Dynamics of peakons and antipeakons in Novikov's equation

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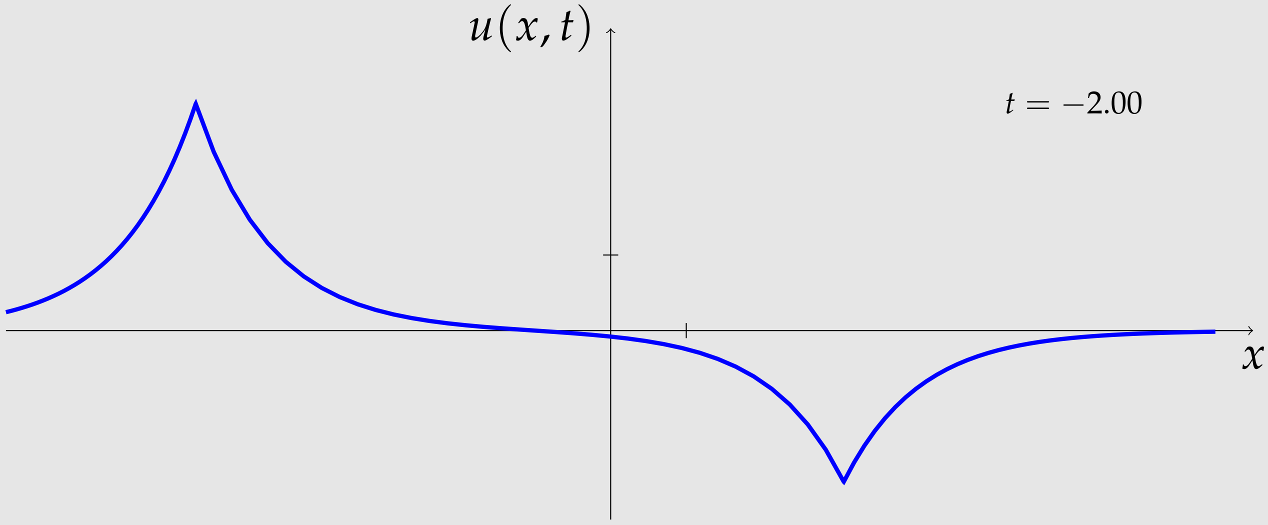
Session "CH – 30 Years Later"

Probably familiar to many in the audience:

Peakon solutions of the Camassa–Holm equation

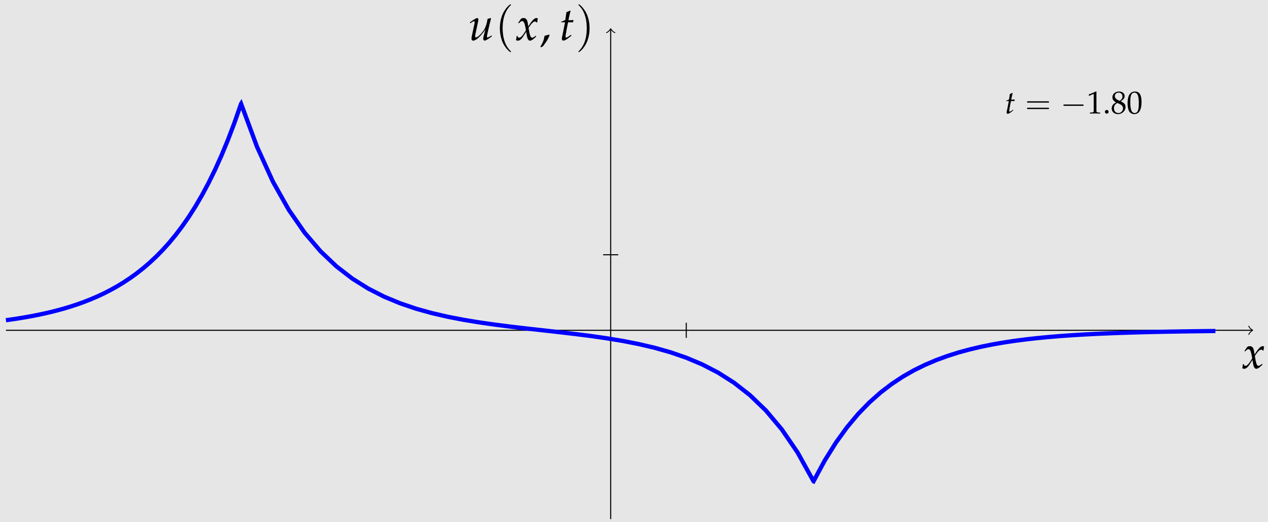
And in particular:

Peakon–antipeakon collisions

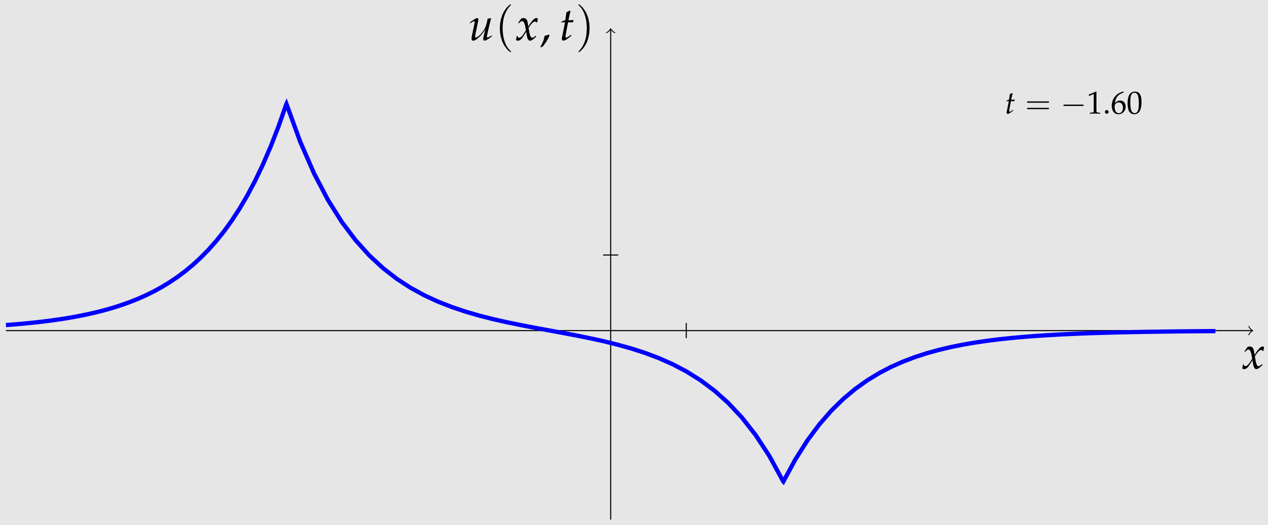


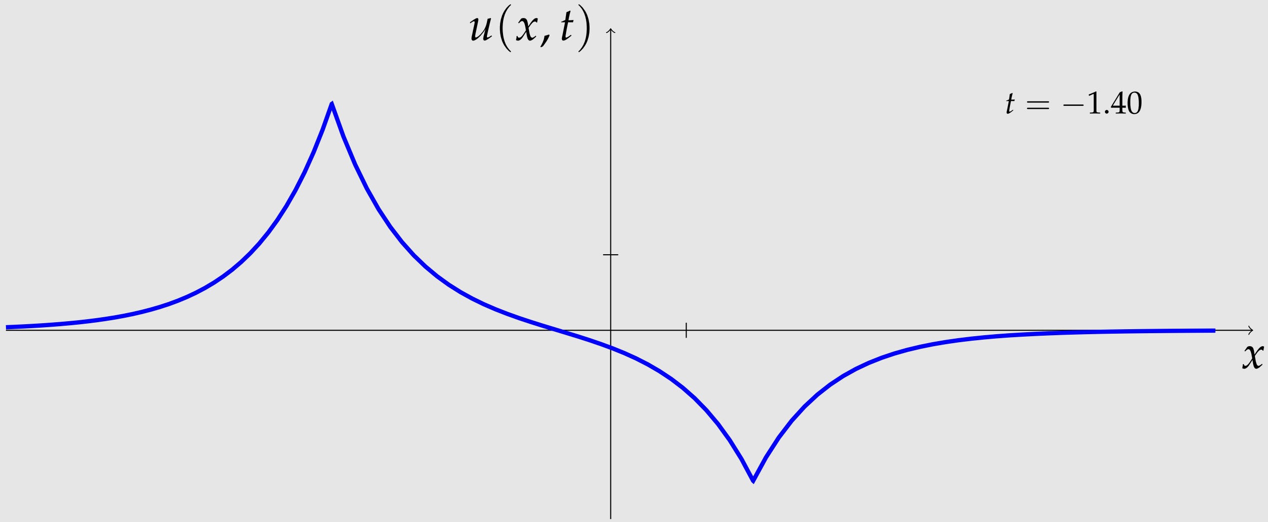
Incoming **peakon**
(asymptotic velocity 3)

Incoming **antipeakon**
(asymptotic velocity -2)

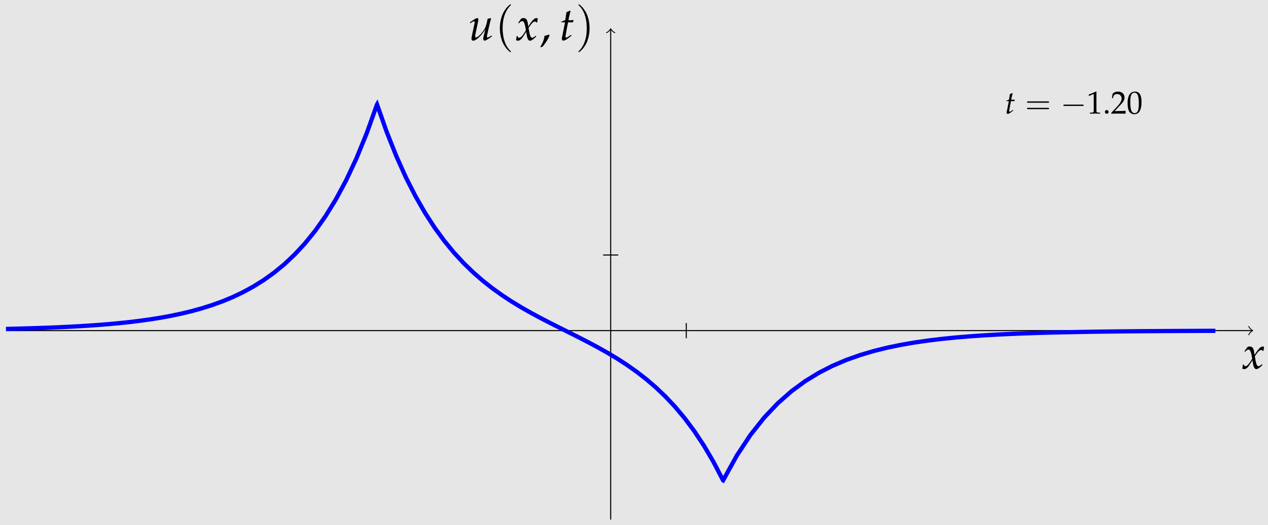


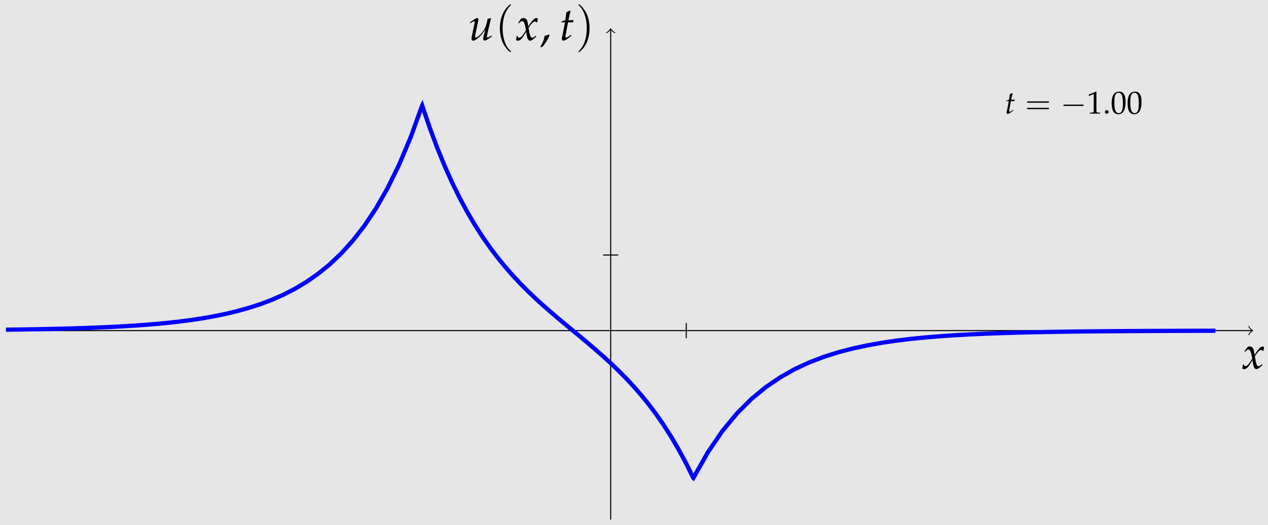
$u(x,t)$
$t = -1.80$
x


$u(x,t)$
$t = -1.60$
x

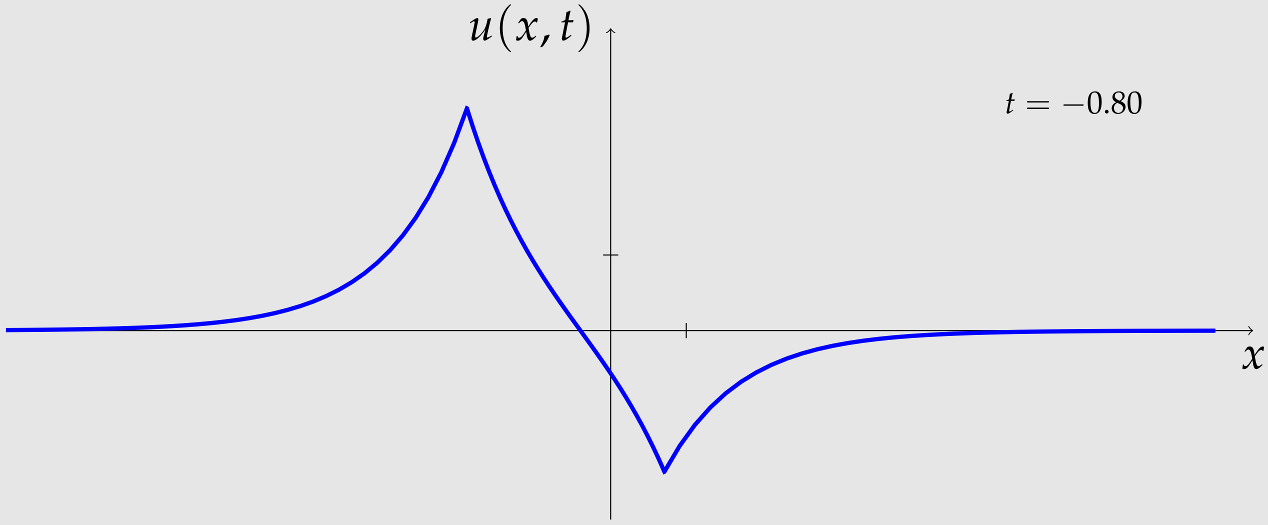


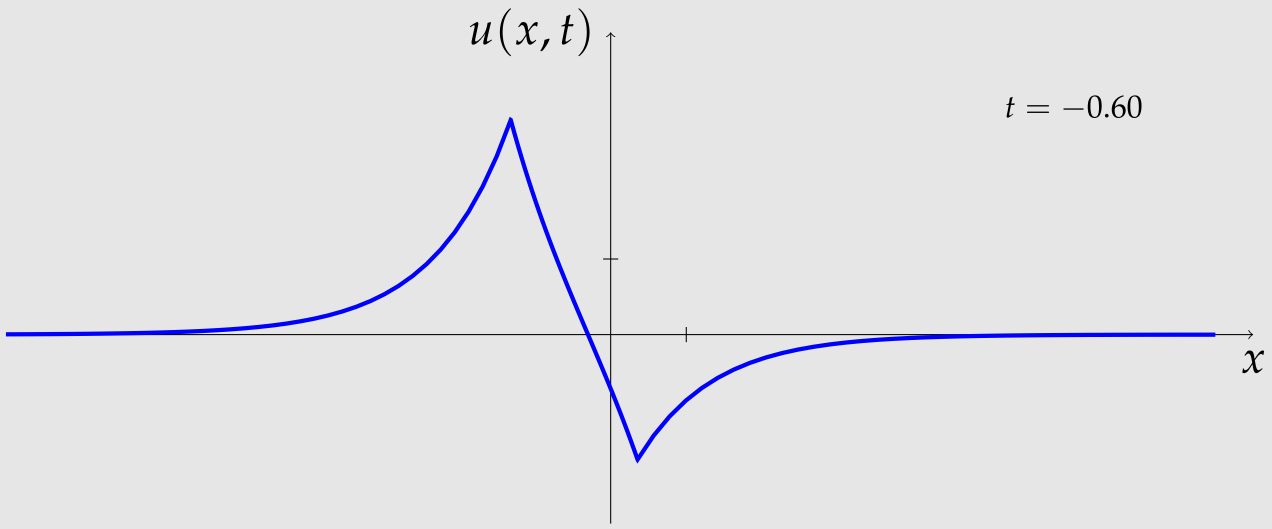
$u(x,t)$
$t = -1.40$
x


$u(x,t)$
$t = -1.20$
x

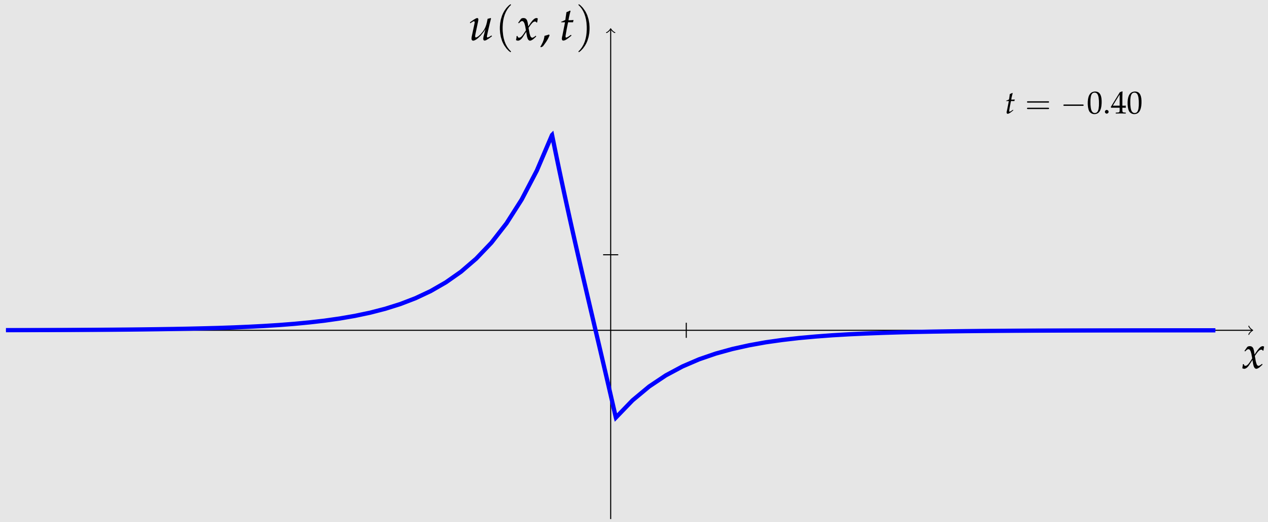


$u(x,t)$
$t = -1.00$
x

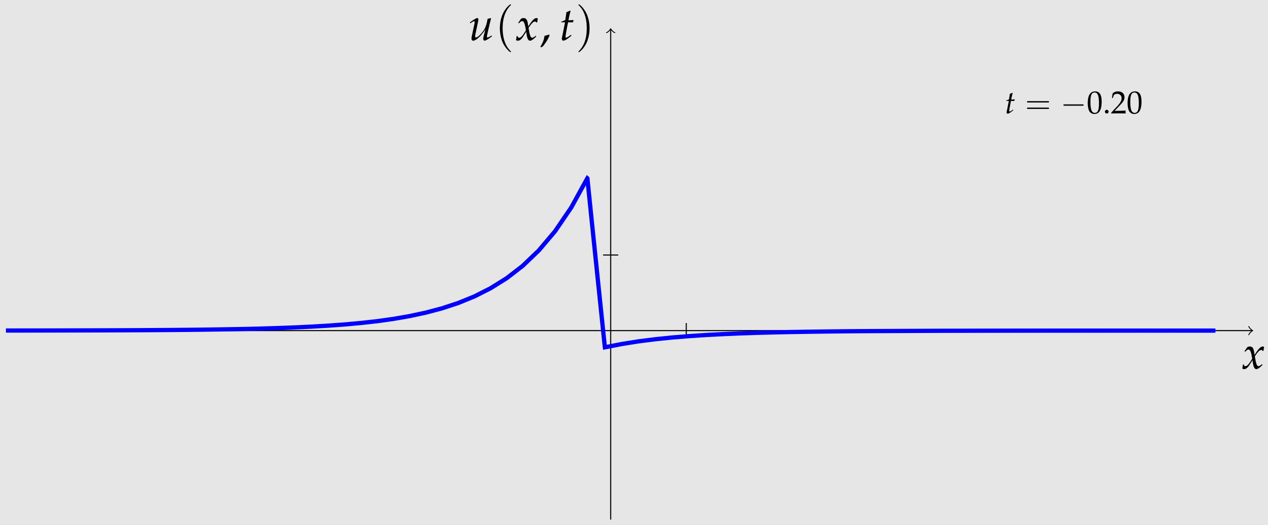

$u(x,t)$
$t = -0.80$
x



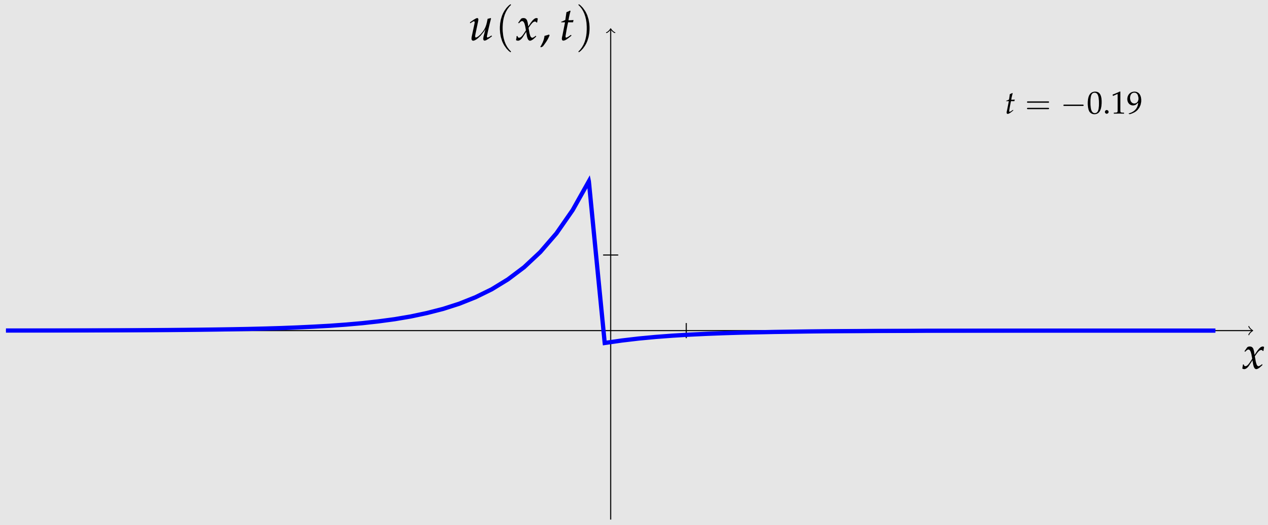
$u(x,t)$
$t = -0.60$
x

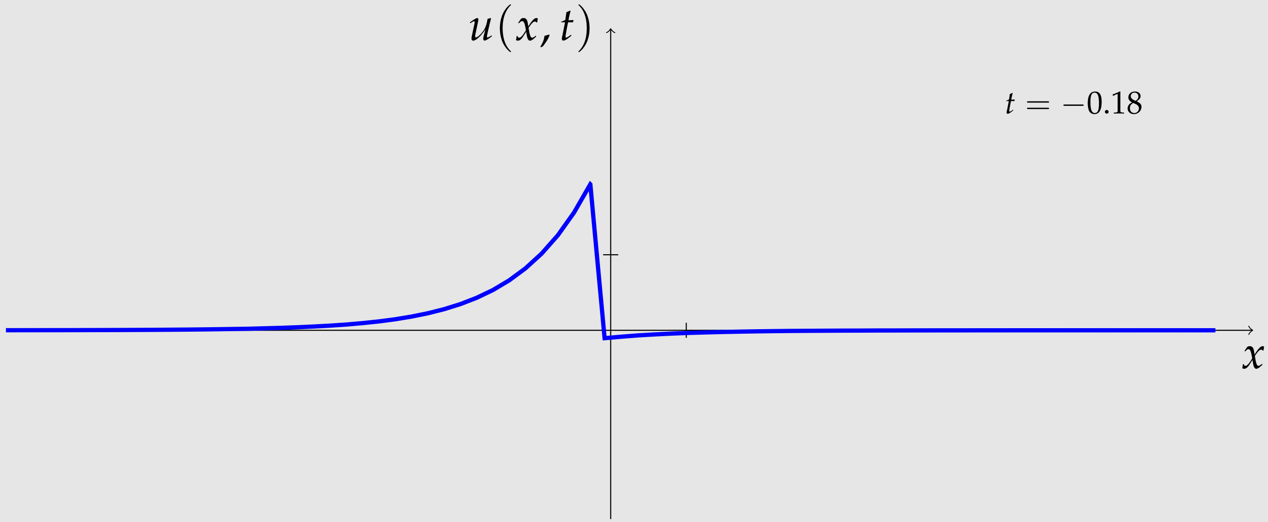


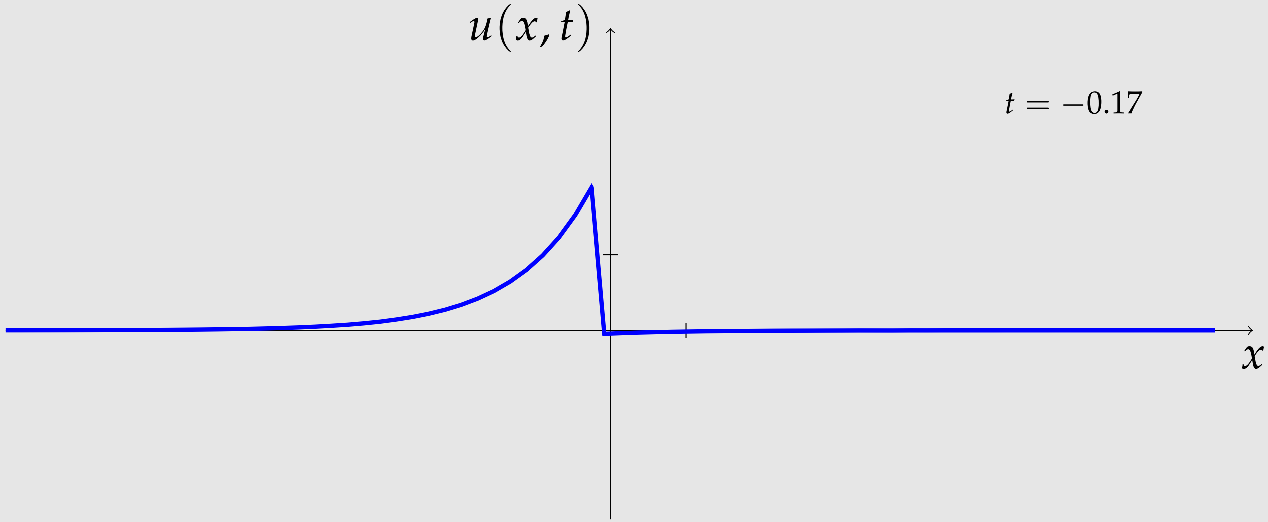
$u(x,t)$
$t = -0.40$
x

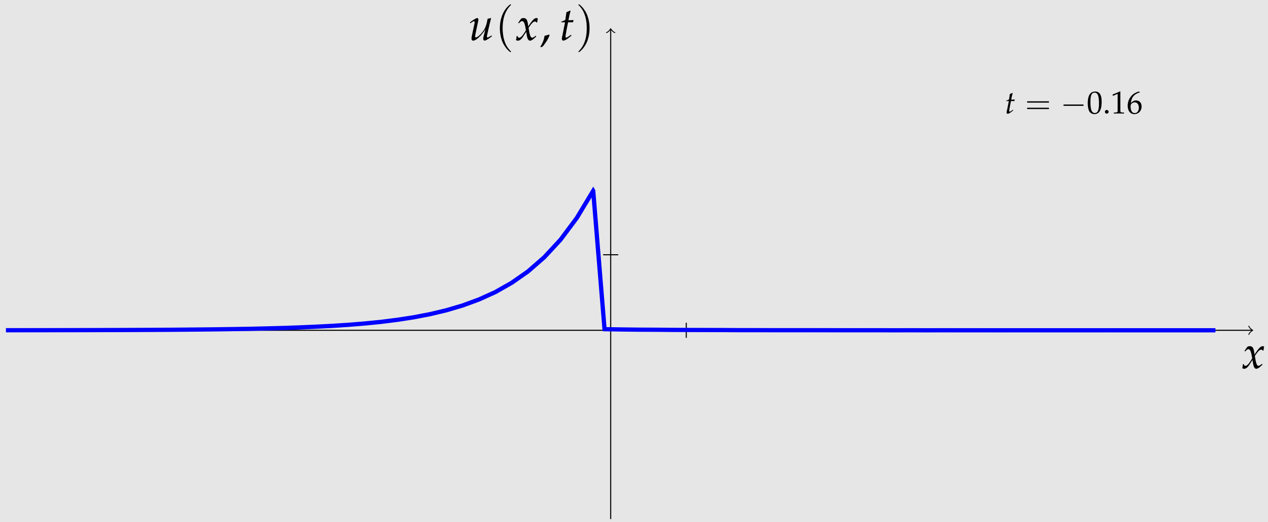


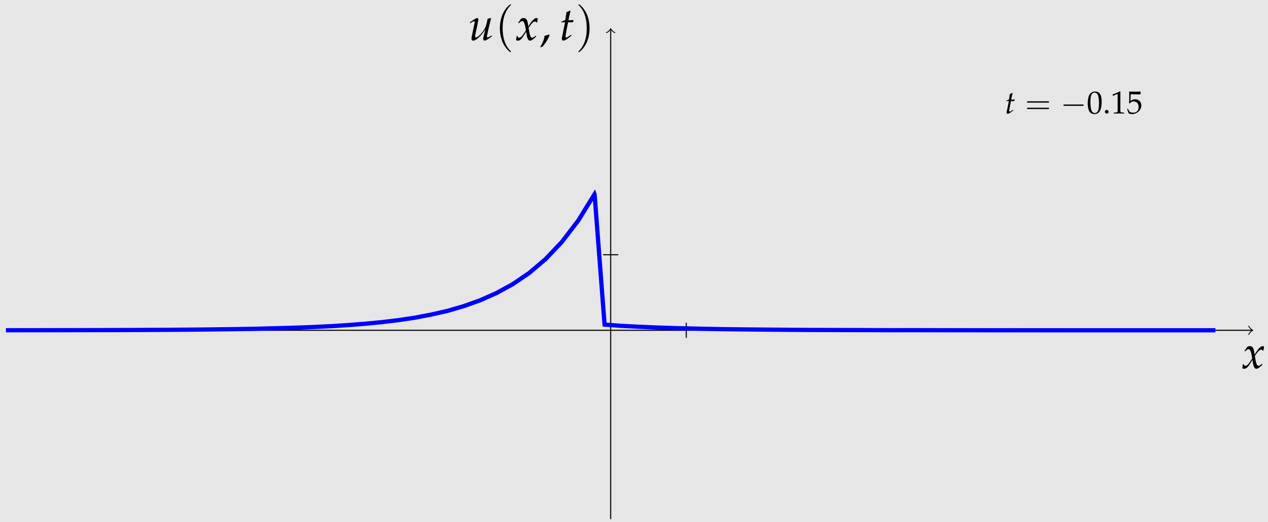
(Smaller timesteps from now on.)

