Peakons and shockpeakons in the Degasperis–Procesi equation

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L'Ametlla de Mar

The Camassa–Holm equation

$$m_t + m_x u + 2mu_x = 0, \qquad m = u - u_{xx}$$

…and its "evil twin":

The Degasperis–Procesi equation

$$m_t + m_x u + 3mu_x = 0, \qquad m = u - u_{xx}$$

Outline:

1. Compressed history of the Camassa–Holm and Degasperis–Procesi equations (some papers relevant to this talk).
2. What are peakons?
3. Explicit solutions of the equations governing the peakon dynamics. What's the difference between the CH case and the DP case?
4. What are shockpeakons? Why do they appear in the DP equation but not in the CH equation?
5. Some properties of shockpeakons.

References 1: CH eqn & peakons

Roberto Camassa and Darryl Holm
An integrable shallow water equation with peaked solitons
Physical Review Letters (1993)

> Derived CH equation in water wave theory. Lax pair.
> Solitons with peaked wave crests – "peakons".
> (Found 2-peakon solution explicitly.)

Richard Beals, David Sattinger, and Jacek Szmigielski
Multipeakons and the classical moment problem
Advances in Mathematics (2000)

> Explicit n-peakon solution of the CH eqn.
> Asymptotics as $t \to \pm\infty$, peakon-antipeakon collisions.
> (Stieltjes continued fractions, orthogonal polynomials, inverse spectral problem for the "discrete string".)

References 2: DP eqn

Antonio Degasperis and Michaela Procesi
Asymptotic integrability
Symmetry and Perturbation Theory (Rome, 1998)

> KdV, CH, and DP are (modulo scaling etc.) the only equations in the family
> $$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x$$
> that satisfy “asymptotic integrability to third order”.

Antonio Degasperis, Darryl Holm, and Andrew Hone
A new integrable equation with peakon solutions
Theoretical and Mathematical Physics (2002)

> Integrability of DP eqn: Lax pair, conservation laws.
> Peakons. (Found 2-peakon solution explicitly.)

References 3: DP peakons

Hans Lundmark and Jacek Szmigielski
Multi-peakon solutions of the Degasperis–Procesi equation
Inverse Problems (2003)

Hans Lundmark and Jacek Szmigielski
Degasperis–Procesi peakons and the discrete cubic string
International Mathematics Research Papers (2005)

Jennifer Kohlenberg, Hans Lundmark, and Jacek Szmigielski
The inverse spectral problem for the discrete cubic string
Inverse Problems (2007)

> Explicit n-peakon solution of the DP eqn.
> (Generalizations of some of the classical concepts from the theory of orthogonal polynomials and continued fractions. “Discrete cubic string”.)

References 4: Discontinuous DP solutions

Giuseppe Coclite and Kenneth Hvistendahl Karlsen
On the well-posedness of the Degasperis–Procesi equation
Journal of Functional Analysis (2006)

> Existence and uniqueness of *entropy weak solutions* of the DP eqn in spaces of discontinuous functions.
> (CH needs H^1 regularity: $u_x \in L^2$, hence u continuous.)

Hans Lundmark
Formation and dynamics of shock waves in the D.–P. equation
Journal of Nonlinear Science (2007)

> Shocks (jump discontinuities in u) form when DP peakons and antipeakons collide (unlike the CH case).
> The resulting “shockpeakons” are explicit examples of discontinuous entropy weak solutions.

What are peakons?

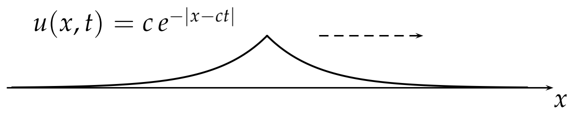
The “b-equation”,

$$m_t + m_x u + b m u_x = 0, \qquad m = u - u_{xx},$$

is integrable iff $b = 2$ (CH) or $b = 3$ (DP).

It admits a particular class of solutions called peakons.

