Peakons and shockpeakons: an introduction to the world of nonsmooth solitons

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Outline:

1. Korteweg–de Vries shallow water wave equation (1895).
   - **Solitons** – waves interacting like particles (1965).
   - Solvable by Inverse Scattering Transform (inverse spectral problem for Schrödinger equation).
2. Camassa–Holm shallow water wave equation (1993).
   - **Peakons** – peak-shaped solitons.
   - **Discrete string** inverse spectral problem (orthogonal polynomials, Stieltjes continued fractions).
3. Degasperis–Procesi equation (1998).
   - The "evil twin" of the Camassa–Holm equation!
   - Peakons. **Discrete cubic string**.
   - **Shockpeakons** – discontinuous solitons.

Mandatory quotation in every talk about solitons:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

John Scott Russell (Scottish naval engineer)

The KdV equation

- Korteweg–de Vries equation for waves in shallow water (1895):

  $$u_t + uu_x + u_{xxx} = 0$$

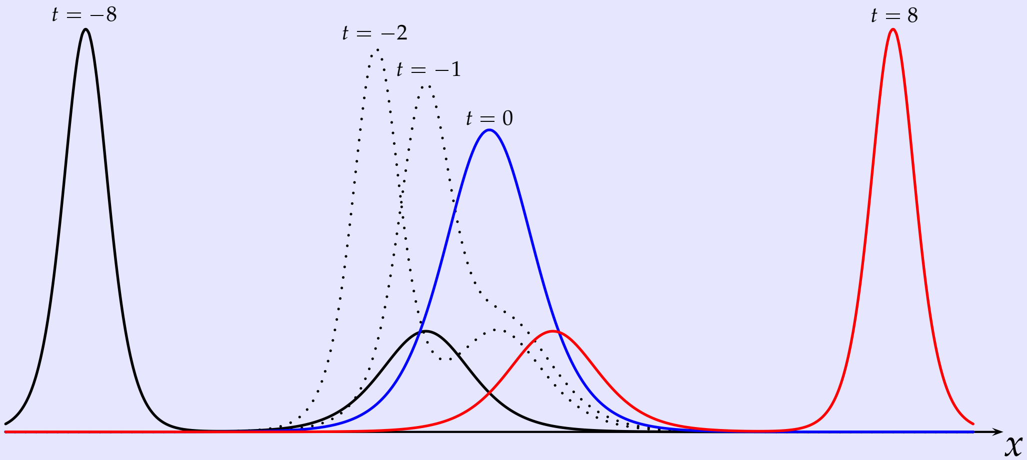
  Solitary wave solution (cf. Russell's wave):

  $$u(x,t) = 3c \,/\, \cosh^2\big(\tfrac{1}{2}\sqrt{c}(x-ct)\big)$$

- Zabusky and Kruskal discovered numerically that several solitary waves can interact and yet maintain their identity – **solitons** (1965).

- Later:
  Inverse Scattering Transform, explicit formula for the n-soliton solution, infinitely many conservation laws, bi-Hamiltonian formulation, Lax pair, etc.

Typical KdV 2-soliton interaction:



A few things to note:

- The individual solitons are "blurred" during the interaction; it's hard to tell exactly where they are.
- If the solitons are nearly equal in size, the two local maxima will not merge into one.
- Several decompositions and interpretations have been proposed. Does the faster soliton overtake the slower one? Or does it slow down and stay behind? A nice review paper by Benes, Kasman, and Young in J. Nonlinear Sci. 2006 suggests an "exchange soliton" that transfers energy from the faster soliton to the slower.

The Camassa–Holm equation

Roberto Camassa and Darryl Holm
An integrable shallow water equation with peaked solitons
Physical Review Letters (1993)

- MathSciNet says: Among top 10 cited papers 2005 & 2006.
- In a shallow water wave approximation to higher order than KdV, they obtained this equation:

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

  or equivalently

$$m_t + m_x u + 2mu_x = 0, \qquad m = u - u_{xx}$$

- Remarkable new feature: **peakons**.
- C & H derived ODEs which describe the n-peakon solution, and solved the cases $n = 1$ and $n = 2$. They also found a Lax pair for the CH equation.

The Degasperis–Procesi equation

Antonio Degasperis and Michaela Procesi
Asymptotic integrability
Symmetry and Perturbation Theory (Rome, 1998)

- They searched for integrable equations similar to the Camassa–Holm equation. In the family

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x$$

  only KdV, CH, and one new equation satisfy the necessary condition of "asymptotic integrability to third order".

