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Some recent advances in the study of peakons

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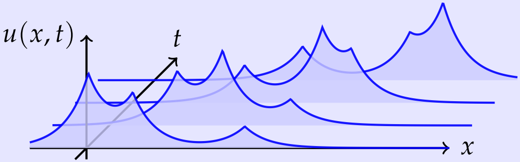
Peakon = peaked soliton



$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x - x_i(t)|}$$

Some PDEs admit (weak) solutions of this form.

Of course, the positions $x_k(t)$ and the amplitudes $m_k(t)$ must have the right time dependence. (Governed by ODEs.)

Examples of integrable PDEs with peakons:

$$m_t + m_x u + 2mu_x = 0$$ Camassa–Holm (1993)

$$m_t + m_x u + 3mu_x = 0$$ Degasperis–Procesi (1998)

$$m_t + (m_x u + 3mu_x)u = 0$$ V. Novikov (2008)

$$\begin{aligned} m_t + (m_x u + 3mu_x)v = 0 \\ n_t + (n_x v + 3nv_x)u = 0 \end{aligned}$$ Geng–Xue (2009)

where $m = u - u_{xx}$ and $n = v - v_{xx}$

In this talk: overview of some recent work

With Marcus Kardell (LiU):

Dynamics of Novikov peakon–antipeakon (conservative) solutions.

With Jacek Szmigielski (Univ. of Saskatchewan, Canada):

Dynamics of interlacing GX peakons.

GX shockpeakons.

With Budor Shuaib (LiU):

“Ghostpeakons” and characteristic curves for CH/DP/Novikov multipeakon solutions.

Application: Dynamics of non-interlacing GX peakons.

Novikov peakons & antipeakons

$$\boxed{\dot{x}_k = u(x_k)^2} \qquad \boxed{\dot{m}_k = -m_k \langle u_x(x_k) \rangle u(x_k)}$$

Compared to CH: extra factor $u(x_k)$ in $\dot{x}_k$ and $\dot{m}_k$.

$\dot{x}_k \geq 0 \implies$ antipeakons move to the right too!

Admits breather-like peakon–antipeakon pairs, and also triples, quadruples, etc. (with periodic or quasi-periodic oscillations).

Also admits less tightly bound clusters, associated to eigenvalues of multiplicity greater than one. (Logarithmic separation as $t \to \pm\infty$.)

Explicit formulas for the N-peakon solution are known. [Hone–Lundmark–Szmigielski 2009]

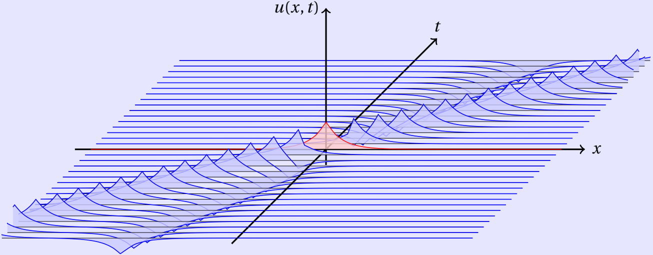
Structure: e^{x_k} and m_k are rational functions of the exponentials

$$b_i(t) = b_i(0)\, e^{t/\lambda_i} \qquad (i = 1, \dots, N)$$

where the N **eigenvalues** λ_i and the N additional parameters $b_i(0)$ are determined by initial data.

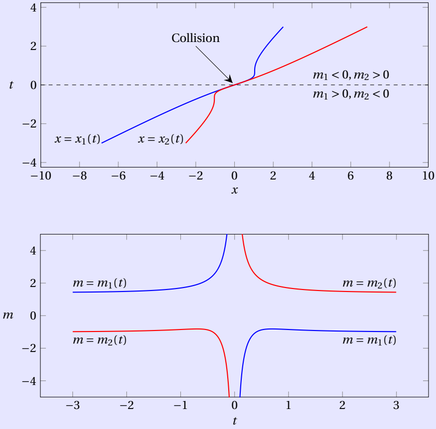
(The eigenvalues must lie in the **right** half of the complex plane.)