A single peakon is a travelling wave of the following shape:



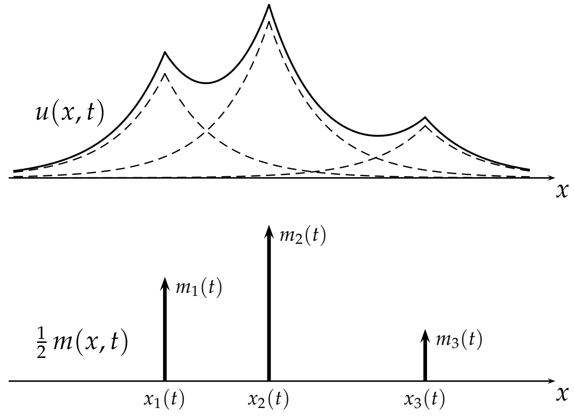
This corresponds to $m(x,t) = 2c\,\delta_{x-ct}$ (Dirac delta).

c = height = speed (momentum).

For $c < 0$ we get an “antipeakon” moving to the left.

The multipeakon solution is simply a superposition of n peakons:

$$u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x-x_i(t)|} \qquad m(x,t) = 2\sum_{i=1}^{n} m_i(t)\, \delta_{x-x_i(t)}$$



The n-peakon superposition $u = \sum m_i\, e^{-|x-x_i|}$ is a solution of the b-equation iff the positions $x_k(t)$ and momenta $m_k(t)$ satisfy the following system of ODEs:

$$\dot{x}_k = \sum_{i=1}^{n} m_i\, e^{-|x_k-x_i|}$$

$$\dot{m}_k = (b-1)\sum_{i=1}^{n} m_k\, m_i\ \operatorname{sgn}(x_k - x_i)\, e^{-|x_k-x_i|}$$

Shorthand notation, with $u_x(x_k) = \dfrac{1}{2}\Big(u_x(x_k^-) + u_x(x_k^+)\Big)$:

$$\dot{x}_k = u(x_k) \qquad \dot{m}_k = -(b-1)\, m_k\, u_x(x_k)$$

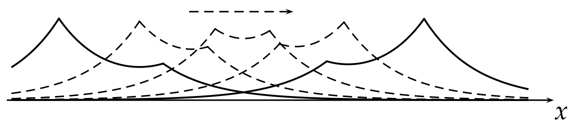
(Note that the speed $\dot{x}_k$ of the kth peakon equals the height of the wave at that point.)

$n = 1$: $\begin{cases} \dot{x}_1 = m_1 \\ \dot{m}_1 = 0 \end{cases}$ Travelling wave $x_1(t) = ct$, $m_1(t) = c$.

$n = 2$: Can be solved in new variables $x_1 \pm x_2$ and $m_1 \pm m_2$.

The integrable cases $b = 2$ (CH peakons) and $b = 3$ (DP peakons) have been solved for arbitrary n using inverse spectral methods.

Typical two-peakon interaction (plotted from CH $n = 2$ solution formulas; see next page):



Asymptotically (as $t \to \pm\infty$) the peakons separate and behave like free particles (travelling waves).

Camassa–Holm peakons

The solution for $n = 2$ is

$$x_1(t) = \log \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2}$$

$$x_2(t) = \log(b_1 + b_2)$$

$$m_1(t) = \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)}$$

$$m_2(t) = \frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2}$$

where $b_k(t) = b_k(0) e^{t/\lambda_k}$. The constants λ_1, λ_2, $b_1(0)$, $b_2(0)$ are uniquely determined by initial conditions.

The *eigenvalues* λ_k are real, simple, nonzero. The number of positive eigenvalues equals the number of positive m_k's.

The quantities b_k (*residues of the Weyl function*) are always positive.

