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Ghostpeakons

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(Joint work with Budor Shuaib)

Integrable PDEs admitting **peakon** solutions:

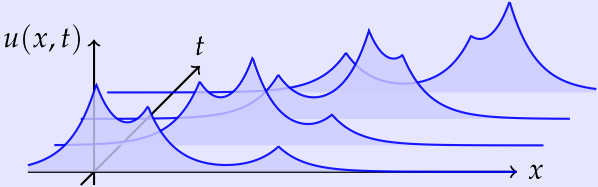
$$m_t + m_x u + 2mu_x = 0$$ Camassa–Holm (1993)

$$m_t + m_x u + 3mu_x = 0$$ Degasperis–Procesi (1998)

$$m_t + (m_x u + 3mu_x)u = 0$$ V. Novikov (2008)

where $$m = u - u_{xx}$$

Peakon = peaked soliton



$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}$$

Positions $x_k(t)$ and amplitudes $m_k(t)$ are governed by a system of ODEs. For CH, it's a Hamiltonian system with $H = \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j \, e^{-\left|x_i - x_j\right|}$:

$$\dot{x}_k = \quad \frac{\partial H}{\partial m_k} = \sum_{i=1}^{N} m_i \, e^{-|x_k - x_i|}$$

$$\dot{m}_k = -\frac{\partial H}{\partial x_k} = \sum_{i=1}^{N} m_k \, m_i \; \operatorname{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

Shorthand notation:

$$\dot{x}_k = u(x_k) \qquad \dot{m}_k = -m_k \langle u_x(x_k) \rangle$$

Special case: The ODEs for three peakons

We can assume $x_1 < x_2 < x_3$.

Write $E_{ij} = e^{-|x_i - x_j|}$ ($= e^{x_i - x_j}$ when $i < j$).

$$\begin{aligned}
\dot{x}_1 &= m_1 + m_2 E_{12} + m_3 E_{13} \\
\dot{x}_2 &= m_1 E_{12} + m_2 + m_3 E_{23} \\
\dot{x}_3 &= m_1 E_{13} + m_2 E_{23} + m_3
\end{aligned}$$

$$\begin{aligned}
\dot{m}_1 &= m_1(-m_2 E_{12} - m_3 E_{13}) \\
\dot{m}_2 &= m_2(m_1 E_{12} - m_3 E_{23}) \\
\dot{m}_3 &= m_3(m_1 E_{13} + m_2 E_{23})
\end{aligned}$$

Beals, Sattinger & Szmigielski (2000) derived the (almost!) general solution for arbitrary N.

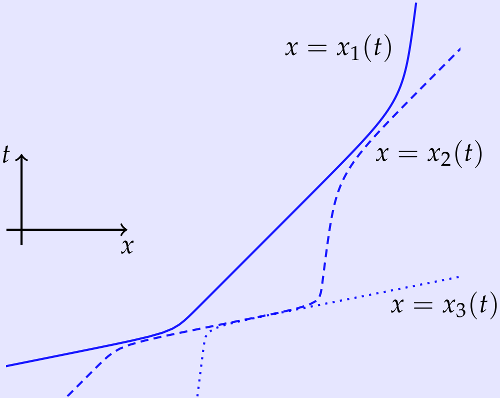
Sketch of method: Form a *discrete string* (point masses connected by weightless thread) where positions and weights of the point masses depend suitably on the positions and amplitudes of the peakons. Consider vibrational eigenmodes. As the peakons move, the string deforms in such a way that the eigenvalues $\{\lambda_k\}_{k=1}^N$ stay constant, and the residues $\{b_k\}_{k=1}^N$ of the Weyl functions satisfy $\frac{d}{dt}b_k = b_k/\lambda_k$. Reconstruct string from these data.

Solution formulas for $N = 3$ are shown on the next page, in terms of λ_k and $b_k(t) = b_k(0)e^{t/\lambda_k}$.

$$
\begin{aligned}
x_1(t) &= \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k} \\
x_2(t) &= \ln \frac{\sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3} \\
x_3(t) &= \ln(b_1+b_2+b_3)
\end{aligned}
$$

$$
\begin{aligned}
m_1(t) &= \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1\lambda_2\lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 b_j b_k} \\
m_2(t) &= \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 b_j b_k} \\
m_3(t) &= \frac{b_1+b_2+b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}
\end{aligned}
$$

Typical spacetime picture:



In general:

$$x_{N+1-k} = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2}$$

$$m_{N+1-k} = \frac{\Delta_k^0 \Delta_{k-1}^2}{\Delta_k^1 \Delta_{k-1}^1}$$

where the deltas are certain determinants.

