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Peakon solutions of the Novikov and Geng–Xue equations

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Main topic in this talk:

- **Explicit formulas** for **multipeakon solutions** of certain integrable PDEs.

Background:

- The Camassa–Holm equation.

  (And the Degasperis–Procesi equation.)

New results:

- The Novikov equation.
- The Geng–Xue equation.

(Based on joint papers with Jacek Szmigielski, Marcus Kardell, Budor Shuaib.)

The Camassa–Holm equation

$$\boxed{u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}}$$

Derived in 1993 as a model for shallow water waves.

- $u(x,t)$ is the fluid velocity in the x direction.
- $\kappa > 0$ is a constant.
- Some controversy regarding derivation and validity.

R. S. Johnson (2002):

- Vertical domain $0 \le z \le 1$.
- Then CH, with $\kappa = \frac{2}{5}\sqrt{\frac{3}{5}}$, describes what happens at $z = \frac{1}{\sqrt{2}}$.

Here we'll only consider the limiting case where $\kappa = 0$:

$$\boxed{u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}}$$

Equivalently (easier to remember):

$$\boxed{m_t + m_x u + 2mu_x = 0} \quad \text{where} \quad \boxed{m = u - u_{xx}}$$

Or rewrite the equation like this:

$$
\begin{aligned}
0 &= (u - u_{xx})_t + 3uu_x - 2u_x u_{xx} - uu_{xxx} \\
&= (1 - \partial_x^2)\left[u_t + \left(\tfrac{1}{2}u^2\right)_x\right] + \left(u^2 + \tfrac{1}{2}u_x^2\right)_x
\end{aligned}
$$

Taking $(1 - \partial_x^2)^{-1}$ to be convolution with $\frac{1}{2}e^{-|x|}$ gives

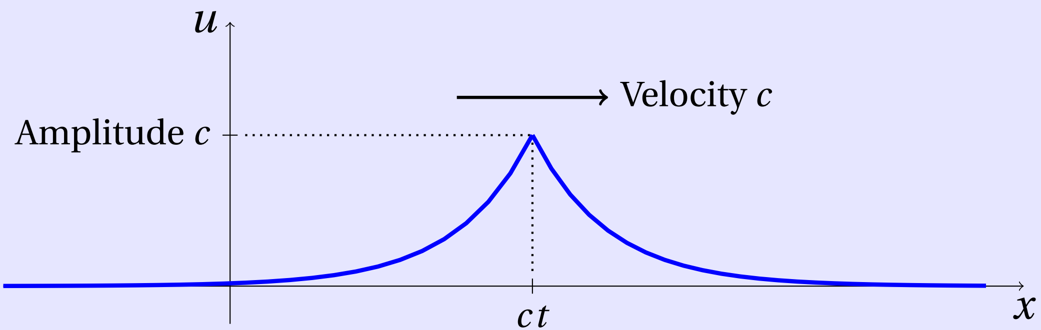
$$
\boxed{u_t + \partial_x\left[\tfrac{1}{2}u^2 + \tfrac{1}{2}e^{-|x|} * \left(u^2 + \tfrac{1}{2}u_x^2\right)\right] = 0}
$$

(Better formulation for rigorously defining weak solutions.)

The travelling wave

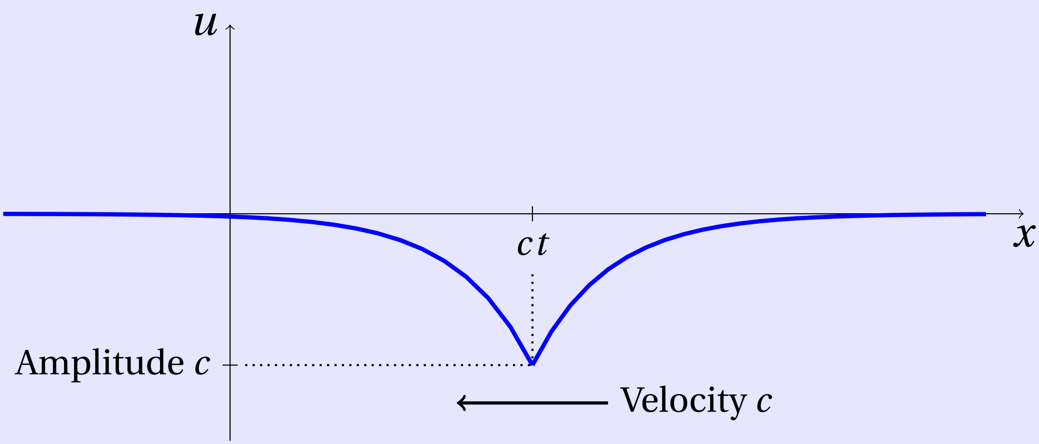
$$\boxed{u(x,t) = c\, e^{-|x-ct|}}$$

is a weak solution of the CH equation (with $\kappa = 0$).

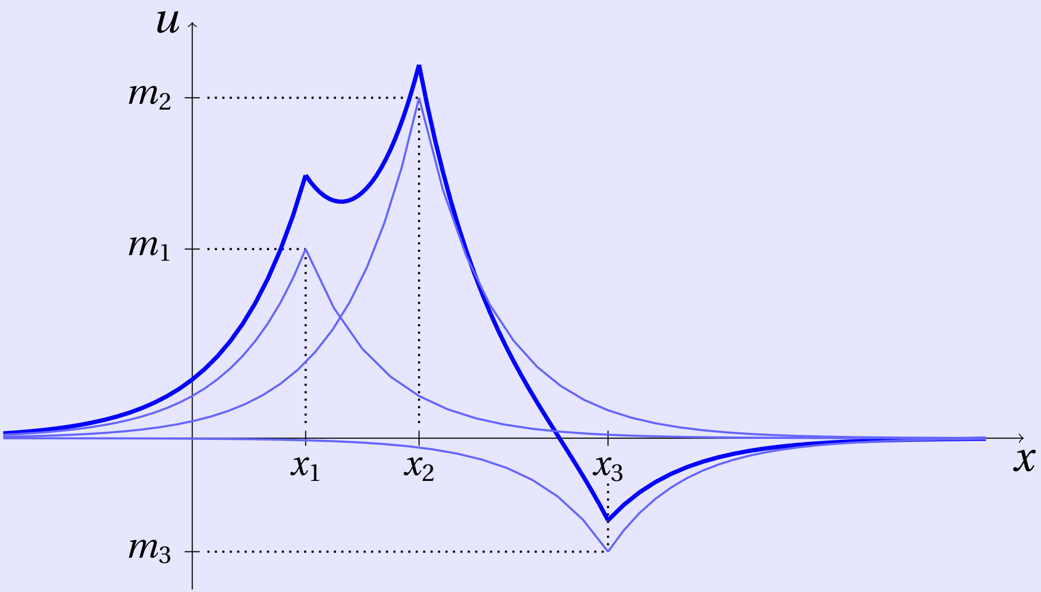


Peakon = peaked soliton

A peakon with $c < 0$ is sometimes called an **antipeakon**:



Multipeakon solutions



$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x - x_i(t)|}$$

The multipeakon *Ansatz*

$$\boxed{u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}}$$

is a weak solution of the CH equation (with $\kappa = 0$) iff

$$\boxed{\dot{x}_k = u(x_k)} \qquad \boxed{\dot{m}_k = -m_k \big\langle u_x(x_k) \big\rangle}$$

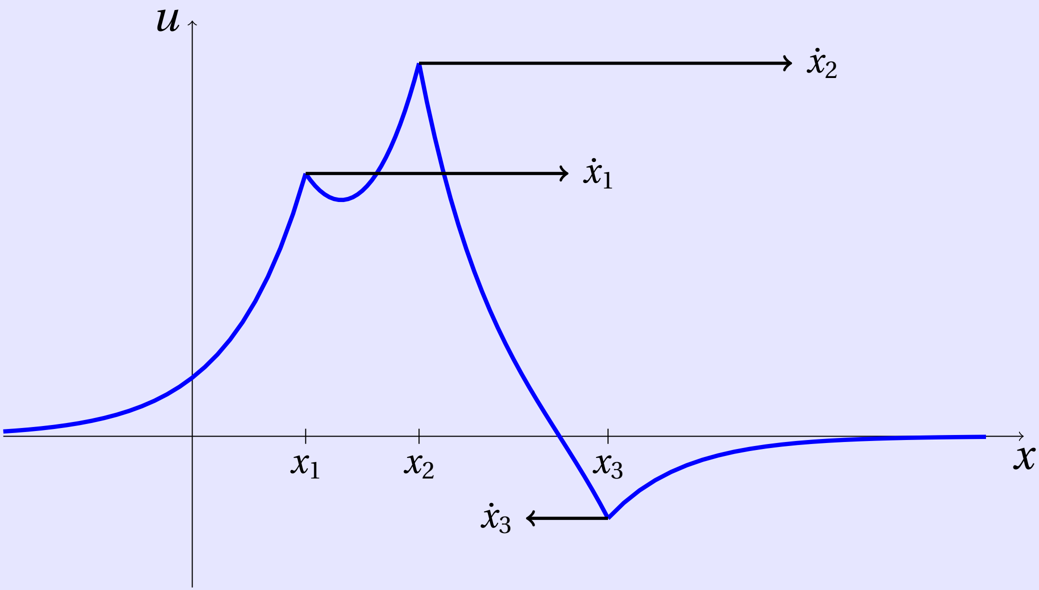
for $k = 1, \dots, N$.