The CH solution for $n = 3$ is

$$x_1(t) = \log \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$x_2(t) = \log \frac{\sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3}$$

$$x_3(t) = \log(b_1 + b_2 + b_3)$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1 \lambda_2 \lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

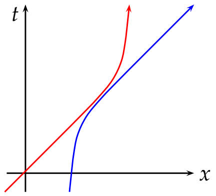
$$m_2(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$m_3(t) = \frac{b_1 + b_2 + b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}$$

(The solution for general n looks similar, but to write it down one needs a bit of notation for symmetric functions.)

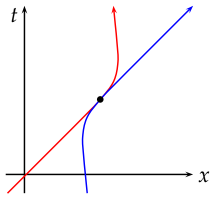
(Camassa–Holm $n = 2$ continued)

Typical plots of $x_1(t)$ and $x_2(t)$ in the (x, t) plane:



$\lambda_1 = 1$ and $\lambda_2 = 10$
(two peakons)

$x_1(t) < x_2(t)$ for all t.

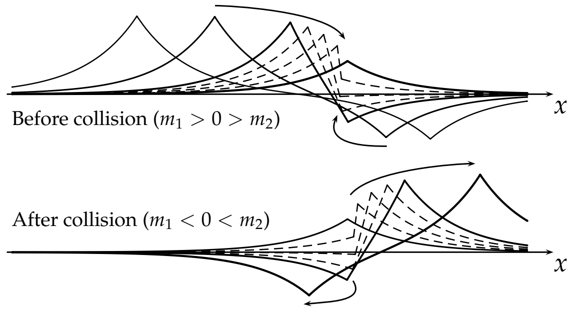


$\lambda_1 = 1$ and $\lambda_2 = -10$
(peakon and antipeakon)

$x_1(t) < x_2(t)$ except at the instant of collision.

The asymptotic speeds as $t \to \pm\infty$ are $1/\lambda_1$ and $1/\lambda_2$.

CH peakon-antipeakon collision: $\left(\frac{1}{\lambda_1} \approx 1.5,\ \frac{1}{\lambda_2} \approx -0.86\right)$



The individual peakon amplitudes $m_1(t)$ and $m_2(t)$ both blow up at the instant of collision, one to $+\infty$ and the other to $-\infty$, but in such a way that the infinities cancel and $u = \sum m_i e^{-|x - x_i|}$ remains continuous. (However, u_x blows up.)

A few words about the method of solution

Consider a string with mass distribution $g(y)$, fixed at the endpoints $y = \pm 1$, governed by the wave equation $g(y) F_{tt} = F_{yy}$. Separation of variables gives the vibrational modes:

$$-\phi''(y) = z\, g(y)\, \phi(y) \qquad \text{for } -1 < y < 1$$
$$\phi(-1) = 0 \qquad \phi(1) = 0$$

To a given peakon configuration $\{x_k, m_k\}$, associate a discrete measure $g(y) = \sum_1^n g_i\, \delta_{y_i}$ with

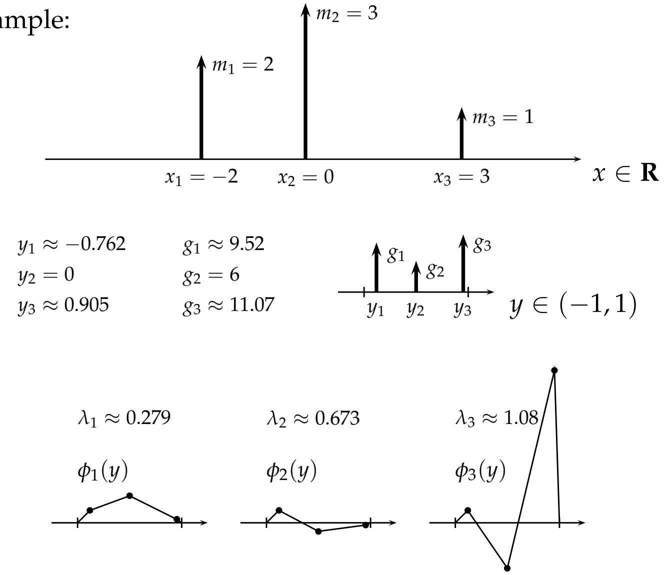
$$y_i = \tanh(x_i/2) \qquad g_i = 2m_i/(1 - y_i^2)$$

(Point masses g_i at positions y_i connected by massless string.)