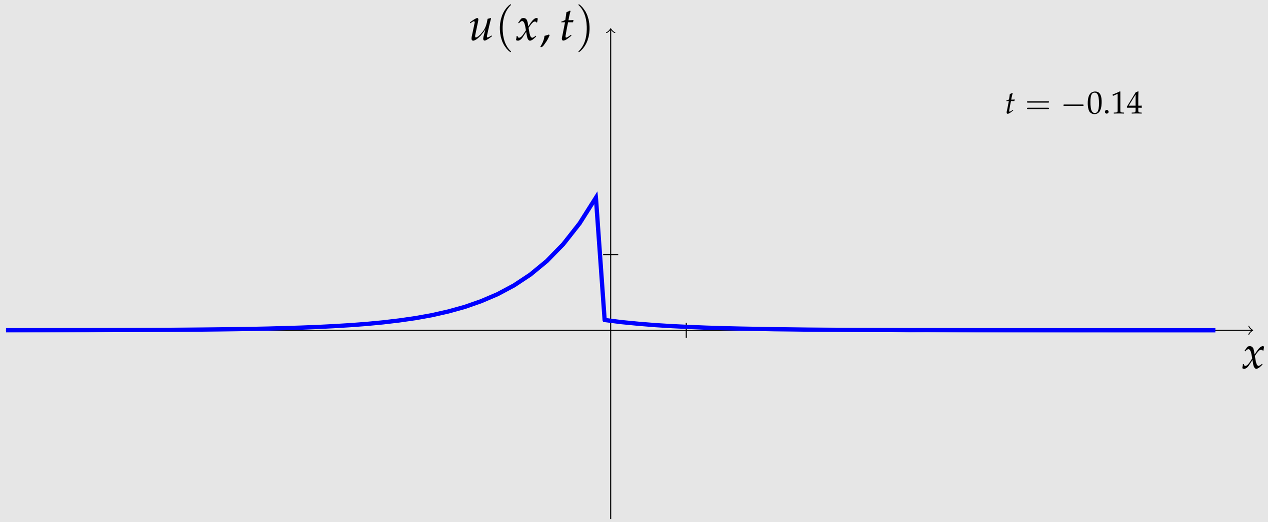

$u(x,t)$
$t = -0.19$
x


$u(x,t)$
$t = -0.18$
x

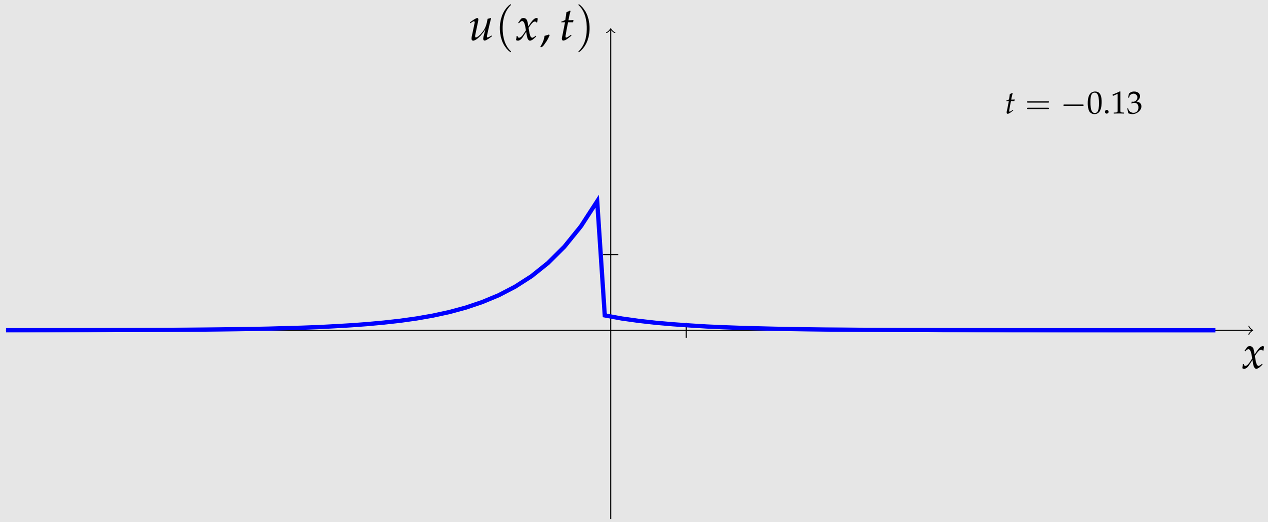

$u(x,t)$
$t = -0.17$
x

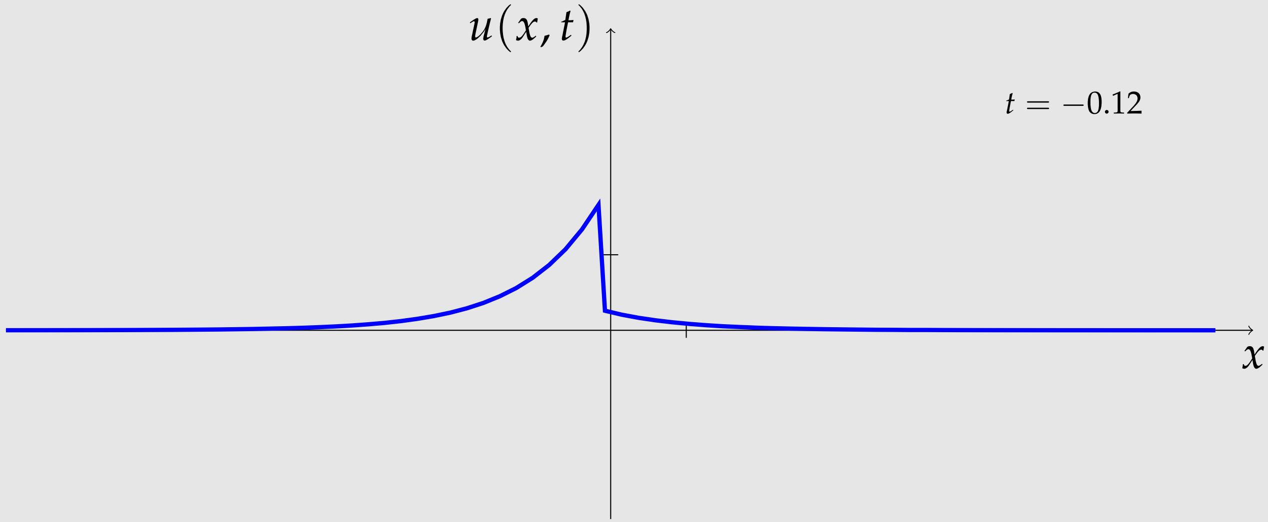

u(x, t)
t = −0.16
x

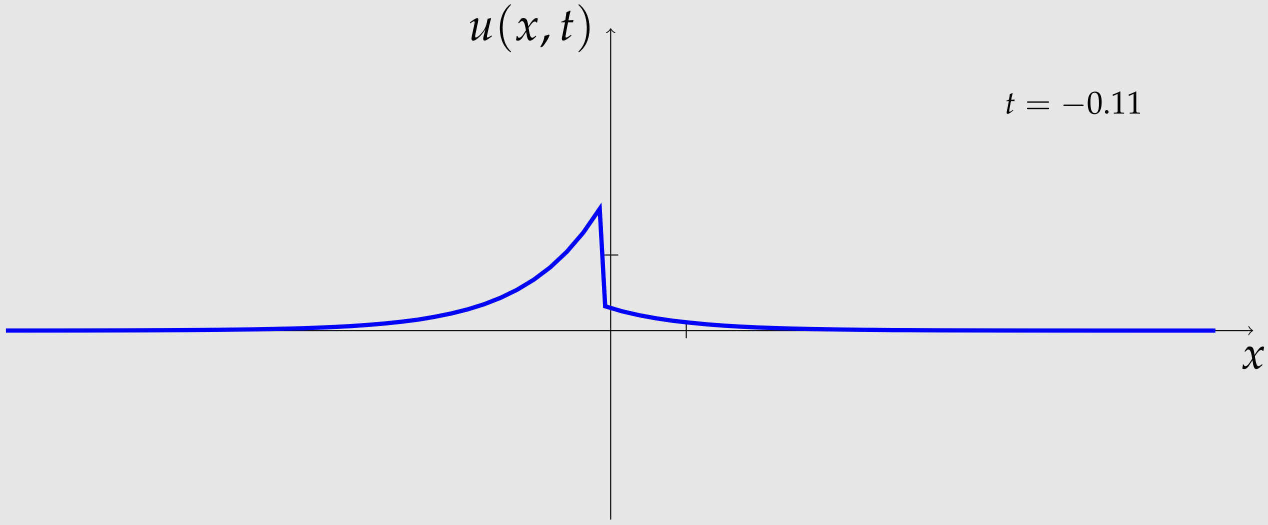

$u(x,t)$
$t = -0.15$
x

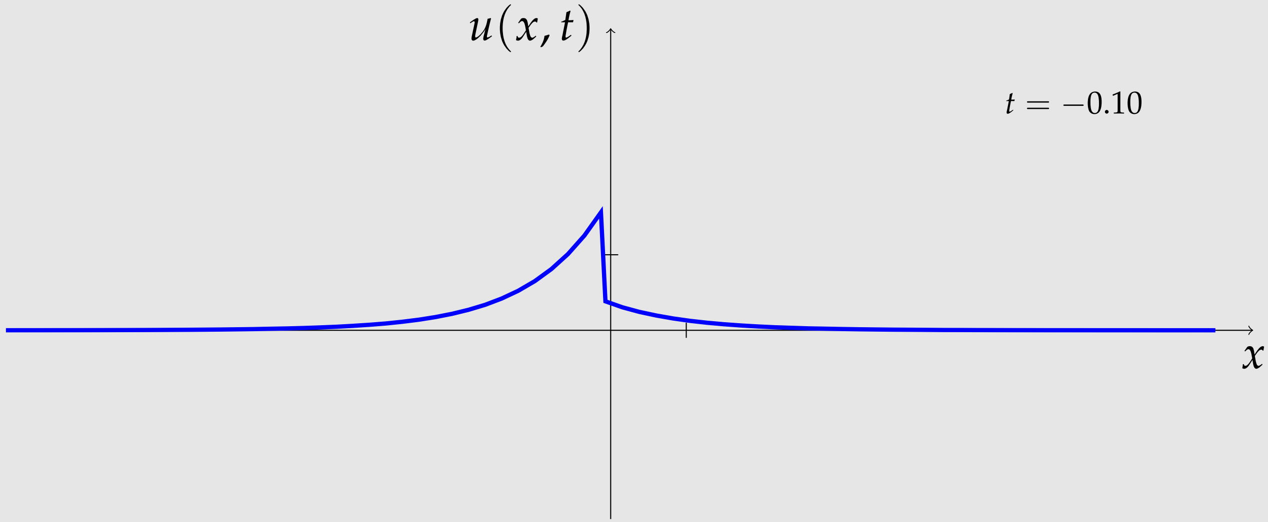


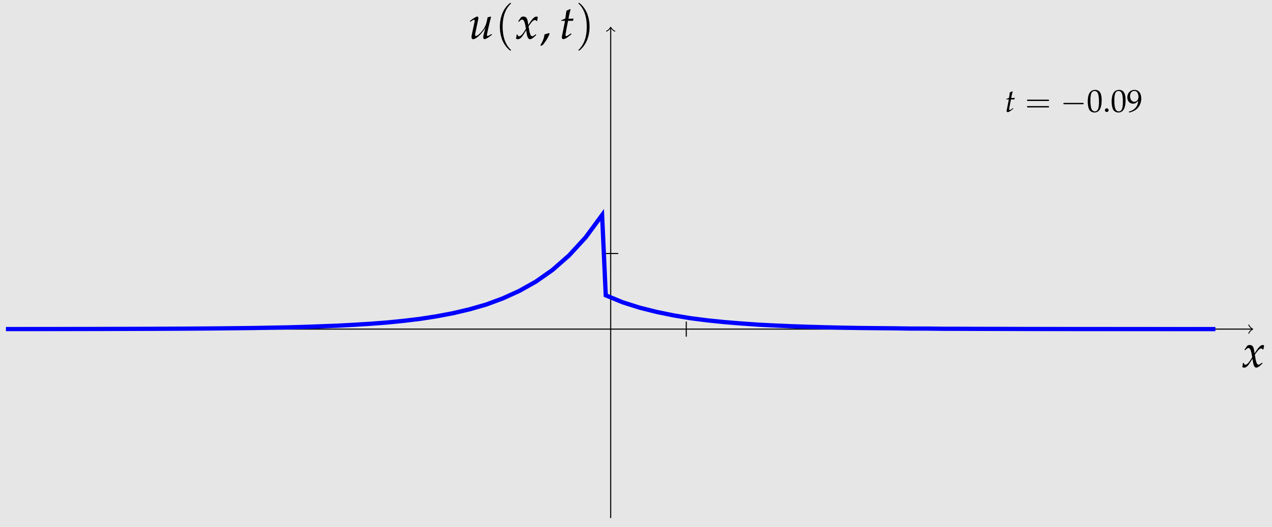
$u(x,t)$
$t = -0.14$
x


$u(x,t)$
$t = -0.13$
x

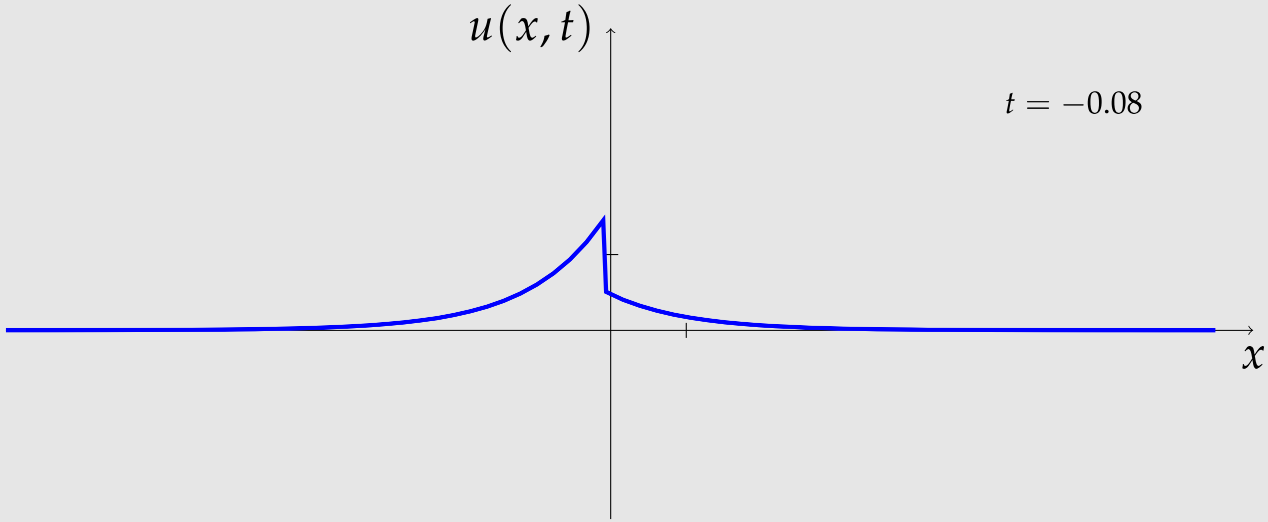

$u(x,t)$
$t = -0.12$
x

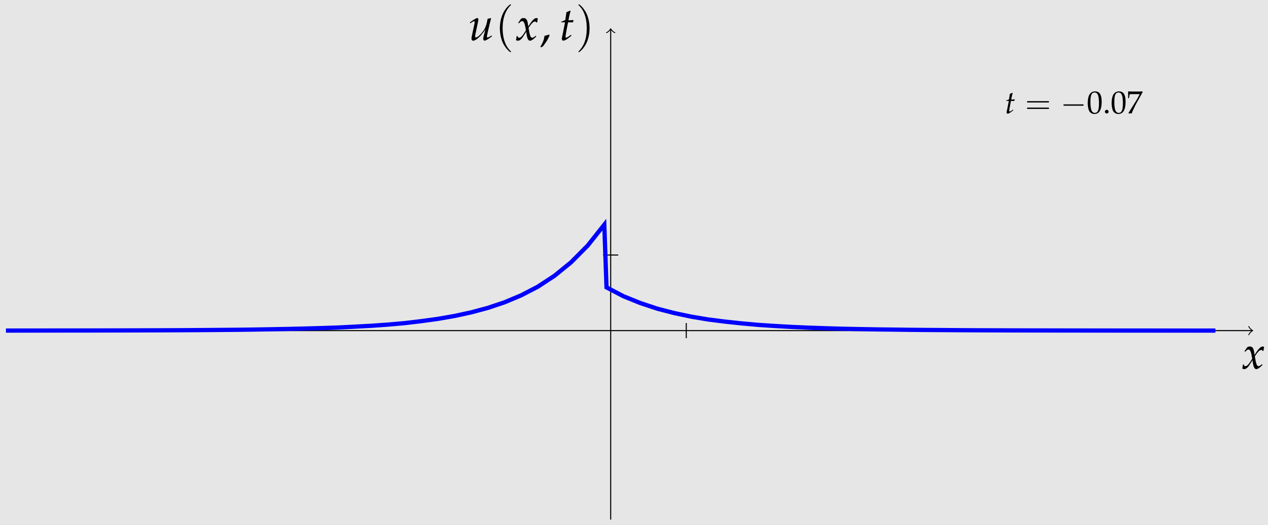

$u(x,t)$
$t = -0.11$
x


$u(x,t)$
$t = -0.10$
x

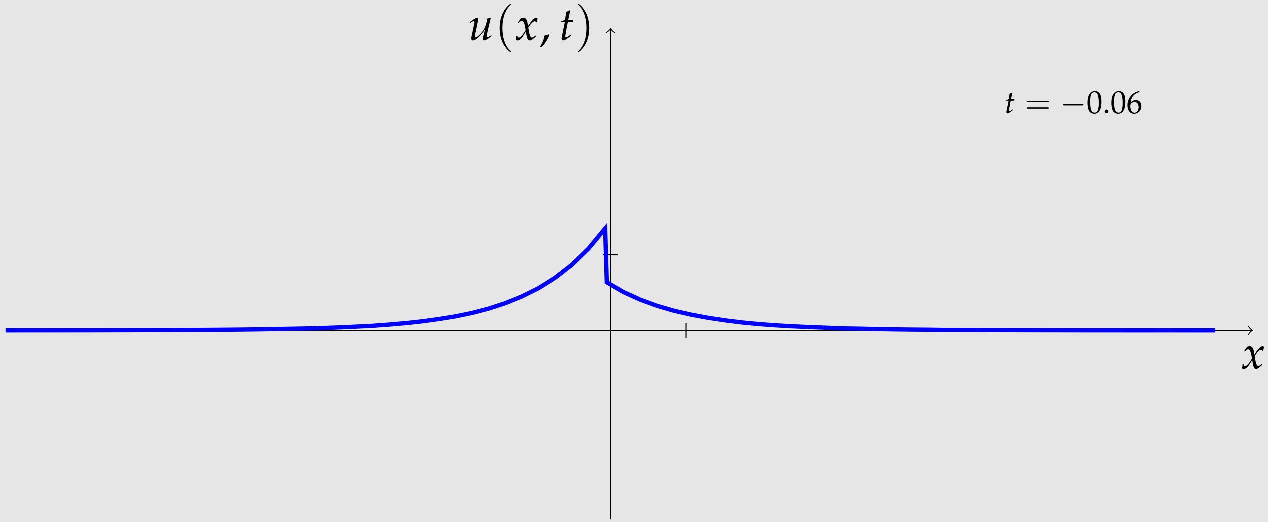


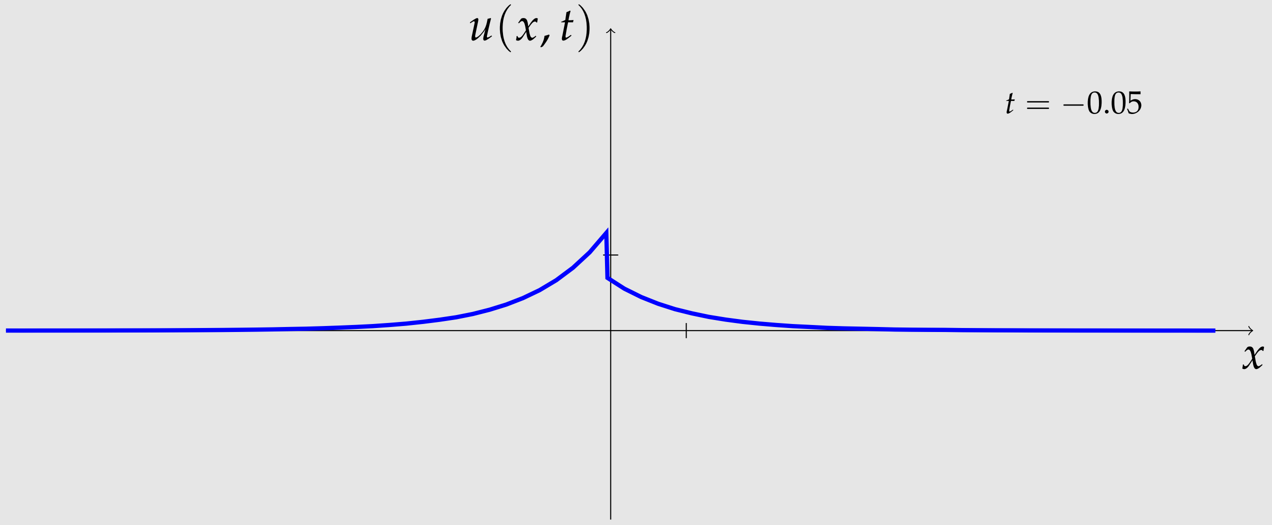
$u(x,t)$
$t = -0.09$
x

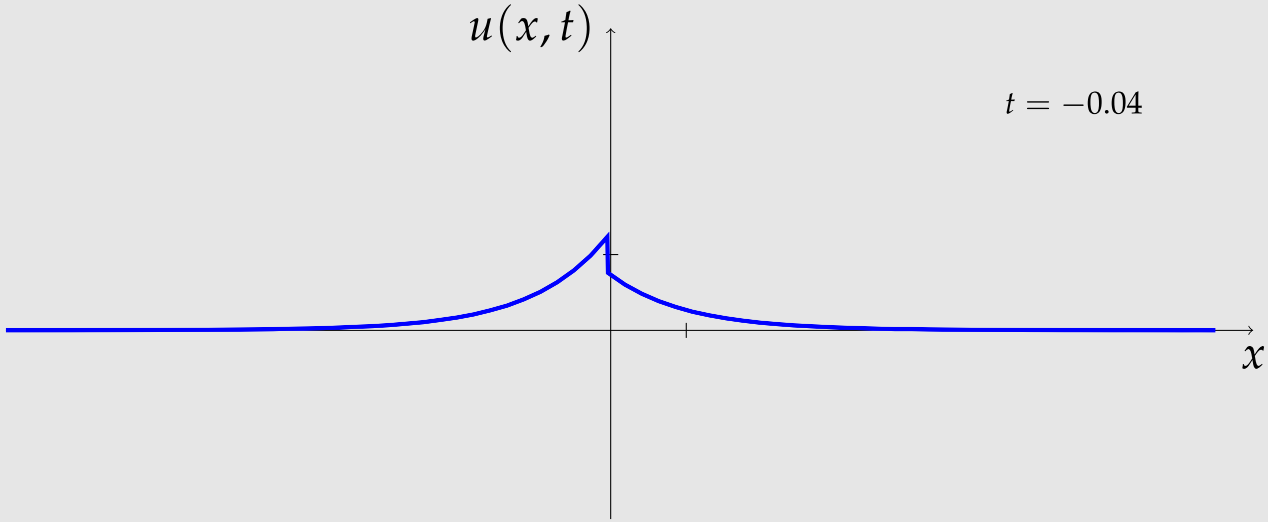

$u(x,t)$
$t = -0.08$
x



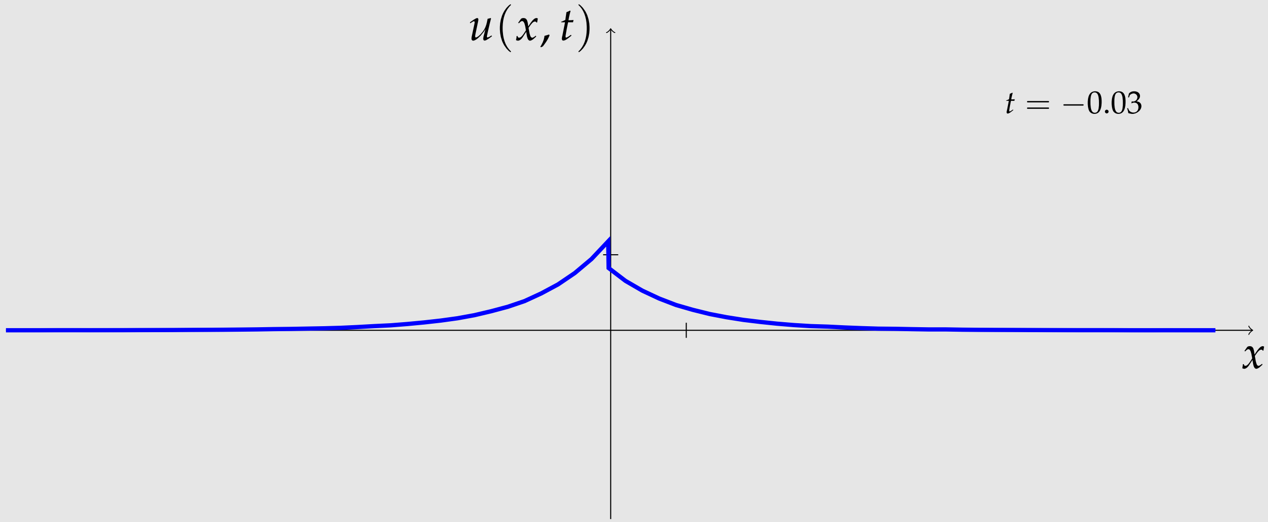
$u(x,t)$
$t = -0.07$
x


$u(x,t)$
$t = -0.06$
x

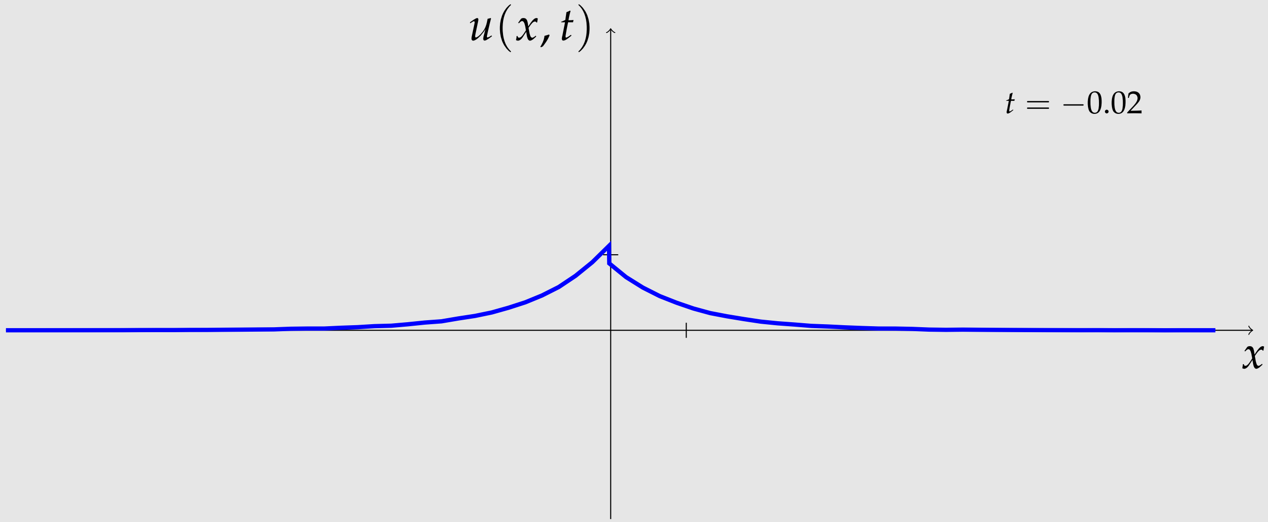

$u(x,t)$
$t = -0.05$
x

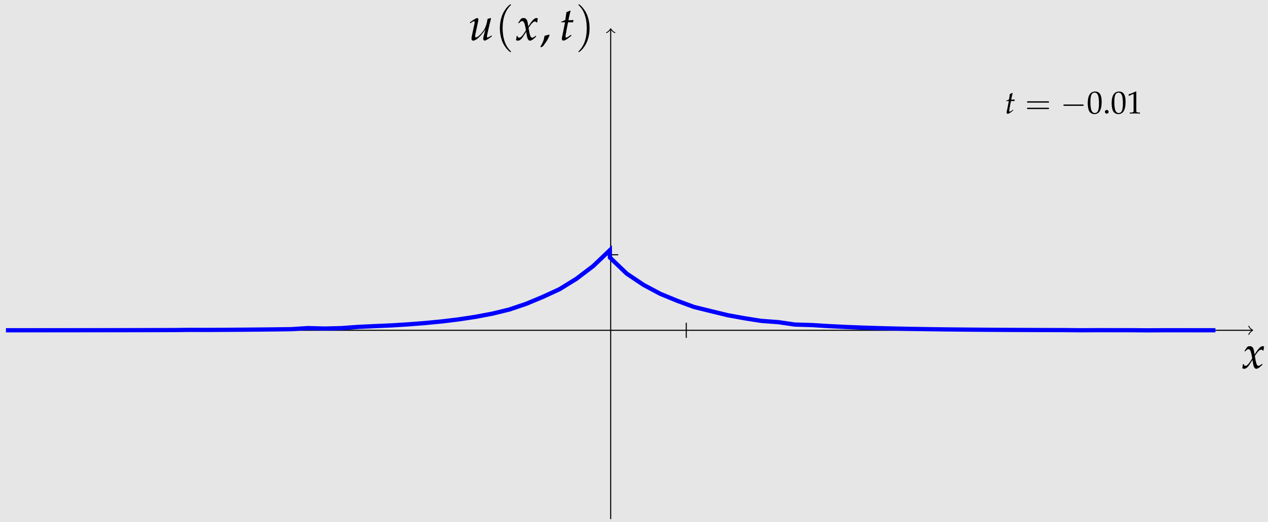


$u(x,t)$
$t = -0.04$
x



u(x, t)
t = −0.03
x


$u(x,t)$
$t = -0.02$
x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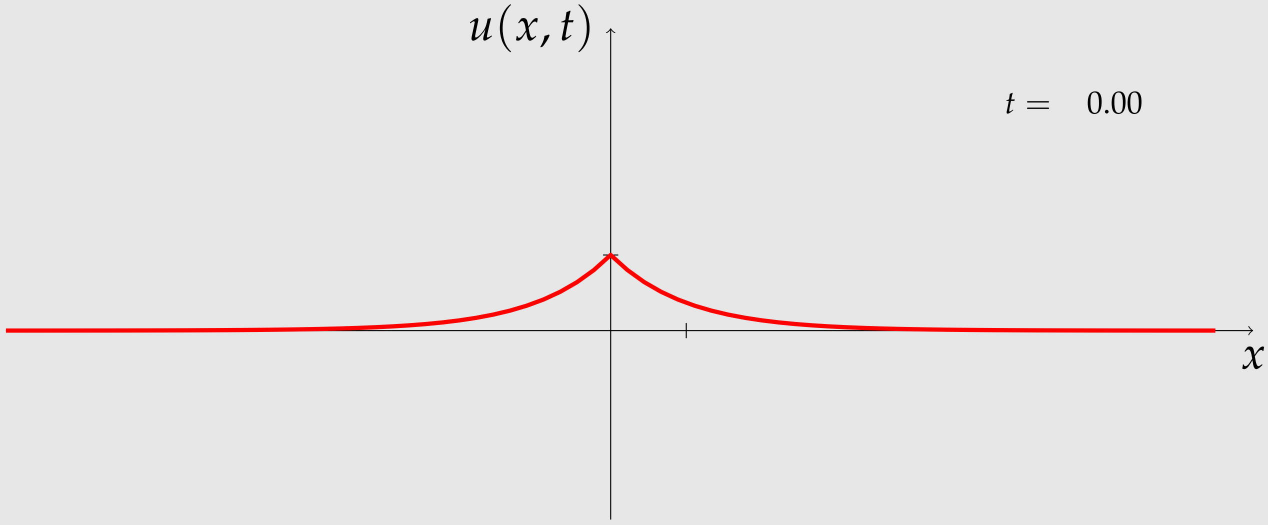
$t = -0.01$
x


$u(x,t)$
$t =$ 0.00
x

In this talk:

- Background material about Camassa–Holm peakons and antipeakons (and a little bit about Degasperis–Procesi).
- Analogous results for Novikov's equation.

  (Old, but still not quite finished, work with Marcus Kardell.)

  Compared to Camassa–Holm, there is a much greater variety of possible behaviours for peakon–antipeakon solutions.