So the PDE is reduced to a **finite-dimensional system of ODEs** for the positions x_k and the amplitudes m_k.

(Hamilton's equations with $H = \frac{1}{2}\sum_{i,j} m_i m_j\, e^{-|x_i - x_j|}$.)

ODE for positions

$$\boxed{\dot{x}_k = u(x_k)}$$

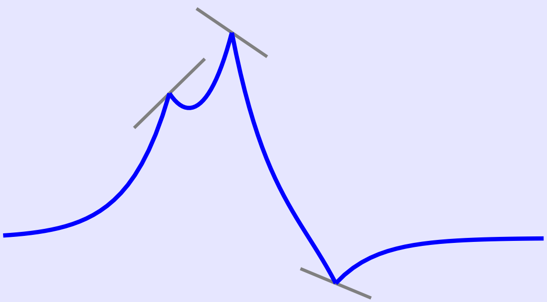


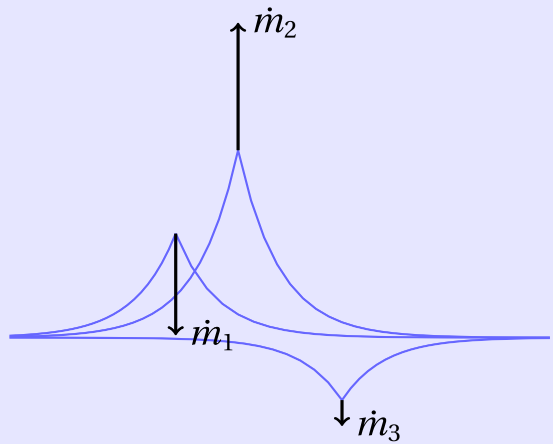
Velocity of kth peakon = elevation of the wave at at that point.

ODE for amplitudes

$$\boxed{\dot{m}_k = -m_k \langle u_x(x_k) \rangle}$$

$$\langle u_x(x_k) \rangle = \frac{u_x(x_k^-) + u_x(x_k^+)}{2} = \text{average slope of the wave at } x_k.$$





$$\boxed{\frac{d}{dt} \ln |m_k| = \frac{\dot{m}_k}{m_k} = -\langle u_x(x_k) \rangle} \qquad (\text{or } m_k(t) \equiv 0)$$

Positive/negative slope $\Longrightarrow$ $|m_k|$ decreasing/increasing.

Example. Peakon ODEs for $N = 3$, with $\boxed{E_{ij} = e^{x_i - x_j}}$:

$$\begin{aligned}
\dot{x}_1 &= m_1 + m_2 E_{12} + m_3 E_{13} \\
\dot{x}_2 &= m_1 E_{12} + m_2 + m_3 E_{23} \\
\dot{x}_3 &= m_1 E_{13} + m_2 E_{23} + m_3
\end{aligned}$$

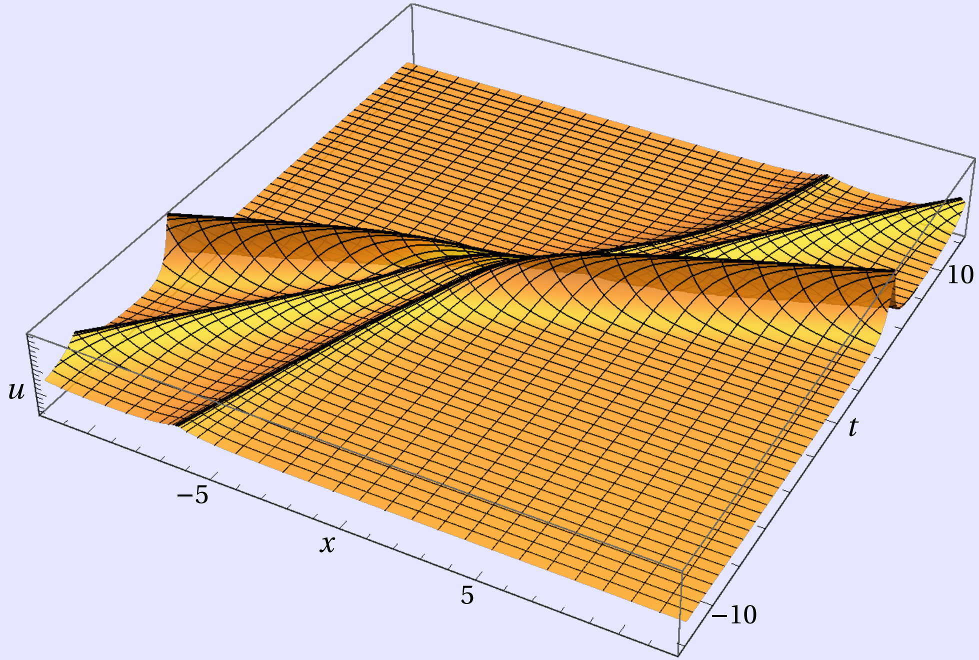
$$\begin{aligned}
\dot{m}_1 &= -m_1(m_2 E_{12} + m_3 E_{13}) \\
\dot{m}_2 &= -m_2(-m_1 E_{12} + m_3 E_{23}) \\
\dot{m}_3 &= -m_3(-m_1 E_{13} - m_2 E_{23})
\end{aligned}$$

We assume $\boxed{x_1 < x_2 < x_3}$. This is preserved at least locally in time, and in fact globally for **pure** peakon solutions (all $m_k > 0$).

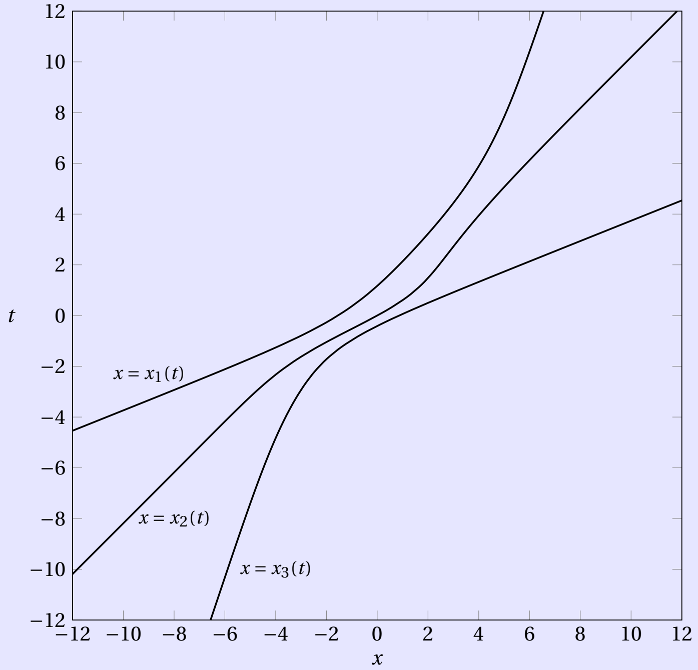
Note that $E_{ij} \approx 0$ if the peakons are far apart, so in that situation each peakon is approximately a travelling wave:

$$\dot{x}_k \approx m_k, \quad \dot{m}_k \approx 0.$$

A pure 3-peakon solution of the CH equation:



Viewing this from above, we see the positions $x = x_k(t)$:



Each peakon has its own asymptotic velocity (= amplitude).
Incoming $\dot{x}_1$ = outgoing $\dot{x}_3$. (Etc.)