Such a ***discrete string*** has exactly n eigenvalues $z = \lambda_1, \ldots, \lambda_n$ and the corresponding eigenfunctions $\phi_k(y)$ are piecewise linear.

The quantity b_k is the *coupling coefficient* $\phi_k'(1)/\phi_k'(-1)$ of the kth eigenfunction, divided by the factor $-2\prod_{j \neq k}(1 - \lambda_k/\lambda_j)$.

Example:



Crucial fact (thanks to the Lax pair of course):

> The CH peakons move in such a way that the spectral data of the corresponding discrete string satisfy
> $$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k/\lambda_k$$

The inverse problem of determining the mass distribution of a discrete string given the spectral data was solved long ago (*analytic continued fractions* T. Stieltjes 1895, *string interpretation* M. Krein 1951).

Using this, one obtains explicit formulas for the general solution $\{x_k(t), m_k(t)\}$ for any n.

J. Moser (1975) showed (in the case $n = 3$) how Stieltjes' results give the solution of the n-particle nonperiodic Toda lattice. The Toda lattice and the CH peakons are special cases of a more general construction due to Beals–Sattinger–Szmigielski (2001).

Degasperis–Procesi peakons

The solution for $n = 2$ is

$$
\begin{aligned}
x_1(t) &= \log \frac{\frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2} \\
x_2(t) &= \log(b_1 + b_2) \\
m_1(t) &= \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1 \lambda_2 \left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2 \right)} \\
m_2(t) &= \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2}
\end{aligned}
$$

with $b_k(t) = b_k(0) e^{t/\lambda_k}$ as before, but with the spectral data now coming from a "discrete cubic string" instead of an ordinary string.

The DP solution for $n = 3$ is

$$
x_1(t) = \log \frac{U_3}{V_2} \qquad x_2(t) = \log \frac{U_2}{V_1} \qquad x_3(t) = \log U_1
$$
$$
m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1} \quad m_3(t) = \frac{(U_1)^2}{W_1}
$$

where

$$
\begin{aligned}
U_1 &= b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 \\
U_2 &= \frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2 + \frac{(\lambda_1-\lambda_3)^2}{\lambda_1+\lambda_3} b_1 b_3 + \frac{(\lambda_2-\lambda_3)^2}{\lambda_2+\lambda_3} b_2 b_3 \\
V_2 &= \frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} \lambda_1\lambda_2 b_1 b_2 + \frac{(\lambda_1-\lambda_3)^2}{\lambda_1+\lambda_3} \lambda_1\lambda_3 b_1 b_3 + \frac{(\lambda_2-\lambda_3)^2}{\lambda_2+\lambda_3} \lambda_2\lambda_3 b_2 b_3 \\
U_3 &= \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} b_1 b_2 b_3 \qquad V_3 = \lambda_1\lambda_2\lambda_3 U_3
\end{aligned}
$$

$$
W_1 = U_1 V_1 - U_2 = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2 + \frac{4\lambda_1\lambda_3}{\lambda_1+\lambda_3} b_1 b_3 + \frac{4\lambda_2\lambda_3}{\lambda_2+\lambda_3} b_2 b_3
$$
$$
W_2 = U_2 V_2 - U_3 V_1 = \frac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2} \lambda_1\lambda_2 (b_1 b_2)^2 + \cdots + \frac{4\lambda_1\lambda_2\lambda_3(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2 b_1^2 b_2 b_3}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} + \dots
$$