(Hankel determinants of the moments of the spectral measure $\mu = \sum_{k=1}^{N} b_k \delta_{\lambda_k}$.)

This assumes that all amplitudes m_k are **nonzero**, which is a reasonable assumption from the point of view of finding solutions of the PDE – a term with $m_k = 0$ is then uninteresting since it is simply absent from the sum $u = \sum m_k e^{-|x-x_k|}$.

However, viewing the peakon ODEs as a finite-dimensional integrable system in its own right, we ought to be able to integrate them regardless of the initial conditions!

(That's why I wrote "*almost* general solution" above.)

If some $m_k(0) = 0$, then we will have $m_k(t) = 0$ for all t, since the equation for m_k has the form

$$\frac{dm_k}{dt} = m_k \cdot (\dots)$$

Then the equations for the other peakons reduce to the $N - 1$ case, and the solution of these equations drives the remaining equation for $x_k(t)$.

Example: $m_3 = 0$ in the case $N = 3$

$$
\begin{aligned}
\longrightarrow \quad \dot{x}_1 &= m_1 + m_2 E_{12} + 0 E_{13} \\
\longrightarrow \quad \dot{x}_2 &= m_1 E_{12} + m_2 + 0 E_{23} \\
\dot{x}_3 &= m_1 E_{13} + m_2 E_{23} + 0
\end{aligned}
$$

$$
\begin{aligned}
\longrightarrow \quad \dot{m}_1 &= m_1(-m_2 E_{12} - 0 E_{13}) \\
\longrightarrow \quad \dot{m}_2 &= m_2(m_1 E_{12} - 0 E_{23}) \\
0 &= 0(m_1 E_{13} + m_2 E_{23})
\end{aligned}
$$

The arrows indicate the 2-peakon ODEs, which we know how to solve (if m_1 and m_2 are nonzero). But there is also an equation for $x_3(t)$ which remains to be solved.

Another example: $m_2 = 0$ in the case $N = 3$

$$\begin{aligned}
\longrightarrow \quad \dot{x}_1 &= m_1 + 0E_{12} + m_3 E_{13} \\
\dot{x}_2 &= m_1 E_{12} + 0 + m_3 E_{23} \\
\longrightarrow \quad \dot{x}_3 &= m_1 E_{13} + 0E_{23} + m_3
\end{aligned}$$

$$\begin{aligned}
\longrightarrow \quad \dot{m}_1 &= m_1(-0E_{12} - m_3 E_{13}) \\
0 &= 0(m_1 E_{12} - m_3 E_{23}) \\
\longrightarrow \quad \dot{m}_3 &= m_3(m_1 E_{13} + 0E_{23})
\end{aligned}$$

Up to relabeling, these are again the 2-peakon ODEs, plus an additional equation for $x_2(t)$ (which is *not* equivalent to the equation for $x_3(t)$ on the previous page).

When $m_k = 0$, the equation for $x_k(t)$ determines the position of a zero-amplitude "ghostpeakon" which is driven by the other peakons but does not affect them.

It is this equation that we would like to solve.

It is not immediately obvious how to do this! Direct integration seems nearly impossible (unless $k = 1$ or $k = N$), and ghostpeakons are invisible to the inverse spectral methods.

Nevertheless, we will see that is a very simple way of doing it.

A few more reason for **why** we're interested in this:

- Ghostpeakons (and ordinary peakons) follow **characteristic curves**. [A characteristic curve associated to a given solution $u(x,t)$ of the CH equation is a curve $x = X(t)$ such that $\frac{d}{dt}X(t) = u(X(t),t)$.]
- We know how to compute **interlacing** peakon solutions of the two-component Geng–Xue equation (2009) by inverse spectral methods. We don't want to go through this for every possible peakon configuration. Idea to handle **non-interlacing** cases: Introduce auxiliary peakons to make the configuration interlacing, then let their amplitudes tend to zero and see what you get in the limit.

Before discussing proofs, let us first have a look at what the results look like (for the Camassa–Holm equation with $N = 3$; one ghostpeakon and two ordinary peakons).

(The results can be generalized to any N, and the corresponding results for DP and Novikov are also known.)

In what follows, c is a positive constant of integration, determined by (or determining, if you like) the position of the ghostpeakon at $t = 0$.

