Peakon equations related to the cubic string

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Terms to be explained:

1. **Ordinary (quadratic) string**
2. **Cubic string**
3. **Peakon** (peaked soliton)

Ordinary string = vibrating cord, linear wave equation:

$$g(x)\, v_{tt} = v_{xx}$$

$$\begin{pmatrix} v(x,t) = \text{amplitude} \\ g(x) = \text{mass density} \end{pmatrix}$$

Separation of variables:

$$\begin{cases} g(x)\, v_{tt} = v_{xx} \\ v(x,t) = X(x)T(t) \end{cases}$$

$$\frac{T''(t)}{T(t)} = \boxed{\frac{X''(x)}{g(x)\, X(x)}} = \text{constant} = \boxed{-z}$$

The string equation:

$$\boxed{X''(x) = -z\, g(x)\, X(x)}$$

Classical eigenvalue problem for $X(x)$.
Boundary conditions

$$X(-1) = 0 = X(1)$$

if attached at $x = \pm 1$. Selfadjoint.

The **cubic string** equation:

$$\boxed{X'''(x) = -z\, g(x)\, X(x)}$$

Third order.

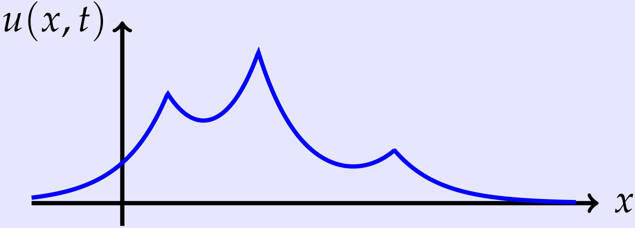
Typical boundary conditions:

$$X(-1) = X'(-1) = 0 = X(1)$$

Not selfadjoint.

Peakons = peaked solitons

Some PDEs have multisoliton solutions looking like this:

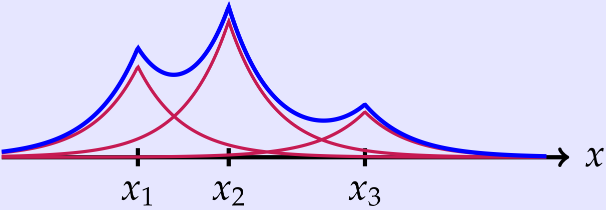


Main example: **Camassa–Holm** shallow water equation (1993)

$$m_t + m_x u + 2 m u_x = 0$$
$$m = u - u_{xx}$$

(Integrable PDE. Bi-Hamiltonian formulation, Lax pair, solitons, etc.)

$$u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x-x_i(t)|}$$



Peakon *Ansatz* satisfies CH eqn iff

$$\dot{x}_k = \sum_{i=1}^{n} m_i \, e^{-|x_k - x_i|}$$

$$\dot{m}_k = \sum_{i=1}^{n} m_k \, m_i \ \mathrm{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

(Geodesics for metric $g^{ij} = e^{-\left|x_i - x_j\right|}$.)

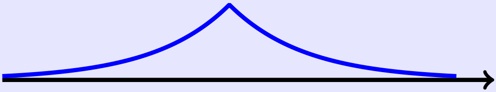
Shorthand notation:

$$\boxed{\dot{x}_k = u(x_k)}$$

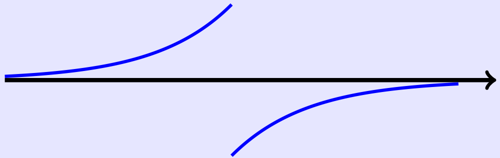
$$\boxed{\dot{m}_k = -\,m_k \left\langle u_x(x_k) \right\rangle}$$

Speed of peakon number k
= **wave height** at that point.