Exact solution formulas (Beals, Sattinger & Szmigielski 2000):

$$x_1(t) = \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 a_1 a_2 a_3}{\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2 a_j a_k}$$

$$x_2(t) = \ln \frac{(\lambda_1-\lambda_2)^2 a_1 a_2 + (\lambda_1-\lambda_3)^2 a_1 a_3 + (\lambda_2-\lambda_3)^2 a_2 a_3}{\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3}$$

$$x_3(t) = \ln(a_1 + a_2 + a_3)$$

$$m_1(t) = \frac{\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2 a_j a_k}{\lambda_1\lambda_2\lambda_3\sum_{j<k}\lambda_j\lambda_k(\lambda_j-\lambda_k)^2 a_j a_k}$$

$$m_2(t) = \frac{\left(\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3\right)\sum_{j<k}(\lambda_j-\lambda_k)^2 a_j a_k}{(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3)\sum_{j<k}\lambda_j\lambda_k(\lambda_j-\lambda_k)^2 a_j a_k}$$

$$m_3(t) = \frac{a_1 + a_2 + a_3}{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}$$

where $\lambda_k = \text{constant}$ and $a_k(t) = a_k(0)\, e^{t/\lambda_k}$.

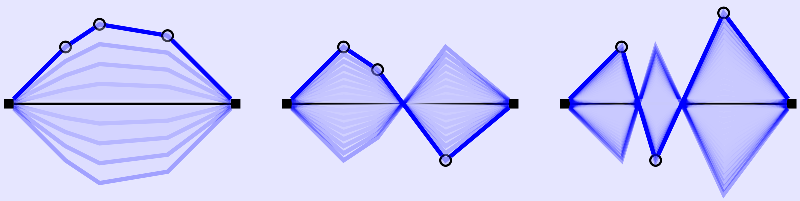
Sketch of derivation of N-peakon solution formulas

Consider a **discrete string** (N point masses connected by weightless thread) where the positions $y_k \in (-1,1)$ and weights g_k of the point masses are given by $y_k = \tanh(x_k/2)$ and $g_k = 2m_k/(1-y_k^2)$.

Separation of variables $U(y,t) = \varphi(y)\,T(\tau)$ in the usual linear wave equation $U_{\tau\tau} = g(y)U_{yy}$ for a string with mass distribution $g(y)$ gives a selfadjoint problem for the vibrational eigenmodes:

$$\boxed{-\varphi''(y) = z\,g(y)\,\varphi(y)} \qquad \boxed{\varphi(-1) = \varphi(1) = 0}$$

Here: discrete mass distribution $g(y) = \sum_{k=1}^{N} g_k\,\delta(y - y_k)$.



Eigenvalues (squared eigenfrequencies): $z = \lambda_1, \ldots, \lambda_N$, real, nonzero, distict.

Residues of Weyl function: $a_1, \ldots, a_N$, positive.

(Number of pos/neg λ_k = number of pos/neg m_k.)

Integrability magic (due to **Lax pair**):

> If $\{x_k, m_k\}_{k=1}^N$ evolve in time according to the Camassa–Holm peakon ODEs, then the associated discrete string deforms **isospectrally**, meaning that the eigenvalues λ_k stay constant. Moreover, the residues a_k satisfy $\dot{a}_k = a_k/\lambda_k$.

The **inverse spectral problem** is to **reconstruct the string** from the spectral data λ_k and a_k. The solution involves Stieltjes continued fractions, orthogonal polynomials, Padé approximation, etc.

(Stieltjes 1894, Krein 1951, Moser 1975)

The time evolution of the spectral data is known, so reconstructing the string gives the time evolution of the string variables $y_k(t)$ and $g_k(t)$, and hence of the peakon variables $x_k(t)$ and $m_k(t)$.

Done!

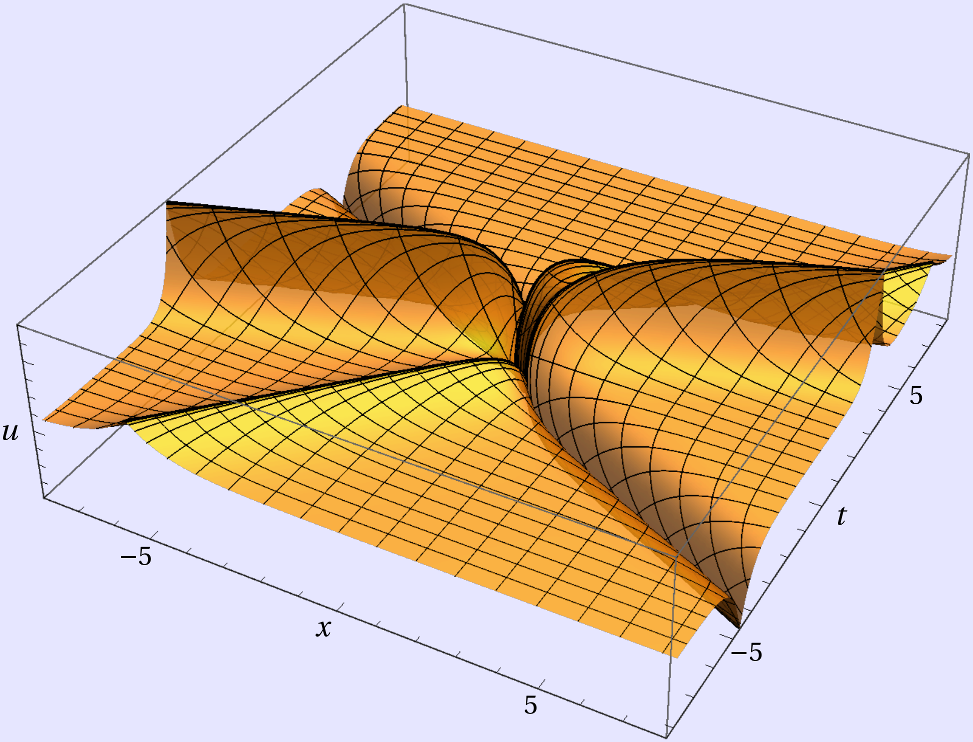
The asymptotic velocities (and amplitudes) are $\left\{\frac{1}{\lambda_k}\right\}_{k=1}^{N}$.

For example, suppose $N = 3$ and $\boxed{\frac{1}{\lambda_1}} > \frac{1}{\lambda_2} > \frac{1}{\lambda_3}$. Then, as $\boxed{t \to +\infty}$:

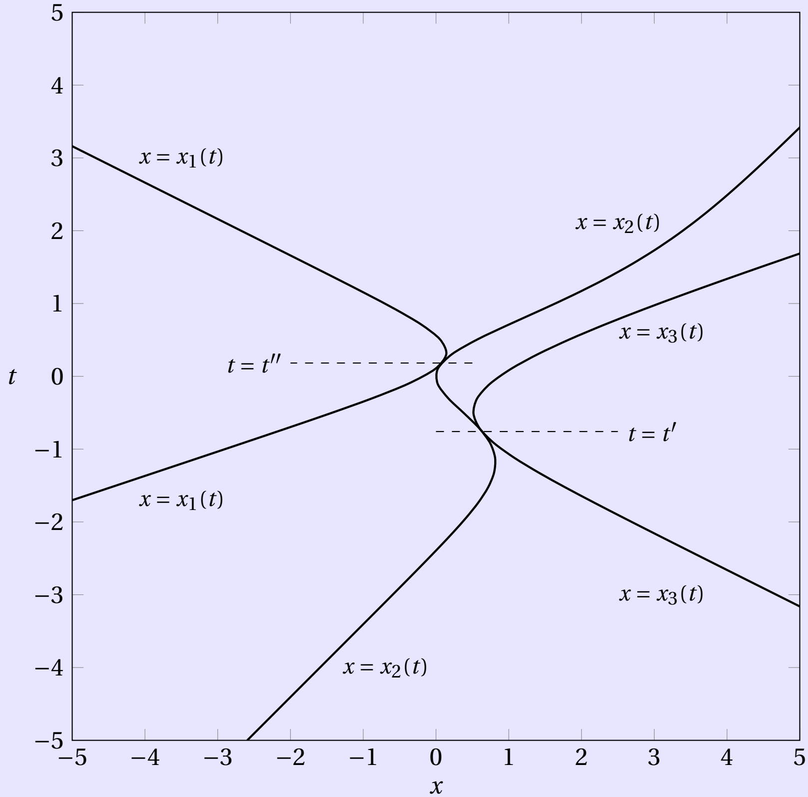
$$\begin{aligned}
\boxed{x_3(t)} &= \ln\Big(a_1(0)\,\boxed{e^{t/\lambda_1}} + a_2(0)\,e^{t/\lambda_2} + a_3(0)\,e^{t/\lambda_3}\Big) \\
&= \ln\Big(a_1(0)\,e^{t/\lambda_1}\big(1 + o(1)\big)\Big) \\
&= \boxed{\frac{t}{\lambda_1}} + \ln a_1(0) + o(1)
\end{aligned}$$

$$\begin{aligned}
\boxed{m_3(t)} &= \frac{a_1(0)\,\boxed{e^{t/\lambda_1}} + a_2(0)\,e^{t/\lambda_2} + a_3(0)\,e^{t/\lambda_3}}{\lambda_1 a_1(0)\,\boxed{e^{t/\lambda_1}} + \lambda_2 a_2(0)\,e^{t/\lambda_2} + \lambda_3 a_3(0)\,e^{t/\lambda_3}} \\
&= \frac{a_1(0)\,e^{t/\lambda_1}\big(1 + o(1)\big)}{\lambda_1 a_1(0)\,e^{t/\lambda_1}\big(1 + o(1)\big)} \\
&= \boxed{\frac{1}{\lambda_1}} + o(1)
\end{aligned}$$

