹ to the q system under the map $r_1 = q_1$, $r_2 = -c_1(q_1^2 + q_2^2)$, $c = -4c_1[\frac{1}{2}(q_1^2 + q_2^2) + V(q_1, q_2)]$. The r system does not have any natural Lagrangian or Hamiltonian formulation and its integrability has previously been studied only through its equivalence with the original gH-H system. We will show here a more direct approach based on the qLN theory.

Beginning with the q system, we insert the right-hand side M from (8.2) into (8.1), identifying (q_1, q_2) with (r, w) as usual. Since the powers $r^i w^j$ are linearly independent, the coefficients at different powers must be individually zero. This gives, after some simplification, that $a = b = 0$, $\alpha = \gamma$ arbitrary, $(6c_1 + c_2)c = 0$, $(c_1 + c_2)\beta = 0$ and $c_0\beta = 0$. This means that we always have a solution $A = tI$ (of course, corresponding to the energy integral, since the system has a potential). Moreover, in two cases there exists a two-parameter solution; when $6c_1 + c_2 = 0$,

$$A = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & -q_2 \\ -q_2 & 2q_1 \end{bmatrix},$$

and, when $c_1 + c_2 = c_0 = 0$,

$$A = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So in this way we have recovered the KdV and S-K ($c_0 = 0$) cases, while the K-K and S-K ($c_0 \neq 0$) cases, with a quartic extra integral, fall outside of the qLN theory.

Performing the same procedure for the r system (8.3), we find that $a = 0$, $4c_1\gamma + b = 0$, $(31c_1 + c_2)c = (3c_1^2 + c_1c_2)c = 0$, $2\alpha - 3c_3c = 0$, and $(6c_1 + c_2)\beta = 0$. Since we have excluded the trivial case $c_1 = 0$, it follows that $c = \alpha = 0$, so that the solution is

$$A = t \begin{bmatrix} -2r_2 & r_1 \\ r_1 & 1/2c_1 \end{bmatrix}$$

except for the KdV case $6c_1 + c_2 = 0$ which admits a two-parameter solution

$$A = t \begin{bmatrix} -2r_2 & r_1 \\ r_1 & \frac{1}{c_1} \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This agrees with the known fact¹¹ that for the r system the second integral is quartic in velocities in the S-K and K-K cases and thus cannot be found by this method.

Suppose, however, that we had found the second integral in these cases by some other method. Then we would still have use for the qLN theory in proving the system's integrability, since the r parametrization admits the nonstandard Hamiltonian formulation (6.2) with $\lambda = 1$ and c_3 playing the role of the fifth variable d . (This coincides with the Hamiltonian formulation that was found in Ref. 11 by transferring the standard Hamiltonian formulation from the q parametrization, except for naming the momenta in reverse order; here $p_i = \dot{r}_i$, while in that paper $s_1 = \dot{r}_2, s_2 = \dot{r}_1$.)

For example, in the S-K case ($c_1 = -c_2 = \frac{1}{2}$) we have

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{c}_3 \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & -1/2 & -r_1/2 & p_1 \\ & 0 & -r_1/2 & r_2 & p_2 \\ & * & 0 & -p_1/2 & r_2 \\ & * & * & 0 & c_3 - 5r_1r_2 - (5/3)r_1^3 \\ * & * & * & * & 0 \end{array} \right] \nabla_{\mathcal{M}} c_3,$$

with the commuting integrals of motion $E = -2r_2p_1^2 + 2r_1p_1p_2 + p_2^2 + 4r_1r_2^2 + \frac{2}{3}r_1^3(r_1^2 + 5r_2) + c_3(-r_1^2 - 2r_2)$, which is a Casimir, the Hamiltonian c_3 , and $F = \frac{3}{2}p_1^4 - 6r_1r_2p_1^2 + (3r_1^2 - r_2)p_1p_2 + r_1p_2^2 + \frac{7}{2}r_1^2r_2^2 + \frac{10}{3}r_1^4r_2 + \frac{5}{6}r_1^6 + \frac{1}{3}r_2^3 + (-2r_1r_2 - r_1^3 + \frac{3}{2}p_1^2)c_3$, which is quartic in momenta.

In order to give an impression of the wealth of different types of nonpotential Newton forces belonging to our theory, we will now examine solutions of the fundamental equation (4.5) for some specified pairs of matrices A and B . Any such solution corresponds to an integrable qLN system, and once one solution has been found, a whole family of solutions can be constructed using the recursion theorem 4.3.

Example 8.4 (One-dimensional complex motion): If we take A and B as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then the fundamental equation reduces to the Laplace equation $K_{rr} + K_{ww} = 0$. Given a solution $K(r, w)$, i.e., a harmonic function, we have $k = K \det(A) = -K$, so k is also harmonic. We find the corresponding l from the relation

$$\nabla l = BA^{-1} \nabla k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla k,$$

which is nothing but the Cauchy–Riemann equations for k and l , so l is the harmonic conjugate of k . The corresponding qLN system $\dot{r} = -k_r/2, \dot{w} = k_w/2$ can be integrated by introducing the complex variable $z = r + iw$ and the complex integral of motion $\mathcal{E} = E + iF = (\dot{r}^2 - \dot{w}^2 + k) + i(2\dot{r}\dot{w} + l) = (\dot{r} + i\dot{w})^2 + (k + il) = \dot{z}^2 + f(z)$, where $f(z) = k(z) + il(z)$ is analytic. We can now determine z , and thus r and w , from $\dot{z} = \pm \sqrt{\mathcal{E} - f(z)}$ by one complex quadrature.