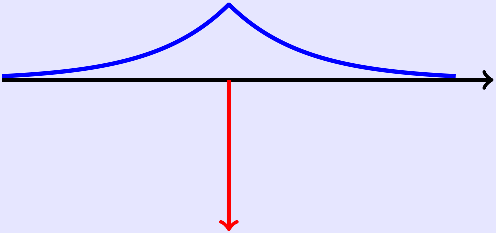
Derivatives of peakons



$$u = e^{-|x|} = \begin{cases} e^{x}, & x \le 0 \\ e^{-x}, & x \ge 0 \end{cases}$$



$$u_x = \begin{cases} +e^{x}, & x < 0 \\ \text{undefined}, & x = 0 \\ -e^{-x}, & x > 0 \end{cases}$$



$$u_{xx} = e^{-|x|} - 2\,\delta(x)$$

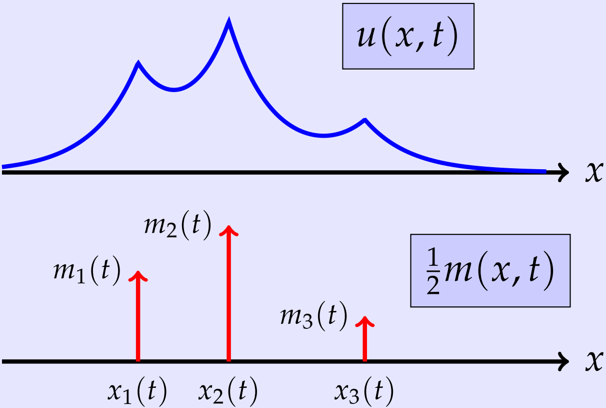
(In the sense of distributions.)

Thus, $u = e^{-|x|}$ implies that

$$m = u - u_{xx} = 2\,\delta$$

For multipeakons, $m = u - u_{xx}$ is a linear combination of Dirac deltas:

$$m(x,t) = 2\sum_{i=1}^{n} m_i(t)\,\delta\big(x - x_i(t)\big)$$


$u(x,t)$
x
$m_2(t)$
$m_1(t)$
$\frac{1}{2}m(x,t)$
$m_3(t)$
x
$x_1(t)$
$x_2(t)$
$x_3(t)$

The Camassa–Holm peakon ODEs are explicitly solvable using inverse spectral methods.

(R. Beals, D. Sattinger, J. Szmigielski 2000)

Starting point: the CH Lax pair

$$(\partial_x^2 - \tfrac{1}{4})\psi = -\tfrac{1}{2}\, z\, m\, \psi$$
$$\partial_t \psi = \dots$$

Liouville transformation:

$$\boxed{y = \tanh \frac{x}{2}} \in (-1, 1)$$

$$\psi(x) = \frac{\phi(y)}{\sqrt{1 - y^2}}$$

$$\frac{1}{2}(1 - y^2)^2 g(y) = m(x)$$

Lax equation for $\psi(x)$

$$(\partial_x^2 - \tfrac{1}{4})\psi = -\tfrac{1}{2}\, z\, m\, \psi$$

transforms into **string equation** for $\phi(y)$:

$$\boxed{\partial_y^2 \phi = -z\, g\, \phi}$$

For multipeakons:

Discrete momentum

$$m = u - u_{xx} = 2\sum m_i\,\delta_{x_i}$$

corresponds to **discrete string**

$$g = \sum g_i\,\delta_{y_i}$$

Discrete string: Point masses g_i at the points $y = y_i$, with

$$-1 < y_1 < \cdots < y_n < 1.$$

Boundary conditions:

$$\phi(\pm 1) = 0$$

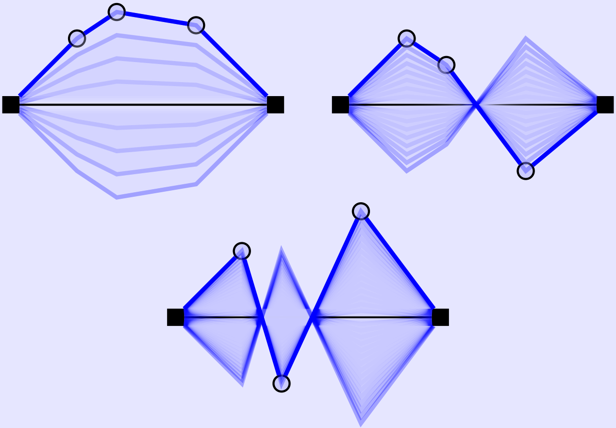
(Compatible with CH evolution.)