A mixed CH solution (two peakons, one antipeakon):

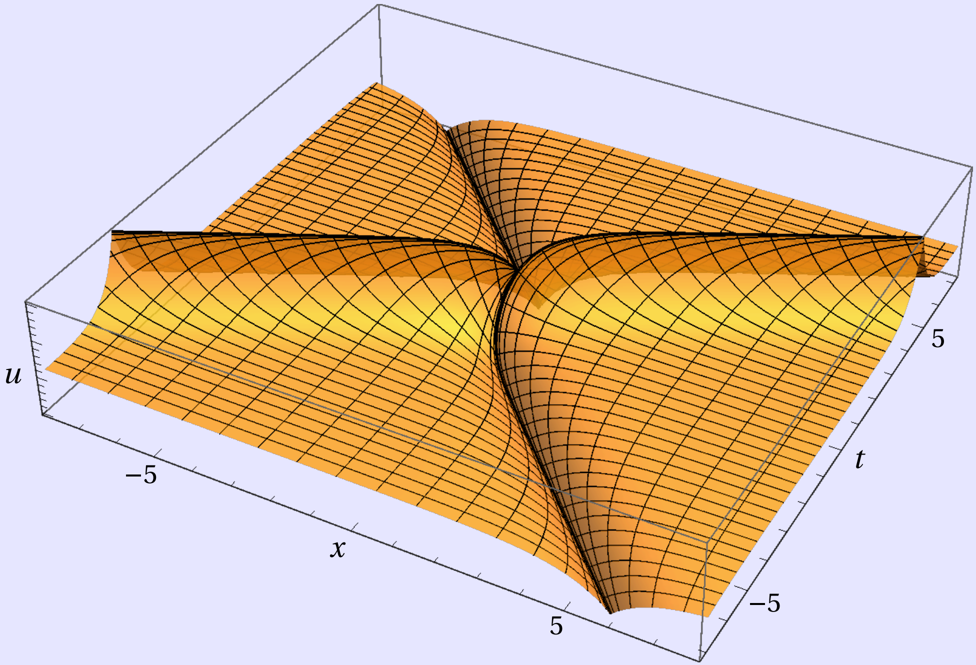


$\frac{1}{\lambda_1} > \frac{1}{\lambda_2} > 0 > \frac{1}{\lambda_3}$



Collisions at $t = t'$ and $t = t''$. Derivative u_x blows up, u doesn't.

Easier to see with just one peakon and one antipeakon:



At the collision, there is only *one* peakon: $u(x_0, t_0) = m_0\, e^{-|x-x_0|}$.

$m_0 = \lim\limits_{t \nearrow t_0} \big(m_1(t) + m_2(t)\big)$, where $m_1(t) \to +\infty$ and $m_2(t) \to -\infty$.

Continuation past the collision is **not unique**.

Above:

- **Conservative** solution (Constantin & Escher 1998).

  Peakon & antipeakon reappear.

  The energy $E(t) = \frac{1}{2}\int_{\mathbf{R}}(u^2 + u_x^2)\,dx$ drops at the instant of collision, then immediately returns to its previous value.

Also possible:

- **Dissipative** solution (Bressan & Constantin 2007).

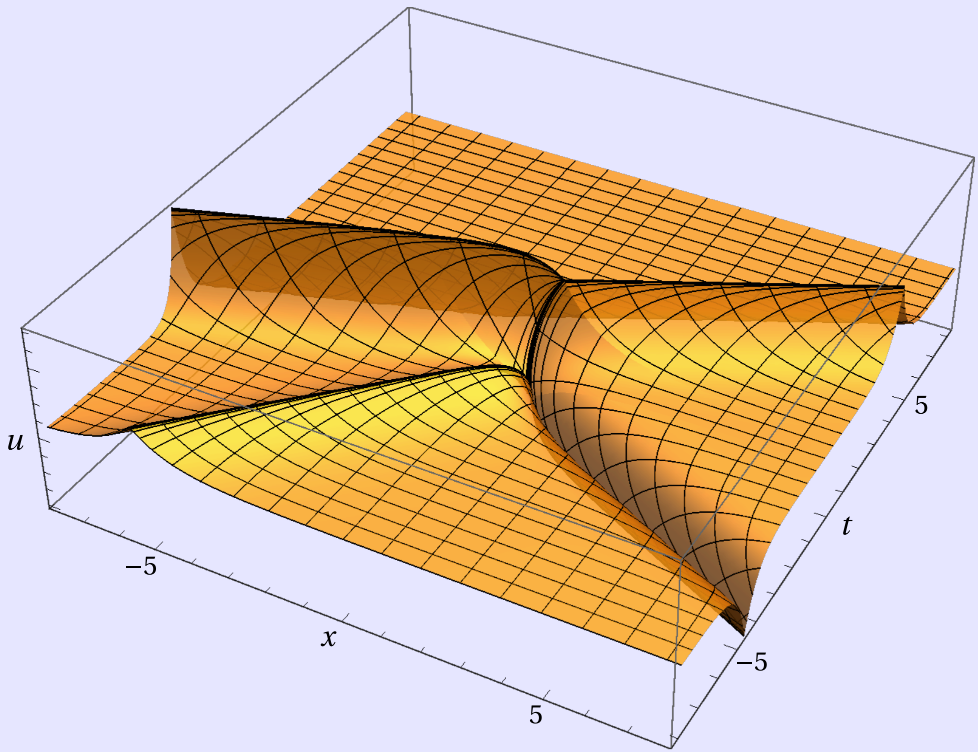
  Peakons stay merged. $E(t)$ only drops, never increases.

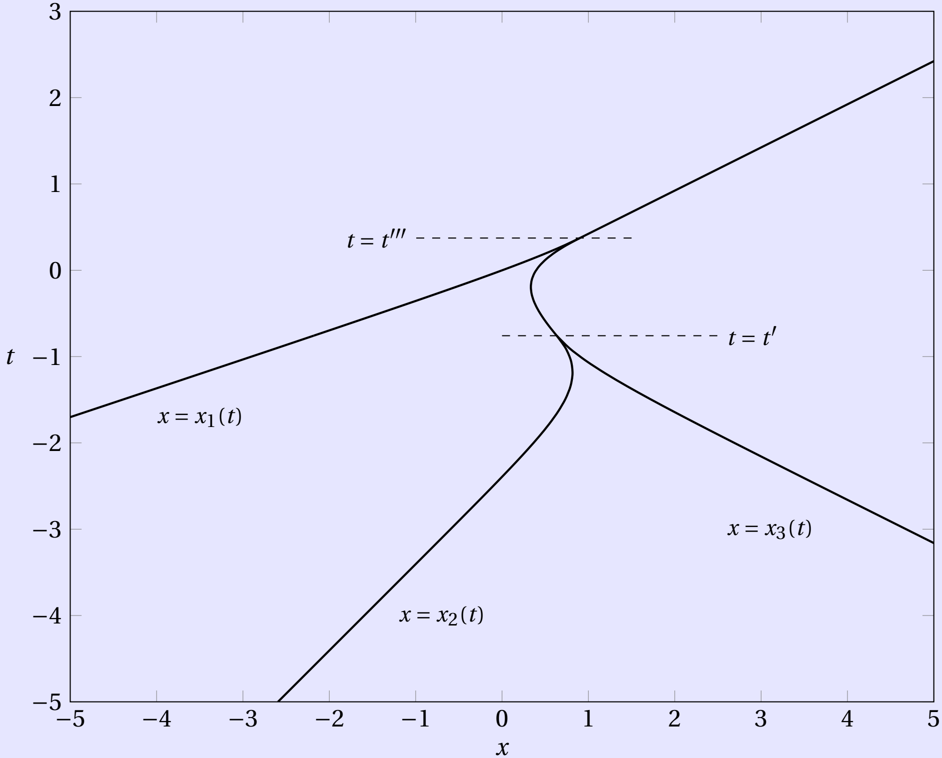
- α**-dissipative** solution (Grunert & Holden 2016).