The Camassa–Holm equation (1993)

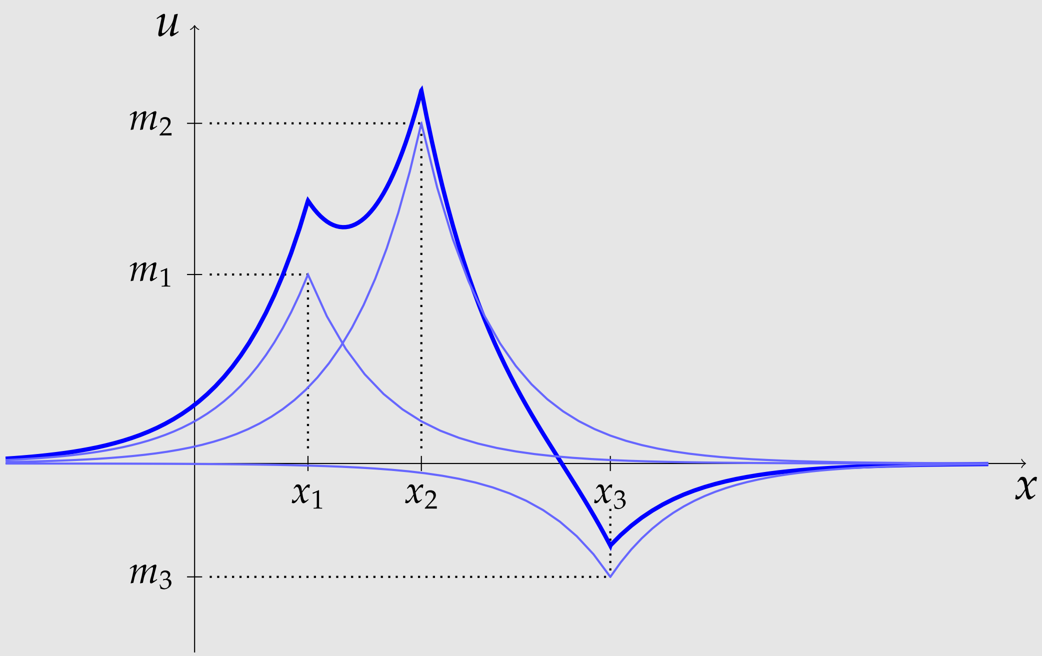
$$m_t + m_x u + 2mu_x = 0 \quad \text{where} \quad m = u - u_{xx}$$

The Degasperis–Procesi equation (1998)

$$m_t + m_x u + 3mu_x = 0 \quad \text{where} \quad m = u - u_{xx}$$

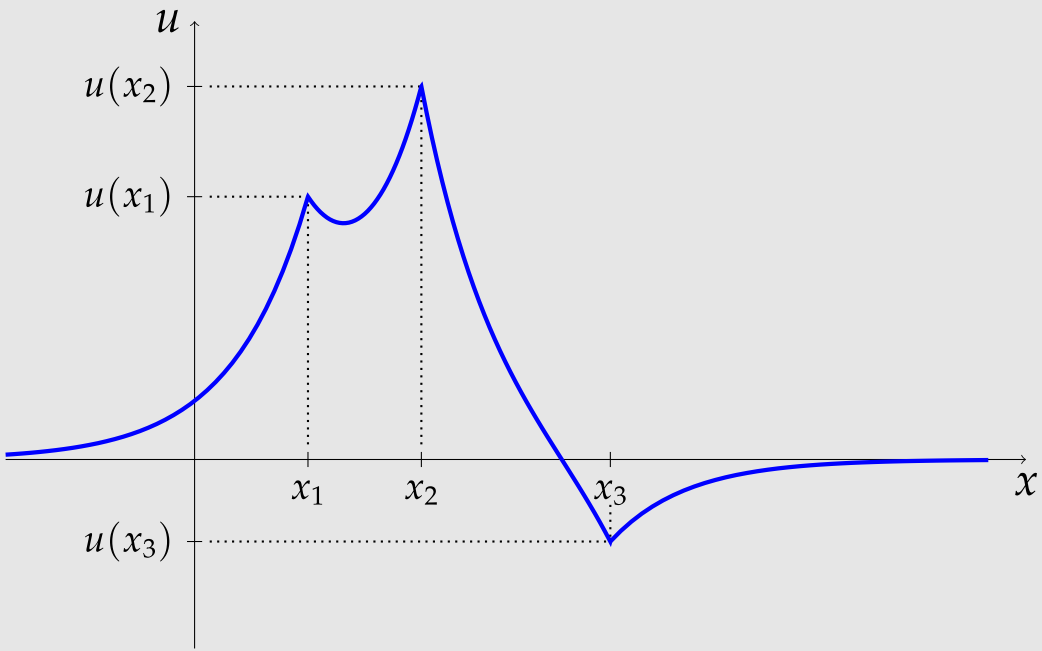
Novikov's equation (2008)

$$m_t + (m_x u + 3mu_x)u = 0 \quad \text{where} \quad m = u - u_{xx}$$

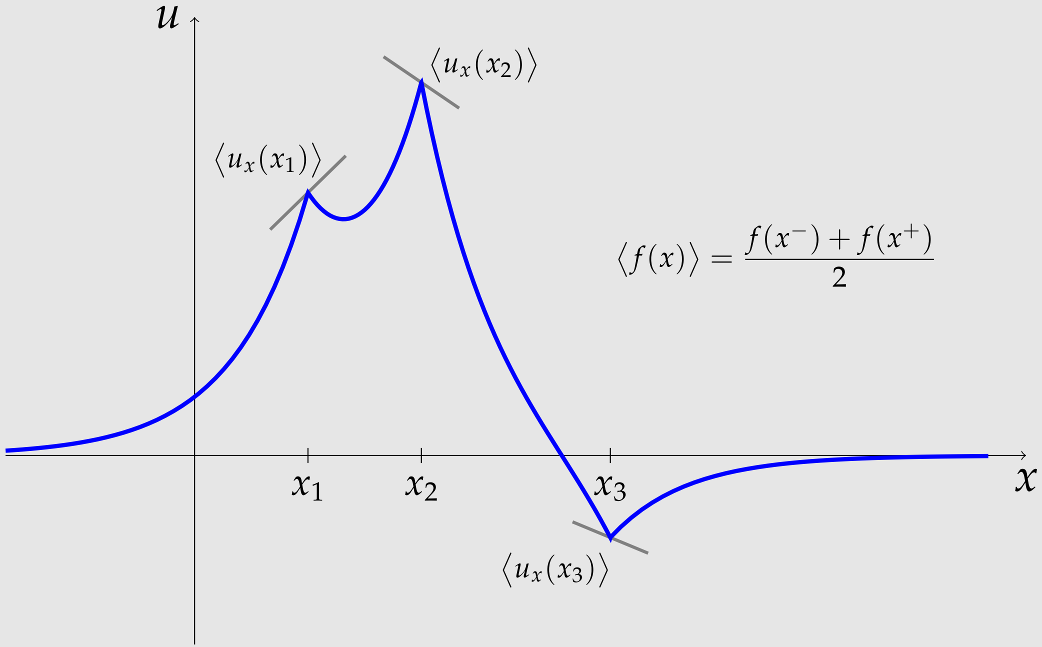


Multipeakon (weak) solutions

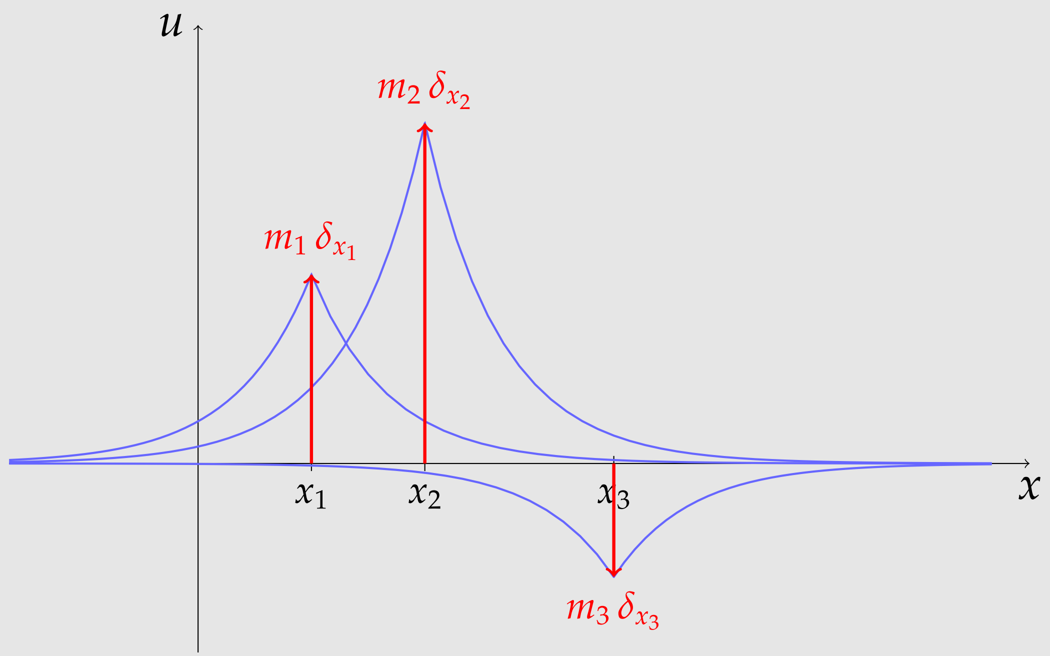
$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x - x_i(t)|}$$



$$u(x) = \sum_{i=1}^{N} m_i \, e^{-|x-x_i|} \quad \Longrightarrow \quad \boxed{u(x_k) = \sum_{i=1}^{N} m_i \, e^{-|x_k-x_i|}}$$



$$u(x) = \sum_{i=1}^{N} m_i\, e^{-|x-x_i|} \quad \Longrightarrow \quad \boxed{\langle u_x(x_k)\rangle = \sum_{i=1}^{N} m_i\, e^{-|x_k-x_i|}\,\mathrm{sgn}(i-k)}$$



$$u(x) = \sum_{i=1}^{N} m_i\, e^{-|x-x_i|} \quad \Longrightarrow \quad \boxed{m = u - u_{xx} = 2\sum_{i=1}^{N} m_i\,\delta_{x_i}}$$

ODEs governing multipeakon solutions

Camassa–Holm

$m_t + m_x u + 2 m u_x = 0$

$$\boxed{\dot{x}_k = u(x_k)} \qquad \boxed{\dot{m}_k = -m_k \left\langle u_x(x_k) \right\rangle}$$

Degasperis–Procesi

$m_t + m_x u + 3 m u_x = 0$

$$\boxed{\dot{x}_k = u(x_k)} \qquad \boxed{\dot{m}_k = -2 m_k \left\langle u_x(x_k) \right\rangle}$$

Novikov

$m_t + (m_x u + 3 m u_x) u = 0$

$$\boxed{\dot{x}_k = u(x_k)^2} \qquad \boxed{\dot{m}_k = -m_k \left\langle u_x(x_k) \right\rangle u(x_k)}$$

(for $k = 1, \dots, N$)

For **Camassa–Holm**:

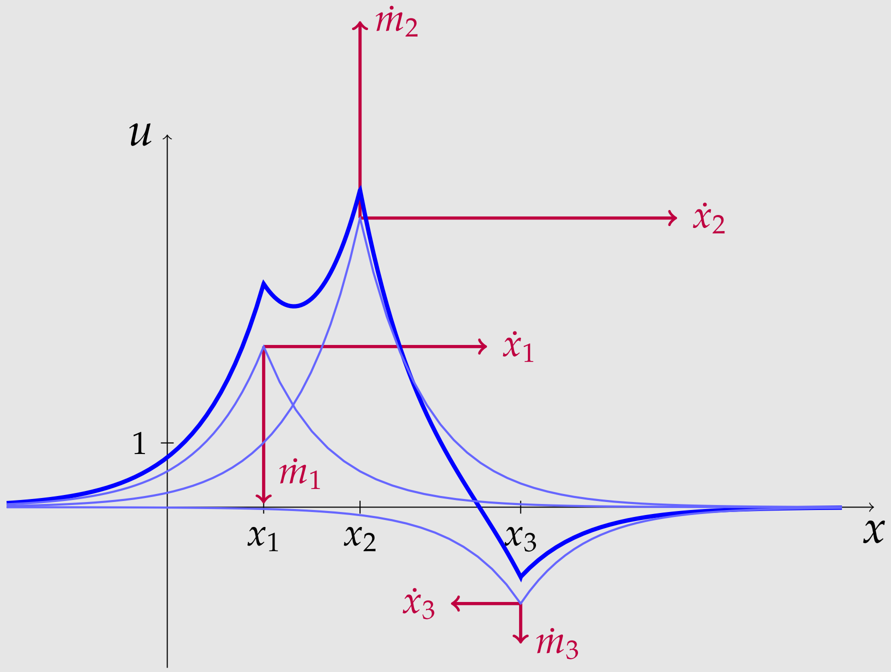
- $\boxed{\dot{x}_k = u(x_k)}$

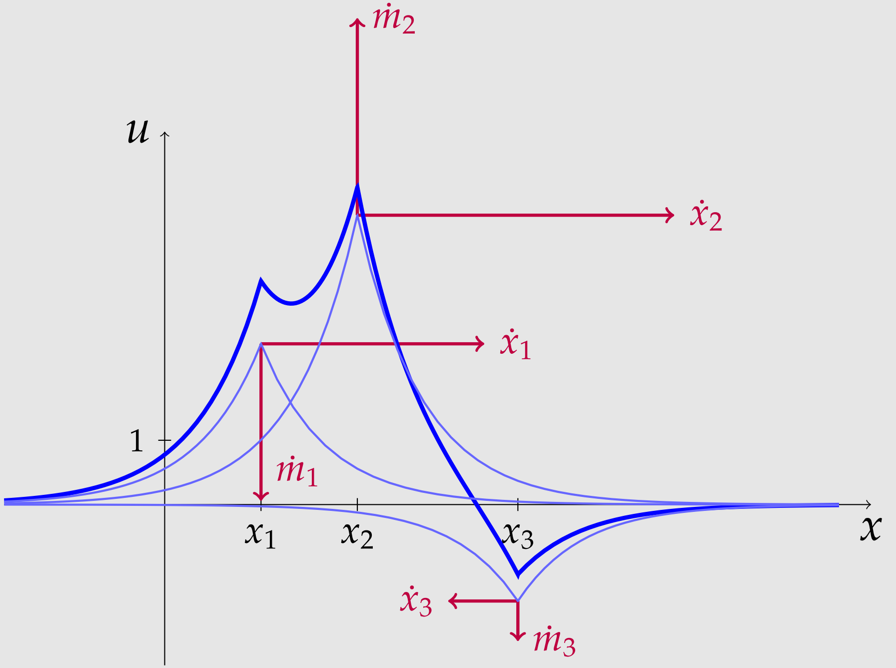
  Velocity of peakon at x_k equals elevation of the total wave at x_k.
  So antipeakons ($m_k < 0$) want to move leftwards.

- $\boxed{\dfrac{d}{dt} \ln |m_k| = \dfrac{\dot{m}_k}{m_k} = -\langle u_x(x_k) \rangle}$ (or $m_k(t) \equiv 0$)

  Slope $\langle u_x(x_k) \rangle$ positive/negative $\implies$ $|m_k|$ decreasing/increasing.

(And similarly for **Degasperis–Procesi** with $\frac{d}{dt} \ln |m_k| = -2\langle u_x(x_k) \rangle$.)


$\dot{m}_2$
u
$\dot{x}_2$
$\dot{x}_1$
1
$\dot{m}_1$
x_1
x_2
x_3
x
$\dot{x}_3$
$\dot{m}_3$



Finite-time blowup of $u_x(x,t)$ due to **peakon–antipeakon collision**:

$$x_3(t) - x_2(t) \to 0 \qquad m_2(t) \to +\infty \qquad m_3(t) \to -\infty$$

Novikov peakons are different:

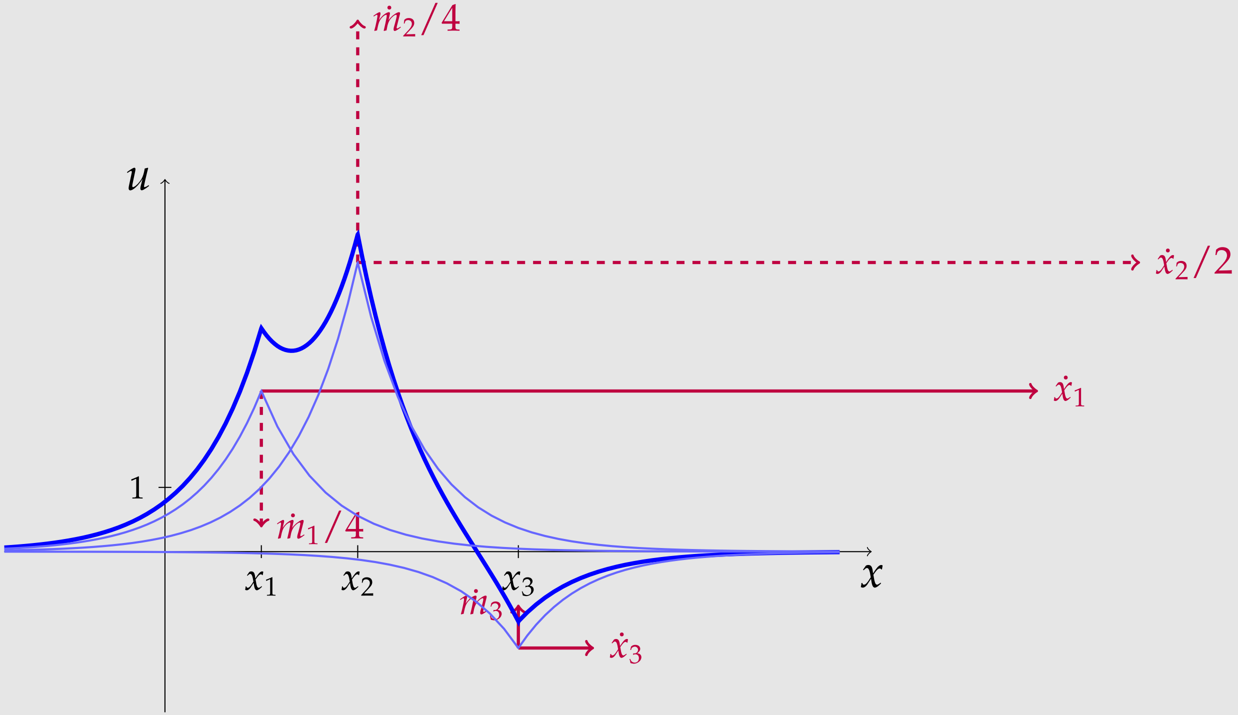
- $\dot{x}_k = u(x_k)^2$

  Velocity of peakon at x_k equals **squared** wave elevation at x_k.

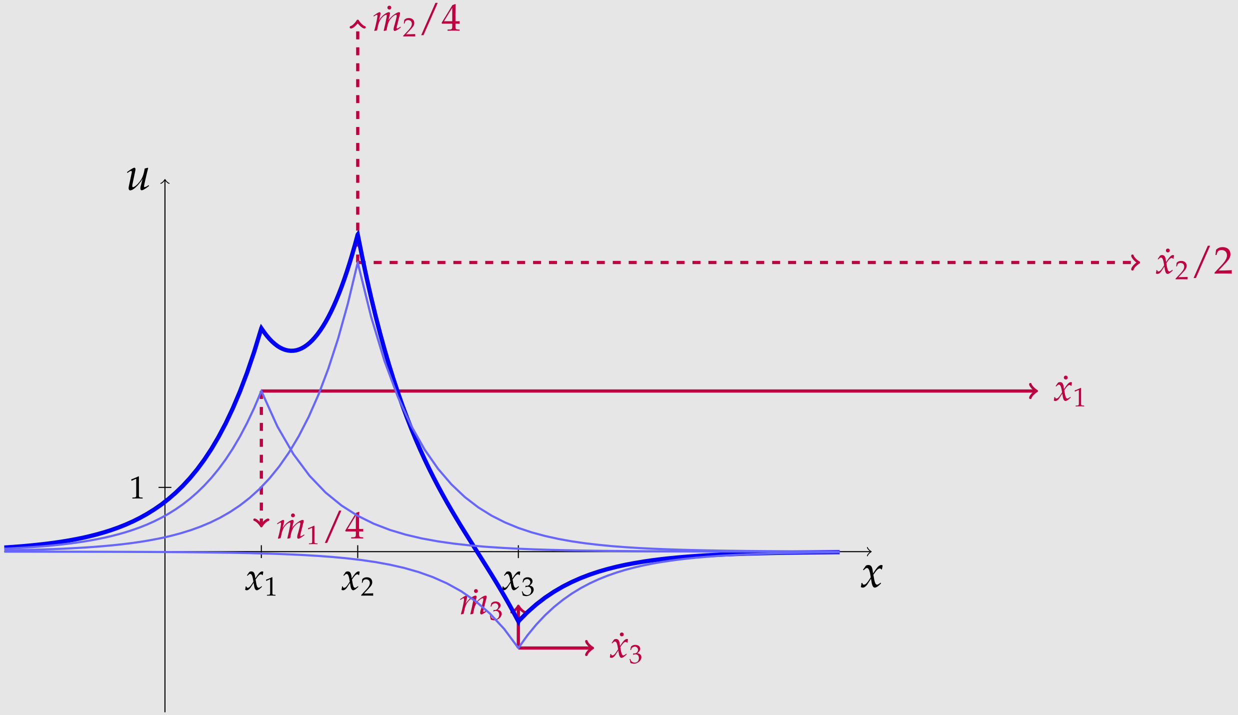
  So everything moves to the right (antipeakons too).

- $\frac{d}{dt} \ln |m_k| = \frac{\dot{m}_k}{m_k} = -u(x_k)\langle u_x(x_k)\rangle$ (or $m_k(t) \equiv 0$)

  Slope and elevation have equal/opposite signs
  $\implies |m_k|$ decreasing/increasing.



$\dot{m}_2/4$
u
$\dot{x}_2/2$
$\dot{x}_1$
1
$\dot{m}_1/4$
x_1
x_2
x_3
x
$\dot{m}_3$
$\dot{x}_3$



Will there be a collision after finite time here too, as in the CH case?

The peakon ODEs for CH/DP/Novikov are explicitly solvable for any N, via Lax pairs and inverse spectral methods.

- Richard Beals, David Sattinger, Jacek Szmigielski. “Multipeakons and the classical moment problem.” Adv. Math. (2000).
- Hans Lundmark, Jacek Szmigielski. “Degasperis–Procesi peakons and the discrete cubic string.” IMRP (2005).
- Andrew Hone, Hans Lundmark, Jacek Szmigielski. “Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa–Holm type equation.” Dynamics of PDE (2009).

Example. Camassa–Holm peakon ODEs for $N = 3$:

$$\begin{aligned} \dot{x}_1 &= u(x_1) = m_1 \phantom{E_{12}} + m_2 E_{12} + m_3 E_{13} \\ \dot{x}_2 &= u(x_2) = m_1 E_{12} + m_2 \phantom{E_{12}} + m_3 E_{23} \\ \dot{x}_3 &= u(x_3) = m_1 E_{13} + m_2 E_{23} + m_3 \end{aligned}$$

$$\begin{aligned} \dot{m}_1 &= -m_1 \langle u_x(x_1) \rangle = -m_1(m_2 E_{12} + m_3 E_{13}) \\ \dot{m}_2 &= -m_2 \langle u_x(x_2) \rangle = -m_2(-m_1 E_{12} + m_3 E_{23}) \\ \dot{m}_3 &= -m_3 \langle u_x(x_3) \rangle = -m_3(-m_1 E_{13} - m_2 E_{23}) \end{aligned}$$

where $E_{ij} = e^{-|x_i - x_j|} = e^{x_i - x_j}$ for $i < j$ (assuming $x_1 < x_2 < x_3$).

If the peakons are **far apart**:

$$E_{ij} \approx 0 \quad \Longrightarrow \quad \dot{x}_k \approx m_k, \quad \dot{m}_k \approx 0.$$

So then each peakon acts nearly like a solitary travelling wave ($\dot{x}_k = m_k =$ constant).

(Ex. cont.) **Constants of motion**

Lax pair provides time-invariant polynomial $1 - M_1\lambda + M_2\lambda^2 - M_3\lambda^3$ with real nonzero distinct roots λ_1, λ_2, λ_3 (**eigenvalues** of a **discrete string**).

Number of positive/negative eigenvalues λ_k = number of peakons/antipeakons.

$$M_1 = m_1 + m_2 + m_3 = \tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_3}$$

$$M_2 = m_1 m_2(1 - E_{12}) + m_1 m_3(1 - E_{13}) + m_2 m_3(1 - E_{23}) = \tfrac{1}{\lambda_1\lambda_2} + \tfrac{1}{\lambda_1\lambda_3} + \tfrac{1}{\lambda_2\lambda_3}$$

$$M_3 = m_1 m_2 m_3(1 - E_{12})(1 - E_{23}) = \tfrac{1}{\lambda_1\lambda_2\lambda_3}$$

$$\Big(H = \tfrac{1}{2}M_1^2 - M_2$$

$$= \tfrac{1}{2}\big(m_1^2 + m_2^2 + m_3^2 + 2E_{12}m_1m_2 + 2E_{13}m_1m_3 + 2E_{23}m_2m_3\big) = \tfrac{1}{2}\big(\tfrac{1}{\lambda_1^2} + \tfrac{1}{\lambda_2^2} + \tfrac{1}{\lambda_3^2}\big)\Big)$$

For **pure peakon solutions** (all m_k positive) **finite-time blowup is impossible**, and ordering $x_1 < x_2 < x_3$ holds for all $t \in \mathbf{R}$, due to the existence of the constants of motion M_1 and M_3.

(Or in general $M_1 = m_1 + m_2 + \cdots + m_N$ and $M_N = m_1 m_2 \cdots m_N (1 - E_{12})(1 - E_{23}) \cdots (1 - E_{N-1,N})$.)

(Ex. cont.) **General solution**

$$x_1(t) = \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 a_1 a_2 a_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 a_j a_k}$$

$$x_2(t) = \ln \frac{(\lambda_1-\lambda_2)^2 a_1 a_2 + (\lambda_1-\lambda_3)^2 a_1 a_3 + (\lambda_2-\lambda_3)^2 a_2 a_3}{\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3}$$

$$x_3(t) = \ln(a_1 + a_2 + a_3)$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 a_j a_k}{\lambda_1\lambda_2\lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 a_j a_k}$$

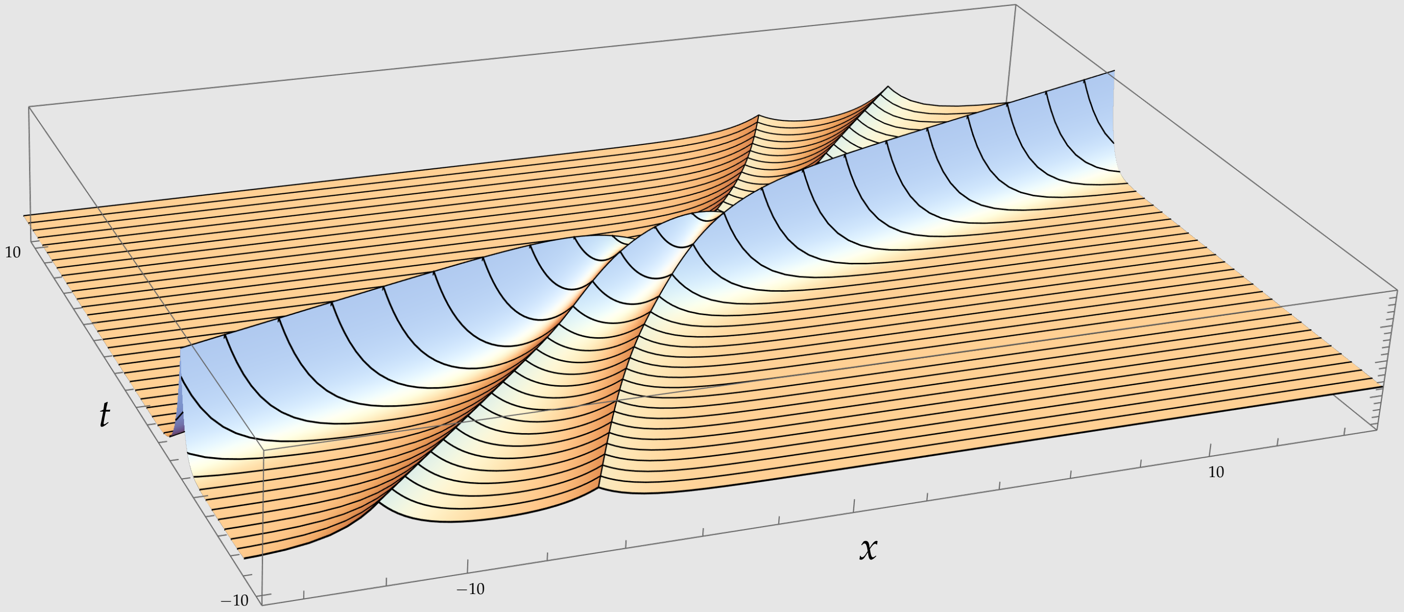
$$m_2(t) = \frac{\left(\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3\right) \sum_{j<k} (\lambda_j-\lambda_k)^2 a_j a_k}{(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 a_j a_k}$$

$$m_3(t) = \frac{a_1 + a_2 + a_3}{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}$$

where $\lambda_k = \text{constant}$ (as on prev. page) and $a_k(t) = a_k(0)\, e^{t/\lambda_k}$ (positive).

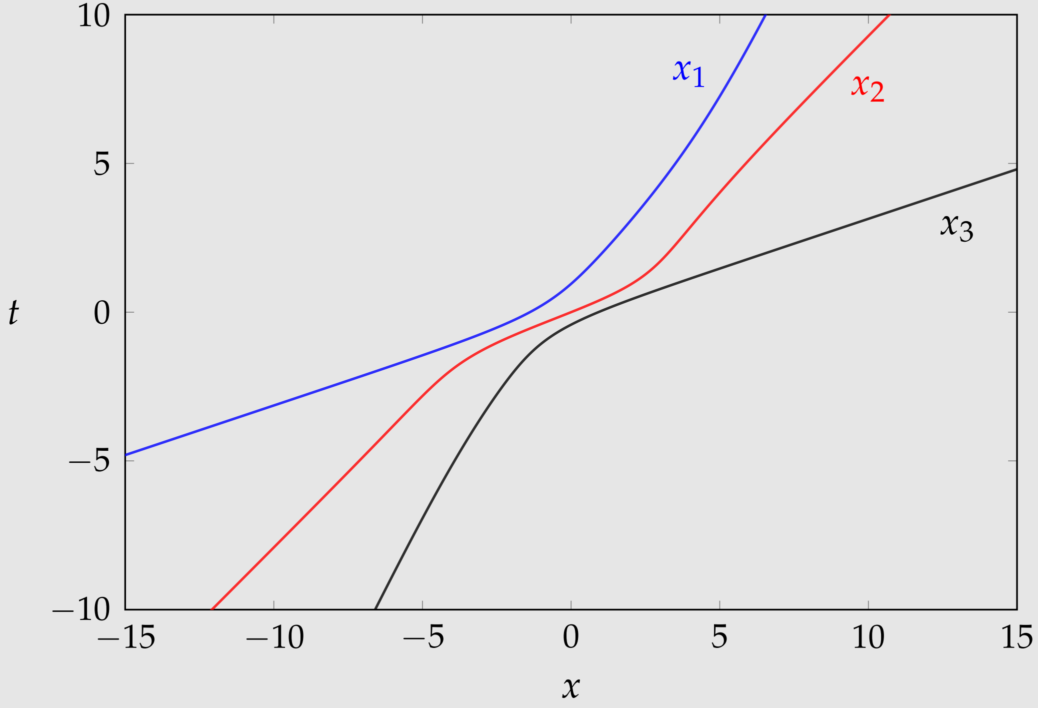
(The quantity a_k is the **residue** at λ_k in the **Weyl function** for the discrete string.)