Such a string has n vibrational modes:

$$z = \lambda_1, \ldots, \lambda_n \in \mathbf{R}$$

The eigenfunctions $\phi_1, \ldots, \phi_n$ are piecewise linear in y.

(Since $\partial_y^2 \phi = 0$ between point masses.)



The spectral information for a given mass distribution $g(y)$ is encoded in the **Weyl function** $W(z)$:

$$\frac{W(z)}{z} = \frac{\partial_y \phi(1;z)}{z\,\phi(1;z)} = \frac{1}{2z} + \sum_{k=1}^{n} \frac{b_k}{z - \lambda_k}$$

CH time evolution for peakons induces:

$$\boxed{\dot{\lambda}_k = 0} \qquad \boxed{\dot{b}_k = b_k/\lambda_k}$$

Now solve **inverse spectral problem**: Reconstruct the point mass distribution from the known spectral data.

Map the reconstructed string data $y_k(t)$, $g_k(t)$ back to the real line.

This gives the explicit general solution $x_k(t)$, $m_k(t)$ to the CH peakon ODEs.

Many connections to classical topics in mathematics:

- Stieltjes continued fractions
- orthogonal polynomials
- the moment problem
- Padé approximation

and so on (but no time for details here).

Example. CH peakons, $n = 3$

Solution formulas on next page.

λ_1, λ_2, λ_3 are constants.

The time dependence is hidden in

$$b_k(t) = b_k(0)\, e^{t/\lambda_k}$$

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$x_2(t) = \ln \frac{\sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3}$$

$$x_3(t) = \ln(b_1 + b_2 + b_3)$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1 \lambda_2 \lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$m_2(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$m_3(t) = \frac{b_1 + b_2 + b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}$$

Next:

Overview of integrable equations with peakon (or similar) solutions

Related to ordinary string:

Camassa–Holm 1993

$$m_t + m_x u + 2mu_x = 0 \quad m = u - u_{xx}$$

Hunter–Saxton 1991

$$m_t + m_x u + 2mu_x = 0 \quad m = u_{xx}$$

(Nematic liquid crystals. Piecewise linear solutions.)

Related to cubic string:

Degasperis–Procesi 1998

$$m_t + m_x u + 3 m u_x = 0 \quad m = u - u_{xx}$$

"Linearly forced inviscid Burgers"

$$m_t + m_x u + 3 m u_x = 0 \quad m = u_{xx}$$

$$\iff \quad (u_t + u u_x)_{xx} = 0$$

(cont.)

V. Novikov 2008

$$m_t + (m_x u + 3 m u_x) u = 0 \quad m = u - u_{xx}$$

Geng–Xue 2009

$$m_t + (m_x u + 3 m u_x) v = 0 \quad m = u - u_{xx}$$
$$n_t + (n_x v + 3 n v_x) u = 0 \quad n = v - v_{xx}$$

Degasperis–Procesi equation

$$m_t + m_x u + 3 m u_x = 0$$

$$m = u - u_{xx}$$

Found by searching for integrable PDEs similar to Camassa–Holm. Later derived as a water wave model.

ODEs for DP peakons:

$$\dot{x}_k = u(x_k)$$
$$\dot{m}_k = -\,2\,m_k \langle u_x(x_k) \rangle$$

Explicitly solvable by inverse spectral methods.