If $m_3(0) = 0$, the solution of the CH 3-peakon ODEs is

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$

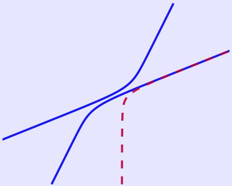
$$x_2(t) = \ln \frac{b_1 + b_2}{1}$$

$$x_3(t) = \ln \frac{(b_1 + b_2) + c \cdot 1}{1 + c \cdot 0} \qquad \leftarrow \text{ghost}$$

$$m_1(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2\right)}{\lambda_1 \lambda_2 \left(\lambda_1 b_1 + \lambda_2 b_2\right)}$$

$$m_2(t) = \frac{b_1 + b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$

$$m_3(t) = 0$$



If $m_2(0) = 0$, the solution is

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$

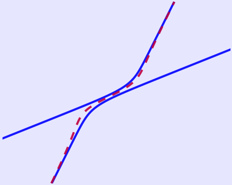
$$x_2(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + c \cdot (b_1 + b_2)}{(\lambda_1^2 b_1 + \lambda_2^2 b_2) + c \cdot 1} \qquad \leftarrow \text{ghost}$$

$$x_3(t) = \ln \frac{b_1 + b_2}{1}$$

$$m_1(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2\right)}{\lambda_1 \lambda_2 \left(\lambda_1 b_1 + \lambda_2 b_2\right)}$$

$$m_2(t) = 0$$

$$m_3(t) = \frac{b_1 + b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$



If $m_1(0) = 0$, the solution is

$$x_1(t) = \ln \frac{0 + c(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 \lambda_2^2 (\lambda_1 - \lambda_2)^2 b_1 b_2 + c(\lambda_1^2 b_1 + \lambda_2^2 b_2)} \quad \leftarrow \text{ghost}$$

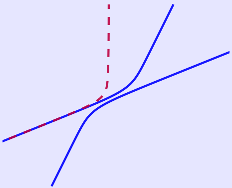
$$x_2(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$

$$x_3(t) = \ln \frac{b_1 + b_2}{1}$$

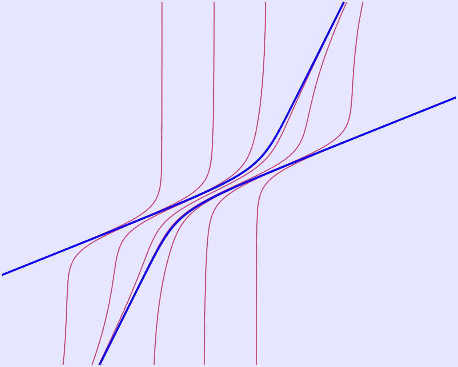
$$m_1(t) = 0$$

$$m_2(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2\right)}{\lambda_1 \lambda_2 \left(\lambda_1 b_1 + \lambda_2 b_2\right)}$$

$$m_3(t) = \frac{b_1 + b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$



Varying c, we obtain the family of characteristic curves covering the whole plane:



Derivation of these results

Idea: Find the formula for the ghostpeakon from the known (non-ghost) formulas through some limiting procedure.

If we could write the solution $\{x_k(t), m_k(t)\}_{k=1}^N$ explicitly in terms of initial data $\{x_k(0), m_k(0)\}_{k=1}^N$, then we could just let some $m_k(0) \to 0$ directly. However, the known solution formulas are parametrized in terms of spectral variables, and the map from the physical variables to spectral variables involves finding the zeros of a polynomial of degree N, so this direct procedure will only work for $N \leq 4$. (In practice, only feasible for $N = 2$.)

Is there an indirect way to see what happens to the spectral data when $m_k(0) \to 0$ for some k?

The eigenvalues $\lambda_1, \dots, \lambda_N$ are the zeros of a polynomial whose highest coefficient is proportional to the product $m_1(0) m_2(0) \cdots m_N(0)$. When one $m_k(0) \to 0$, the degree drops to $N - 1$, so one eigenvalue (say λ_N) tends to infinity.

It's not as obvious what happens to $b_N(0)$!

But, nicely enough, there is a very simple approach which lets us avoid having to deal with this problem.

We **abandon** the idea of letting $m_k(0)$ tend to zero while keeping all other initial values $x_i(0)$ and $m_i(0)$ fixed.

Instead, we will keep (most) things fixed **on the spectral side**, and see what happens to the physical variables as some parameter tends to zero.

We will reparametrize the solution space to introduce parameters $\varepsilon \neq 0$ and c which *all* $x_i(t)$ and $m_i(t)$ depend on, and which are **suitably chosen** such that in the limit $\varepsilon \to 0$ the following happens:

- $m_k(t)$ reduces to $m_k \equiv 0$,
- $\{x_i(t), m_i(t)\}_{i \neq k}$ reduce to the $N-1$ peakon solution (with no dependence on c).