Peakon and antipeakon with $0 < \lambda_1 < \lambda_2$:



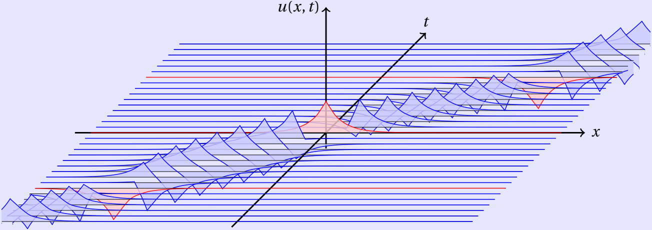
Scattering as $t \to \pm\infty$.

Asymptotic velocities $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$.



Collision
$m_1 < 0, m_2 > 0$
$m_1 > 0, m_2 < 0$
$x = x_1(t)$
$x = x_2(t)$
t
x
$m = m_1(t)$
$m = m_2(t)$
$m = m_2(t)$
$m = m_1(t)$
m
t

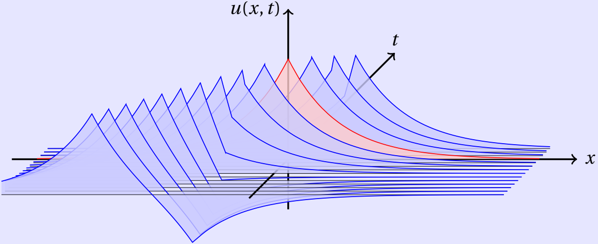
Peakon and antipeakon with $\frac{1}{\lambda_{1,2}} = \alpha \pm i\beta$:



Breather-like peakon–antipeakon pair with velocity α.

Periodic oscillations with period $\frac{2\pi}{\beta}$.

Closeup of the collision at the origin:

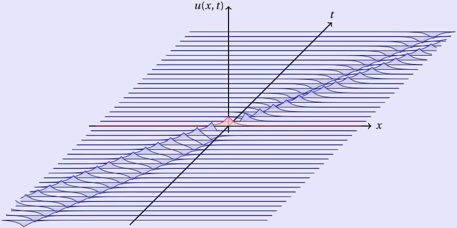


u stays continuous, u_x blows up. (Like for CH.)

$x_2(t) - x_1(t) \sim \text{const.} \times t^4$. (Unlike t^2 for CH.)

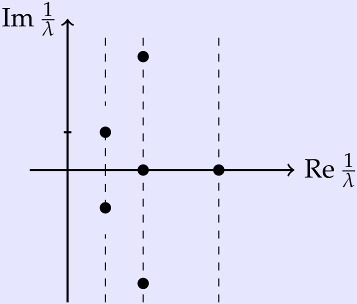
(With a larger number of peakons, it is also possible to have "higher-order collisions" with $x_2 - x_1 \propto t^8, t^{12}, t^{16}$, and so on.)

Peakon and antipeakon, double eigenvalue $\lambda_{1,2} = \mu$:



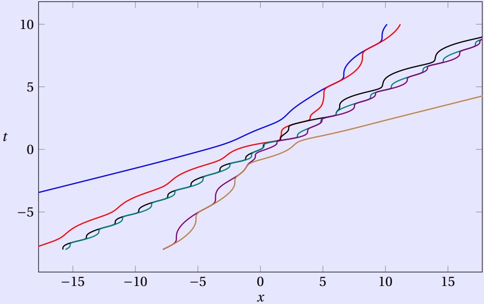
$$x_1(t) = \frac{t}{\mu} - \frac{1}{2}\ln\Big(\frac{2t^2}{\mu^2} - \frac{2t}{\mu} + 1\Big)$$
$$x_2(t) = \frac{t}{\mu} + \frac{1}{2}\ln\Big(\frac{2t^2}{\mu^2} + \frac{2t}{\mu} + 1\Big)$$

Velocity $\sim \frac{1}{\mu}$, logarithmic separation as $t \to \pm\infty$.