Repeated application of the recursion formula (4.10) yields in this case the standard cycle of conjugate harmonic pairs $(k, l) \rightarrow (l, -k) \rightarrow (-k, -l) \rightarrow (-l, k) \rightarrow (k, l)$.

Example 8.5 (Fundamental equation separable in polar coordinates): Let

$$A = \begin{bmatrix} -2w & r \\ r & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & w \\ w & -2r \end{bmatrix}.$$

Then the fundamental equation becomes $0 = 2(r^2K_{rr} + 2rwK_{rw} + w^2K_{ww}) + 9(rK_r + wK_w) + 6K$, which in polar coordinates ($r = R \cos \phi, w = R \sin \phi$) transforms into $0 = 2R^2K_{RR} + 9RK_R + 6K$.

The general solution of this equation is $K(R, \phi) = f_0(\phi)R^{-2} + g_0(\phi)R^{-3/2}$, for some arbitrary functions f_0 and g_0 . Changing back to r and w , we find that the general solution of the fundamental equation in this case is

$$K(r, w) = f_0\left(\arctan \frac{w}{r}\right) \frac{r^{-2}}{1 + (w/r)^2} + g_0\left(\arctan \frac{w}{r}\right) \frac{r^{-3}}{(1 + (w/r)^2)^{3/2}} = f\left(\frac{w}{r}\right) r^{-2} + g\left(\frac{w}{r}\right) r^{-3},$$

where f and g are arbitrary functions.

Finally, let us conclude with an example of a three-dimensional qLN system. A detailed treatment of higher-dimensional qLN systems is presented in a separate article.³

Example 8.6: The Newton system

$$\begin{aligned} \ddot{r}_1 &= -10r_1^2 + 4r_2, \\ \ddot{r}_2 &= -16r_1r_2 + 10r_1^3 + 4r_3, \\ \ddot{r}_3 &= -20r_1r_3 - 8r_2^2 + 30r_1^2r_2 - 15r_1^4 + d \end{aligned}$$

was found in Ref. 12 as a parametrization of the seventh-order stationary KdV flow. It has three integrals of motion

$$\begin{aligned} E_1 &= \dot{r}_1\dot{r}_3 + \frac{\dot{r}_2^2}{2} + 10r_1^2r_3 - 4r_2r_3 + 8r_1r_2^2 - 10r_1^3r_2 + 3r_1^5 - dr_1, \\ E_2 &= r_3\dot{r}_1^2 - r_1\dot{r}_2^2 + r_2\dot{r}_1\dot{r}_2 - \dot{r}_2\dot{r}_3 - r_1\dot{r}_1\dot{r}_3 + 4r_1^2r_2^2 + 5r_1^4r_2 - \frac{5}{2}r_1^6 \\ &\quad - 4r_2^3 + 2r_3^2 - 12r_1r_2r_3 + \frac{dr_1^2}{2} + dr_2, \\ E_3 &= \frac{1}{8}(r_2^2\dot{r}_1^2 + r_1^2\dot{r}_2^2 + \dot{r}_3^2 - (2r_1r_2 + 4r_3)\dot{r}_1\dot{r}_2 + 2r_1\dot{r}_2\dot{r}_3 + 2r_2\dot{r}_1\dot{r}_3) + r_1r_2^3 - 3r_1^3r_2^2 \\ &\quad + \frac{5}{4}r_1^5r_2 + 2r_1r_3^2 + \frac{5}{4}r_1^4r_3 - r_1^2r_2r_3 + r_2^2r_3 - \frac{d}{4}(r_1r_2 - r_3), \end{aligned}$$

which all are quadratic in velocities. This means that the system is generated by any of them through the quasi-Lagrangian equations. From the velocity-dependent parts we find

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2r_3 & r_2 & -r_1 \\ r_2 & -2r_1 & -1 \\ -r_1 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} r_2^2 & -r_1r_2 - 2r_3 & r_2 \\ -r_1r_2 - 2r_3 & r_1^2 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix}$$

as examples of 3×3 -matrices satisfying the cyclic conditions (2.3).

IX. CONCLUSIONS

In this article we have developed a new theory—the theory of quasi-Lagrangian Newton equations. It was originally inspired by interesting properties of the second stationary flow of the Harry Dym hierarchy, which led us to a broad theory which encompasses the classical separability theory but goes far beyond the classical results—the classical Bertrand–Darboux theory of separability for two-dimensional potential forces depends on one essential free parameter while our theory depends on five parameters.

The main part of this work has been focused on two-dimensional qLN systems which admit two integrals of motion E and F quadratic in velocities. These systems have only a nonstandard Hamiltonian formulation and are completely integrable by embedding into five-dimensional Liouville integrable systems. All such qLN systems are characterized by a single PDE called here the fundamental equation. We have shown that there is a one-to-one correspondence between fundamental equations and linear pencils $\lambda A + \mu B$ of matrices A and B . These linear pencils have been classified in Sec. V. In Sec. VII the class of driven systems has been studied in detail and new types of separation variables (non-confocal conics) have been found. We have also shown that any given force can be effectively tested for the existence of qLN formulation, which can further be used for unveiling its complete integrability and for solving the corresponding Newton equation. We have illustrated this by several examples including the generalized Hénon-Heiles system (Sec. VIII).

There are several natural directions of development of the theory of qLN systems. The n -dimensional versions of our main theorems on fundamental equation and on complete integrability have already been formulated and proved in Ref. 3.

The great wealth of different types of integrable Newton equations contained in the fundamental equation remains to be studied. Here we have only discussed two special cases: separable systems and driven systems. However, one of the most challenging questions yet to be answered is the existence of separation variables for the fundamental equation in its most general form. It can lead to new and interesting connections with the classical theory of separability of the Hamilton–Jacobi equation and of linear PDEs.

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