(Ex. cont.) Graph of a pure 3-peakon solution $u(x,t)$:



Parameters: $\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right) = \left(3, 1, \frac{1}{2}\right)$ $\quad (a_1, a_2, a_3)_{t=0} = \left(\frac{9}{5}, \frac{1}{2}, \frac{1}{5}\right)$

(Ex. cont.) Peakon trajectories $x = x_k(t)$ for solution on prev. page:



(Ex. cont.) **Asymptotics**

Say $1/\lambda_1 > 1/\lambda_2 > 1/\lambda_3$. Then $a_1 \gg a_2 \gg a_3$ as $t \to +\infty$, since $a_k(t) = a_k(0)\, e^{t/\lambda_k}$.

$$x_1(t) = \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 a_1 a_2 a_3}{\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2 a_1 a_2 + \dots} \approx t/\lambda_3 + \text{const.}$$

$$x_2(t) = \ln \frac{(\lambda_1-\lambda_2)^2 a_1 a_2 + (\lambda_1-\lambda_3)^2 a_1 a_3 + (\lambda_2-\lambda_3)^2 a_2 a_3}{\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3} \approx t/\lambda_2 + \text{const.}$$

$$x_3(t) = \ln(a_1 + a_2 + a_3) \approx t/\lambda_1 + \text{const.}$$

$$m_1(t) = \frac{\left(\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2 a_1 a_2 + \dots\right)}{\lambda_1\lambda_2\lambda_3\left(\lambda_1\lambda_2(\lambda_1-\lambda_2)^2 a_1 a_2 + \dots\right)} \approx 1/\lambda_3$$

$$m_2(t) = \frac{\left(\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3\right)\left((\lambda_1-\lambda_2)^2 a_1 a_2 + \dots\right)}{\left(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3\right)\left(\lambda_1\lambda_2(\lambda_1-\lambda_2)^2 a_1 a_2 + \dots\right)} \approx 1/\lambda_2$$

$$m_3(t) = \frac{a_1 + a_2 + a_3}{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3} \approx 1/\lambda_1$$

(And similarly, but in the opposite order, as $t \to -\infty$.)

Peakon–antipeakon collisions in the CH equation

- Can be analyzed in detail using the explicit solution formulas.
- Always in pairs, $x_{k-1} < x_k = x_{k+1} < x_{k+2}$. No triple collisions.
- As $t \nearrow t_{\text{coll}}$ we have, for some $\alpha > 0$, $\beta > 0$, $\gamma_1 \in \mathbf{R}$, $\gamma_2 \in \mathbf{R}$:

$$x_{k+1}(t) - x_k(t) = \alpha(t - t_{\text{coll}})^2 + O\big((t - t_{\text{coll}})^3\big)$$
$$m_k(t) = \frac{-\beta}{t - t_{\text{coll}}} + \gamma_1 + O(t - t_{\text{coll}})$$
$$m_{k+1}(t) = \frac{\beta}{t - t_{\text{coll}}} + \gamma_2 + O(t - t_{\text{coll}})$$

  So $m_k \to +\infty$ and $m_{k+1} \to -\infty$, but $m_k + m_{k+1} \to \gamma_1 + \gamma_2$.
- $u_x(x,t) \to -\infty$ on the interval $x \in \big(m_k(t), m_{k+1}(t)\big)$.
- $u(x,t)$ for $t < t_{\text{coll}}$ extends continuously to $t = t_{\text{coll}}$.

Camassa–Holm 2-peakon solution formulas

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 a_1 a_2}{\lambda_1^2 a_1 + \lambda_2^2 a_2} \qquad m_1(t) = \frac{\lambda_1^2 a_1 + \lambda_2^2 a_2}{(\lambda_1 a_1 + \lambda_2 a_2)\,\lambda_1 \lambda_2}$$

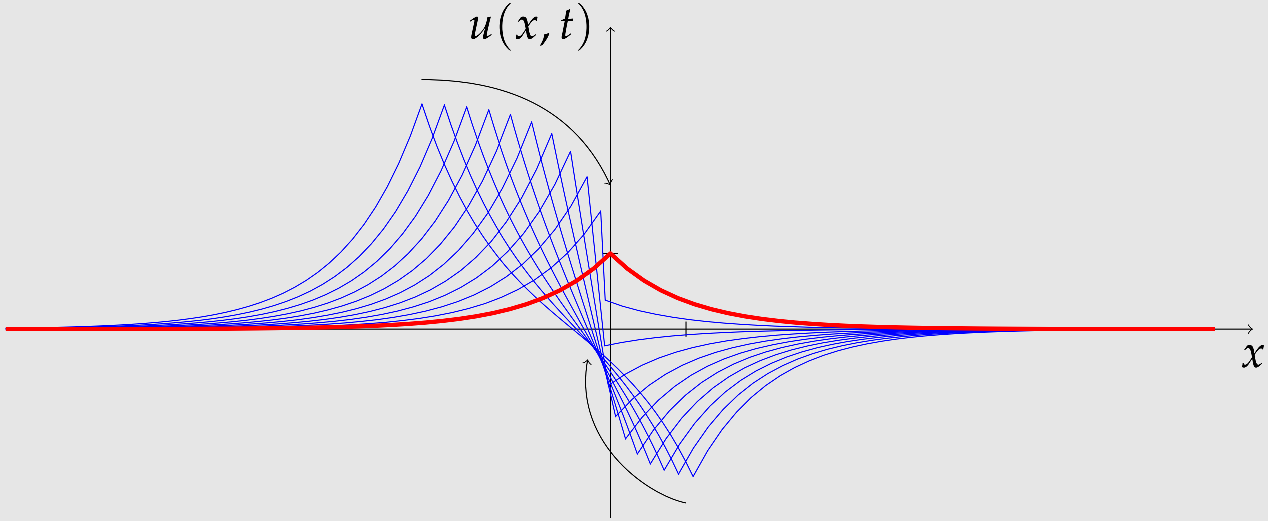
$$x_2(t) = \ln(a_1 + a_2) \qquad m_2(t) = \frac{a_1 + a_2}{\lambda_1 a_1 + \lambda_2 a_2}$$

To obtain a peakon–antipeakon solution with asymptotic velocities $c_1 = \frac{1}{\lambda_1} > 0$ and $-c_2 = \frac{1}{\lambda_2} < 0$ and the collision occuring at $(x,t) = (0,0)$, take $a_1(0) = \frac{c_1}{c_1+c_2}$ and $a_2(0) = \frac{c_2}{c_1+c_2}$:

$$x_1(t) = -\ln \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{c_1 + c_2} \qquad m_1(t) = \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{e^{-c_1 t} - e^{c_2 t}}$$

$$x_2(t) = \ln \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{c_1 + c_2} \qquad m_2(t) = \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{e^{c_1 t} - e^{-c_2 t}}$$

Recall the solution illustrated in the “movie” earlier (with $c_1 = 3$, $c_2 = 2$):

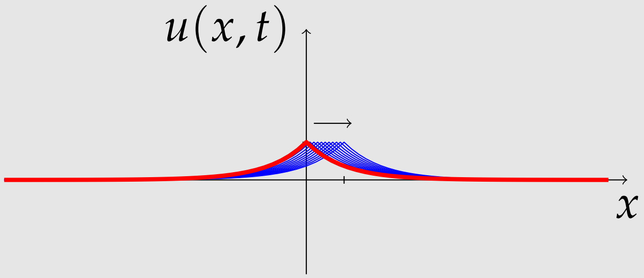


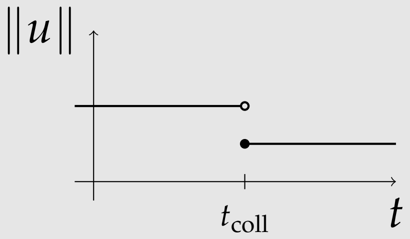
How to continue after the collision?

Two main scenarios

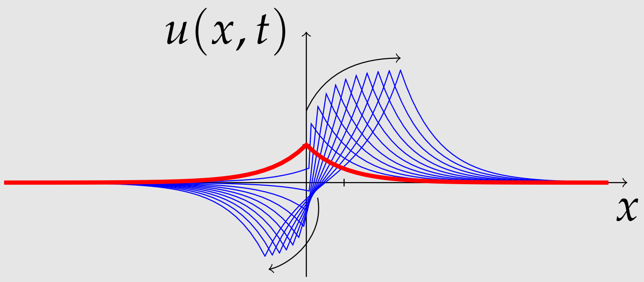
H^1-norm $\|u\|^2 = \int_{\mathbf{R}} (u^2 + u_x^2)\, dx$

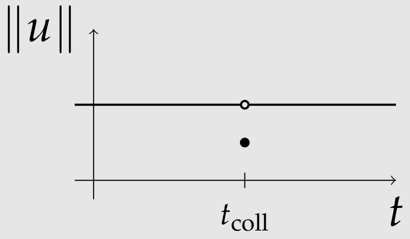
- **Dissipative**



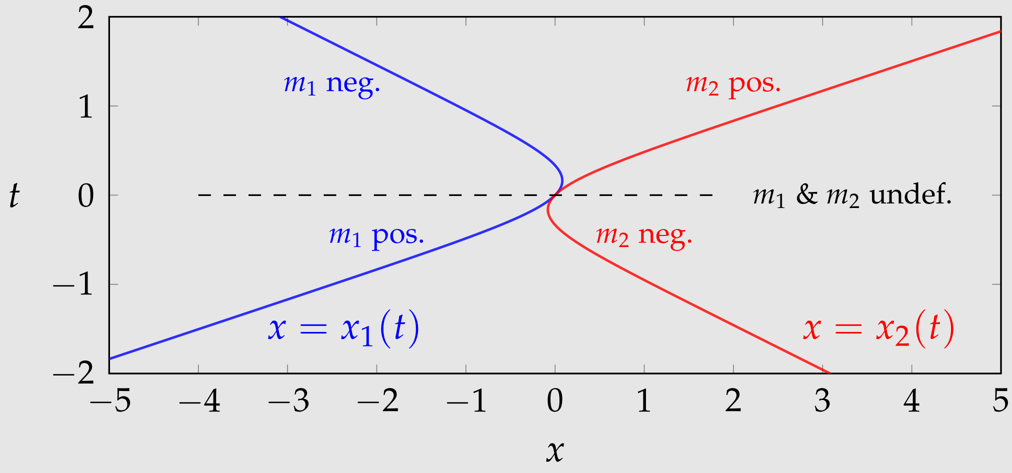


- **Conservative** (given by the explicit formulas above, for $t \neq t_{\text{coll}}$)





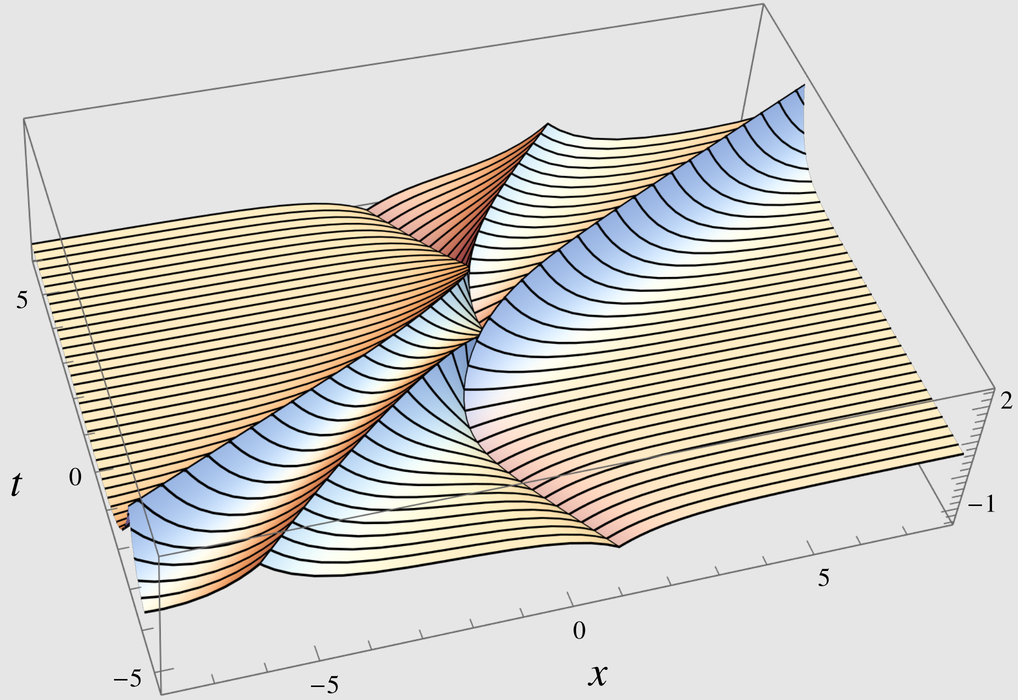
Peakon trajectories $x = x_k(t)$ for the **conservative** continuation of the CH peakon–antipeakon solution above:



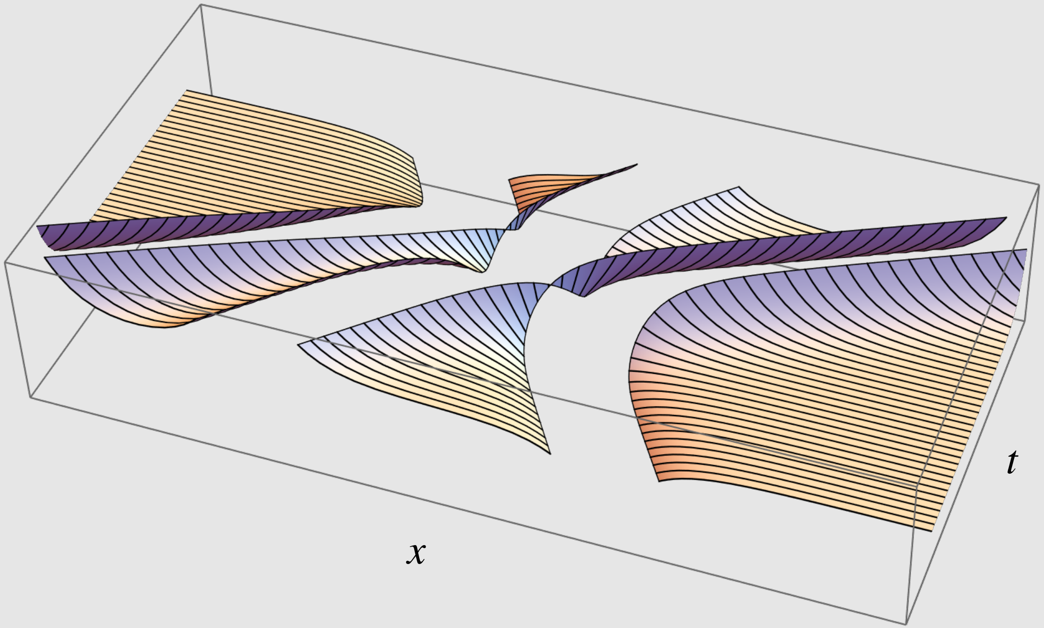
$$x_1(t) = -\ln \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{c_1 + c_2} \qquad \text{(for } t \in \mathbf{R}\text{)}$$
$$x_2(t) = \ln \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{c_1 + c_2}$$

$$m_1(t) = \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{e^{-c_1 t} - e^{c_2 t}} \qquad \text{(for } t \neq 0\text{)}$$
$$m_2(t) = \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{e^{c_1 t} - e^{-c_2 t}}$$

Graph of a conservative CH solution with 2 peakons & 1 antipeakon:



The same surface $u(x, t)$, cut along the curves $x = x_k(t)$ and pulled apart:



(Source: Hans Lundmark, Jacek Szmigielski. “A view of the peakon world through the lens of approximation theory.” Physica D (2022).)

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- Hong Cai, Geng Chen, Hongwei Mei. “Uniqueness of dissipative solution for Camassa–Holm equation with peakon–antipeakon initial data.” Appl. Math. Lett (2021).

Conservative solutions

- Alberto Bressan, Adrian Constantin. "Global conservative solutions of the Camassa–Holm equation." Arch. Rational Mech. Anal. (2007).
- Helge Holden, Xavier Raynaud. "Global conservative solutions of the Camassa–Holm equation – a Lagrangian point of view." Comm. PDE (2007). "Global conservative multipeakon solutions of the Camassa–Holm equation." JHDE (2007).
- Alberto Bressan, Geng Chen, Qingtian Zhang. "Uniqueness of conservative solutions to the Camassa–Holm equation via characteristics." DCDS (2015).

α-dissipative solutions $(0 \leq \alpha \leq 1$, with $\alpha = 0$ conservative and $\alpha = 1$ dissipative)

- Katrin Grunert, Helge Holden, Xavier Raynaud. "A continuous interpolation between conservative and dissipative solutions for the two-component Camassa–Holm system." Forum Math. Sigma (2015).
- Katrin Grunert, Helge Holden. "The general peakon–antipeakon solution for the Camassa–Holm equation." JHDE (2016).

Over to **Novikov’s equation** now!

Some references (origins)

- Andrew Hone, Jing Ping Wang. "Integrable peakon equations with cubic nonlinearity." J. Phys. A (2008).
- Vladimir Novikov. "Generalisations of the Camassa–Holm equation." J. Phys. A (2009).

Some references (explicit N-peakon solution formulas)

- Andrew Hone, Hans Lundmark, Jacek Szmigielski. “Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa–Holm type equation.” Dynamics of PDE (2009).
- Keivan Mohajer, Jacek Szmigielski. “On an inverse problem associated with an integrable equation of Camassa–Holm type: Explicit formulas on the real axis.” Inverse Problems (2012).
- Xiang-Ke Chang, Xing-Biao Hu, Shi-Hao Li, Jun-Xiao Zhao. “An application of Pfaffians to multipeakons of the Novikov equation and the finite Toda lattice of BKP type.” Adv. Math. (2018).
- Xiang-Ke Chang. “Hermite–Padé approximations with Pfaffian structures: Novikov peakon equation and integrable lattices.” Adv. Math. (2022)

Some references (misc.)

- Alex Himonas, Curtis Holliman, Carlos Kenig. "Construction of 2-peakon solutions and ill-posedness for the Novikov equation." SIAM J. Math. Anal. (2018).
- Robin Ming Chen, Dmitry Pelinovsky. "$W^{1,\infty}$ instability of H^1-stable peakons in the Novikov equation." Dynamics of PDE (2021).
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Novikov 2-peakon solution formulas

$$x_1(t) = \frac{1}{2}\ln \frac{\dfrac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\, a_1^2 a_2^2}{\lambda_1\, a_1^2 + \lambda_2\, a_2^2 + \dfrac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, a_1 a_2}$$

$$x_2(t) = \frac{1}{2}\ln\left(\frac{a_1^2}{\lambda_1} + \frac{a_2^2}{\lambda_2} + \frac{4}{\lambda_1+\lambda_2}\, a_1 a_2\right)$$

$$m_1(t) = \frac{\left(\dfrac{(\lambda_1-\lambda_2)^4\, a_1^2 a_2^2}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\left(\lambda_1\, a_1^2 + \lambda_2\, a_2^2 + \dfrac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, a_1 a_2\right)\right)^{1/2}}{\dfrac{(\lambda_1-\lambda_2)^2\, a_1 a_2}{\lambda_1+\lambda_2}\,(a_1+a_2)}$$

$$m_2(t) = \frac{\left(\dfrac{a_1^2}{\lambda_1} + \dfrac{a_2^2}{\lambda_2} + \dfrac{4}{\lambda_1+\lambda_2}\, a_1 a_2\right)^{1/2}}{a_1+a_2} \qquad \left(\text{where } a_k(t) = a_k(0)\, e^{t/\lambda_k}\right)$$

The constants λ_1 and λ_2 are **eigenvalues** of a **discrete dual cubic string** (depending parametrically on x_1, x_2, m_1, m_2).

They are the roots of the polynomial $A(\lambda) = 1 - H_1\lambda + H_2\lambda^2$, where

$$
\begin{aligned}
H_1 &= m_1^2 + m_2^2 + 2E_{12}m_1m_2 \\
H_2 &= m_1^2m_2^2(1 - E_{12}^2)
\end{aligned}
$$

are constants of motion for the Novikov peakon ODEs.

(Recall the abbreviation $E_{12} = e^{-|x_1 - x_2|} = e^{x_1 - x_2}$, assuming $x_1 < x_2$.)

The time-varying quantities a_k are the **residues** in the Weyl function

$$
W(\lambda) = -\frac{B(\lambda)}{A(\lambda)} = \frac{a_1}{\lambda - \lambda_1} + \frac{a_2}{\lambda - \lambda_2}
$$

where

$$
B(\lambda) = m_1e^{x_1} + m_2e^{x_2} - \lambda m_1^2m_2e^{x_2}(1 - E_{12}^2).
$$

Pure peakon case: λ_1 and λ_2 positive, distinct.
a_1 and a_2 positive.

Asymptotic velocities $\dfrac{1}{\lambda_k}$ and amplitudes $\sqrt{\dfrac{1}{\lambda_k}}$.

(Not very different from CH pure peakon solutions, qualitatively.)

Pure antipeakon case: λ_1 and λ_2 positive, distinct.
a_1 and a_2 negative.

Asymptotic velocities $\dfrac{1}{\lambda_k}$ and amplitudes $-\sqrt{\dfrac{1}{\lambda_k}}$.

(Simply upside-down pure peakon solutions. If u satisfies the Novikov equation, then so does $-u$.)

Peakon–antipeakon case:

Type 1. λ_1 and λ_2 positive and distinct.
a_1 and a_2 real, nonzero, of opposite signs.

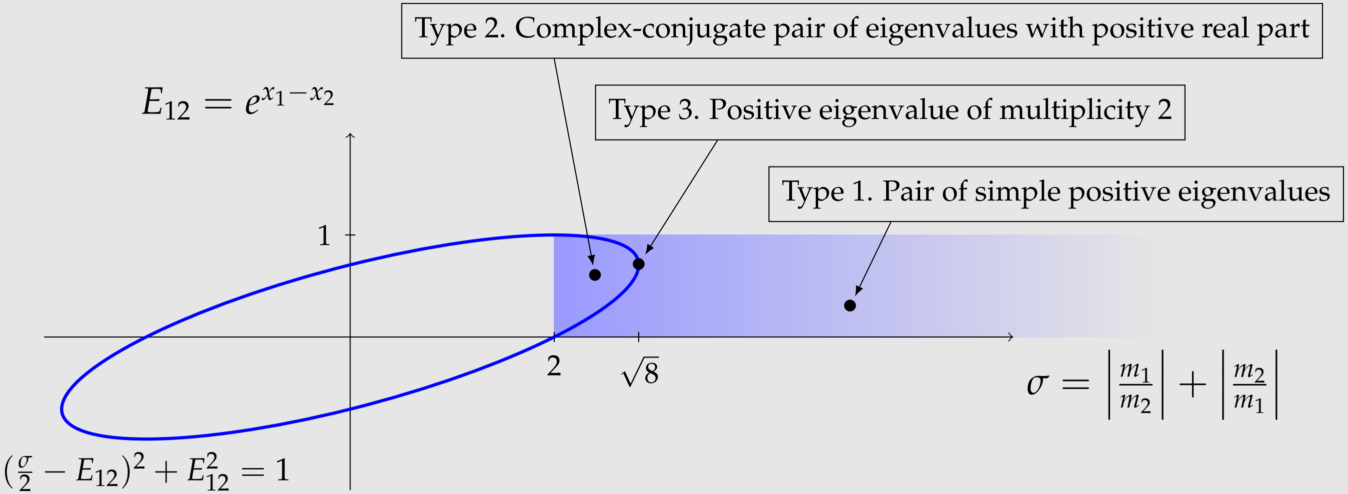
Type 2. $\lambda_2 = \overline{\lambda_1}$ with positive real part (and nonzero imaginary part).
$a_2 = \overline{a_1}$ nonzero.

Type 3. $\lambda_1 = \lambda_2 = \mu > 0$.
Limiting case, described by other solution formulas (see below).

The Weyl function has a different form in this case:

$$W(\lambda) = \frac{a_1}{\lambda - \mu} + \frac{\mu a_2}{(\lambda - \mu)^2} \qquad (a_2 \neq 0)$$

Classification in terms of m_1 and m_2 (of opposite signs) and $x_1 < x_2$:



Type 1. Real distinct eigenvalues

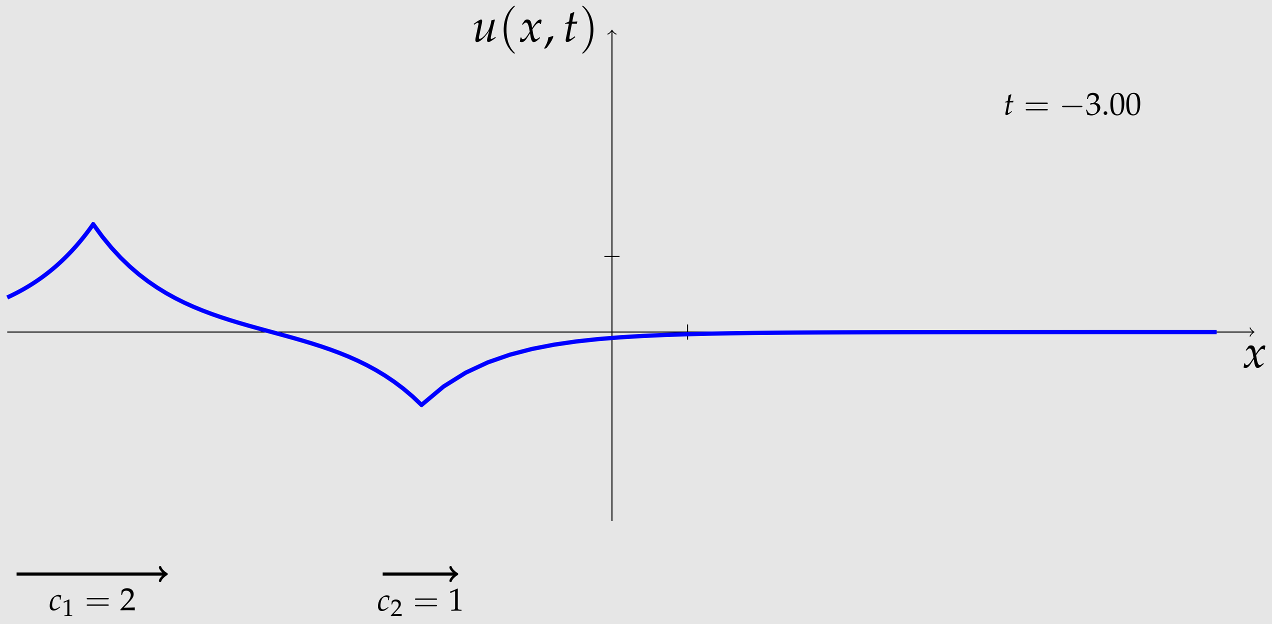
For asymptotic velocities $c_1 = \frac{1}{\lambda_1} > 0$ and $c_2 = \frac{1}{\lambda_2} > 0$, and collision at $(x,t) = (0,0)$, take $a_1(0) = -a_2(0) = \frac{\sqrt{c_1+c_2}}{c_1-c_2}$:

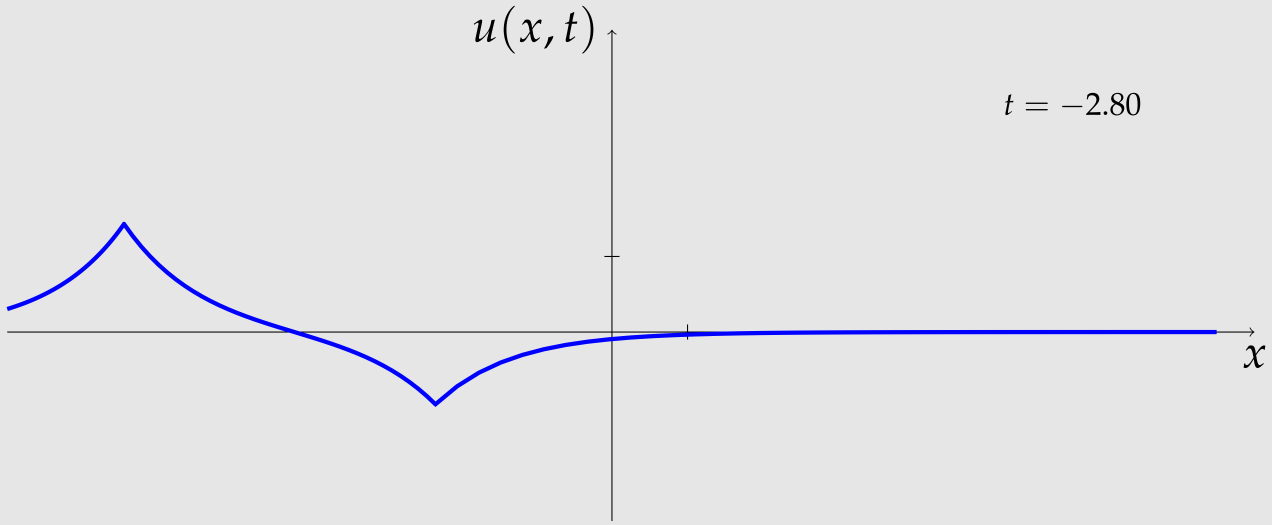
$$x_1(t) = -x_2(-t)$$

$$x_2(t) = \frac{1}{2}\ln\left(c_1\, e^{2c_1 t} + c_2\, e^{2c_2 t} - \frac{4\, c_1 c_2\, e^{(c_1+c_2)t}}{c_1+c_2} \right) - \frac{1}{2}\ln\frac{(c_1-c_2)^2}{c_1+c_2}$$

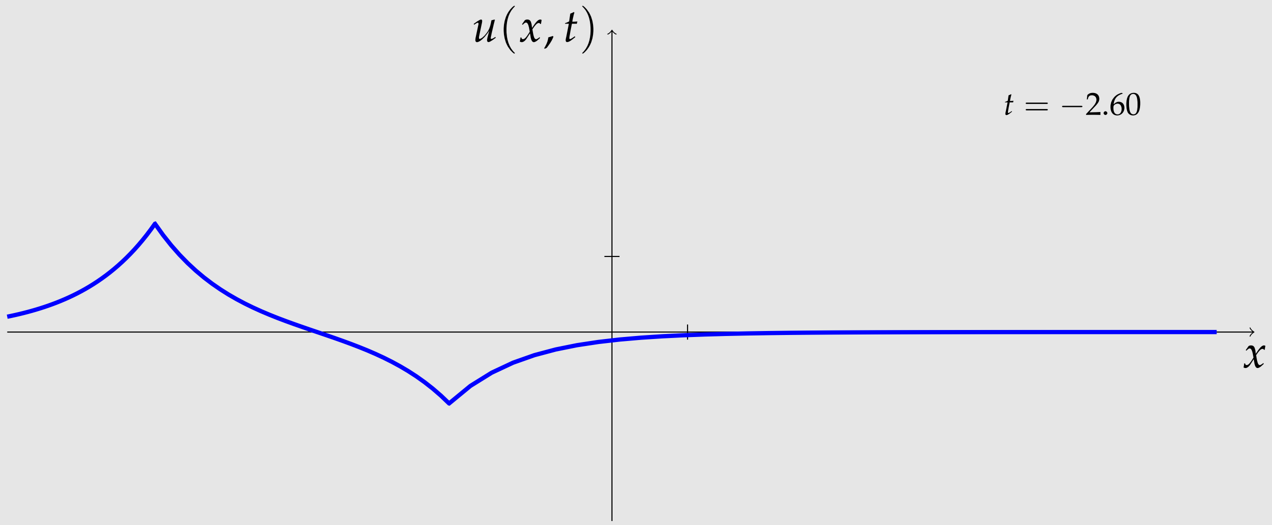
$$m_1(t) = m_2(-t)$$

$$m_2(t) = \frac{1}{e^{c_1 t} - e^{c_2 t}} \left(c_1\, e^{2c_1 t} + c_2\, e^{2c_2 t} - \frac{4\, c_1 c_2\, e^{(c_1+c_2)t}}{c_1+c_2} \right)^{1/2}$$