(H. Lundmark, J. Szmigielski 2003, 2005)

Degasperis–Procesi Lax pair:

$$(\partial_x^3 - \partial_x)\psi = -z\, m\, \psi$$
$$\partial_t \psi = \dots$$

Liouville trf gives **cubic** string:

$$\boxed{\partial_y^3 \phi = -z\, g\, \phi}$$

$$\left[y = \tanh\frac{x}{2} \qquad \psi(x) = \frac{2\,\phi(y)}{1-y^2} \qquad \tfrac{1}{8}(1-y^2)^3 g(y) = m(x) \right]$$

Cubic string:

$$\partial_y^3 \phi(y) = -z\, g(y)\, \phi(y)$$

Boundary conditions consistent with the DP time evolution:

$$\phi(-1) = \partial_y \phi(-1) = 0 = \phi(1)$$

For peakons: **discrete** cubic string

$$g = \sum_{k=1}^{n} g_k \, \delta_{y_k}$$

Eigenvalues $z = \lambda_1, \ldots, \lambda_n$.

Real (positive) eigenvalues if all $g_k > 0$.
(*Total positivity, Krein–Gantmacher theory.*)

Weyl functions:

$$W(z) = \frac{\partial_y \phi(1; z)}{\phi(1; z)}$$

$$Z(z) = \frac{\partial_y^2 \phi(1; z)}{\phi(1; z)}$$

Time evolution of spectral data:

$$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k / \lambda_k$$

where

$$\frac{W(z)}{z} = \frac{\partial_y \phi(1;z)}{z\,\phi(1;z)} = \frac{1}{z} + \sum_{k=1}^{n} \frac{b_k}{z - \lambda_k}$$

The second Weyl function $Z(z)$ is actually redundant, since it is not independent of $W(z)$.

Proof. Let $\eta(y;z) = \phi(y;-z)$ and prime $= \partial_y$. Then $\phi''' = -zg\phi$ and $\eta''' = +zg\eta$ so that

$$0 = \eta\phi''' + \eta'''\phi = (\eta\phi'' - \eta'\phi' + \eta''\phi)'.$$

Integration over $-1 \le y \le 1$ gives

$$0 = \eta(1)\,\phi''(1) - \eta'(1)\,\phi'(1) + \eta''(1)\,\phi(1)$$

since the boundary conditions kill contributions from $y = -1$.

Division by $\eta(1)\,\phi(1)$ gives

$$Z(z) - W(-z)\,W(z) + Z(-z) = 0. \qquad \square$$

With $\frac{W(z)}{z} = \frac{1}{z} + \sum \frac{b_k}{z-\lambda_k}$ and $\frac{Z(z)}{z} = \frac{1}{2z} + \sum \frac{c_k}{z-\lambda_k}$ the formula on the previous page determines c_k in terms of b_k:

$$Z(z) - W(-z)\,W(z) + Z(-z) = 0$$

$$\operatorname*{res}_{z=\lambda_k} \left(\frac{Z(z)}{z} - W(-z)\,\frac{W(z)}{z} + \frac{Z(-z)}{z} \right) = 0$$

$$c_k - W(-\lambda_k)\,b_k + 0 = 0$$

$$c_k = \left(1 + \sum_{j=1}^{n} \frac{\lambda_k b_j}{\lambda_k + \lambda_j} \right) b_k$$

Here appears the **Cauchy kernel**

$$\frac{1}{x+y}$$

which plays an important role in the inverse spectral problem for the discrete cubic string.

Solution of inverse spectral problem for the discrete cubic string in terms of determinants of **bimoments** (with respect to the Cauchy kernel) of the spectral measure $\mu = \sum_{k=1}^{n} b_k \, \delta_{\lambda_k}$:

$$I_{ab} = \iint \frac{x^a y^b}{x+y} \, d\mu(x) \, d\mu(y)$$

Curious simultaneous approximation of Weyl functions

$$W(z) = \frac{\phi'(1;z)}{\phi(1;z)} \qquad Z(z) = \frac{\phi''(1;z)}{\phi(1;z)}$$

by rational functions with a common denominator.

(Similar to Hermite–Padé, but W & Z are not independent.)