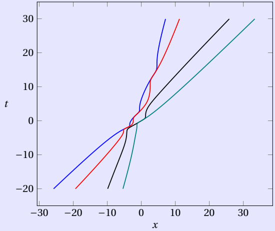
With $N = 6$ and $\frac{1}{\lambda_k} \in \{1 \pm i,\ 2,\ 2 \pm 3i,\ 4\}$ we get…

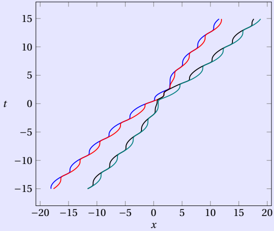
…solutions like this (only positions $x = x_k(t)$ shown):



Scattering into **clusters** as $t \to \pm\infty$.
Asymptotic velocities of the clusters: $\operatorname{Re} \frac{1}{\lambda_k} \in \{1, 2, 4\}$.

The old formulas don't apply if there are non-simple eigenvalues. The formulas for such cases can be obtained via a (somewhat involved) limiting procedure.





Left: two real double eigenvalues ($0 < \lambda_1 = \lambda_2 < \lambda_3 = \lambda_4$).

Right: two complex double eigenvalues ($\lambda_1 = \lambda_2 = \overline{\lambda_3} = \overline{\lambda_4} \in \mathbf{C} \setminus \mathbf{R}$).

Geng–Xue (shock)peakons

$$\begin{aligned} m_t + (m_x u + 3 m u_x) v &= 0 \\ n_t + (n_x v + 3 n v_x) u &= 0 \end{aligned} \iff$$

$$\begin{aligned} m_t + v \cdot (4 - \partial_x^2) \partial_x (\tfrac{1}{2} u^2) &= 0, \\ n_t + u \cdot (4 - \partial_x^2) \partial_x (\tfrac{1}{2} v^2) &= 0. \end{aligned}$$

One component can be **nasty** (even discontinuous) at some points, but then the **other** component must be **nice** there. (Distribution times smooth function is OK.)

So peakon solutions must be **non-overlapping**:

$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}$$

$$v(x,t) = \sum_{i=1}^{N} n_i(t)\, e^{-|x-x_i(t)|}$$

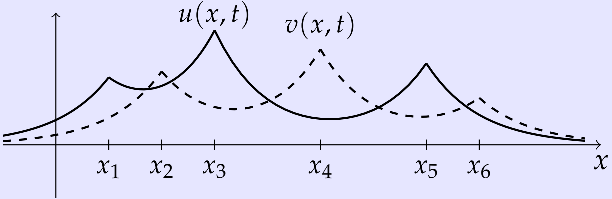
where for each $i = 1, \ldots, N$ exactly one of m_i and n_i is nonzero.

(Similarly for shockpeakons.)

First: **interlacing** peakon solutions.

$N = 2K$ even.

Odd-numbered m_i and even-numbered n_i nonzero.



Peakon ODEs:

$$\dot{x}_k = u(x_k)\,v(x_k) \qquad \dot{m}_k = \dots \qquad \dot{n}_k = \dots$$

Explicit solution formulas for the interlacing case can be derived using inverse spectral metods.

A rather long and technical procedure! Similar to the "discrete cubic string" used for DP and Novikov, but now with two Lax pairs and two spectra. Involves Cauchy biorthogonal polynomials with respect to two independent spectral measures:

$$\alpha = \sum_{i=1}^{K} a_i\,\delta_{\lambda_i} \qquad \beta = \sum_{j=1}^{K-1} b_j\,\delta_{\mu_j}$$

Dynamics of $K + K$ interlacing GX peakons

(With $K \geq 2$ and no antipeakons.)