  Peakon & antipeakon reappear, but a fraction $0 < \alpha < 1$ of the energy concentrated at the collision is lost).

  (Conservative if $\alpha = 0$, dissipative if $\alpha = 1$.)

Dissipative continuation of the solution starting out as above:





Peakons merge at collisions.

Some other integrable PDEs with peakon solutions:

$$\boxed{m_t + m_x u + 2 m u_x = 0}$$ Camassa–Holm (1993)

$$\boxed{m_t + m_x u + 3 m u_x = 0}$$ Degasperis–Procesi (1998)

$$\boxed{m_t + (m_x u + 3 m u_x) u = 0}$$ V. Novikov (2008)

$$\boxed{\begin{gathered} m_t + (m_x u + 3 m u_x) v = 0 \\ n_t + (n_x v + 3 n v_x) u = 0 \end{gathered}}$$ Geng–Xue (2009)

where $\boxed{m = u - u_{xx}}$ and $\boxed{n = v - v_{xx}}$

Multipeakon solutions have the form $u = \sum_{i=1}^{N} m_i\, e^{-|x-x_i|}$ for all these equations, but the ODEs differ:

CH	$\boxed{\dot{x}_k = u(x_k)}$	$\boxed{\dot{m}_k = -m_k \langle u_x(x_k)\rangle}$
DP	$\boxed{\dot{x}_k = u(x_k)}$	$\boxed{\dot{m}_k = -2m_k \langle u_x(x_k)\rangle}$
Novikov	$\boxed{\dot{x}_k = u(x_k)^2}$	$\boxed{\dot{m}_k = -m_k\, u(x_k) \langle u_x(x_k)\rangle}$

GX

$$\boxed{\dot{x}_k = u(x_k)\, v(x_k)} \quad \boxed{\dot{m}_k = m_k \big(u(x_k) \langle v_x(x_k)\rangle - 2\langle u_x(x_k)\rangle\, v(x_k)\big)}$$

$$\boxed{\dot{y}_k = u(y_k)\, v(y_k)} \quad \boxed{\dot{n}_k = n_k \big(\langle u_x(y_k)\rangle\, v(y_k) - 2u(y_k) \langle v_x(y_k)\rangle\big)}$$

where $u = \sum_{i=1}^{N_1} m_i\, e^{-|x-x_i|}$, $v = \sum_{j=1}^{N_2} n_j\, e^{-|x-y_j|}$, $x_i \neq y_j$.

Degasperis–Procesi peakons

Solution formulas (Lundmark & Szmigielski 2005).

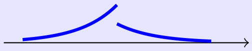
Discrete **cubic** string instead of ordinary string:

$$-\varphi'''(y) = z\, g(y)\, \varphi(y)$$
$$\varphi(-1) = \varphi'(-1) = 0 \qquad \varphi(1) = 0$$

Positive & simple eigenvalues for **pure** peakon solutions (Gantmacher–Krein theory of oscillatory kernels). Eigenvalues can be complex in general.

Cauchy biorthogonal polynomials (Bertola, Gekhtman & Szmigielski 2009).

Discontinuous **shockpeakon** solutions form at peakon–antipeakon collisions (Lundmark 2007).



Open problem: Are the shockpeakon ODEs integrable?

Novikov peakons

Solution formulas (Hone, Lundmark & Szmigielski 2009).

Dual discrete cubic string (swap distances and masses).

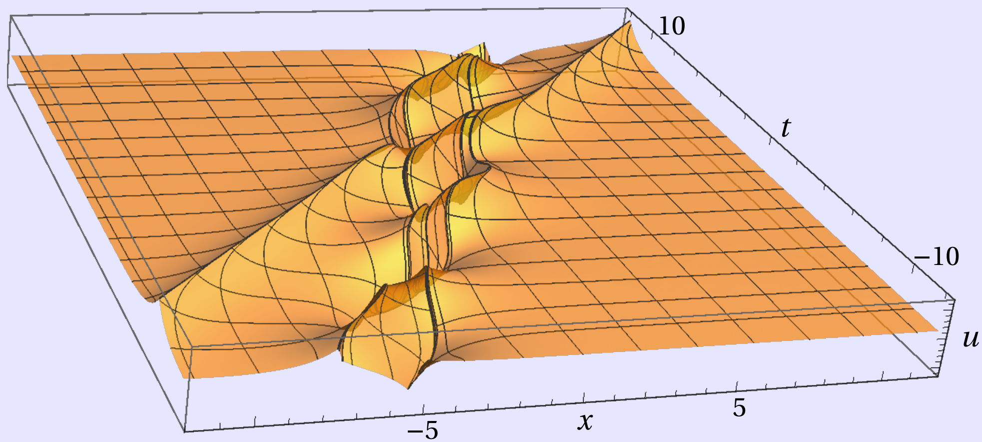
Positive & simple eigenvalues for pure peakon solutions.

Can have complex and/or multiple eigenvalues for mixed peakon–antipeakon solutions. The real part is always nonnegative.

Because of $\dot{x}_k = u(x_k)^2$, both peakons and antipeakons move to the right. But they still collide. At collisions, u stays continuous, and $E(t) = \frac{1}{2}\int_{\mathbf{R}}(u^2 + u_x^2)\,dx$ is preserved.

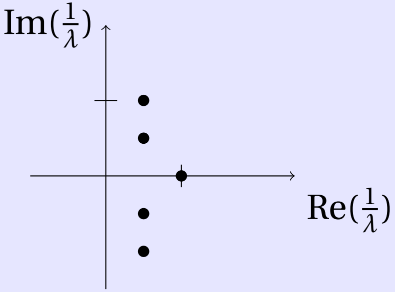
Conservative vs. dissipative distinction through a conserved (or not) quantity of degree 4. (Chen, Chen & Liu, preprint 2015.)

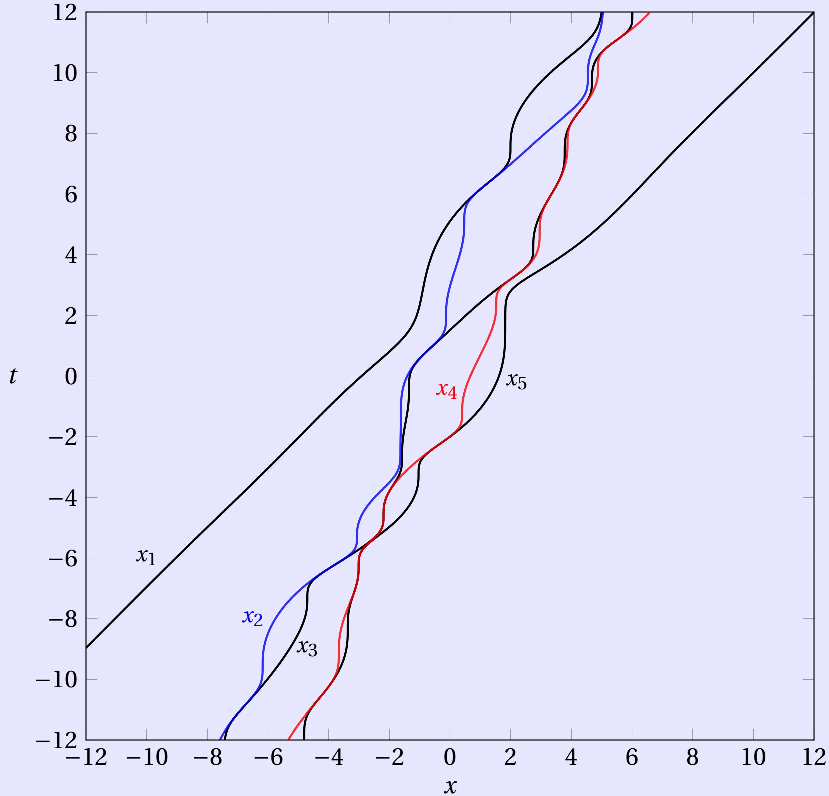
Example. A conservative solution of the Novikov equation:



Three peakons, two antipeakons.