Gives rise to theory of **Cauchy biorthogonal polynomials**.

Four-term recurrence, Riemann–Hilbert problems, random matrix models.

(M. Bertola, M. Gekhtman, J. Szmigielski 2009)

Example. DP peakons, $n = 3$

$$x_1(t) = \ln \frac{U_3}{V_2} \quad m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2}$$
$$x_2(t) = \ln \frac{U_2}{V_1} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1}$$
$$x_3(t) = \ln U_1 \quad m_3(t) = \frac{(U_1)^2}{W_1}$$

with abbreviations explained on next page.
(Time evolution $b_k(t) = b_k(0)\, e^{t/\lambda_k}$ as before.)

$$U_1 = b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$U_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3$$

$$V_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3$$

$$U_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3 \quad V_3 = \lambda_1 \lambda_2 \lambda_3 U_3$$

$$W_1 = U_1 V_1 - U_2 \qquad W_2 = U_2 V_2 - U_3 V_1$$

Vladimir Novikov's equation

Cubic nonlinearity:

$$m_t + (m_x u + 3mu_x)\, u = 0$$

$$m = u - u_{xx}$$

The Novikov peakon ODEs look slightly different:

$$\dot{x}_k = u(x_k)^2$$
$$\dot{m}_k = -m_k \left\langle u_x(x_k) \right\rangle u(x_k)$$

Speed = **square** of amplitude, so even antipeakons ($m_k < 0$) travel to the right.

Again, this system is explicitly solvable using similar methods as before.

(A. Hone, H. Lundmark, J. Szmigielski 2009)

Matrix Lax pair (A. Hone & J. P. Wang):

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zm & 1\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$

$$\frac{\partial}{\partial t}(\dots) = \dots$$

The Liouville transformation

$$\begin{aligned} y &= \tanh x \\ \phi_1(y) &= \psi_1(x)\cosh x - \psi_3(x)\sinh x \\ \phi_2(y) &= z\,\psi_2(x) \\ \phi_3(y) &= z^2\,\psi_3(x)/\cosh x \\ g(y) &= m(x)\ \cosh^3 x \\ \lambda &= -z^2 \end{aligned}$$

gives …

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} 0 & g(y) & 0 \\ 0 & 0 & g(y) \\ -\lambda & 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

(Notice that the 1 in the upper right corner is gone.)

"Dual cubic string"

For comparison, the "primal" cubic string

$$\partial_y^3 \phi = -\lambda g \phi$$

can be written as

$$\frac{\partial}{\partial y} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda g(y) & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

by letting $(\phi_1, \phi_2, \phi_3) = (\phi, \phi_y, \phi_{yy})$.

Duality:

One maps to the other under the transformation

$$\frac{d\tilde{y}}{dy} = g(y) = \frac{1}{\tilde{g}(\tilde{y})}$$

(In the discrete case, interchange of masses g_k and distances $l_k = y_{k+1} - y_k$.)

Because of the duality we can reuse results from the DP case to derive n-peakon solution formulas for Novikov's equation.

See next page for two-peakon solution. (Three-peakon formulas are too large!)

$$x_1 = \frac{1}{2} \ln \frac{\dfrac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2 \lambda_1 \lambda_2} b_1^2 b_2^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2}$$

$$x_2 = \frac{1}{2} \ln \left(\frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1 + \lambda_2} b_1 b_2 \right)$$

$$m_1 = \frac{\left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)^{1/2}}{\sqrt{\lambda_1 \lambda_2}(b_1 + b_2)}$$

$$m_2 = \frac{\left(\dfrac{b_1^2}{\lambda_1} + \dfrac{b_2^2}{\lambda_2} + \dfrac{4}{\lambda_1 + \lambda_2} b_1 b_2 \right)^{1/2}}{b_1 + b_2}$$

The constants of motion $H_1, \dots, H_n$ for the Novikov peakon ODEs have an interesting combinatorial structure (sums of minors of a certain matrix).