Distinct asymptotic velocitites as $t \to \pm\infty$, except that

$$\lim_{t\to-\infty} \big(x_2(t) - x_1(t)\big) = 0 = \lim_{t\to+\infty} \big(x_{2K}(t) - x_{2K-1}(t)\big).$$

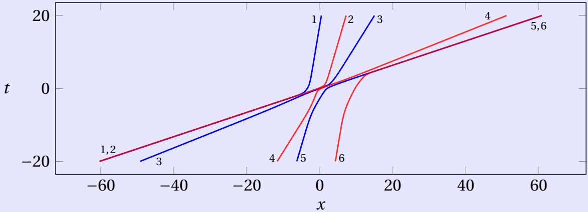
Each amplitude asymptotically grows or decays exponentially (or tends to a constant, but only in borderline cases).

Hence hard to plot $u(x,t)$ and $v(x,t)$. But their *product* $u(x,t)\,v(x,t)$ stays bounded.

Logarithms of amplitudes asymptotically look like straight lines, and exhibit phase shifts, just like the positions.

Positions $x = x_k(t)$ for a $3 + 3$ interlacing GX solution

Blue = odd k: $x = x_1(t), x = x_3(t), x = x_5(t)$
Red = even k: $x = x_2(t), x = x_4(t), x = x_6(t)$



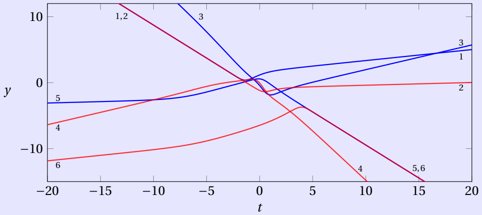
Eigenvalues: $\lambda_1 = \frac{1}{2} < \lambda_2 = 1 < \lambda_3 = 3 \quad \mu_1 = \frac{1}{4} < \mu_2 = 4$

Asymptotic velocitites:

$$\frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\mu_1}\right) > \frac{1}{2}\left(\frac{1}{\lambda_2} + \frac{1}{\mu_1}\right) > \frac{1}{2}\left(\frac{1}{\lambda_2} + \frac{1}{\mu_2}\right) > \frac{1}{2}\left(\frac{1}{\lambda_3} + \frac{1}{\mu_2}\right) > \frac{1}{2}\frac{1}{\lambda_3}$$

$\pm$ Logarithms of amplitudes for the same solution:

Blue = odd k: $y = +\ln m_1(t), y = +\ln m_3(t), y = +\ln m_5(t)$
Red = even k: $y = -\ln n_2(t), y = -\ln n_4(t), y = -\ln n_6(t)$



Asymptotic slopes:

$\frac{1}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\frac{1}{\lambda_3}$

Example of a Geng–Xue peakon–antipeakon case

If m_1, n_2, m_3 are positive and n_4 is negative, then after finite time there will be a collision

$$x_1 < x_2 = x_3 = x_4$$

where

$$n_2 \to +\infty \qquad n_4 \to -\infty$$

in such a way that $v(x,t)$ forms a shockpeakon, and

$$m_3 \to 0$$

so that $u(x,t)$ automatically becomes smooth at the point where $v(x,t)$ jumps.

What about **non-interlacing** peakons?

We don't want to go through the intricate process of solving the inverse spectral problem for every possible non-interlacing configuration.

(The article about the interlacing case is 80 pages…)

What's worse, we **can't** do it again, even if we wanted to! The Lax pairs don't provide sufficiently many constants of motion in non-interlacing cases.

To be continued…

Ghostpeakons

It is often said that we know the "general solution" of the N-peakon ODEs.