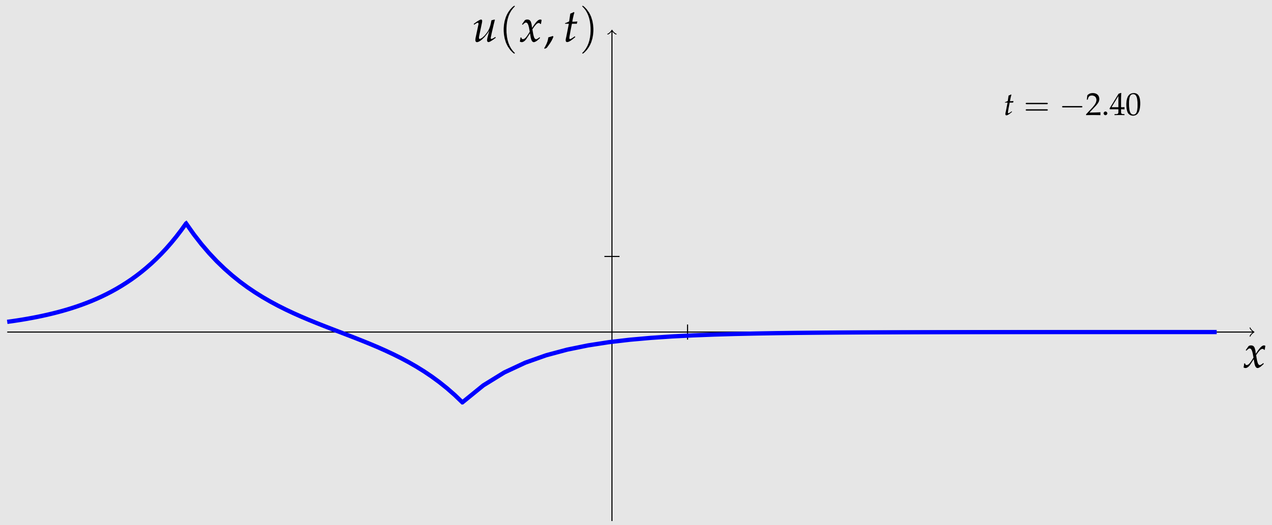

$u(x,t)$
$t = -3.00$
x
$c_1 = 2$
$c_2 = 1$

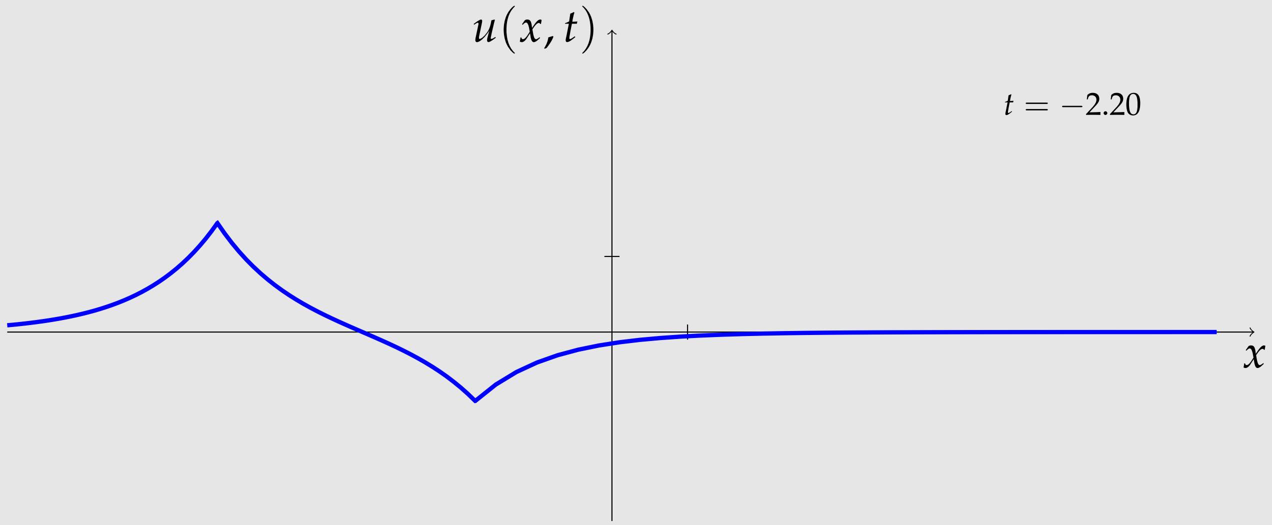


$u(x,t)$
$t = -2.80$
x

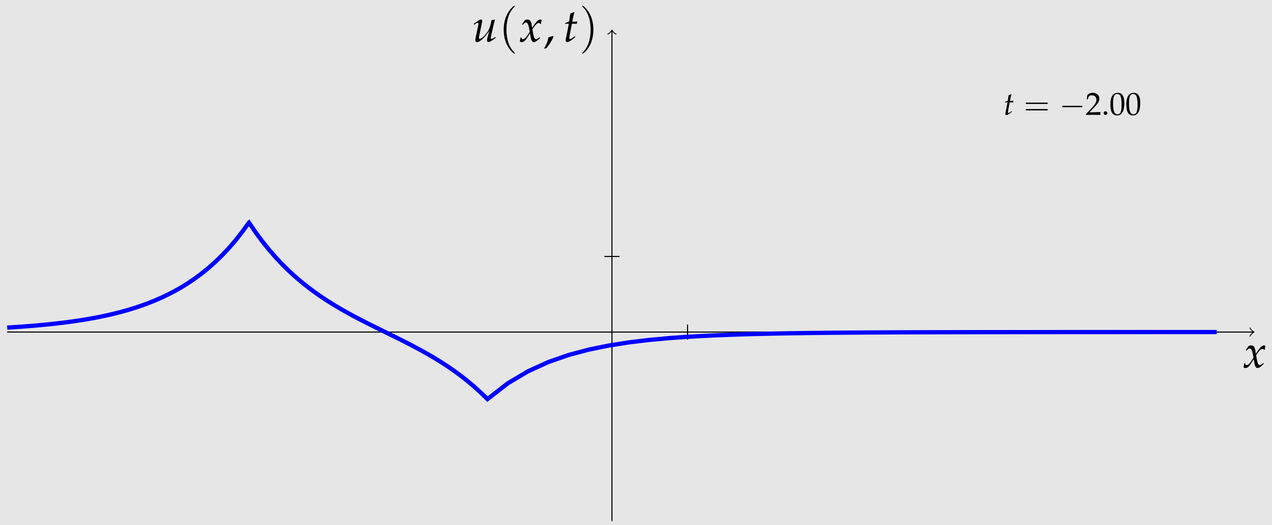


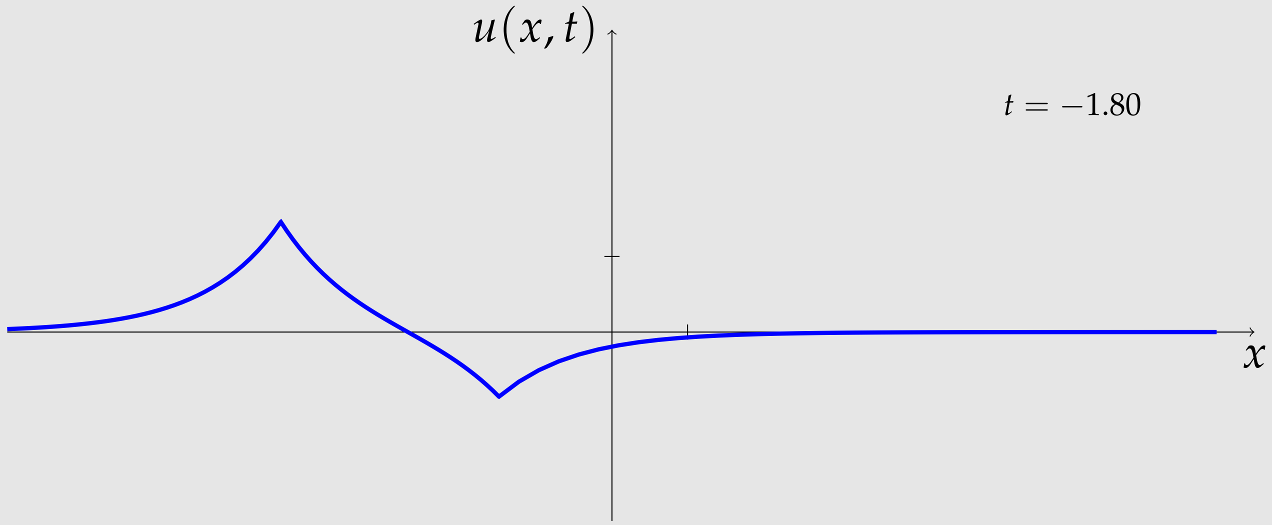
$u(x,t)$
$t = -2.60$
x

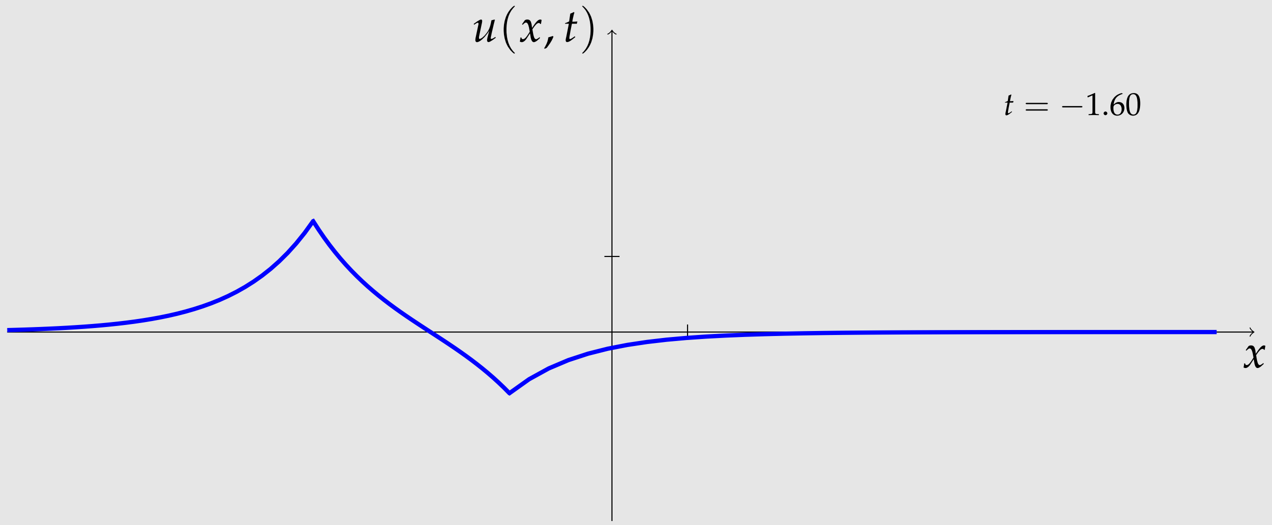

$u(x,t)$
$t = -2.40$
x



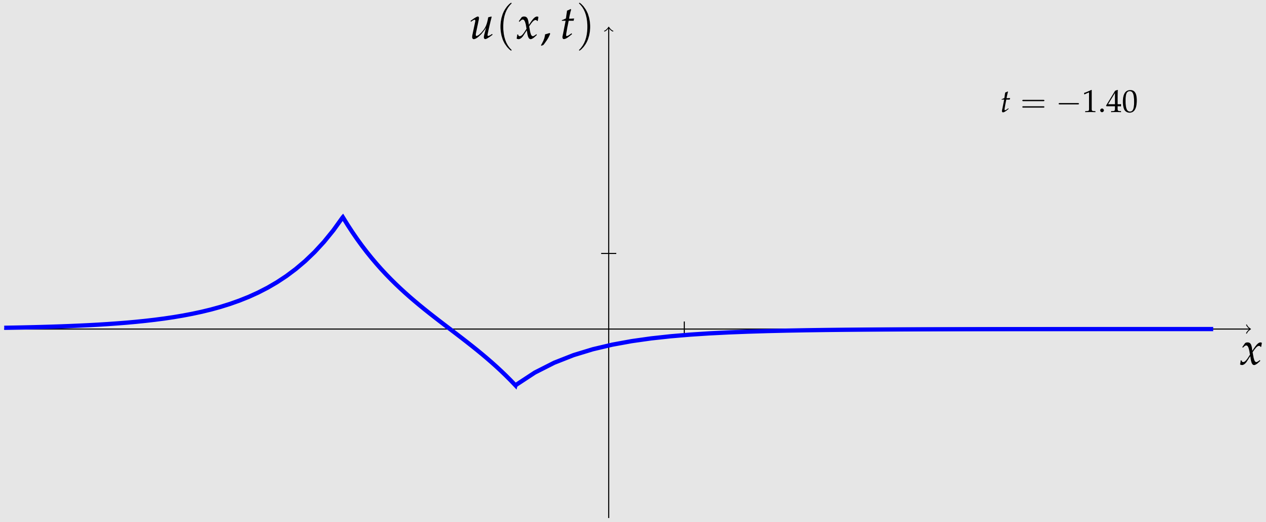
$u(x,t)$
$t = -2.20$
x

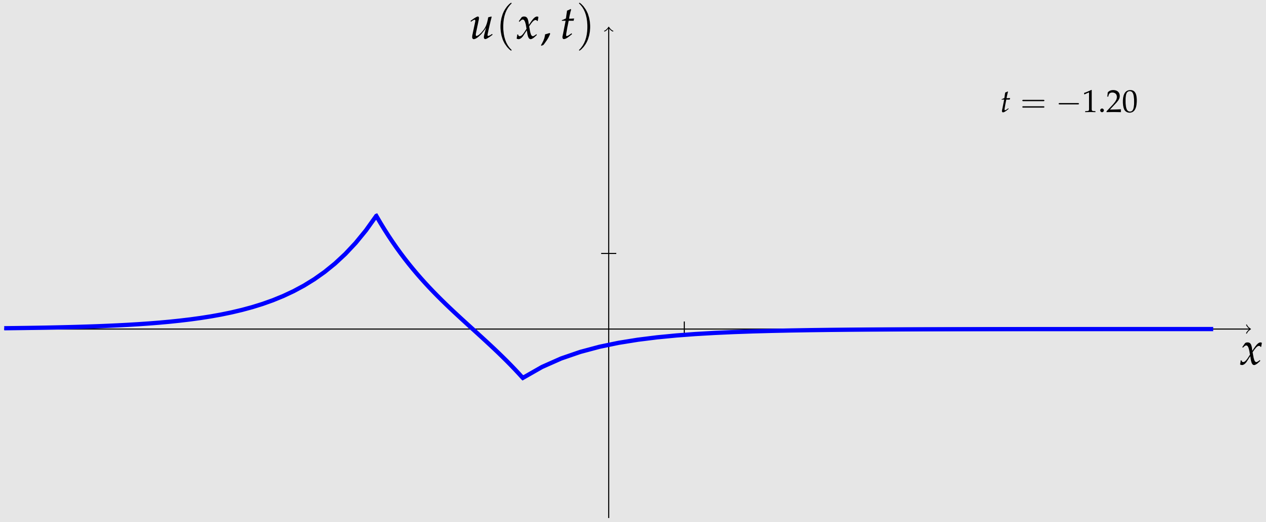

u(x, t)
t = −2.00
x


$u(x,t)$
$t = -1.80$
x

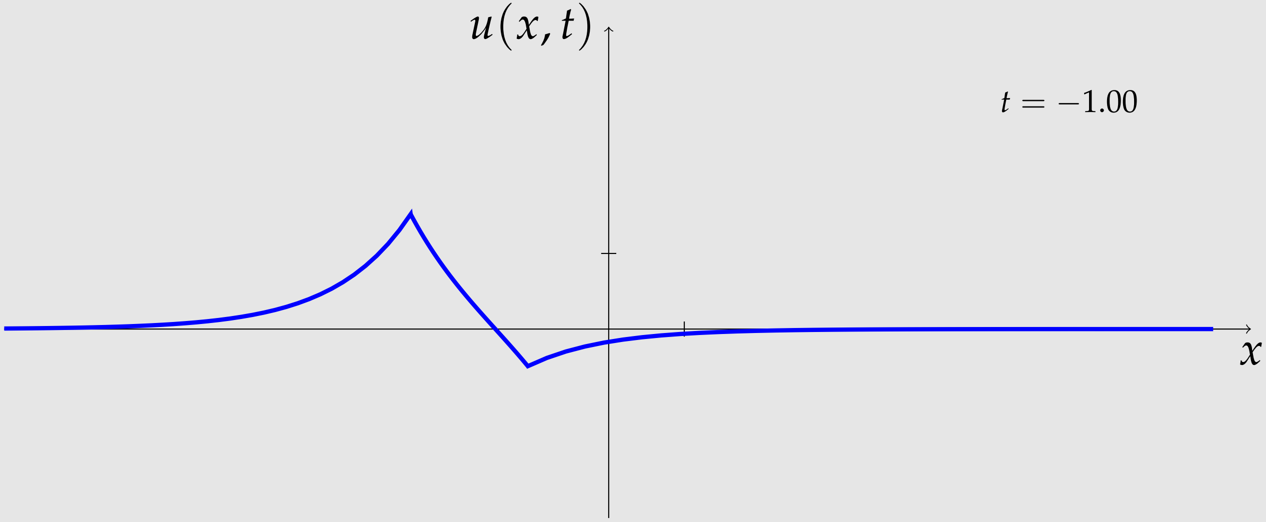


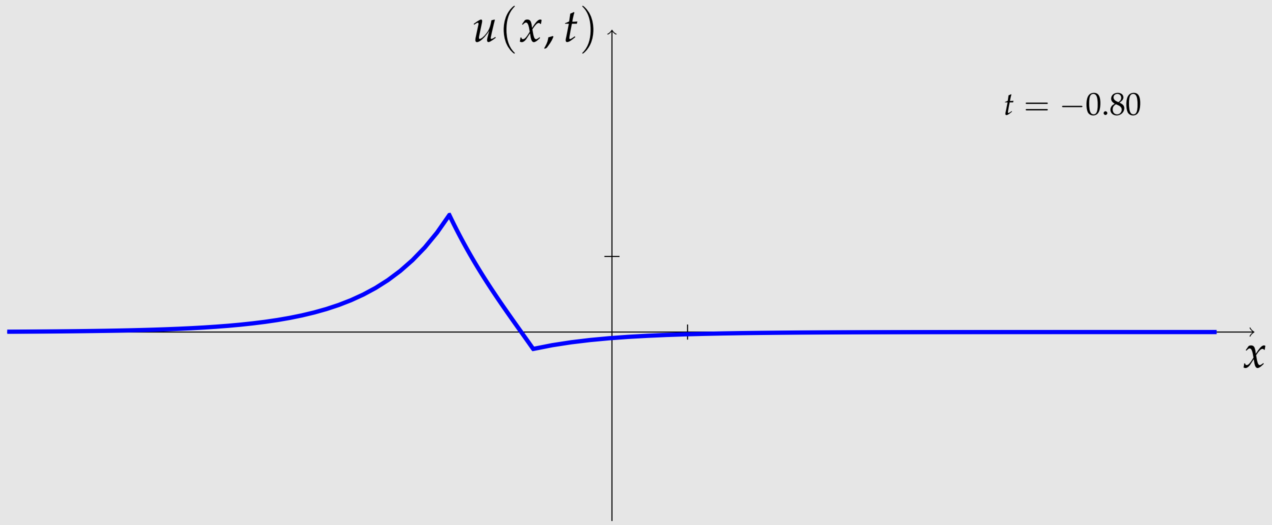
$u(x,t)$
$t = -1.60$
x


$u(x,t)$
$t = -1.40$
x

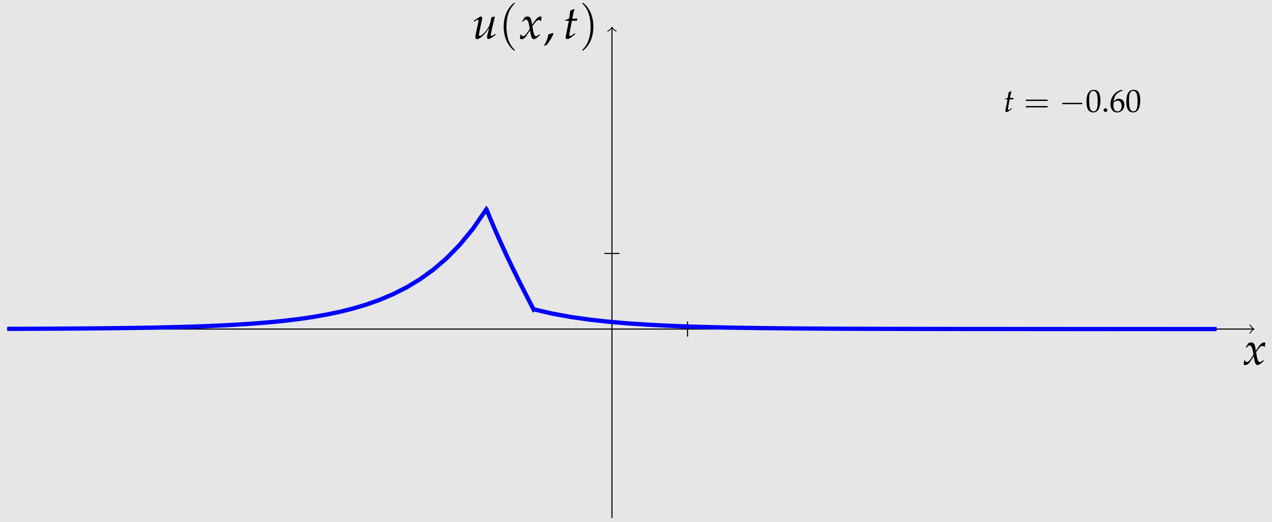


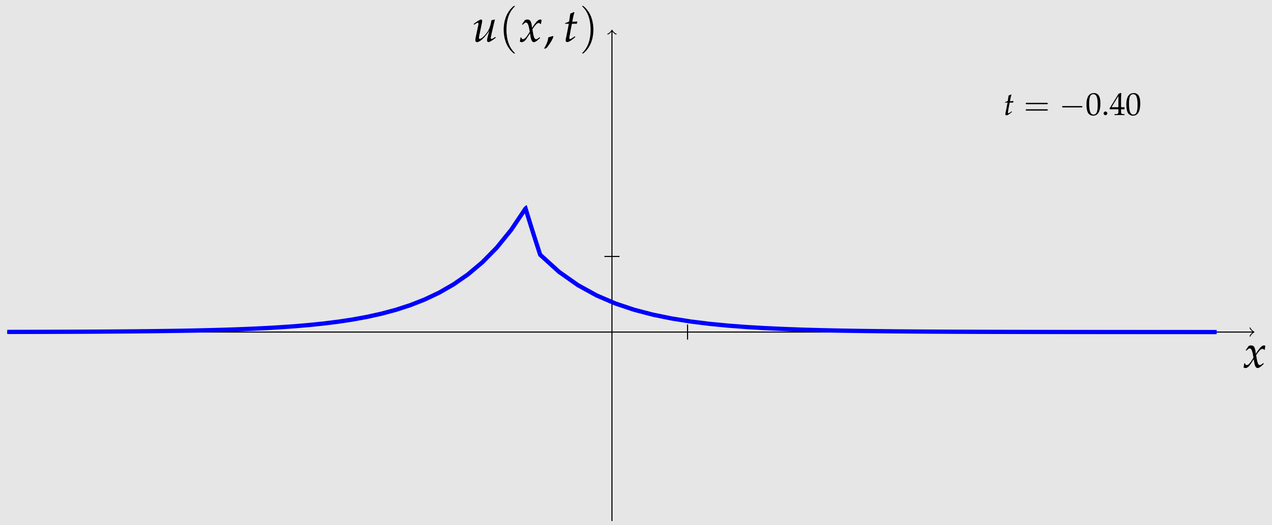
$u(x,t)$
$t = -1.20$
x


$u(x,t)$
$t = -1.00$
x

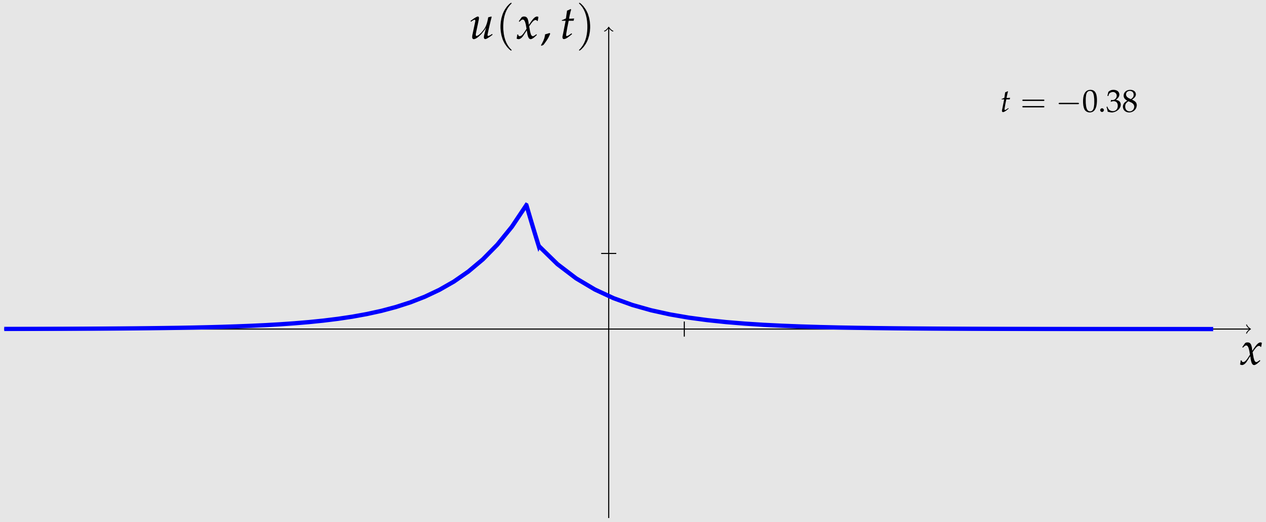


$u(x,t)$
$t = -0.80$
x

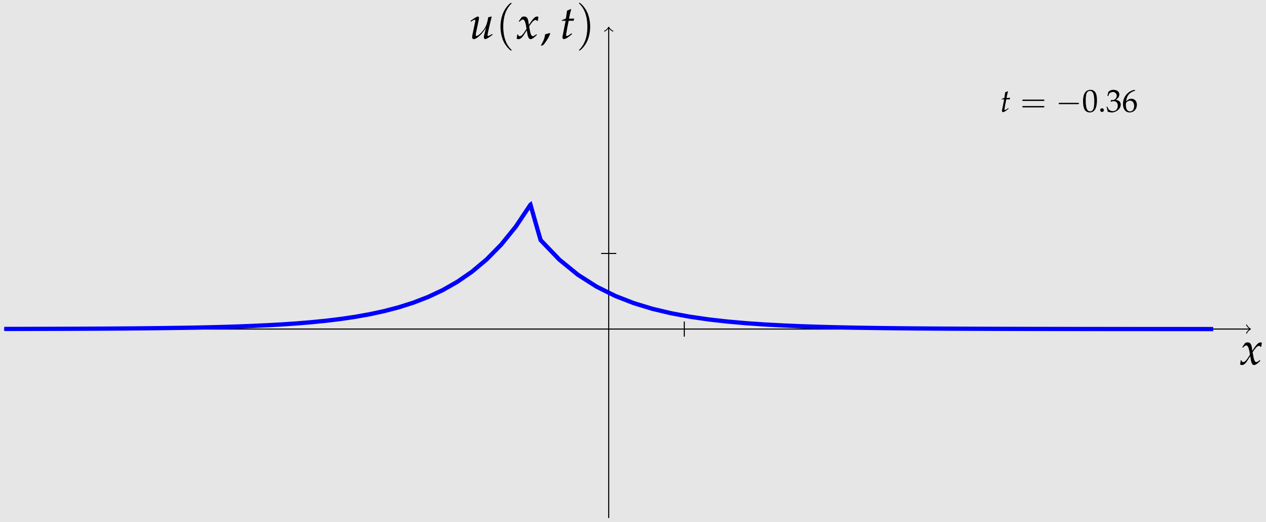

$u(x,t)$
$t = -0.60$
x



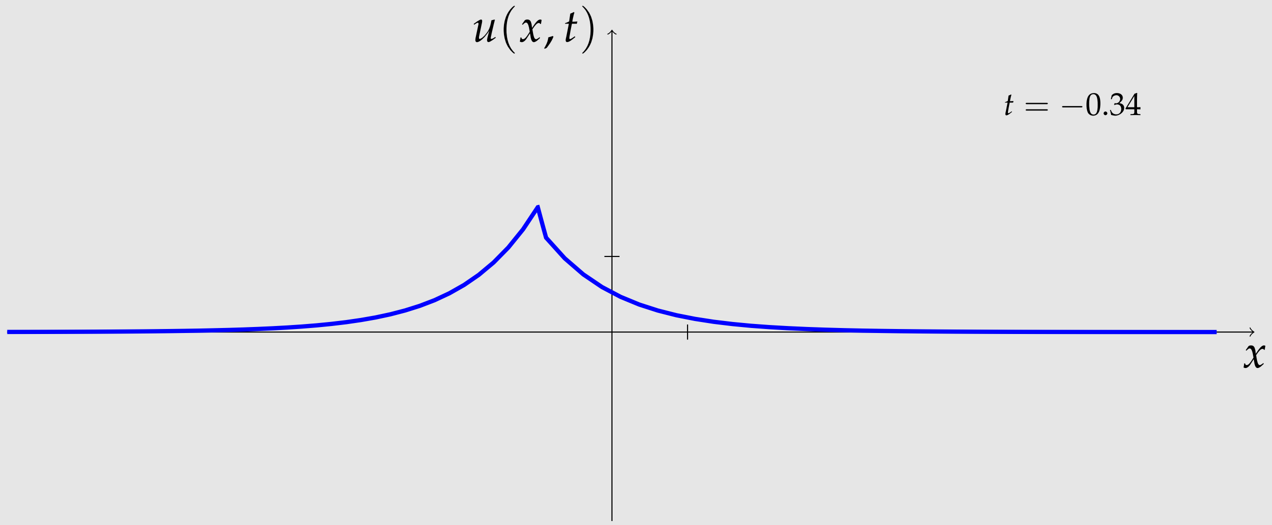
(Smaller timestep.)



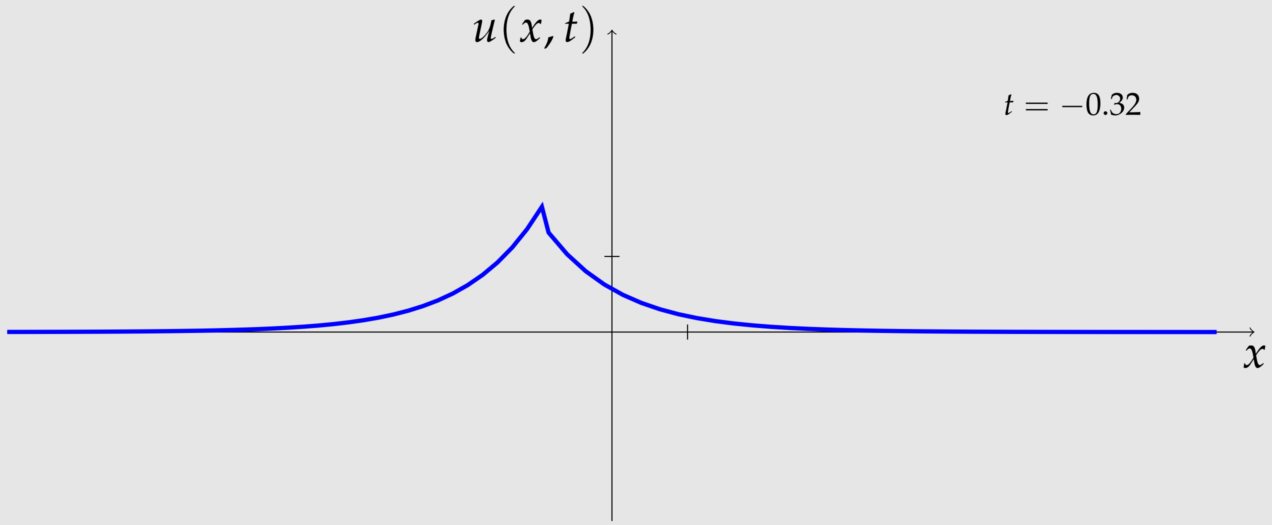
(Smaller timestep.)



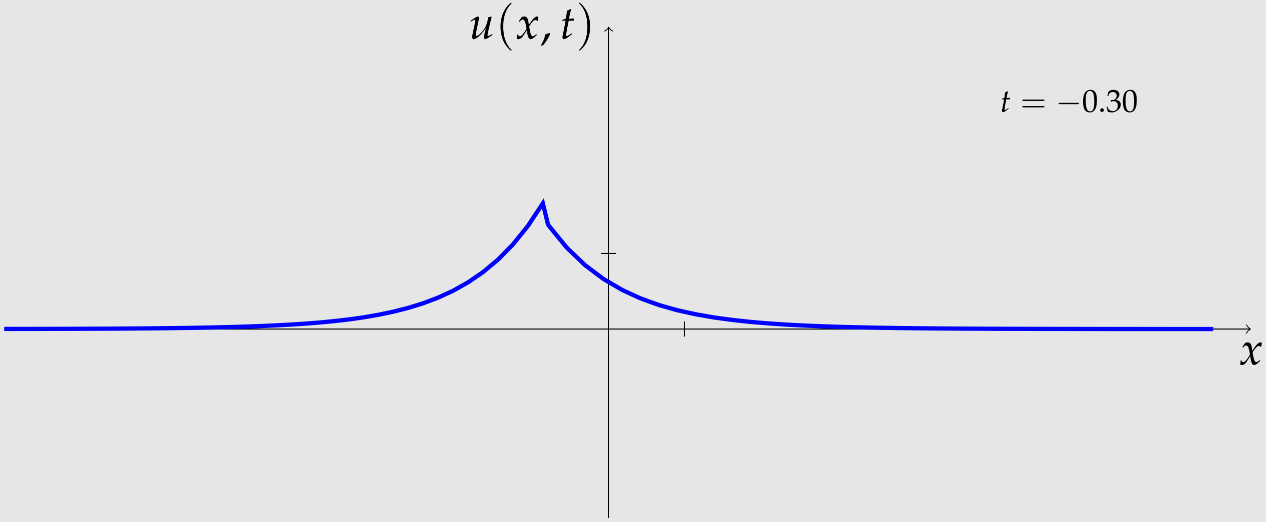
(Smaller timestep.)