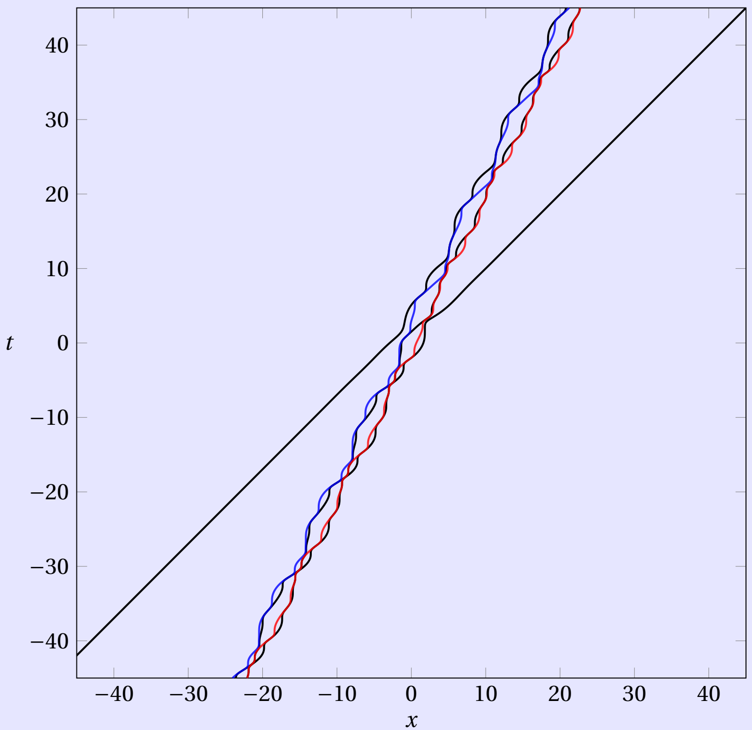
One real eigenvalue. Two complex-conjugate eigenvalue pairs with common value of $\mathrm{Re}(1/\lambda)$.





$x_{k+1}(t) - x_k(t) = O\big((t - t_0)^4\big)$ at a typical collision. Powers $4k$ can also occur.
(The power is always 2 for Camassa–Holm collisions.)

Zoom out:



Asymptotically, one solitary peakon (velocity $\frac{1}{\lambda} = 1$) and a four-peakon cluster with overall drift velocity $\mathrm{Re}(\frac{1}{\lambda}) = \frac{1}{2}$ and two frequencies $\mathrm{Im}(\frac{1}{\lambda}) \in \{\frac{1}{2}, 1\}$.

Lots of possibilities:

- Arbitrarily many clusters, each with arbitrarily many peakons.
- Frequencies commensurable or not.

  (Periodic or quasi-periodic.)
- Eigenvalues of higher multiplicity.

  (Peakons separate at logarithmic rate as $t \to \pm\infty$.)
- Purely imaginary eigenvalues.

  (Peakons slow to a halt, amplitude tends to zero.)

(Kardell & Lundmark, in preparation.)

Geng–Xue peakons

Peakons in u and v must be **non-overlapping**.

First: Solution formulas for **interlacing** $K+K$ case

$$\boxed{u(x,t)} = \sum_{i=1}^{K} m_i(t)\, e^{-|x-x_i(t)|}$$

$$\boxed{v(x,t)} = \sum_{i=1}^{K} n_i(t)\, e^{-|x-y_i(t)|}$$

where

$$\boxed{x_1} < \boxed{y_1} < \boxed{x_2} < \boxed{y_2} < \cdots < \boxed{x_K} < \boxed{y_K}$$

(Lundmark & Szmigielski 2016, 2017.)

The GX equation has **two** Lax pairs (swap u and v), leading to two spectral problems of cubic string type.

The solution formulas for the $K+K$ interlacing case contain two sets of constant eigenvalues

$$\{\lambda_i\}_{i=1}^{K}, \qquad \{\mu_j\}_{j=1}^{K-1}$$

with associated residues $\{a_i\}_{i=1}^{K}$ and $\{b_j\}_{j=1}^{K-1}$ such that

$$a_i(t) = a_i(0)\, e^{t/\lambda_i}, \qquad b_j(t) = b_j(0)\, e^{t/\mu_j},$$

plus two additional constants C and D also coming from the spectral problems.

($4K$ parameters in total, as it should be.)

Example. The solution formulas for the $3+3$ interlacing case:

$$X_1 = \tfrac{1}{2} e^{2x_1} = \frac{\mathrm{J}^{00}_{32}}{\mathrm{J}^{11}_{21} + C\,\mathrm{J}^{10}_{22}} \qquad Y_1 = \tfrac{1}{2} e^{2y_1} = \frac{\mathrm{J}^{00}_{32}}{\mathrm{J}^{11}_{21}}$$

$$X_2 = \tfrac{1}{2} e^{2x_2} = \frac{\mathrm{J}^{00}_{22}}{\mathrm{J}^{11}_{11}} \qquad Y_2 = \tfrac{1}{2} e^{2y_2} = \frac{\mathrm{J}^{00}_{21}}{\mathrm{J}^{11}_{10}}$$

$$X_3 = \tfrac{1}{2} e^{2x_3} = \mathrm{J}^{00}_{11} \qquad Y_3 = \tfrac{1}{2} e^{2y_3} = \mathrm{J}^{00}_{11} + D\,\mathrm{J}^{00}_{10}$$

and

$$Q_1 = 2m_1 e^{-x_1} = \frac{\mu_1\mu_2}{\lambda_1\lambda_2\lambda_3}\left(\frac{\mathrm{J}^{11}_{21}}{\mathrm{J}^{10}_{22}} + C\right) \qquad P_1 = 2n_1 e^{-y_1} = \frac{\mathrm{J}^{11}_{21}\,\mathrm{J}^{10}_{22}}{\mathrm{J}^{01}_{21}\,\mathrm{J}^{01}_{32}}$$

$$Q_2 = 2m_2 e^{-x_2} = \frac{\mathrm{J}^{11}_{11}\,\mathrm{J}^{01}_{21}}{\mathrm{J}^{10}_{11}\,\mathrm{J}^{10}_{22}} \qquad P_2 = 2n_2 e^{-y_2} = \frac{\mathrm{J}^{11}_{10}\,\mathrm{J}^{10}_{11}}{\mathrm{J}^{01}_{10}\,\mathrm{J}^{01}_{21}}$$

$$Q_3 = 2m_3 e^{-x_3} = \frac{\mathrm{J}^{01}_{10}}{\mathrm{J}^{10}_{11}} \qquad P_3 = 2n_3 e^{-y_3} = \frac{1}{\mathrm{J}^{00}_{10}}$$

where, for instance,

$$\begin{aligned}\mathrm{J}^{01}_{21} = {} & \frac{(\lambda_1-\lambda_2)^2\,\mu_1}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}\,a_1a_2b_1 + \frac{(\lambda_1-\lambda_3)^2\,\mu_1}{(\lambda_1+\mu_1)(\lambda_3+\mu_1)}\,a_1a_3b_1 + \frac{(\lambda_2-\lambda_3)^2\,\mu_1}{(\lambda_2+\mu_1)(\lambda_3+\mu_1)}\,a_2a_3b_1 \\ & + \frac{(\lambda_1-\lambda_2)^2\,\mu_2}{(\lambda_1+\mu_2)(\lambda_2+\mu_2)}\,a_1a_2b_2 + \frac{(\lambda_1-\lambda_3)^2\,\mu_2}{(\lambda_1+\mu_2)(\lambda_3+\mu_2)}\,a_1a_3b_2 + \frac{(\lambda_2-\lambda_3)^2\,\mu_2}{(\lambda_2+\mu_2)(\lambda_3+\mu_2)}\,a_2a_3b_2\end{aligned}$$

Collisions lead to shockpeakon formation, so here we assume **pure** peakon solutions (no antipeakons).

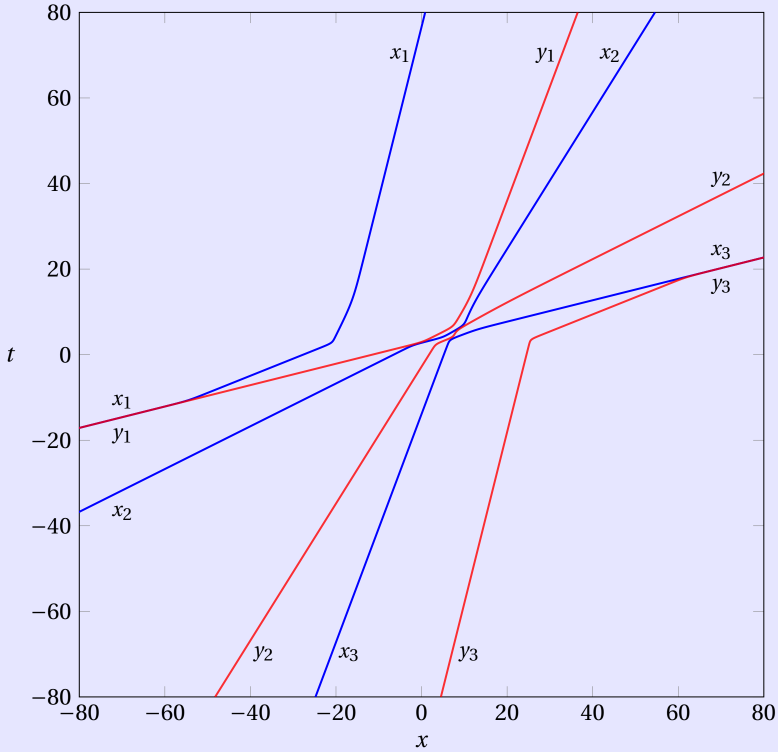
Then the eigenvalues are positive and simple:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K, \qquad 0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1}$$

Asymptotic velocities as $t \to \pm\infty$ for $3+3$ interlacing solution (from fastest to slowest):

$$\underbrace{\frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\mu_1}\right)}_{\text{twice}}, \quad \frac{1}{2}\left(\frac{1}{\lambda_2} + \frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2} + \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3} + \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3}\right)$$

Positions: $x = x_k(t)$ and $x = y_k(t)$



Incoming $\dot{x}_1$ & $\dot{y}_1$ = outgoing $\dot{x}_3$ & $\dot{y}_3$. Incoming $\dot{x}_2$ = outgoing $\dot{y}_2$. (Etc.)