Take CH peakons, for instance:

$$\boxed{\dot{x}_k = u(x_k)} \quad \boxed{\dot{m}_k = -m_k \left\langle u_x(x_k) \right\rangle} \quad \Longleftrightarrow$$

$$\boxed{x_{N+1-k}(t) = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2}} \quad \boxed{m_{N+1-k}(t) = \frac{\Delta_k^0 \, \Delta_{k-1}^2}{\Delta_k^1 \, \Delta_{k-1}^1}}$$

[Beals–Sattinger–Szmigielski 2000]

But this is only half true, since these formulas are only valid under the assumption that **all amplitudes m_k are nonzero**.

If $m_k(0) = 0$, then $m_k(t) = 0$ for all t, but the equation for $x_k(t)$ is still a nontrivial driven ODE.

It describes the trajectory of a "**ghostpeakon**" with zero amplitude, which is influenced by the remaining $N - 1$ peakons (if they have nonzero amplitude) but does not affect them back.

Equivalently, it is a **characteristic curve** of the multipeakon solution $u(x, t)$, i.e. a solution of

$$\dot{X}(t) = u(X(t), t).$$

Characteristic curves play a central role when studying the question of continuation of solutions past singularities. [Bressan–Constantin 2007, Holden–Raynaud 2007, …]

Ghostpeakons are **not directly accessible** via inverse spectral methods, since zero-amplitude peakons become **zero-weight point masses** in the corresponding discrete string problem, and these **leave no trace in the spectral data**.

Nevertheless, it is possible to obtain explicit formulas for the ghostpeakons as **limiting cases** of **reparametrized** versions of the not-quite-general formulas above.

(Set $b_N(0) = c\,\lambda_N^{2(1-i)}$ and let $\lambda_N \to \infty$. Then $m_{N+1-i}(t) \to 0$ and $x_{N+1-i}(t)$ tends to a ghostpeakon curve parametrized by c.)

Result (for CH):

The family of characteristic curves between

$$x_{N-k}(t) = \ln \frac{\Delta_{k+1}^0}{\Delta_k^2} \quad \text{and} \quad x_{N+1-k}(t) = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2}$$

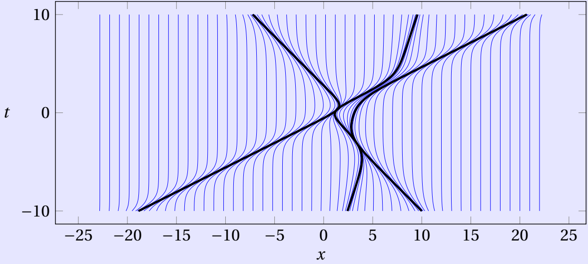
is given by

$$X(t;c) = \ln \frac{\Delta_{k+1}^0 + c\,\Delta_k^0}{\Delta_k^2 + c\,\Delta_{k-1}^2}$$

where c ranges over all positive numbers.

(Similar formulas apply to DP and Novikov.)

Example: Characteristic curves for a (conservative) CH solution with two peakons and one antipeakon.



Thick curves: peakon/antipeakon trajectories $x = x_{1,2,3}(t)$.

Thin curves: ghostpeakon trajectories.

We can use similar methods (although it's more technical) to find the formulas for **non-interlacing** Geng–Xue peakon configurations.

Start with the formulas for interlacing peakons, and take suitable limits to make amplitudes go to zero one after another, until only the desired configuration remains.

$$\begin{array}{cc|c|c|c|c|c|c|c|c|c}
 & m_1 & n_2 & m_3 & n_4 & m_5 & n_6 & m_7 & n_8 & m_9 & n_{10} \\
\to & m_1 & n_2 & m_3 & n_4 & m_5 & n_6 & m_7 & n_8 & 0 & n_{10} \\
\to & m_1 & n_2 & m_3 & n_4 & m_5 & 0 & m_7 & n_8 & 0 & n_{10} \\
\to & m_1 & n_2 & m_3 & 0 & m_5 & 0 & m_7 & n_8 & 0 & n_{10}
\end{array}$$

After renaming: $\quad m_1 \mid n_2 \mid m_3 \quad m_4 \quad m_5 \mid n_6 \quad n_7$

Thank you for listening!