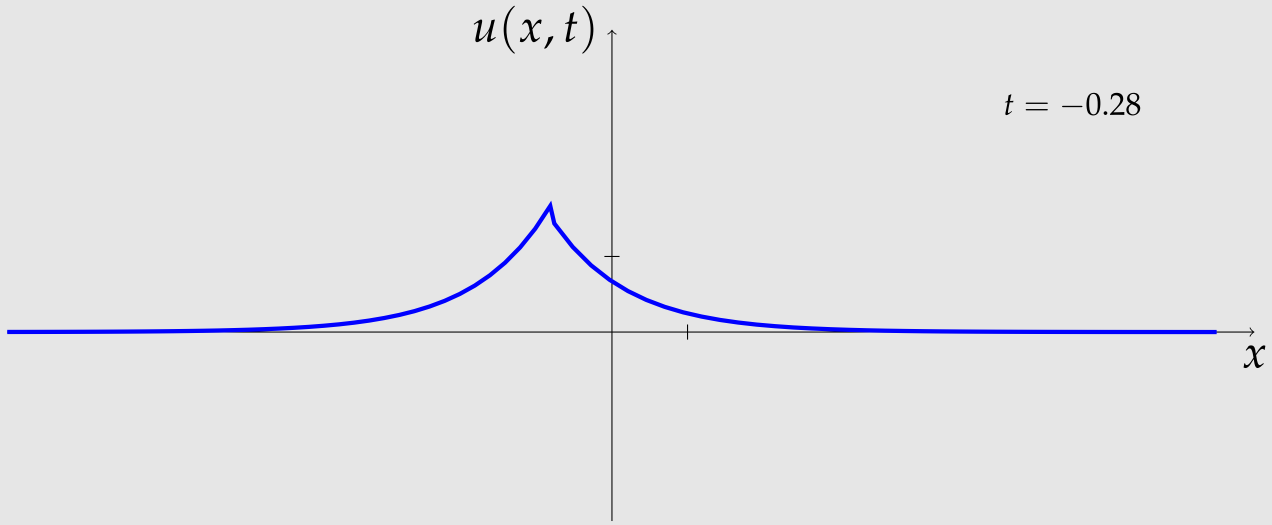
(Smaller timestep.)



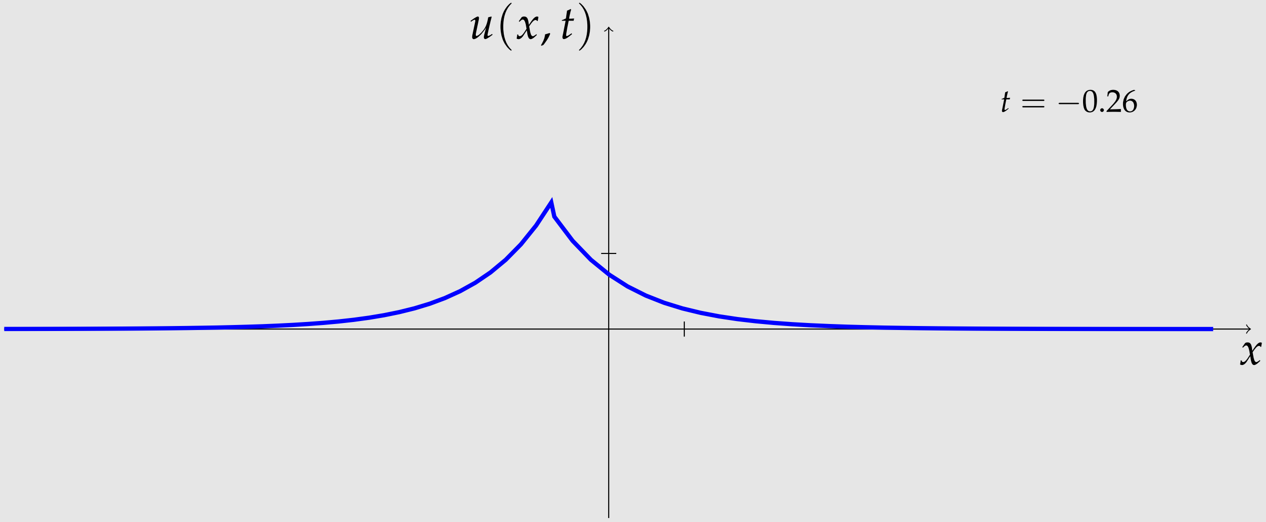
(Smaller timestep.)



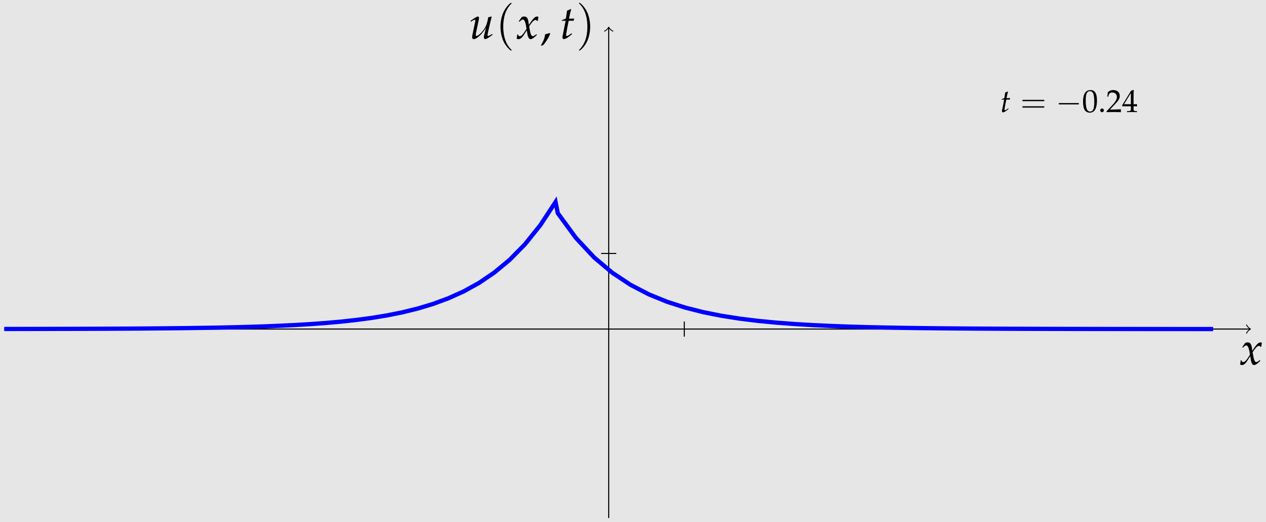
(Smaller timestep.)



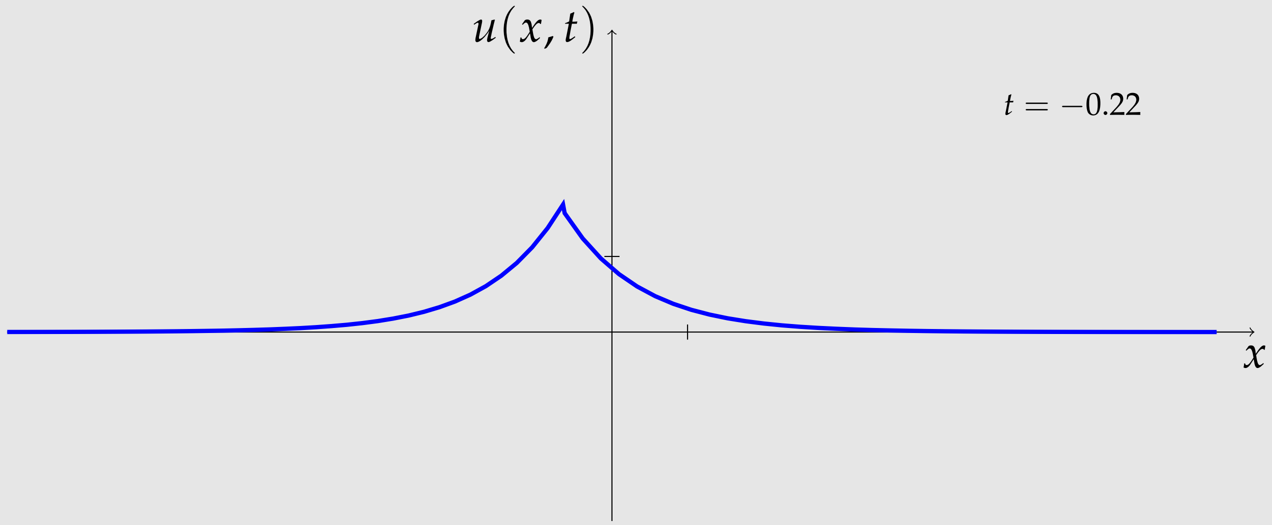
(Smaller timestep.)



(Smaller timestep.)



(Smaller timestep.)



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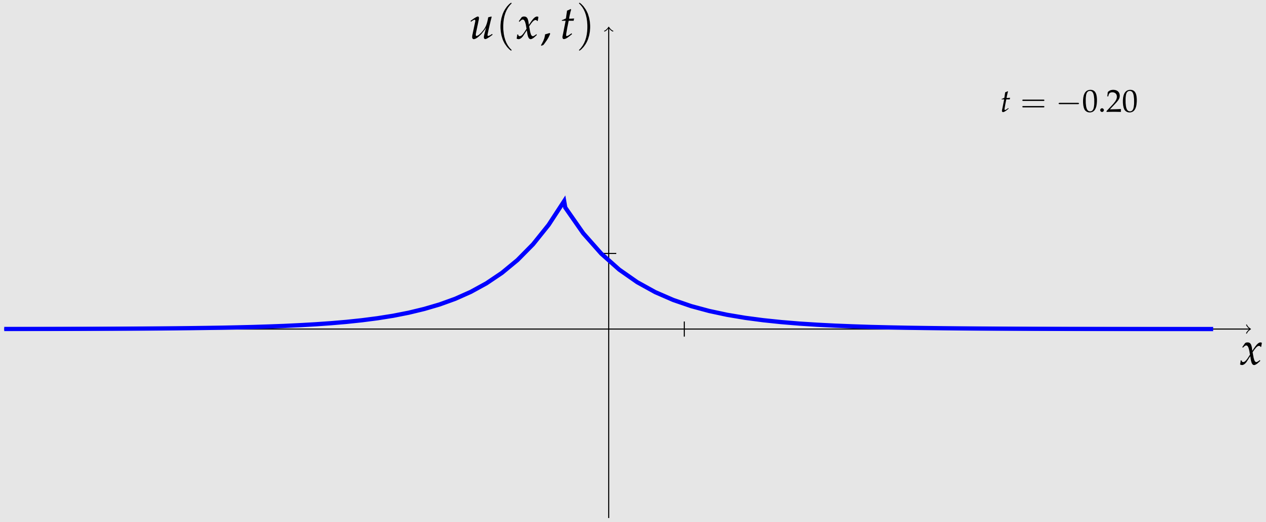

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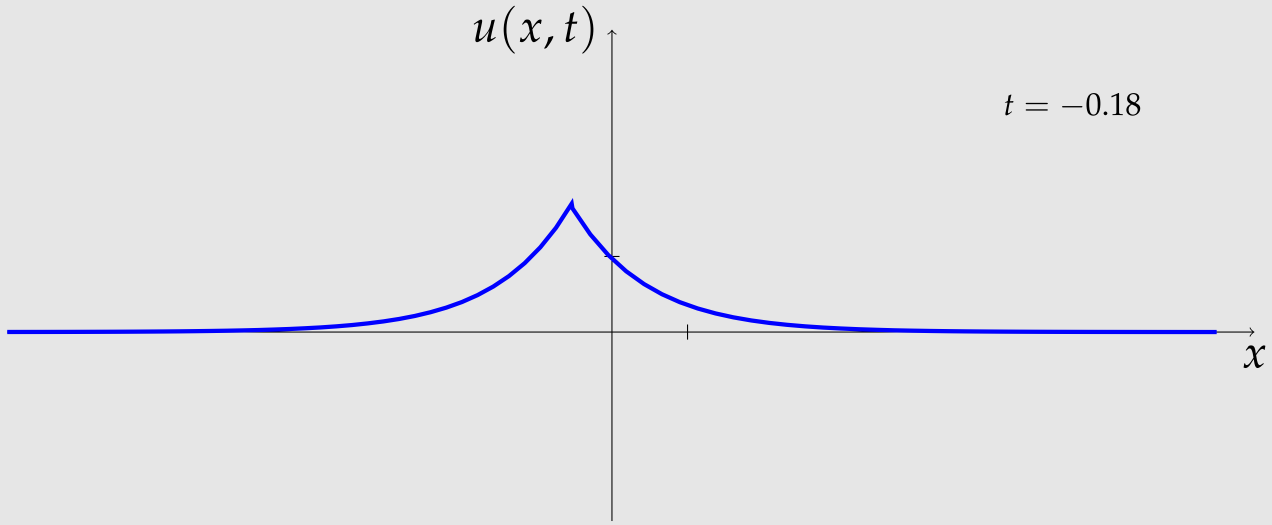
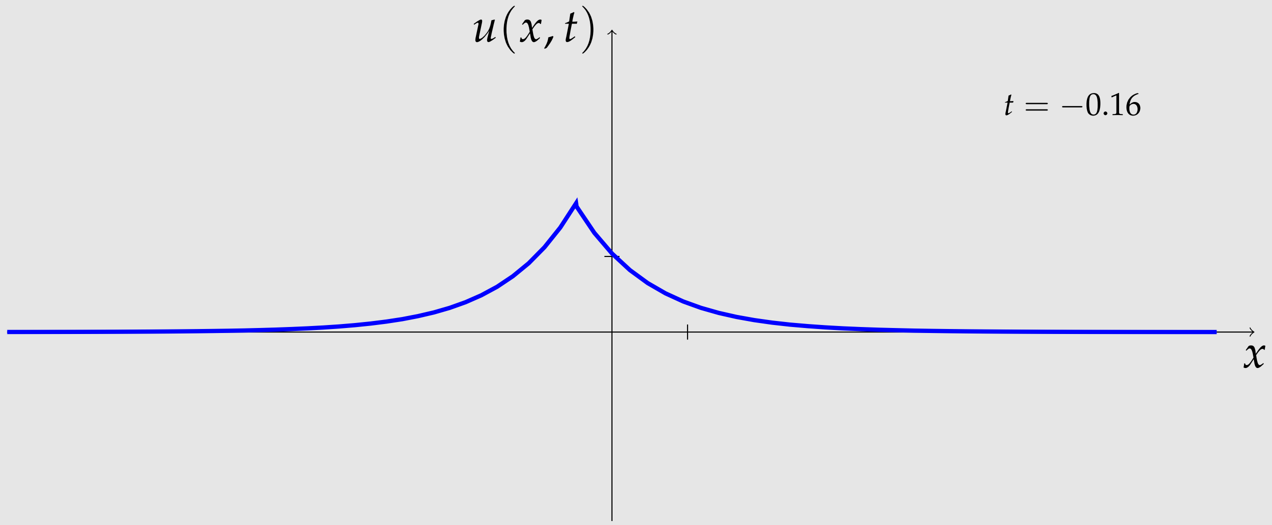


(Smaller timestep.)



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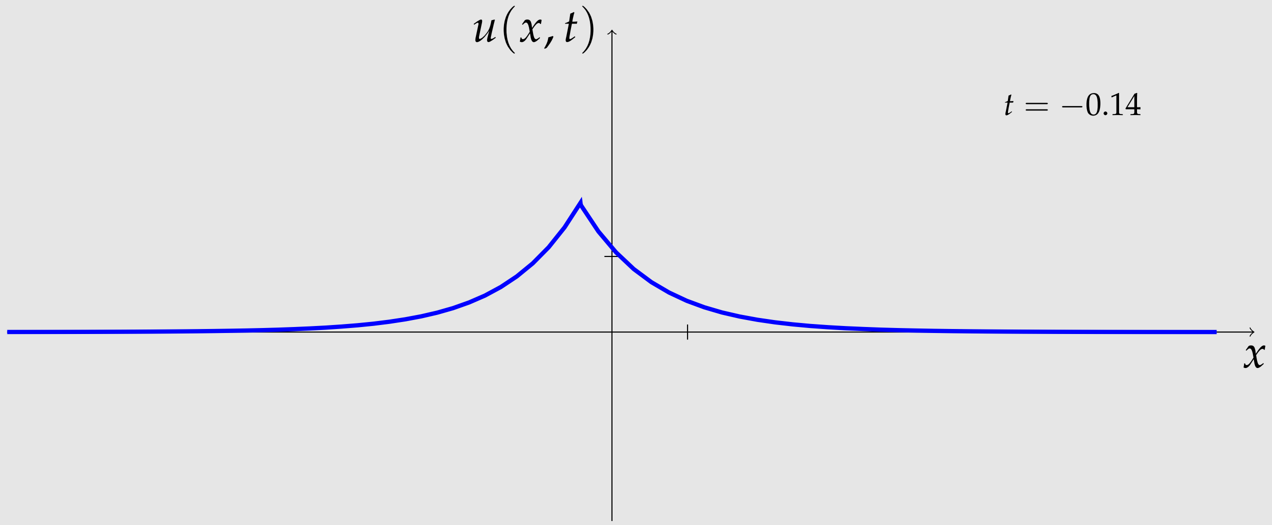

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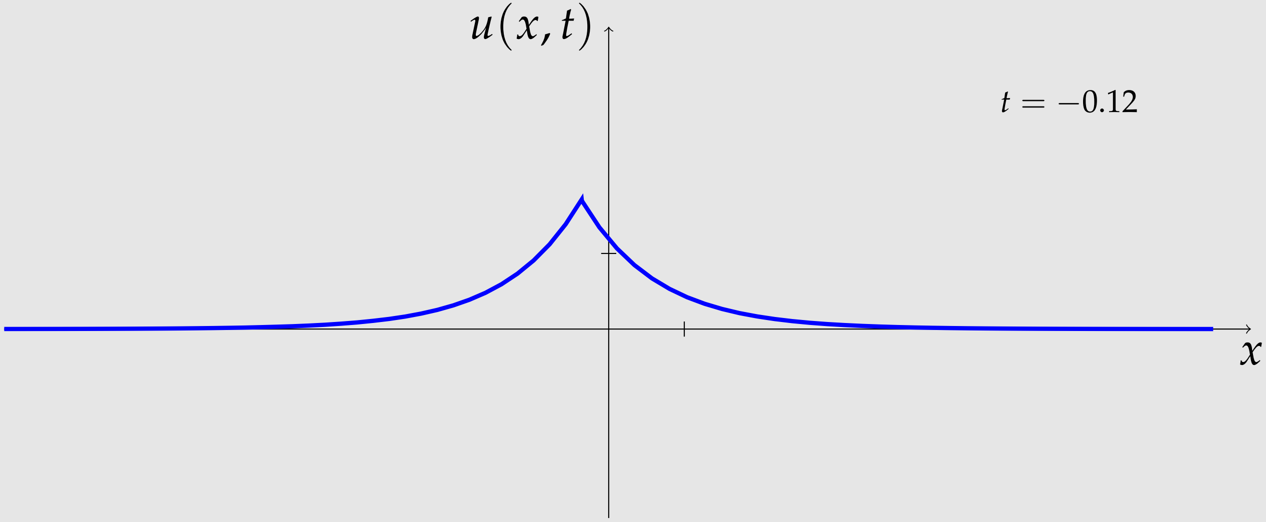

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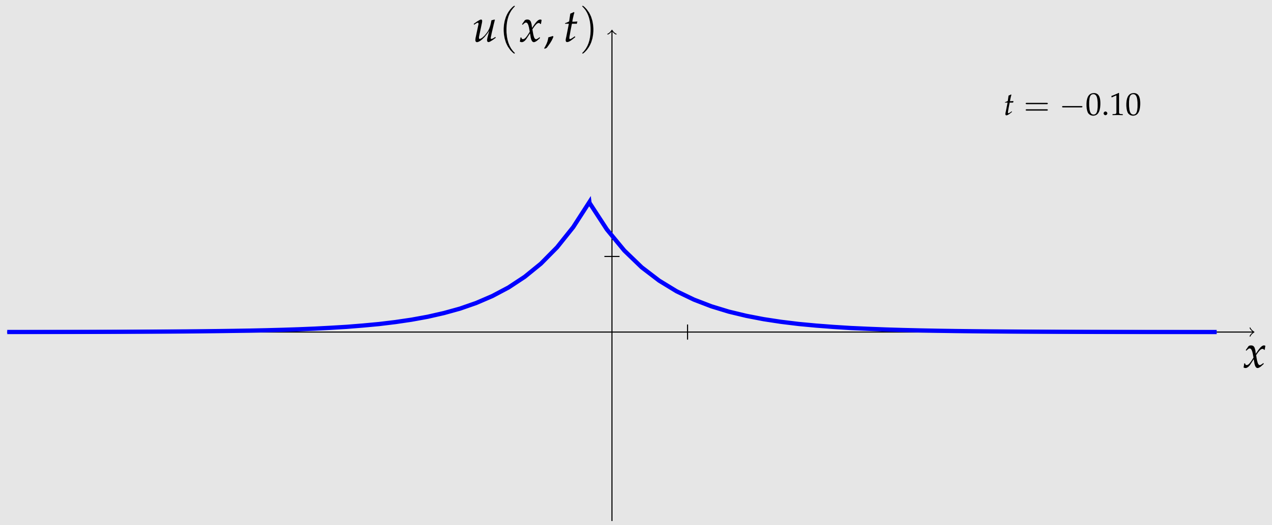

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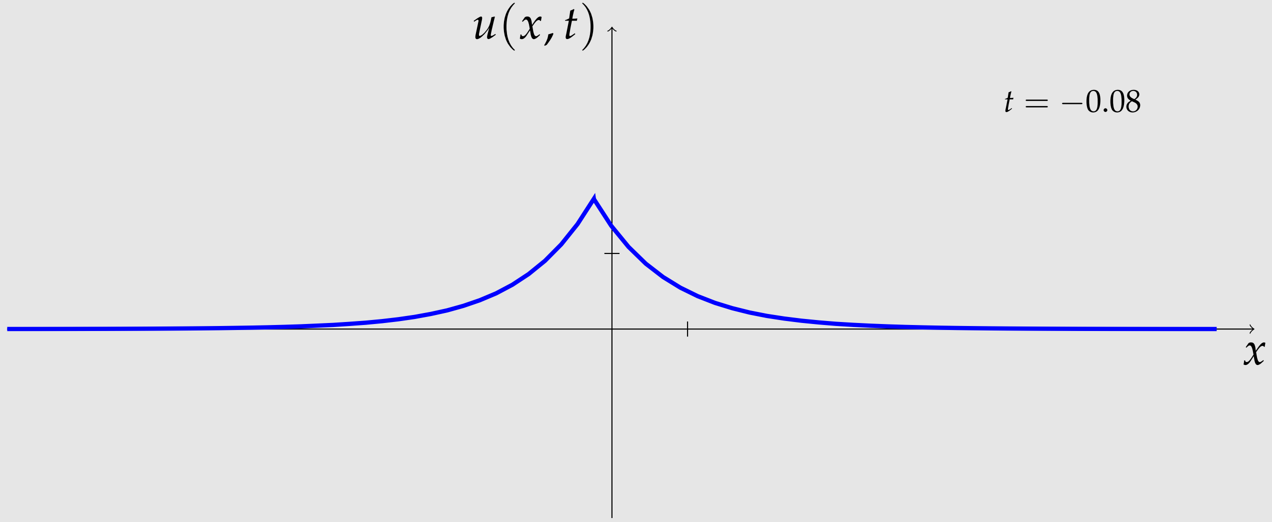
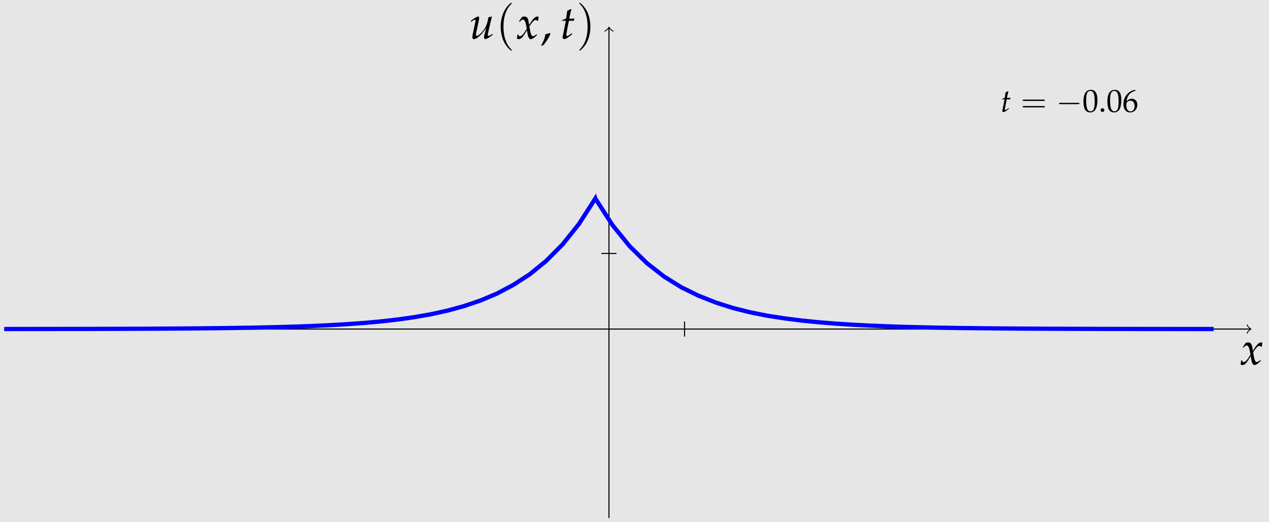


(Smaller timestep.)



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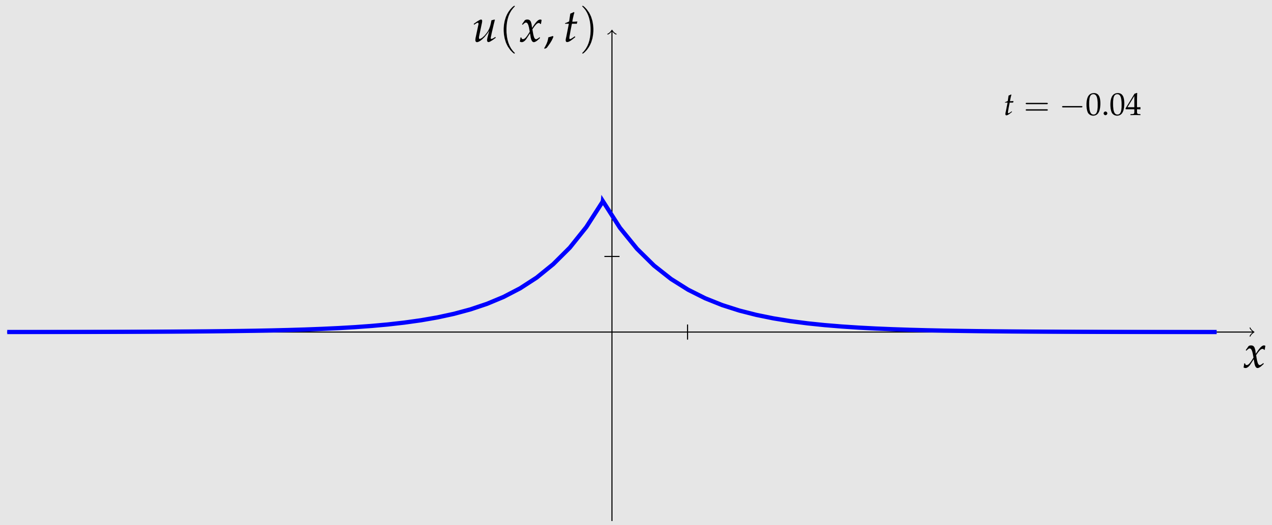
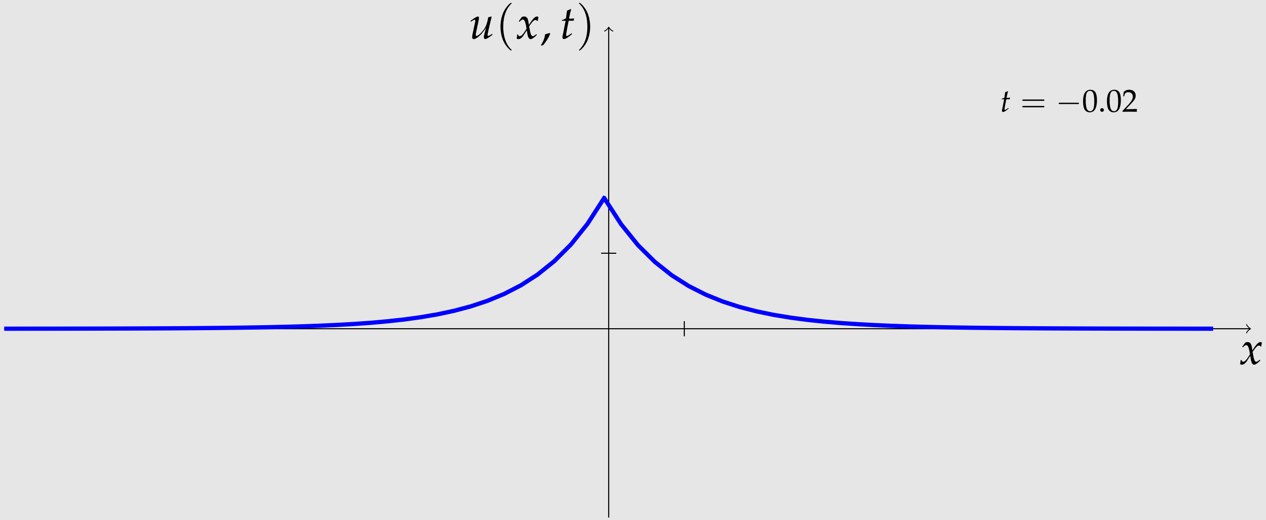
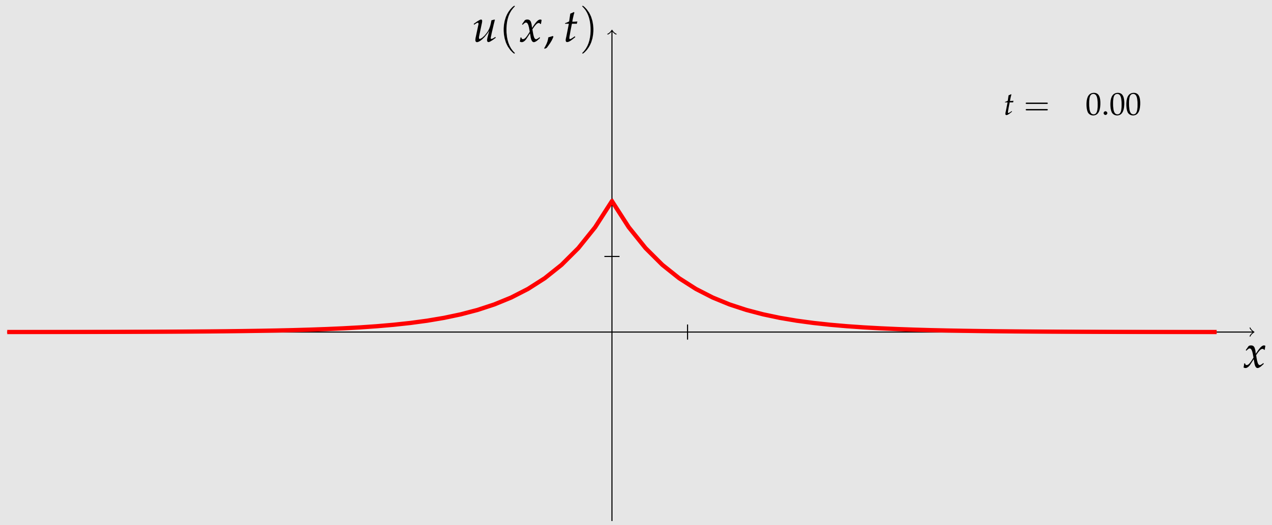


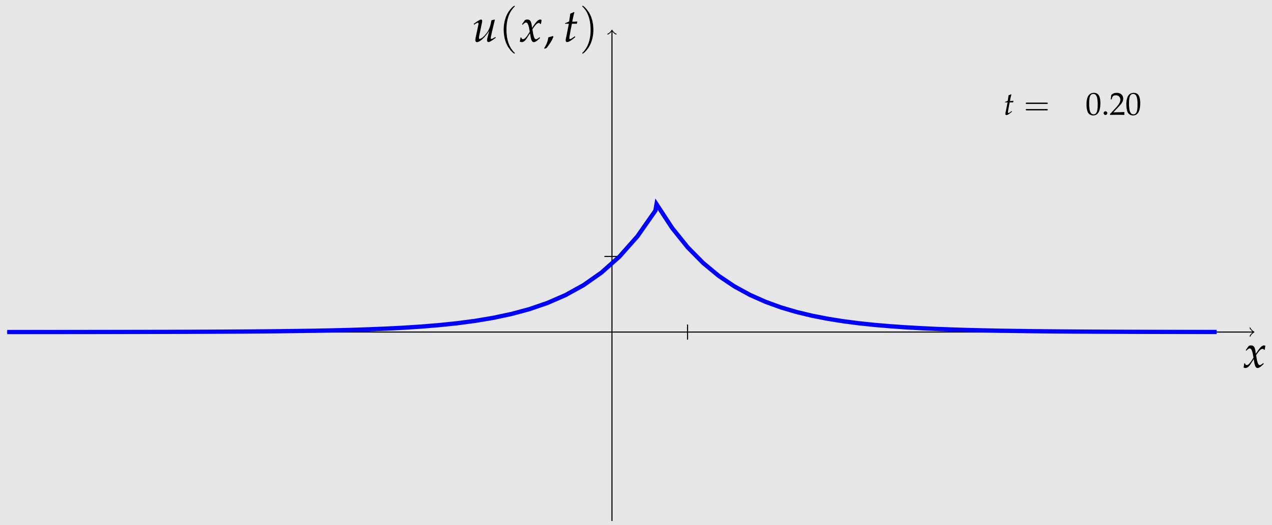
(Smaller timestep.)