This investigation led to a curious by-product, the **“Canada Day Theorem”** about minors in symmetric matrices.

The Canada Day Theorem

For any symmetric $n \times n$ matrix X, the sum of the $k \times k$ **principal** minors of TX equals the sum of **all** $k \times k$ minors of X.

Here T is the lower triangular $n \times n$ matrix defined by $T_{ij} = 1 + \text{sgn}(i - j)$.

($T_{ij} = 0/1/2$ above/on/below the diagonal.)

The Geng–Xue equation

$$
\begin{aligned}
m_t + (m_x u + 3 m u_x) v &= 0 \\
n_t + (n_x v + 3 n v_x) u &= 0 \\
m &= u - u_{xx} \\
n &= v - v_{xx}
\end{aligned}
$$

Found by fiddling around with the Lax pair: try

$$\begin{pmatrix} 0 & zm & 1 \\ 0 & 0 & zn \\ 1 & 0 & 0 \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} 0 & zm & 1 \\ 0 & 0 & zm \\ 1 & 0 & 0 \end{pmatrix}$$

and look for a suitable time evolution to go along with that.

(Xianguo Geng, Bo Xue 2009)

The Geng–Xue equation has coupled peakon solutions:

$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}$$

$$v(x,t) = \sum_{i=1}^{N} n_i(t)\, e^{-|x-x_i(t)|}$$

where exactly one of m_i and n_i is zero for each i. (Disjoint support.)

The Geng–Xue peakons are governed by a system of $2N$ ODEs for the N positions and the N nonzero momenta.

Explicitly solvable using similar methods as for DP & Novikov.

(H. Lundmark, J. Szmigielski – *work in progress*)

In order to obtain sufficiently many constants of motion, one needs not only the Lax pair given by Geng & Xue, but also its “twin Lax pair” obtained by interchanging m and n, plus one extra constant of motion which does not seem to be encoded in the Lax pairs.

Two spectral measures, one from each Lax pair.

Bimoments with respect to the Cauchy kernel and both spectral measures:

$$I_{ab} = \iint \frac{x^a y^b}{x+y} \, d\mu(x) \, d\widetilde{\mu}(y)$$

Next page: GX $(2+2)$-peakon solution (interlacing case m_1, n_2, m_3, n_4).

$$\lambda_1, \lambda_2, \widetilde{\lambda}_1, \widetilde{\lambda}_2, b_\infty, F = \text{const.}$$

$$b_k(t) = b_k(0) \, e^{t/\lambda_k} \qquad \widetilde{b}_k(t) = \widetilde{b}_k(0) \, e^{t/\widetilde{\lambda}_k}$$

$$\frac{1}{2}e^{2x_1} = \frac{\dfrac{\left(\widetilde{\lambda}_1 - \widetilde{\lambda}_2\right)^2}{(\lambda_1 + \widetilde{\lambda}_1)(\lambda_1 + \widetilde{\lambda}_2)} b_1 \widetilde{b}_1 \widetilde{b}_2}{\widetilde{\lambda}_1 \widetilde{b}_1 + \widetilde{\lambda}_2 \widetilde{b}_2 - \dfrac{1}{F b_\infty}\left(\dfrac{b_1\, \widetilde{\lambda}_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \dfrac{b_1\, \widetilde{\lambda}_2 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2}\right)}$$

$$\frac{1}{2}e^{2x_2} = \frac{\dfrac{\left(\widetilde{\lambda}_1 - \widetilde{\lambda}_2\right)^2}{(\lambda_1 + \widetilde{\lambda}_1)(\lambda_1 + \widetilde{\lambda}_2)} b_1 \widetilde{b}_1 \widetilde{b}_2}{\widetilde{\lambda}_1 \widetilde{b}_1 + \widetilde{\lambda}_2 \widetilde{b}_2}$$

$$\frac{1}{2}e^{2x_3} = \frac{b_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \frac{b_1 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2}$$