The **amplitudes** m_k and n_k (typically) do **not** tend to constants as $t \to \pm\infty$!

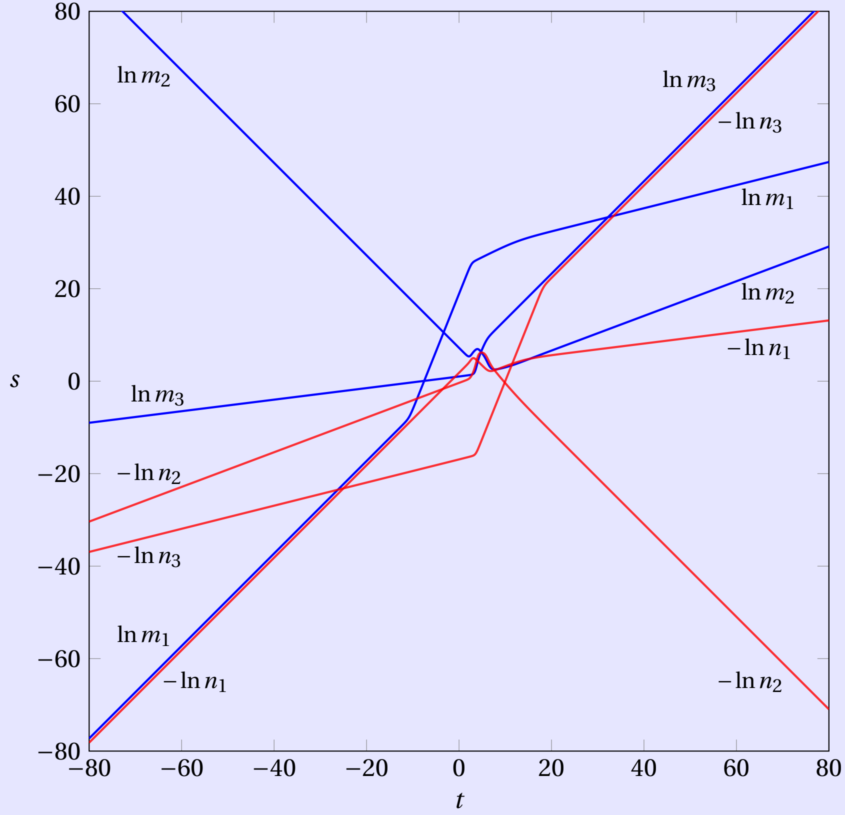
Instead they **grow or decay exponentially**.

The curves $s = \ln m_k(t)$ and $s = -\ln n_k(t)$ asymptotically approach straight lines as $t \to \pm\infty$.

Slopes:

$$\underbrace{\frac{1}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\mu_1}\right)}_{\text{twice}}, \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3}\right)$$

Logarithms of amplitudes: $s = \ln m_k(t)$ and $s = -\ln n_k(t)$



Next: Solution formulas for **arbitrary configuration**.

(Shuaib & Lundmark,in preparation.)

Notation for positions:

$$\underbrace{x_{1,1} < x_{1,2} < \cdots < x_{1,N_1^X}}_{\text{First } X\text{-group}} < \underbrace{y_{1,1} < y_{1,2} < \cdots < y_{1,N_1^Y}}_{\text{First } Y\text{-group}} < \cdots$$

$$< \underbrace{x_{j,1} < x_{j,2} < \cdots < x_{j,N_j^X}}_{j\text{th } X\text{-group}} < \underbrace{y_{j,1} < y_{j,2} < \cdots < y_{j,N_j^Y}}_{j\text{th } Y\text{-group}} < \cdots$$

$$< \underbrace{x_{K,1} < x_{K,2} < \cdots < x_{K,N_K^X}}_{\text{Last } X\text{-group}} < \underbrace{y_{K,1} < y_{K,2} < \cdots < y_{K,N_K^Y}}_{\text{Last } Y\text{-group}}$$

Similarly for the amplitudes $m_{j,i}$ and $n_{j,i}$.

Inverse spectral technique doesn't work directly.

(For non-interlacing configurations, the Lax pairs yield too few constants of motion!)

Instead: use **ghostpeakon** technique.

Also useful for deriving exact formulas for the **characteristic curves** $x = \xi(t)$ associated with a peakon solution $u(x,t)$:

$$\dot{\xi}(t) = u\big(\xi(t), t\big) \quad \text{for CH \& DP}$$
$$\dot{\xi}(t) = u\big(\xi(t), t\big)^2 \quad \text{for Novikov}$$

These curves were used for making the 3D plots of $u(x,t)$ above.

(Lundmark & Shuaib, preprint 2018.)

- An arbitrary configuration is given.
- Pad it with auxiliary peakons to obtain a $K+K$ interlacing configuration. In the known solution formulas for that configuration, make a substitution of the form

$$\lambda_K = \frac{\text{constant}}{\varepsilon} \qquad a_K(0) = \text{constant} \times \varepsilon^{k_1}$$

$$\mu_{K-1} = \frac{\text{constant}}{\varepsilon} \qquad b_{K-1}(0) = \text{constant} \times \varepsilon^{k_2}$$

and let $\varepsilon \to 0^+$.

- With the powers k_1 & k_2 **suitably chosen**, this will turn one of the inserted auxiliary peakons into a “ghostpeakon” with amplitude zero.
- Repeat this, to “kill” all the inserted peakons, one by one.

Example. Solution sought for this config with $3+3$ groups:

$$x_1 < \underbrace{y_{1,1} < y_{1,2} < y_{1,3}}_{Y\text{-group}} < x_2 < y_2 < \underbrace{x_{3,1} < x_{3,2} < x_{3,3}}_{X\text{-group}} < \underbrace{y_{3,1} < y_{3,2} < y_{3,3} < y_{3,4}}_{Y\text{-group}}$$

Schematically: ●○○○●○●●●○○○○

Steps to obtain solution formulas:

Start $(10+10$ interlacing)	●○●○●○●○●○●○●○●○●○●○,
Step 1a	●○●○●○●○●○●○●○●○●○ ○,
Step 1b	●○●○●○●○●○●○●○●○ ○ ○,
Step 1c	●○●○●○●○●○●○●○ ○ ○ ○,
Step 2a	●○●○●○●○●○● ●○ ○ ○ ○,
Step 2b	●○●○●○●○● ● ●○ ○ ○ ○,
Step 3a	●○●○ ○●○● ● ●○ ○ ○ ○,
Step 3b (finish)	●○ ○ ○●○● ● ●○ ○ ○ ○.

Need to keep track of what happens to all the solution formulas at each step. At the end, the desired formulas remain.

Results

- Singletons obey the formulas from the interlacing case.
- Each group with $N \geq 2$ peakons has internal parameters

  $$\tau_1, \dots, \tau_{N-1} > 0 \qquad 0 < \sigma_1 < \cdots < \sigma_{N-1}$$

  appearing only in the solution formulas for that group.
- The two spectral problems only sense the **effective position and amplitude** of each group:

  $$\tilde{m}_j \, e^{\tilde{x}_j} = \sum_{i=1}^{N} m_{j,i} e^{x_{j,i}} \qquad \tilde{m}_j \, e^{-\tilde{x}_j} = \sum_{i=1}^{N} m_{j,i} e^{-x_{j,i}}$$

  The additional constants of motion τ_i and σ_i (not coming from the Lax pairs) are needed in order to determine what happens inside each group.