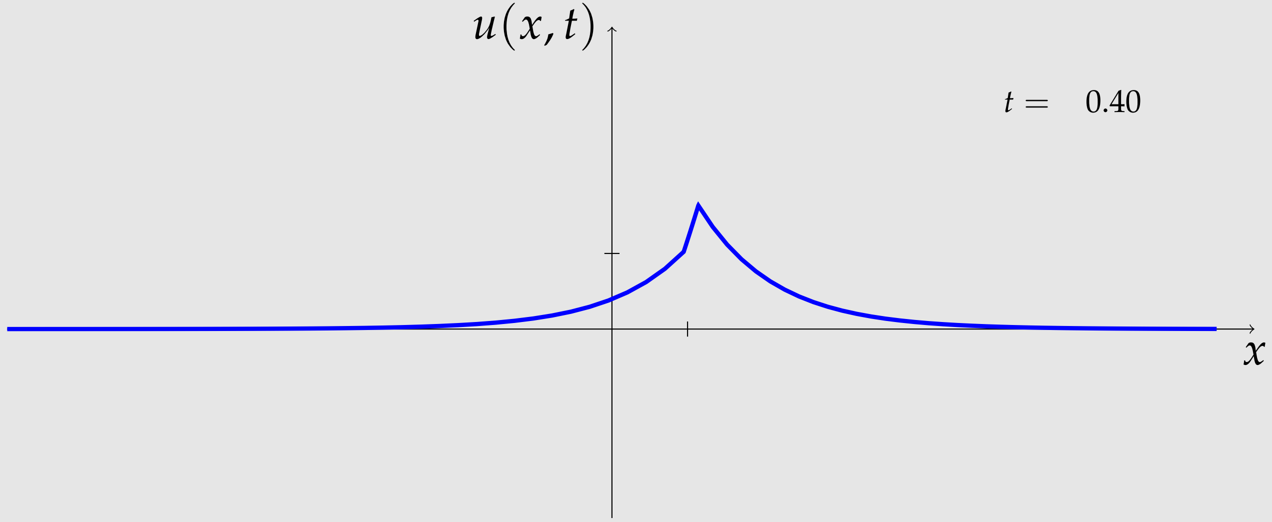
(Smaller timestep.)

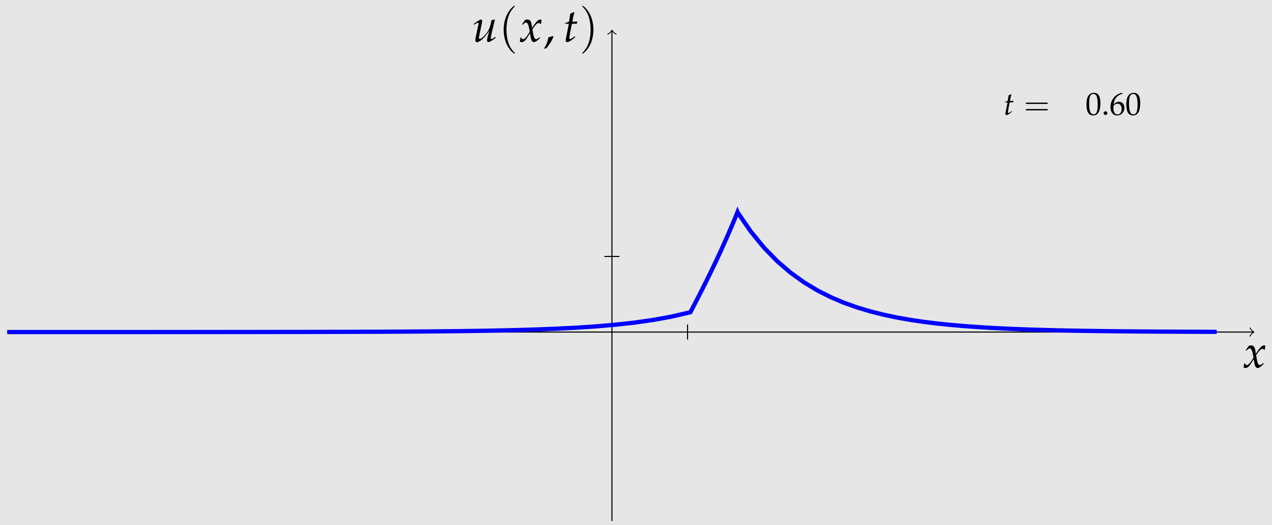


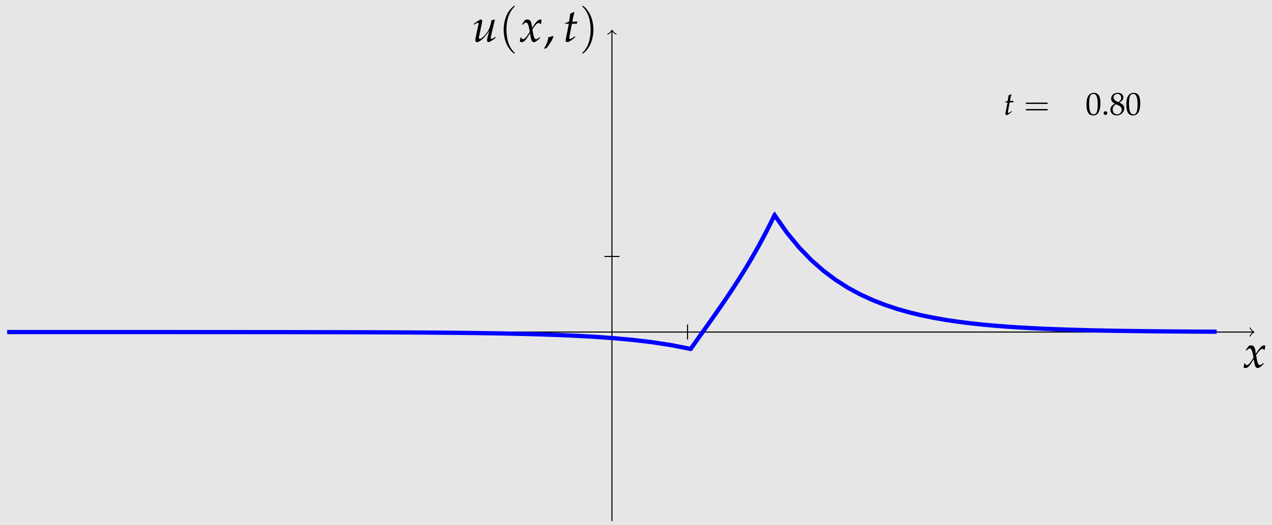
Limiting shape: $u(x,0) = \sqrt{c_1 + c_2}\, e^{-|x|}$

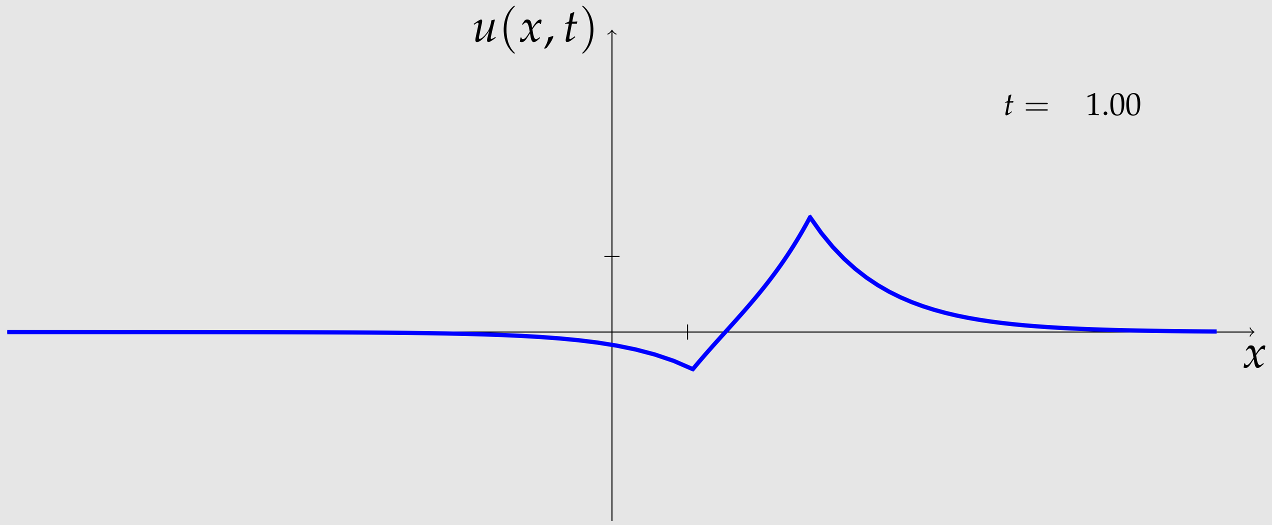


$u(x,t)$
$t = \quad 0.20$
x

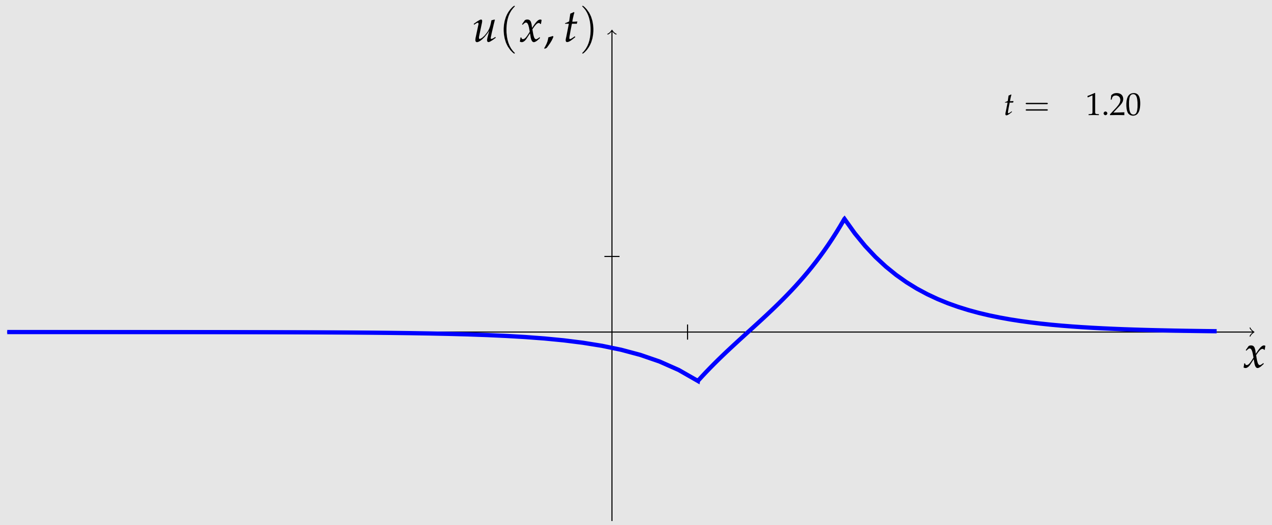

$u(x,t)$
$t = \quad 0.40$
x


u(x, t)
t = 0.60
x

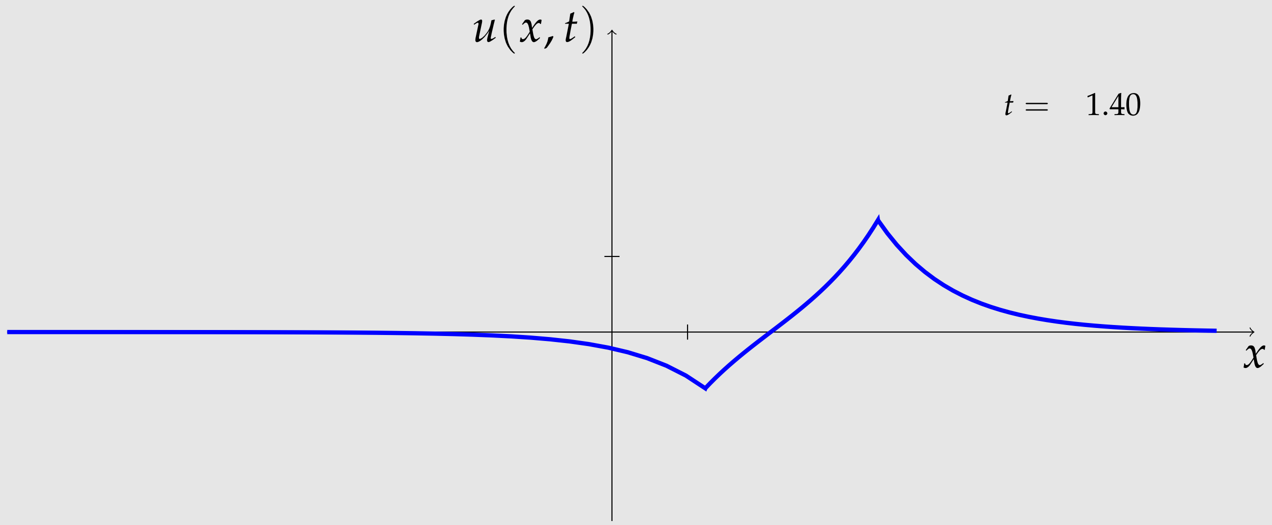

$u(x,t)$
$t = \quad 0.80$
x



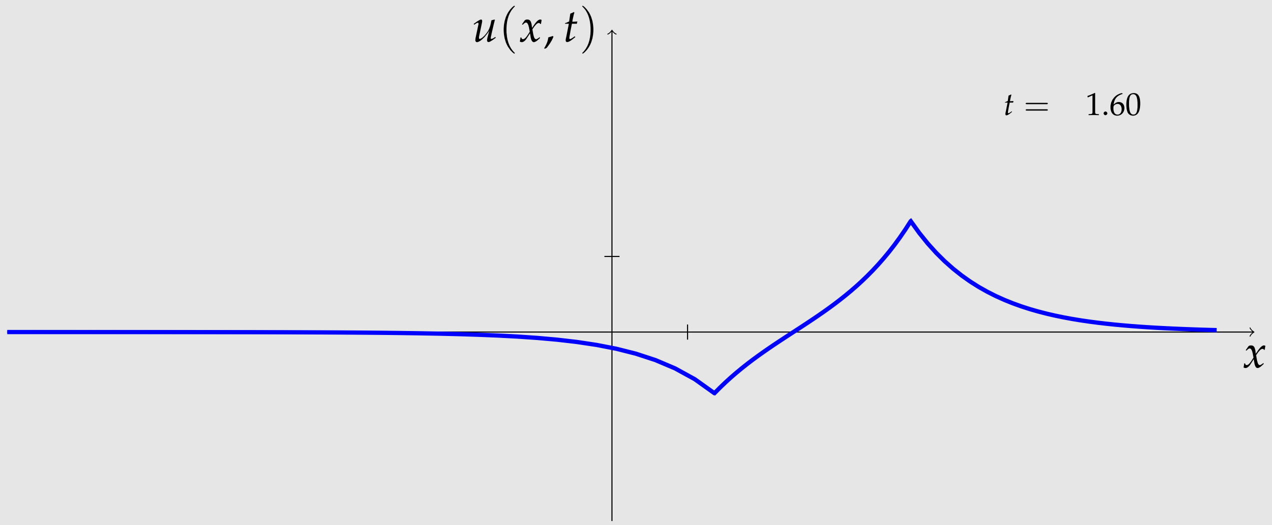
$u(x,t)$
$t = 1.00$
x



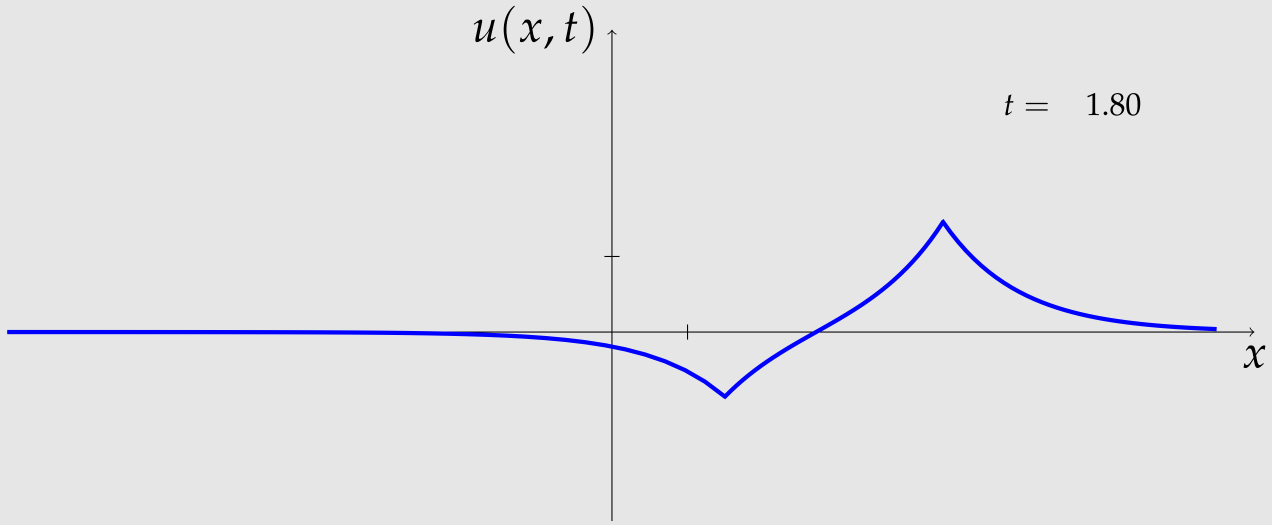
$u(x,t)$
$t = 1.20$
x



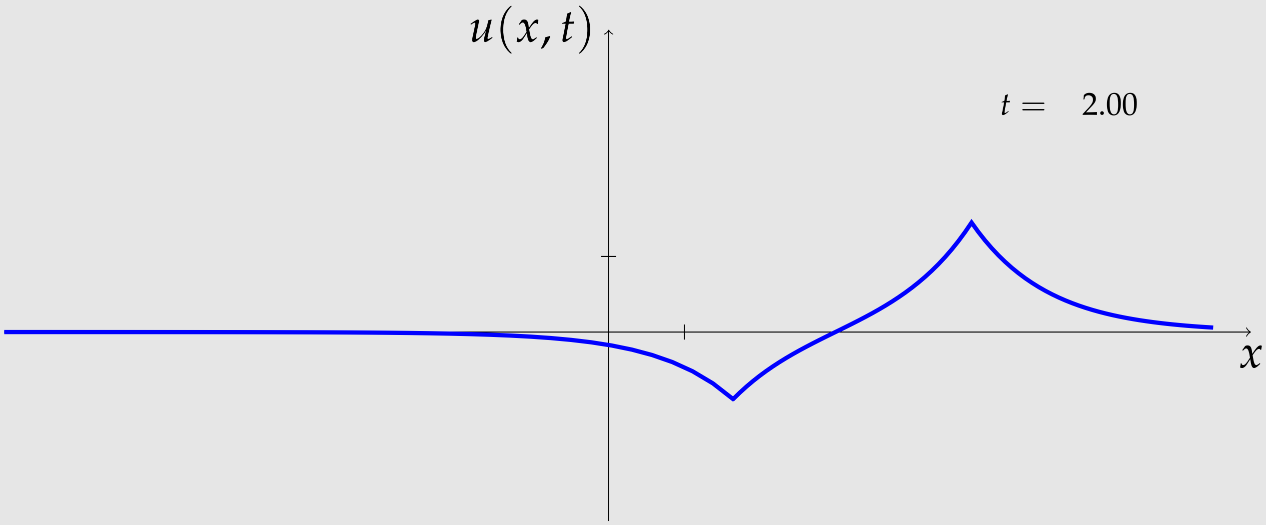
$u(x,t)$
$t = \quad 1.40$
x

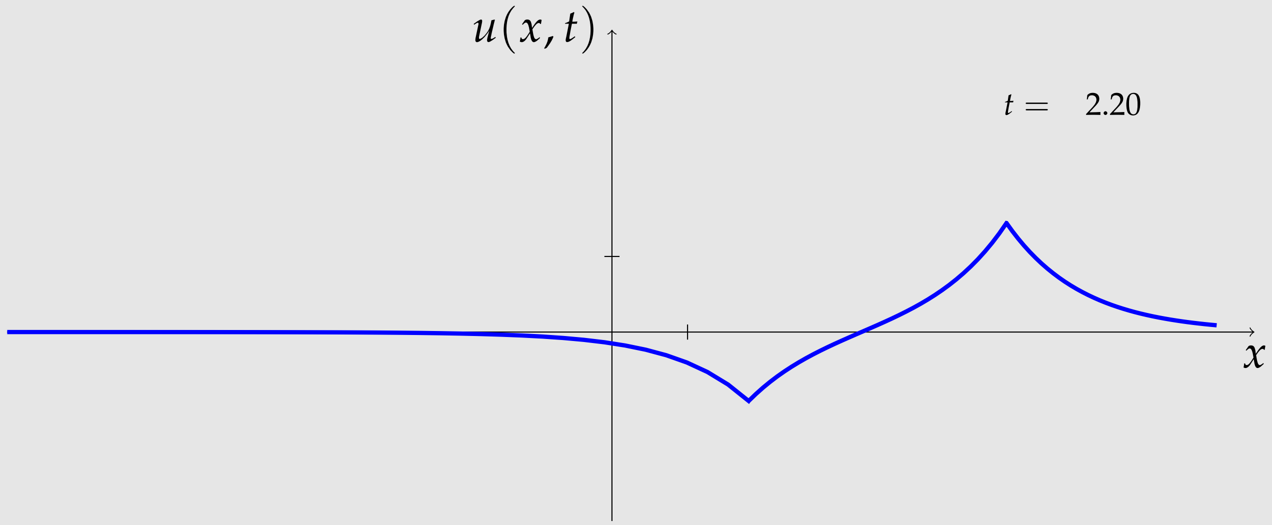


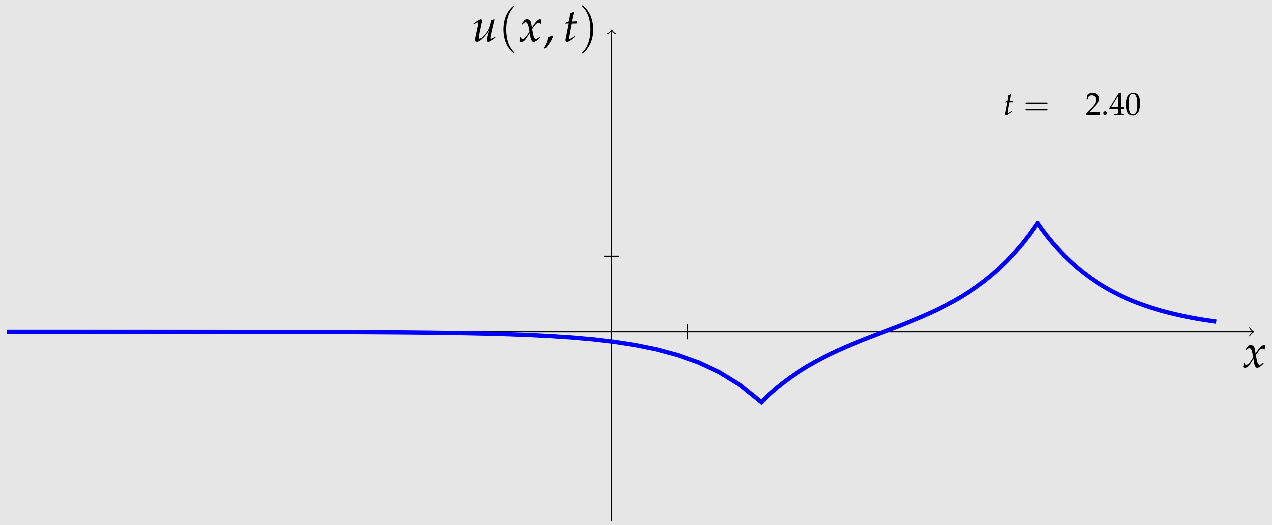
$u(x,t)$
$t = 1.60$
x

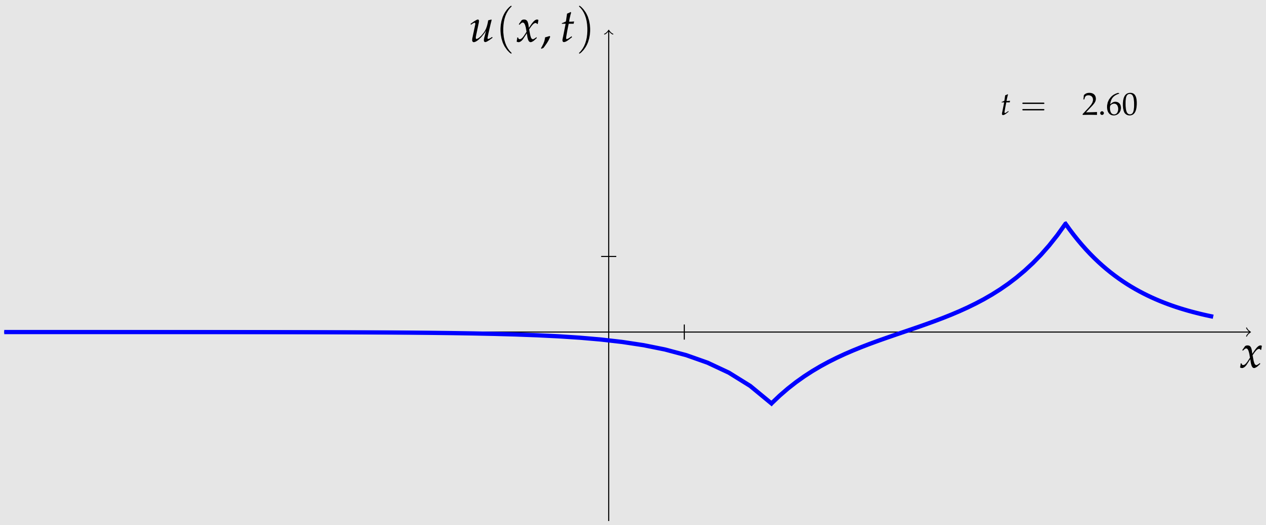


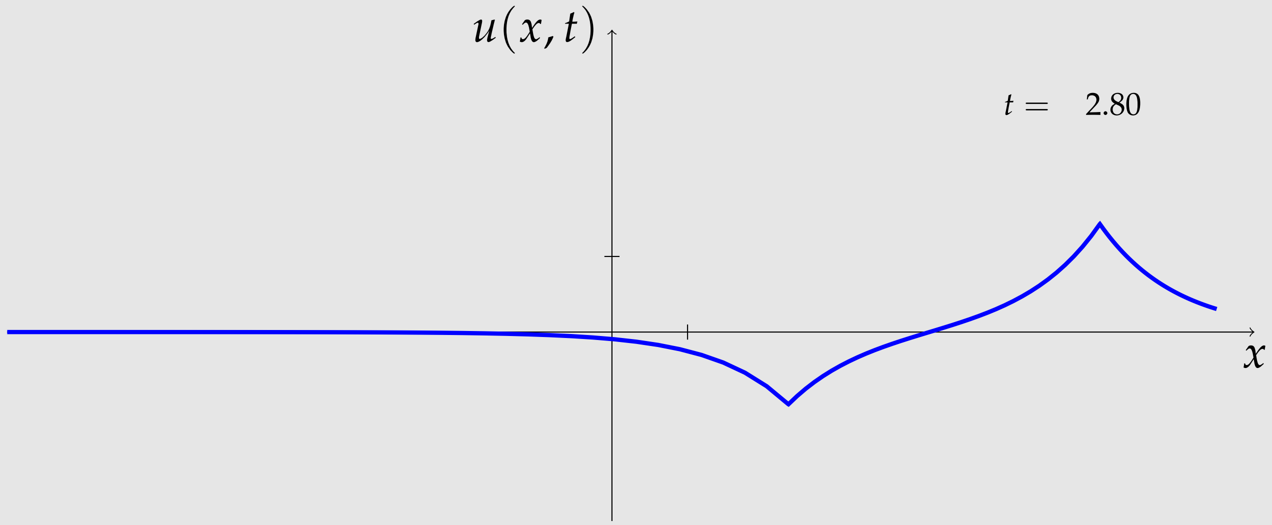
$u(x,t)$
$t = 1.80$
x


$u(x,t)$
$t = 2.00$
x

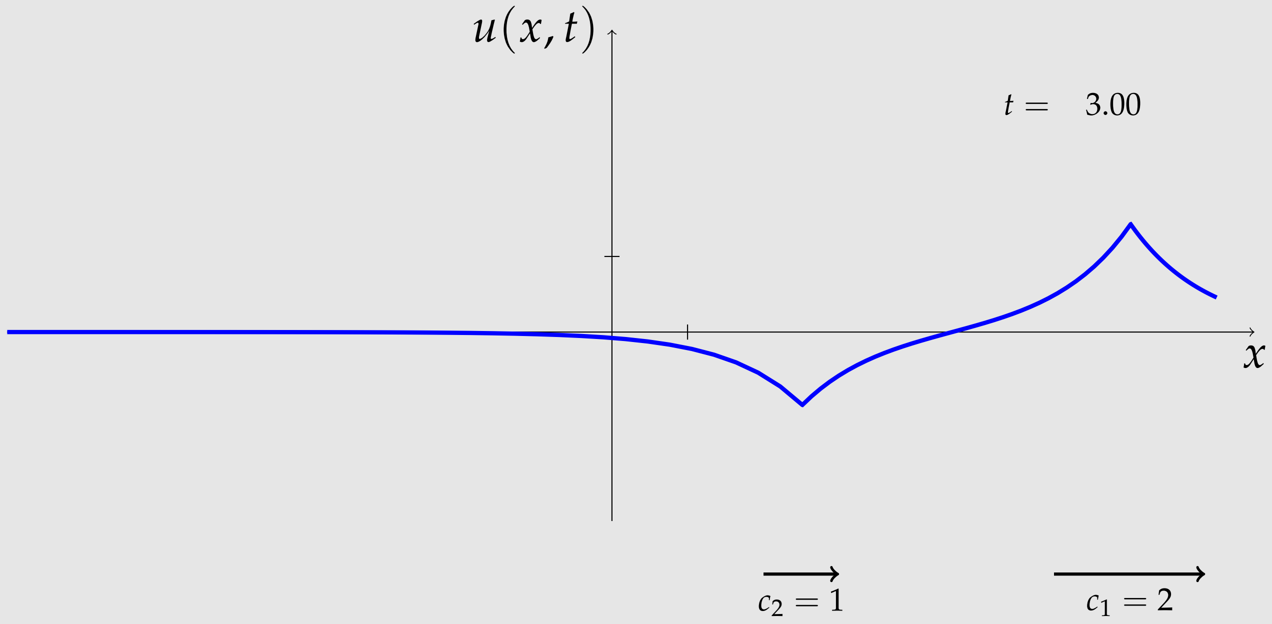

$u(x,t)$
$t = \quad 2.20$
x


$u(x,t)$
$t = \quad 2.40$
x


$u(x, t)$
$t = \quad 2.60$
x

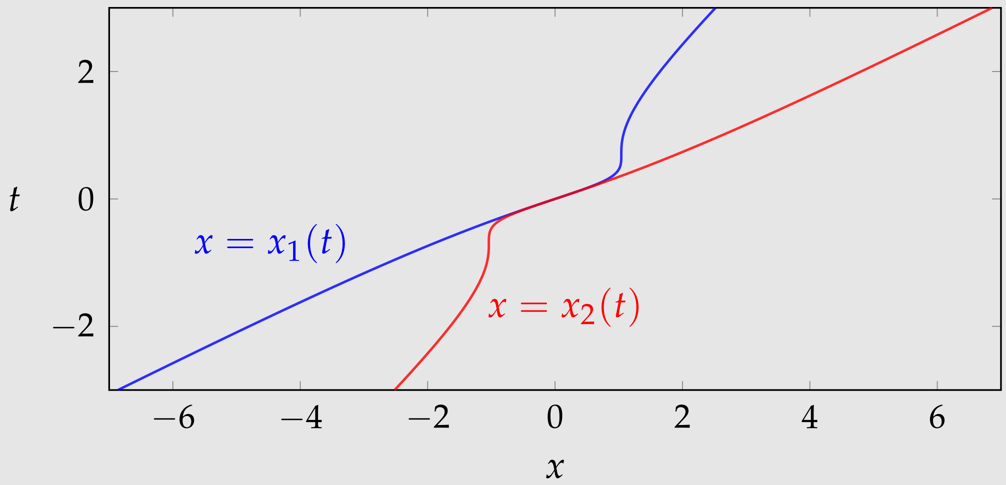


u(x, t)
t = 2.80
x



$u(x,t)$
$t = \quad 3.00$
x
$c_2 = 1$
$c_1 = 2$

Peakon trajectories $x = x_k(t)$ for the solution illustrated above:



$$x_2(t) - x_1(t) = \alpha t^4 + O(t^5) \quad \text{as } t \to 0 \qquad (\alpha > 0)$$

Unlike for Camassa–Holm,

$$E = \int_{\mathbf{R}} (u^2 + u_x^2)\, dx$$

is conserved for **all** times t (including the instant of collision).