$$\frac{1}{2}e^{2x_4} = \frac{b_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \frac{b_1 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2} - b_\infty(\widetilde{b}_1 + \widetilde{b}_2)$$

$$2m_1 e^{-x_1} = \frac{\lambda_1}{\widetilde{\lambda}_1 \widetilde{\lambda}_2} \left(\frac{\widetilde{\lambda}_1 \widetilde{b}_1 + \widetilde{\lambda}_2 \widetilde{b}_2}{\dfrac{b_1\, \widetilde{\lambda}_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \dfrac{b_1\, \widetilde{\lambda}_2 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2}} - \frac{1}{F b_\infty} \right)$$

$$2n_2 e^{-x_2} = \frac{\left(\widetilde{\lambda}_1 \widetilde{b}_1 + \widetilde{\lambda}_2 \widetilde{b}_2\right) \left(\dfrac{b_1\, \widetilde{\lambda}_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \dfrac{b_1\, \widetilde{\lambda}_2 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2} \right)}{\left(\widetilde{b}_1 + \widetilde{b}_2\right) \dfrac{\lambda_1 \big(\widetilde{\lambda}_1 - \widetilde{\lambda}_2\big)^2}{(\lambda_1 + \widetilde{\lambda}_1)(\lambda_1 + \widetilde{\lambda}_2)} b_1 \widetilde{b}_1 \widetilde{b}_2}$$

$$2m_3 e^{-x_3} = \frac{\widetilde{b}_1 + \widetilde{b}_2}{\dfrac{b_1\, \widetilde{\lambda}_1 \widetilde{b}_1}{\lambda_1 + \widetilde{\lambda}_1} + \dfrac{b_1\, \widetilde{\lambda}_2 \widetilde{b}_2}{\lambda_1 + \widetilde{\lambda}_2}}$$

$$2n_4 e^{-x_4} = \frac{1}{\widetilde{b}_1 + \widetilde{b}_2}$$

THE END

(Unless there is time for the *bonus material*…)

Bonus: Shockpeakons

The "b-equation"

$$m_t + m_x u + b m u_x = 0 \qquad m = u - u_{xx}$$

can be rewritten as

$$(1-\partial_x^2)u_t + (b+1-\partial_x^2)\left(\frac{u^2}{2}\right)_x + \left(\frac{3-b}{2}\,u_x^2\right)_x = 0$$

in order to rigorously define weak solutions.
DP case $b=3 \quad \Longrightarrow \quad$ no u_x^2 term!

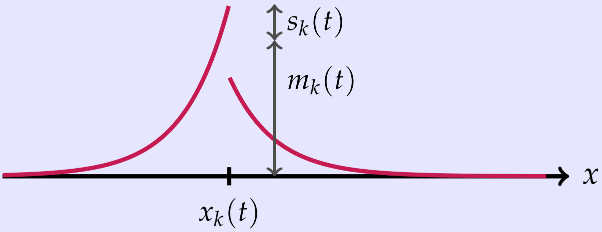
The DP equation admits very weak solutions where u itself is discontinuous, and not just the derivative u_x.

(K. H. Karlsen, G. M. Coclite 2006)

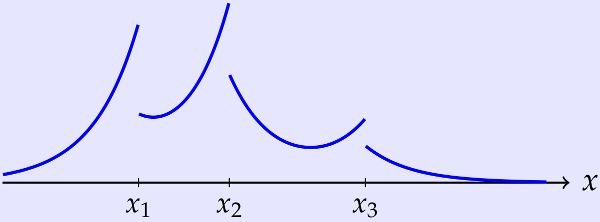
In particular: "**multi-shockpeakon**" solutions (linear combination of e^x and e^{-x} in each interval, but the pieces don't fit together).

(H. Lundmark 2007)

A single shockpeakon:



Multi-shockpeakon solution:



System of $3n$ ODEs for $\{x_k(t), m_k(t), s_k(t)\}_{k=1}^{n}$.

(Integrable or not?!?)