Then the conclusion is that

- $x_k(t; \varepsilon)$ reduces to the ghostpeakon solution (parametrized by c).

To be concrete, let us go back to the space of (non-ghost) 3-peakon solutions, which is parametrized by the constants λ_1, λ_2, λ_3 (nonzero, distinct) and $b_1(0)$, $b_2(0)$, $b_3(0)$ (positive).

$$
\begin{aligned}
x_1(t) &= \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k} \\
x_2(t) &= \ln \frac{\sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3} \\
x_3(t) &= \ln(b_1+b_2+b_3)
\end{aligned}
$$

$$
\begin{aligned}
m_1(t) &= \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1\lambda_2\lambda_3 \sum_{j<k} \lambda_j\lambda_k (\lambda_j-\lambda_k)^2 b_j b_k} \\
m_2(t) &= \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j\lambda_k (\lambda_j-\lambda_k)^2 b_j b_k} \\
m_3(t) &= \frac{b_1+b_2+b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}
\end{aligned}
$$

Substitute $\lambda_3 = \varepsilon^{-1}$ and $b_3(0) = \varepsilon^2 c$ (and hence $b_3(t) = b_3(0)e^{t/\lambda_3} = \varepsilon^2 c e^{\varepsilon t} =: \varepsilon^2 C(t;\varepsilon)$). This means that we parametrize the same solution space with the constants λ_1, λ_2, $b_1(0)$, $b_2(0)$, c, ε instead:

$$
\begin{aligned}
x_1(t) &= \ln(\cdots) \\
x_2(t) &= \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + (\lambda_1 - \varepsilon^{-1})^2 b_1 \varepsilon^2 C + (\lambda_2 - \varepsilon^{-1})^2 b_2 \varepsilon^2 C}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + (\varepsilon^{-1})^2 \varepsilon^2 C} \\
x_3(t) &= \ln(b_1 + b_2 + \varepsilon^2 C)
\end{aligned}
$$

$$
\begin{aligned}
m_1(t) &= (\cdots) \\
m_2(t) &= (\cdots)\,\varepsilon \\
m_3(t) &= \frac{b_1 + b_2 + \varepsilon^2 C}{\lambda_1 b_1 + \lambda_2 b_2 + \varepsilon^{-1} \varepsilon^2 C}
\end{aligned}
$$

We know that these formulas satisfy the peakon ODEs identically for all $\varepsilon \neq 0$, and everything in sight is analytic in ε (with **removable singularity** at $\varepsilon = 0$). So the peakon ODEs are still satisfied if we set $\varepsilon = 0$. (Note that $C(t;0) = c$.)

$$x_1(t) = \ln(\ldots) \quad \to \quad \boxed{\ln(\cdots)}$$

$$x_2(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + (\lambda_1 - \varepsilon^{-1})^2 b_1 \varepsilon^2 C + (\lambda_2 - \varepsilon^{-1})^2 b_2 \varepsilon^2 C}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + (\varepsilon^{-1})^2 \varepsilon^2 C}$$

$$\to \quad \boxed{\ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + c \cdot (b_1 + b_2)}{(\lambda_1^2 b_1 + \lambda_2^2 b_2) + c \cdot 1}}$$

$$x_3(t) = \ln(b_1 + b_2 + \varepsilon^2 C) \quad \to \quad \boxed{\ln(b_1 + b_2)}$$

$$m_1(t) = (\cdots) \quad \to \quad \boxed{(\cdots)}$$

$$m_2(t) = (\cdots)\,\varepsilon \quad \to \quad \boxed{0}$$

$$m_3(t) = \frac{b_1 + b_2 + \varepsilon^2 C}{\lambda_1 b_1 + \lambda_2 b_2 + \varepsilon^{-1} \varepsilon^2 C} \quad \to \quad \boxed{\frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2}}$$

This particular reparametrization was chosen to make m_2 reduce to zero, so that we obtain the ghostpeakon solution formulas with $m_2 = 0$.

To get the ghostpeakon solution with $m_3 = 0$, do the same but take $b_3(0) = c$ instead of $b_3(0) = \varepsilon^2 c$.

And to get the ghostpeakon solution with $m_1 = 0$, take $b_3(0) = \varepsilon^4 c$.

In general, the ghostpeakon solution with $m_{N+1-r} = 0$ is obtained by taking $\lambda_N = \varepsilon^{-1}$ and $b_N(0) = \varepsilon^{2(r-1)} c$ in the N-peakon solution formulas and letting $\varepsilon \to 0$. □

(Similar, but slightly different, rules give the ghostpeakon solutions for DP and Novikov.)