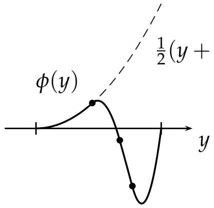
By the *cubic string* we mean the following spectral problem:

$$-\phi'''(y) = z\, g(y)\, \phi(y) \qquad \text{for } -1 < y < 1$$
$$\phi(-1) = \phi'(-1) = 0 \qquad \phi(1) = 0$$

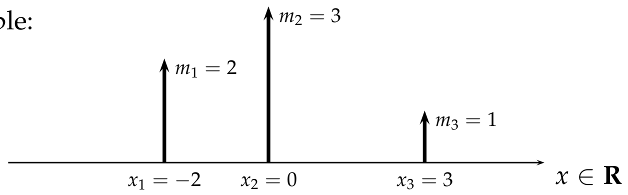
The *discrete cubic string* associated to a DP peakon configuration $\{x_k, m_k\}$ has $g(y) = \sum_1^n g_i\, \delta_{y_i}$ with

$$y_i = \tanh \frac{x_i}{2} \qquad g_i = \frac{8m_i}{(1-y_i^2)^2}$$

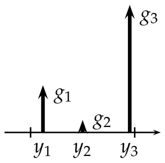
The eigenfunctions are now piecewise quadratic polynomials in y, since $\phi''' = 0$ away from the support of g.

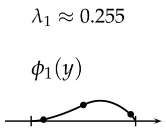


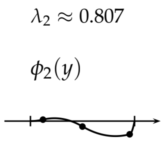
Example:

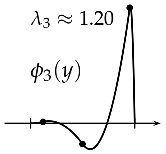


$y_1 \approx -0.762$	$g_1 \approx 90.7$
$y_2 = 0$	$g_2 = 24$
$y_3 \approx 0.905$	$g_3 \approx 245$
(as before)	(different)









The DP peakons move such that the spectral data of the corresponding discrete cubic string satisfy

$$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k / \lambda_k$$

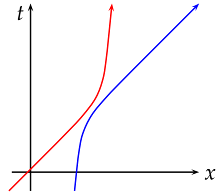
(Here b_k equals the relevant coupling coefficient $\phi_k'(1)/\phi_k''(-1)$ divided by the factor $-2\prod_{j\neq k}(1 - \lambda_k/\lambda_j)$.)

The solution formulas for $x_k(t)$ and $m_k(t)$ hence follow from the solution of the inverse problem for the discrete cubic string, which is much more involved than for the ordinary string.

(No time for details here! But see Jacek Szmigielski's talk.)

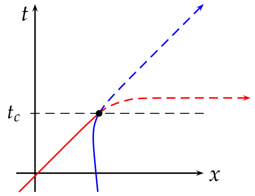
Even the *forward* spectral problem is more complicated, since it is not selfadjoint. (The Gantmacher–Krein theory of oscillatory kernels shows that the spectrum is positive and simple, at least for positive mass distributions.)

(Degasperis–Procesi $n = 2$ continued)



$\lambda_1 = 1$ and $\lambda_2 = 10$
(two peakons)

$x_1(t) < x_2(t)$ for all t.

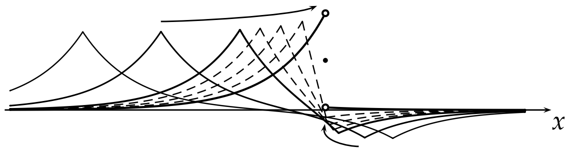


$\lambda_1 = 1$ and $\lambda_2 = -10$
(peakon and antipeakon)

Transversal collision!
(Not tangential as for CH.)

The solution formulas are only valid up to the time of collision t_c since they were derived under the assumption that $|x_1 - x_2|$ can be replaced by $x_2 - x_1$ in the ODEs. Can the solution be continued past the collision?

DP peakon-antipeakon collision:



We see that $u(x,t)$ tends to a discontinuous function as $t \nearrow t_c$. In other words, a *shock* is formed.

- Why is the DP case different from the CH case?
- How does the solution continue?