Example. 3+3 groups, all singletons except one 5-peakon group.

$$X_1 = \frac{\mathrm{J}^{00}_{32}}{\mathrm{J}^{11}_{21} + C\ \mathrm{J}^{10}_{22}} \qquad Y_1 = \frac{\mathrm{J}^{00}_{32}}{\mathrm{J}^{11}_{21}} \qquad X_2 = \frac{\mathrm{J}^{00}_{22}}{\mathrm{J}^{11}_{11}}$$

$$\boxed{Y_{2,1}} = \frac{\mathrm{J}^{00}_{22} + \tau_1\ \mathrm{J}^{00}_{21}}{\mathrm{J}^{11}_{11} + \tau_1\ \mathrm{J}^{11}_{10}} \qquad \left(\tau_i = \tau^Y_{2,i} \quad \& \quad \sigma_i = \sigma^Y_{2,i}\right)$$

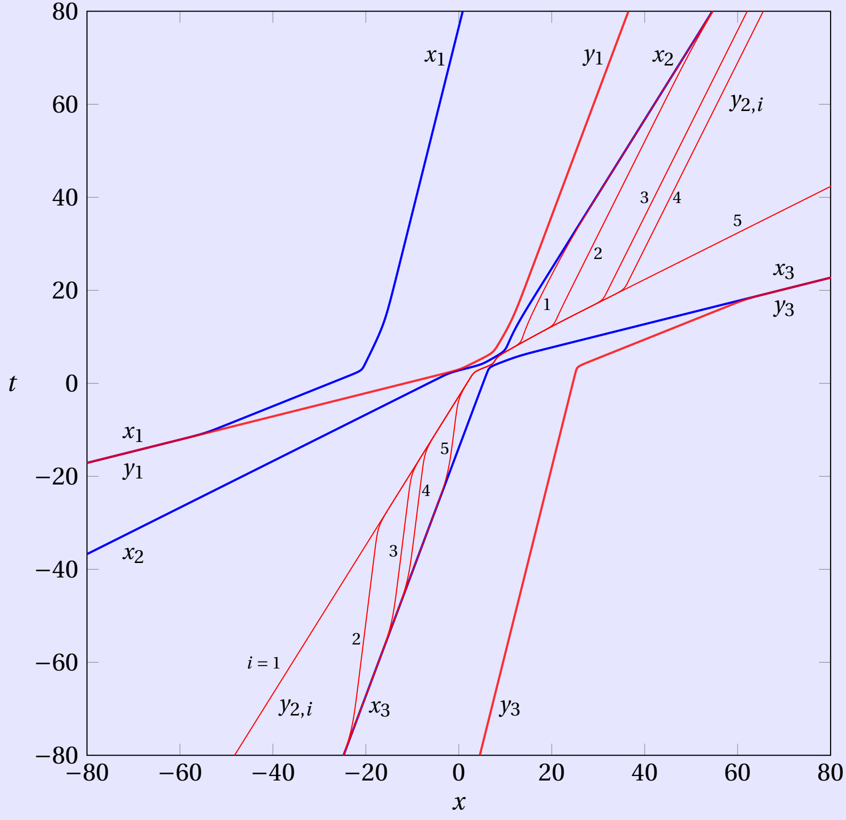
$$\boxed{Y_{2,2}} = \frac{\mathrm{J}^{00}_{22} + (\tau_1 + \tau_2)\ \mathrm{J}^{00}_{21} + (\tau_2\sigma_1)\ \mathrm{J}^{00}_{11}}{\mathrm{J}^{11}_{11} + (\tau_1 + \tau_2)\ \mathrm{J}^{11}_{10} + (\tau_2\sigma_1)\ \mathrm{J}^{11}_{00}}$$

$$\boxed{Y_{2,3}} = \frac{\mathrm{J}^{00}_{22} + (\tau_1 + \tau_2 + \tau_3)\ \mathrm{J}^{00}_{21} + (\tau_2\sigma_1 + \tau_3\sigma_2)\ \mathrm{J}^{00}_{11}}{\mathrm{J}^{11}_{11} + (\tau_1 + \tau_2 + \tau_3)\ \mathrm{J}^{11}_{10} + (\tau_2\sigma_1 + \tau_3\sigma_2)\ \mathrm{J}^{11}_{00}}$$

$$\boxed{Y_{2,4}} = \frac{\mathrm{J}^{00}_{22} + (\tau_1 + \tau_2 + \tau_3 + \tau_4)\ \mathrm{J}^{00}_{21} + (\tau_2\sigma_1 + \tau_3\sigma_2 + \tau_4\sigma_3)\ \mathrm{J}^{00}_{11}}{\mathrm{J}^{11}_{11} + (\tau_1 + \tau_2 + \tau_3 + \tau_4)\ \mathrm{J}^{11}_{10} + (\tau_2\sigma_1 + \tau_3\sigma_2 + \tau_4\sigma_3)\ \mathrm{J}^{11}_{00}}$$

$$\boxed{Y_{2,5}} = \frac{\mathrm{J}^{00}_{21} + \sigma_4\ \mathrm{J}^{00}_{11}}{\mathrm{J}^{11}_{10} + \sigma_4\ \mathrm{J}^{11}_{00}} \qquad X_3 = \frac{\mathrm{J}^{00}_{11}}{\mathrm{J}^{11}_{00}} = \mathrm{J}^{00}_{11} \qquad Y_3 = \mathrm{J}^{00}_{11} + D\ \mathrm{J}^{00}_{10}$$

Positions for the config on the previous page: $\bullet\circ\bullet\underbrace{\circ\circ\circ\circ\circ}\bullet\circ$



Two main cases

- Even case: $K+K$ groups. $\{\lambda_i\}_{i=1}^{K}$ and $\{\mu_j\}_{j=1}^{K-1}$
- Odd case: $(K+1)+K$ groups. $\{\lambda_i\}_{i=1}^{K}$ and $\{\mu_j\}_{j=1}^{K}$

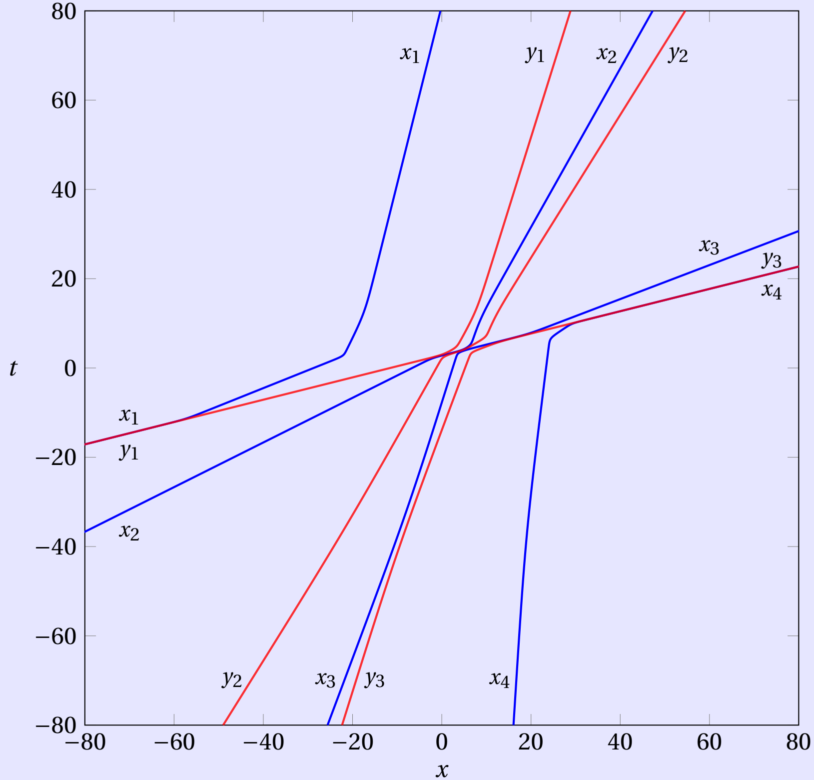
Already the **interlacing** odd case is a bit surprising:

Asymptotic velocities

(4 + 3 interlacing case)

$t \to -\infty$		$t \to +\infty$	
$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_1}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_1}\right)$	(twice)
$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_1}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_2}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_2}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_2}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_2}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_3}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_3}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_3}\right)$	
$\frac{1}{2}\left(0+\frac{1}{\mu_3}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_3}+0\right)$	

Example. Positions for a $4+3$ interlacing configuration:



Incoming $\dot{x}_1$ & $\dot{y}_1$ = outgoing $\dot{y}_3$ & $\dot{x}_4$. But incoming $\dot{x}_2 \neq$ outgoing $\dot{x}_3$. (Etc.)

THE END