But there is another conserved quantity

$$F = \int_{\mathbf{R}} (u^4 + 2u^2 u_x^2 - \tfrac{1}{3} u_x^4)\, dx$$

which jumps (upwards) at the collision, and then immediatedly returns to its previous value.

- Geng Chen, Robin Ming Chen, Yue Liu. "Existence and uniqueness of the global conservative weak solutions for the integrable Novikov equation." Indiana Univ. Math. J. (2018).

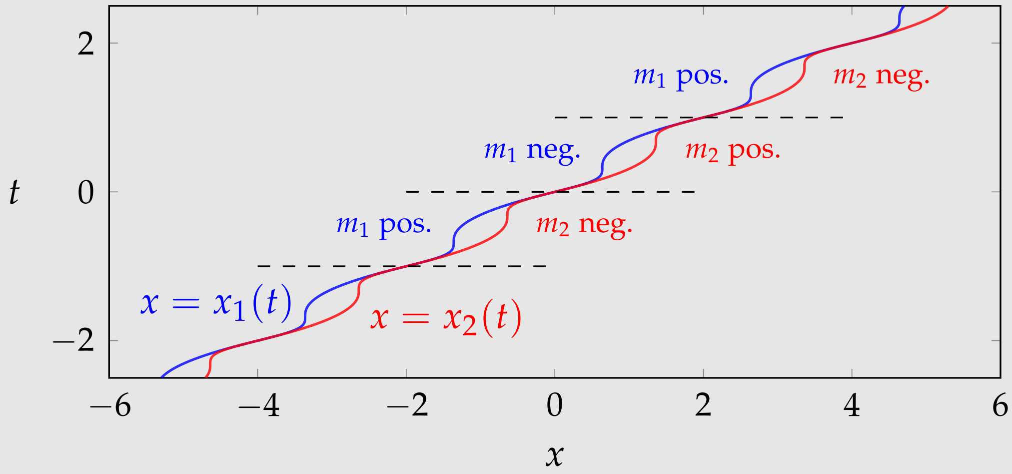
Chen–Chen–Liu use the quantity F in order to define **conservative** solutions of Novikov's equation, in the style of the conservative CH solutions of Bressan–Constantin and Bressan–Chen–Zhang.

Reasonable hypothesis (yet to be rigorously verified): The N-peakon solutions given by our explicit solution formulas are indeed conservative solutions in the sense of Chen–Chen–Liu.

Type 2. Complex eigenvalues

$$\frac{1}{\lambda_1} = \alpha + i\beta \qquad \frac{1}{\lambda_2} = \alpha - i\beta \qquad (\alpha > 0,\ \beta > 0)$$

Periodic oscillations of period $2\pi/\beta$ (two collisions per period) on top of an overall drift with velocity α. For example, with $\alpha = 2$ and $\beta = \pi$:



Type 3. Double real eigenvalue

Recall that for simple eigenvalues λ_1 and λ_2 we had the partial fractions decomposition

$$W(\lambda) = \frac{a_1}{\lambda - \lambda_1} + \frac{a_2}{\lambda - \lambda_2} \qquad (a_1 \neq 0,\, a_2 \neq 0)$$

but for a double eigenvalue $\lambda_1 = \lambda_2 = \mu$ it takes the form

$$W(\lambda) = \frac{a_1}{\lambda - \mu} + \frac{\mu a_2}{(\lambda - \mu)^2} \qquad (a_2 \neq 0)$$

instead. So we need separate solution formulas for this case (see next page).

Note: The notation a_1 and a_2 is being reused with a slightly different meaning.

We will have $a_2(t) = a_2(0)\, e^{t/\mu}$ like before, but

$$a_1(t) = \big(a_1(0) - a_2(0)\, t/\mu\big) e^{t/\mu}$$

gets a different time-dependence.

Let $\dfrac{1}{\lambda_1} = \dfrac{1}{\lambda_2} = c > 0.$

Solution formulas for this case:

$$x_1(t) = \tfrac{1}{2} \ln Q_1 \qquad m_1(t) = P_1 \sqrt{Q_1}$$
$$x_2(t) = \tfrac{1}{2} \ln Q_2 \qquad m_2(t) = P_2 \sqrt{Q_2}$$

where

$$Q_1 = \frac{c a_2^4}{4\left(a_1^2 + a_1 a_2 + \frac{1}{2} a_2^2\right)} \qquad a_1(t) = \big(a_1(0) - a_2(0)\, ct\big) e^{ct}$$
$$Q_2 = c\Big(a_1^2 - a_1 a_2 + \tfrac{1}{2} a_2^2\Big) \qquad a_2(t) = a_2(0)\, e^{ct}$$

$$P_1 = \frac{-2(a_1^2 + a_1 a_2 + \frac{1}{2} a_2^2)}{a_1 a_2^2} \qquad a_1(0) \in \mathbf{R}$$
$$P_2 = \frac{1}{a_1} \qquad a_2(0) \in \mathbf{R} \setminus \{0\}$$

For collision at the origin, take $a_1(0) = 0$ and $a_2(0) = -\sqrt{2/c}$:

$$x_1(t) = ct - \tfrac{1}{2}\ln\bigl(2(ct)^2 - 2ct + 1\bigr)$$

$$x_2(t) = ct + \tfrac{1}{2}\ln\bigl(2(ct)^2 + 2ct + 1\bigr)$$

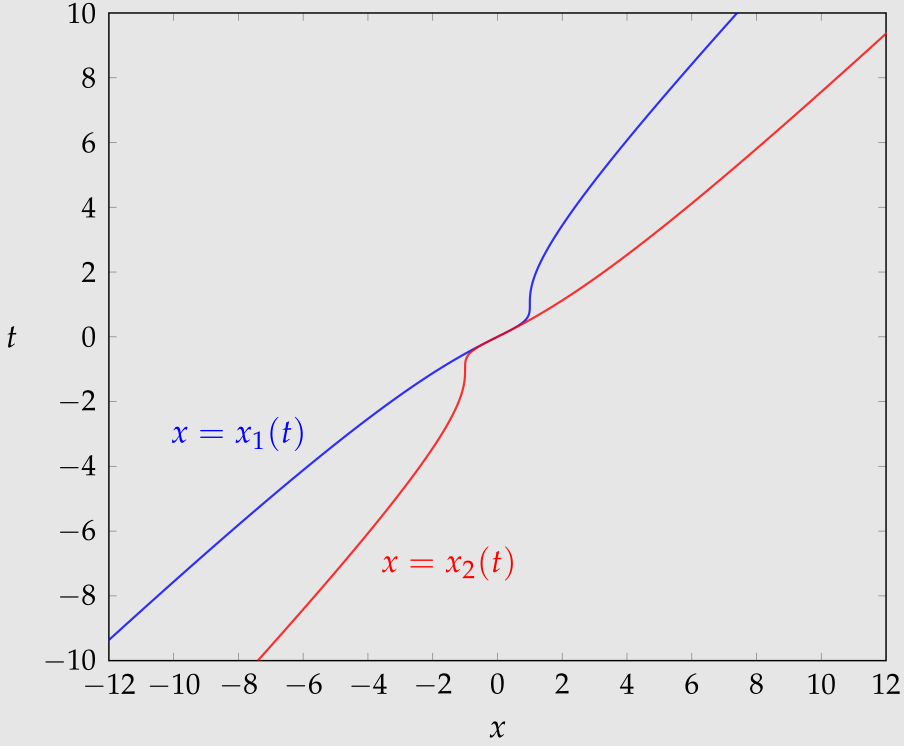
$$m_1(t) = \frac{\sqrt{2(ct)^2 - 2ct + 1}}{-t\sqrt{2c}}$$

$$m_2(t) = \frac{\sqrt{2(ct)^2 + 2ct + 1}}{t\sqrt{2c}}$$

Common asymptotic velocity c as $t \to \pm\infty$, but logarithmic separation:

$$x_2(t) - x_1(t) = \ln(2(ct)^2) + O\bigl(\tfrac{1}{t}\bigr)$$

For example, with $c = 1$:



Peakon–antipeakon solutions with $N \geq 3$

- The **formulas for pure N-peakon solutions** look similar those for the case $N = 2$ shown above, but more complicated (of course).

  They involve positive and distinct eigenvalues $\lambda_1, \ldots, \lambda_N$ of the discrete dual cubic string, and positive residues $a_1, \ldots, a_N$ in the Weyl function, with $a_k(t) = a_k(0)\, e^{t/\lambda_k}$.

  (Pure *antipeakon* solutions if all a_k are *negative* instead.)

- The **same formulas** also produce peakon–antipeakon solutions, provided that:
  - The eigenvalues λ_k are **simple** and have **positive real parts**.
  - The residues a_k are all nonzero.
  - The residues have mixed signs (if all eigenvalues are real) or form complex-conjugate pairs (if the corresponding eigenvalues do).

- Via limiting procedures we obtain **separate solution formulas** describing solutions with eigenvalues of **higher multiplicity** and/or **zero real part**.

Easiest case: **Positive simple eigenvalues**

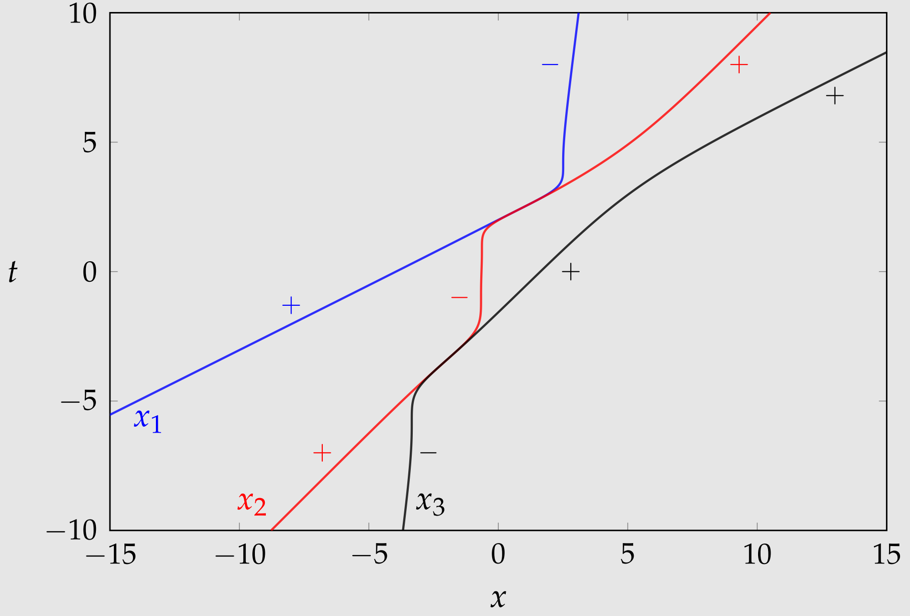
$$\frac{1}{\lambda_1} > \cdots > \frac{1}{\lambda_N} > 0$$

Asymptotic velocities & amplitudes:

$$\left.\begin{aligned} \dot{x}_k &\sim \frac{1}{\lambda_k} \\ m_k &\sim \sqrt{\frac{1}{\lambda_k}}\,\operatorname{sgn} a_k(0) \end{aligned}\right\} \quad \text{as } t \to -\infty$$

(And the same in opposite order as $t \to +\infty$.)

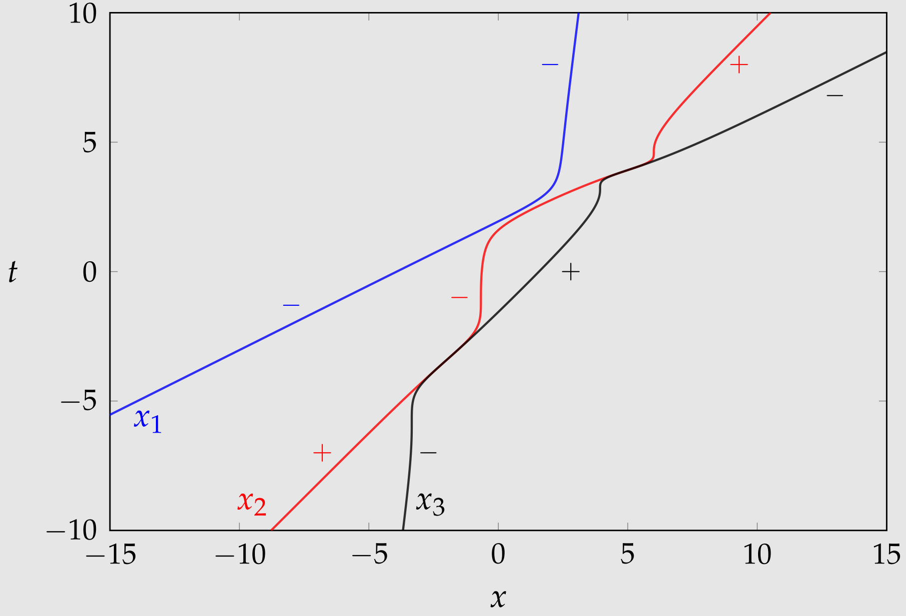
Example.



(± = sign of m_k)

$1/\lambda_1 = 2$	$a_1(0) = +1/10$	pos.
$1/\lambda_2 = 1$	$a_2(0) = +5$	pos.
$1/\lambda_3 = 1/6$	$a_3(0) = -1/4$	neg.

Example.



(± = sign of m_k)

$1/\lambda_1 = 2$ $\quad a_1(0) = -1/10$ neg.

$1/\lambda_2 = 1$ $\quad a_2(0) = +5$ pos.

$1/\lambda_3 = 1/6$ $\quad a_3(0) = -1/4$ neg.

Simple complex eigenvalues with positive real part

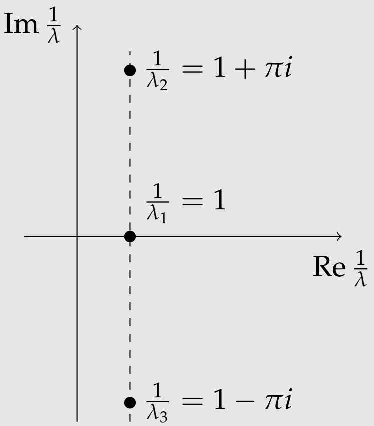
Consider the reciprocal eigenvalues $1/\lambda_k \in \mathbf{C}$.

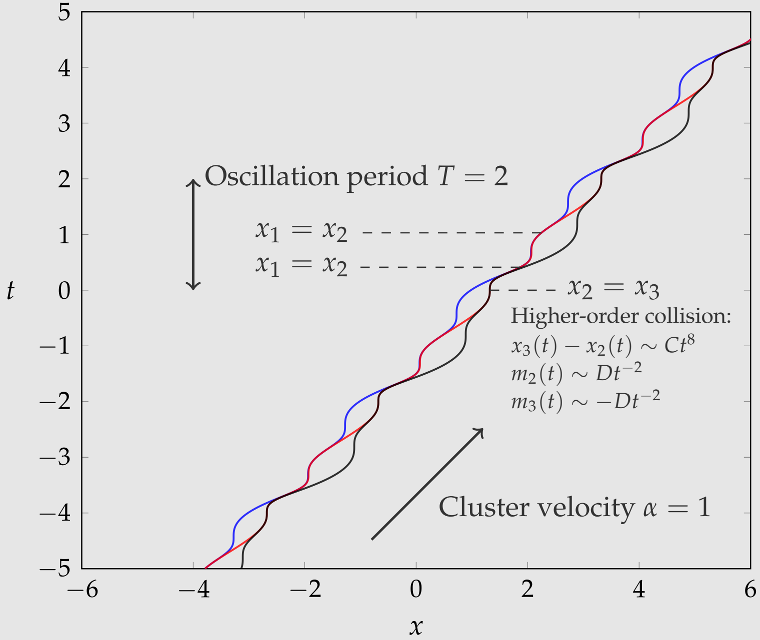
When m of these numbers have the same real part $\alpha > 0$ we get a **cluster** of m peakons and antipeakons travelling together.

- Common velocity α for the cluster as a whole.
- Periodic or quasi-periodic oscillations on top of this, with frequencies determined by the imaginary parts of $1/\lambda_k$.

$$\frac{1}{\lambda_k} = \alpha + i\beta \quad \Longrightarrow \quad a_k(t) = a_k(0)\, e^{t/\lambda_k} = a_k(0)\, e^{\alpha t} \underbrace{e^{i\beta t}}_{\text{period } T = 2\pi/\beta}$$

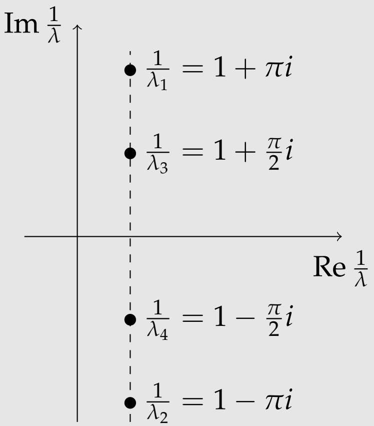
Example. Three eigenvalues with $\alpha = \mathrm{Re}(1/\lambda) = 1$.

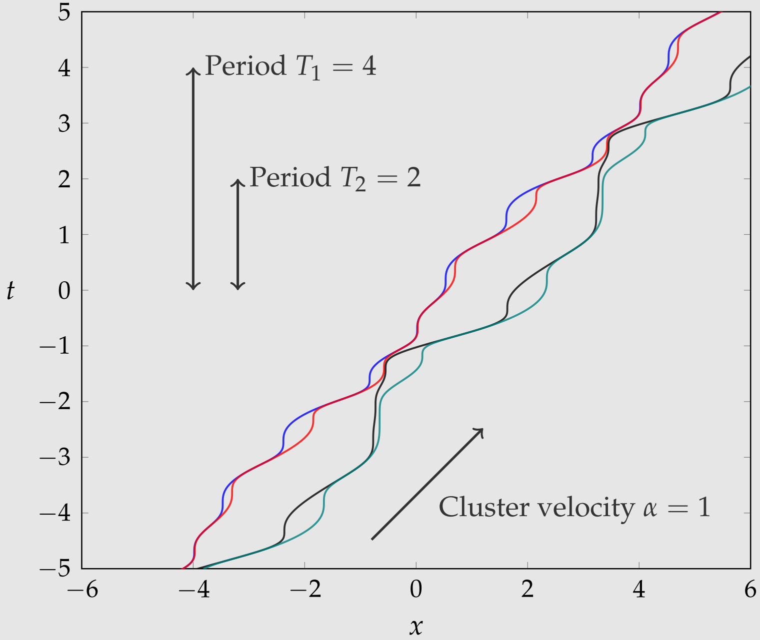




$a_1(0) = -2 \quad a_{2,3}(0) = 1 \quad$ (These particular values give a **higher-order collision** at $t = 0$.)

Example. Four eigenvalues with $\alpha = \mathrm{Re}(1/\lambda) = 1$.

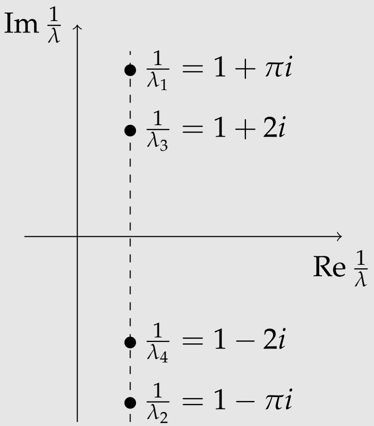


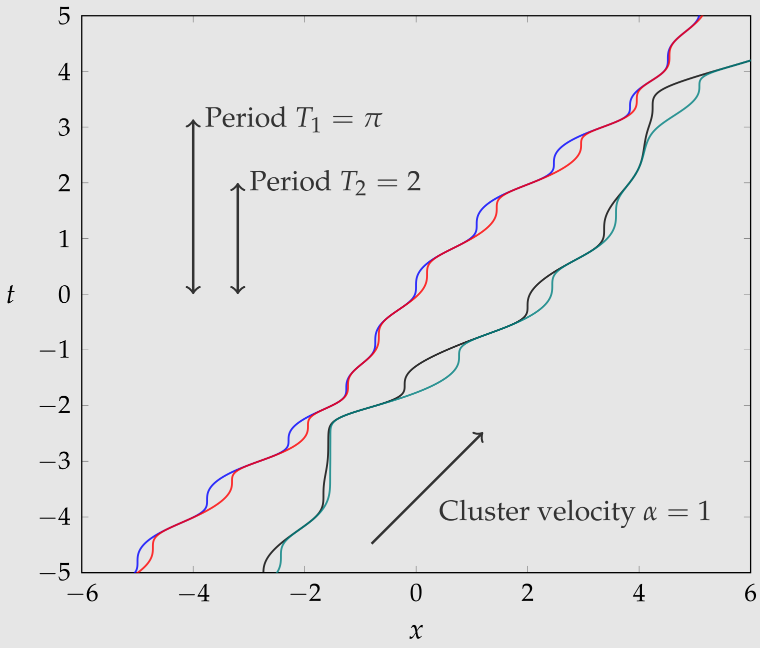


$a_{1,2}(0) = 1 \quad a_{3,4}(0) = 2$

Commensurable imaginary parts $\implies$ **periodic** (with period 4)

Example. Four eigenvalues with $\alpha = \mathrm{Re}(1/\lambda) = 1$.

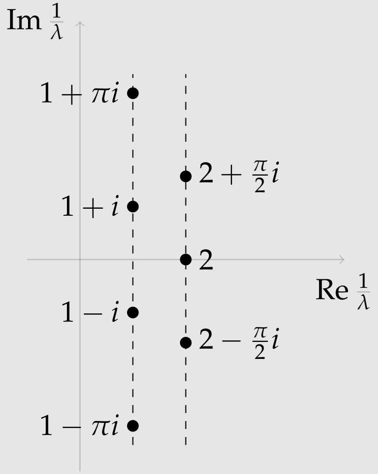


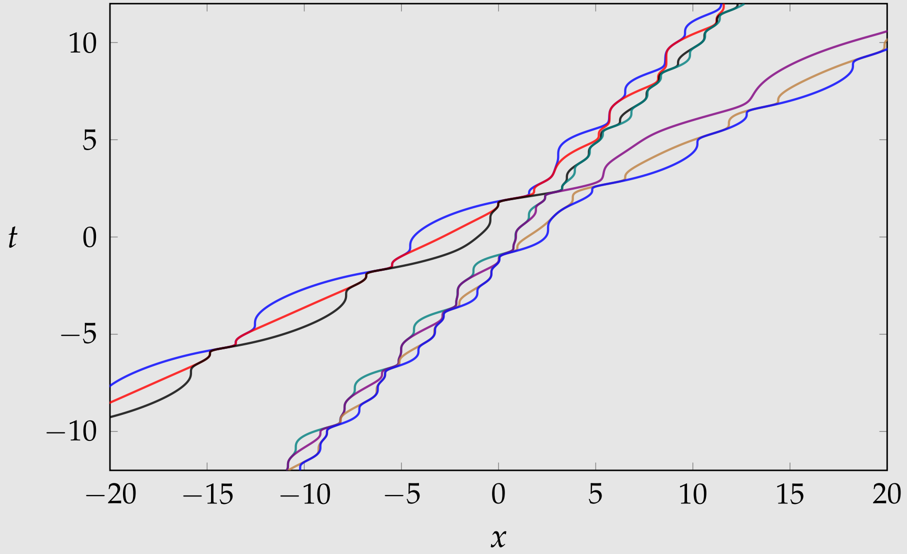


$a_{1,2}(0) = 1 \quad a_{3,4}(0) = 2$

Incommensurable imaginary parts $\Longrightarrow$ **quasiperiodic**

Example. Interaction between a 3-cluster and a 4-cluster.





All $a_k(0) = 1$.

Merely describing the precise **asymptotics** for this 7-peakon solution requires the **exact** formulas for 3-peakon and 4-peakon solutions!

Eigenvalues on the imaginary axis

All eigenvalues must satisfy $\operatorname{Re}(\lambda) \geq 0$ and $\lambda \neq 0$.

Some eigenvalues may actually have $\operatorname{Re}(\lambda) = 0$.

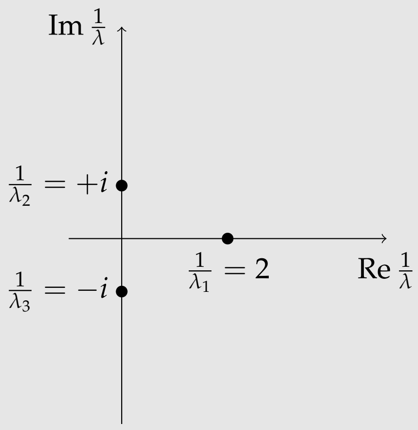
(But not *all* of them, so this can only happen when $N \geq 3$.)

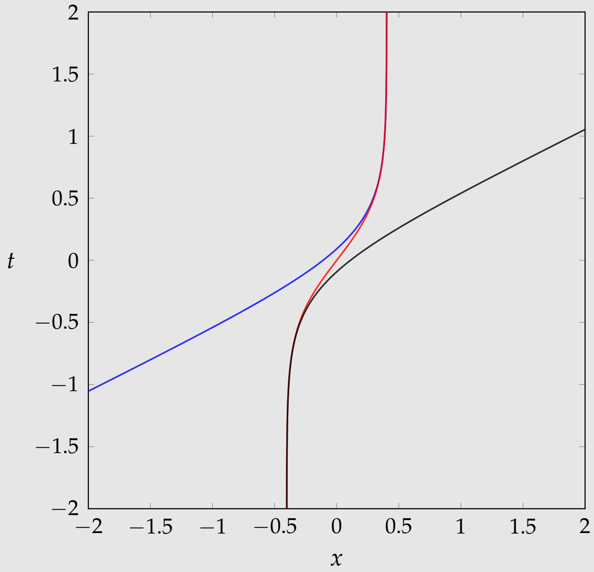
The solution formulas used so far involve divisions by $\lambda_j + \lambda_k$. But this gives division by zero when $(\lambda_j, \lambda_k) = (iL, -iL)$ is a complex-conjugate purely imaginary pair, so those formulas only work for simple eigenvalues strictly in the right half-plane.

To obtain solution formulas for cases involving simple imaginary eigenvalues:

- Write down the (old) solution formulas for perturbed spectral data where the imaginary eigenvalues λ_k have been replaced with $\varepsilon^2 + \lambda_k$ (to give them a positive real part) and their corresponding residues a_k have been replaced with εa_k.
- Let $\varepsilon \to 0$.

Example. A positive eigenvalue + a complex-conjugate imaginary pair.





$a_1(0) = \sqrt{2/5} \qquad a_{2,3}(0) = 20^{-1/4}$

No oscillations (oscillatory terms from $e^{t/\lambda_{2,3}} = e^{\pm it}$ cancel from the solution formulas).

"Asymptotic collisions":

As $t \to +\infty$ we have $x_{1,2}(t) \to D$, $m_{1,2}(t) = \pm Ce^{t/\lambda_1} + o(1)$ and $u(x_{1,2}(t)) \to 0$.

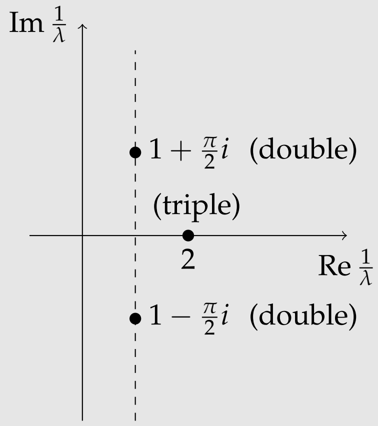
Similarly as $t \to -\infty$ for $x_{2,3}$ and $m_{2,3}$.

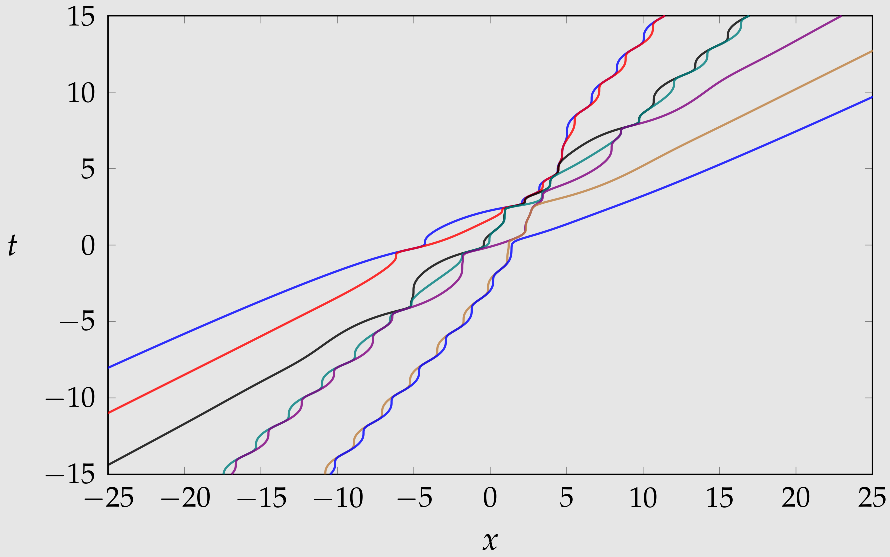
Non-simple eigenvalues

Solution formulas for non-simple eigenvalues are obtained by limiting procedures where simple eigenvalues are made to coalesce.

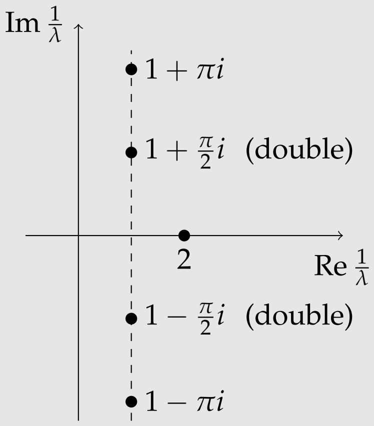
This makes it possible to have clusters together with the phenomenon of logarithmic separation as $t \to \pm\infty$.

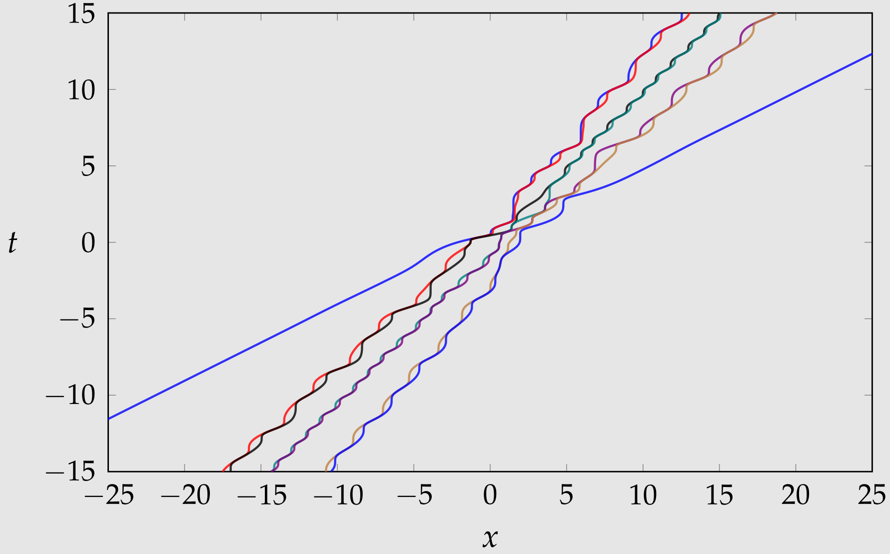
Example. 3-cluster + oscillatory 4-cluster, both with logarithmic separation





Example. Single peakon + oscillatory 6-cluster with logarithmic separation





THE END