Inverting $m = u - u_{xx}$ as $u = \frac{1}{2} G * m$ where $G(x) = e^{-|x|}$, one can formally rewrite the b-equation as a conservation law:

$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(\tfrac{b}{2} u^2 + \tfrac{3-b}{2} u_x^2 \right) \right] = 0$$

After multiplying by a test function and integrating by parts, one obtains a rigorous definition of what weak solutions (including peakons) really mean for this family of equations.

Now a difference between CH and DP emerges:

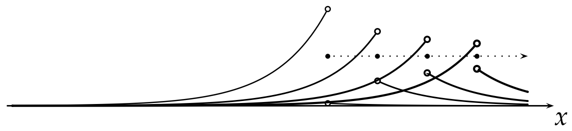
$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(\ u^2 + \tfrac{1}{2} u_x^2 \right) \right] = 0 \qquad (\text{CH}, b = 2)$$
$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(\tfrac{3}{2} u^2 \qquad \right) \right] = 0 \qquad (\text{DP}, b = 3)$$

Since DP does not involve u_x explicitly it is reasonable that it also admits solutions where u (and not just u_x) has jumps.

Coclite and Karlsen: For initial data $u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R})$ there is a unique $u \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}))$ which satisfies DP (in the above weak sense) together with an additional "entropy condition".

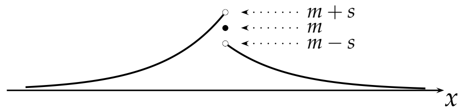
DP shockpeakons

Here is the unique entropy solution with the shock formed at the DP peakon-antipeakon collision as initial data:



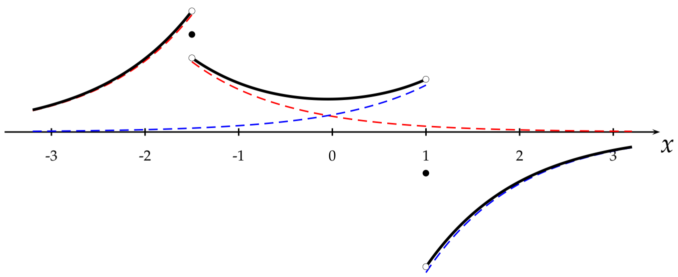
It's a single *shockpeakon*, with shape given by

$$m\, G(x) + s\, G'(x) = m\, e^{-|x|} + s \operatorname{sgn}(-x)\, e^{-|x|} = \begin{cases} (m+s)\, e^{x} & (x < 0) \\ m & (x = 0) \\ (m-s)\, e^{-x} & (x > 0) \end{cases}$$



Natural idea: try superposition!

Superposition (solid curve) of two shockpeakons (dashed curves) with $x_1 = -\frac{3}{2}$, $m_1 = 1$, $s_1 = \frac{1}{4}$ and $x_2 = 1$, $m_2 = -\frac{1}{2}$, $s_2 = 1$ looks like this:



Plug a shockpeakon superposition Ansatz into the DP eqn and compute, and you will get. . .

Theorem: The n-shockpeakon superposition

$$u(x,t) = \sum_{i=1}^{n} m_k(t)\, G(x - x_k(t)) + \sum_{i=1}^{n} s_k(t)\, G'(x - x_k(t))$$

satisfies the DP equation iff

$$\begin{aligned} \dot{x}_k &= u(x_k) \\ \dot{m}_k &= 2\big(s_k\{u_{xx}(x_k)\} - m_k\{u_x(x_k)\}\big) \\ \dot{s}_k &= -s_k\{u_x(x_k)\} \end{aligned}$$

(The entropy condition holds iff $s_k \geq 0$ for all k.)

Here $G(x) = e^{-|x|}$ with $G'(0) := 0$, and curly brackets denote the nonsingular part:

$$u(x_k) = \{u_{xx}(x_k)\} = \sum_{i=1}^{n} m_i\, G(x_k - x_i) + \sum_{i=1}^{n} s_i\, G'(x_k - x_i)$$

$$\{u_x(x_k)\} = \sum_{i=1}^{n} m_i\, G'(x_k - x_i) + \sum_{i=1}^{n} s_i\, G(x_k - x_i)$$

For $n = 1$ we get

$$\dot{x}_1 = m_1 \qquad \dot{m}_1 = 0 \qquad \dot{s}_1 = -s_1^2$$

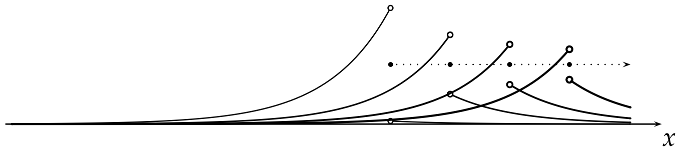
which is a shock wave with constant speed (equal to the average height m_1 at the jump; cf. Rankine–Hugoniot condition).

The jump is $[u] = -2s_1$ where

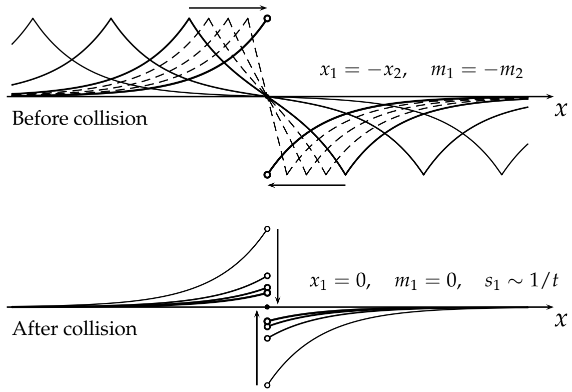
$$s_1(t) = \frac{s_1(t_0)}{1 + (t - t_0)\, s_1(t_0)}$$

so that the shock "dissipates away" like $1/t$ as $t \to \infty$.

This is of course the single shockpeakon shown earlier:



The totally symmetric DP peakon-antipeakon collision results in a stationary shockpeakon (zero momentum):



The only obvious constant of motion is $M = \sum m_k$ so we are still quite far from finding an explicit solution of the shock-peakon ODEs, even in the case $n = 2$:

$$\begin{aligned} \dot{x}_1 &= m_1 + (m_2 + s_2)R \\ \dot{x}_2 &= m_2 + (m_1 - s_1)R \\ \dot{m}_1 &= -2(m_1 - s_1)(m_2 + s_2)R \\ \dot{m}_2 &= +2(m_1 - s_1)(m_2 + s_2)R \\ \dot{s}_1 &= -s_1^2 - s_1(m_2 + s_2)R \\ \dot{s}_2 &= -s_2^2 + s_2(m_1 - s_1)R \end{aligned}$$

(Assume $x_1 < x_2$ and set $R = e^{x_1 - x_2}$)

Is this system even integrable?

(The DP Lax pair involves $m = u - u_{xx}$ and doesn't seem to make sense in this weak setting.)

Numerical experiments show: small shocks $\Rightarrow$ business as usual, large shocks $\Rightarrow$ new phenomena appear. A little bit more can be said in particular cases:

- Antisymmetric 2-shockpeakon case.

  $0 = x_1 + x_2 = m_1 + m_2 = s_1 - s_2$.
  Found additional constant of motion K. In the subcase $K = 0$ the system can be integrated in terms of the inverse of the function $x \mapsto \int_1^{\exp x} (r^2 - 1) e^{r^2/2} dr$.
  Moral: Can't hope for solution formulas as simple as in the shockless case.

- Symmetric peakon-antipeakon with stationary shockpeakon in the middle ($\Rightarrow$ triple collision).

  Test case used in Coclite–Karlsen–Risebro: *Numerical schemes for computing discontinuous solutions of the Degasperis–Procesi equation* (preprint 2006).

THE END