- The new equation that they found was the DP equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}$$

  or equivalently

$$m_t + m_x u + 3mu_x = 0, \qquad m = u - u_{xx}$$

Antonio Degasperis, Darryl Holm, and Andrew Hone
A new integrable equation with peakon solutions
Theoretical and Mathematical Physics (2002)

- They proved that the DP equation is indeed integrable, by finding a Lax pair and conservation laws.
- Moreover, the showed that this equation also admits peakon solutions, and solved the ODEs for $n = 1$ and $n = 2$.

So what are peakons then?

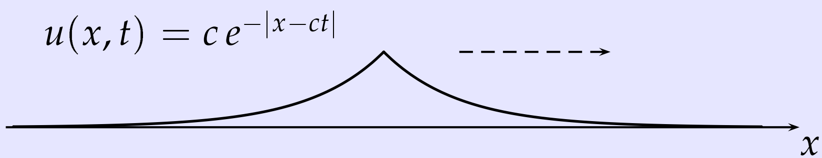
The “b-equation”,

$$m_t + m_x u + b m u_x = 0, \qquad m = u - u_{xx},$$

is integrable iff $b = 2$ (CH case) or $b = 3$ (DP case).

It admits a particular class of solutions called peakons.

A single peakon is a travelling wave of the following shape:



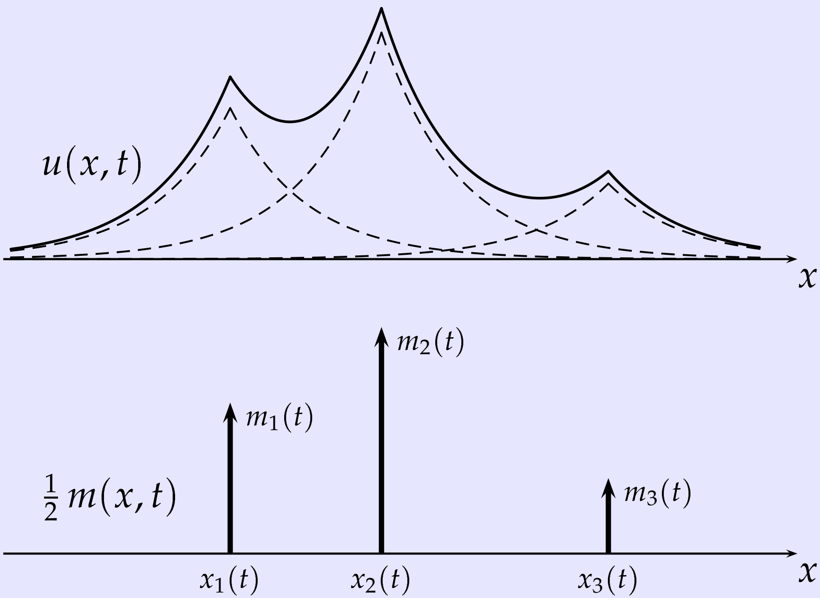
This corresponds to $m(x,t) = 2c\,\delta_{x-ct}$ (Dirac delta).

c = height = speed (momentum).

For $c < 0$ we get an “antipeakon” moving to the left.

The multipeakon solution is simply a superposition of n peakons:

$$u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x-x_i(t)|} \qquad m(x,t) = 2\sum_{i=1}^{n} m_i(t)\, \delta_{x-x_i(t)}$$



The n-peakon superposition $u = \sum m_i \, e^{-|x-x_i|}$ is a solution of the b-equation iff the positions $x_k(t)$ and momenta $m_k(t)$ satisfy the following system of ODEs:

$$\dot{x}_k = \sum_{i=1}^{n} m_i \, e^{-|x_k - x_i|}$$

$$\dot{m}_k = (b-1) \sum_{i=1}^{n} m_k \, m_i \; \mathrm{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

Shorthand notation, with $u_x(x_k) = \dfrac{1}{2}\Big(u_x(x_k^-) + u_x(x_k^+)\Big)$:

$$\dot{x}_k = u(x_k) \qquad\qquad \dot{m}_k = -(b-1) \, m_k \, u_x(x_k)$$

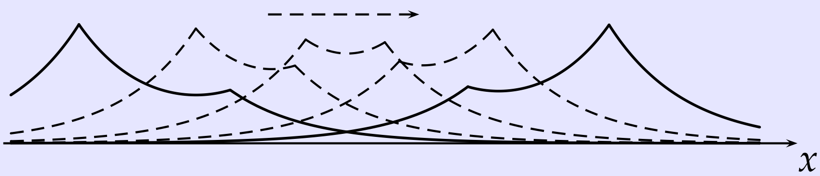
(Note that the speed $\dot{x}_k$ of the kth peakon equals the height of the wave at that point.)

$n = 1$: $\begin{cases} \dot{x}_1 = m_1 \\ \dot{m}_1 = 0 \end{cases}$ Travelling wave $x_1(t) = ct$, $m_1(t) = c$.

$n = 2$: Can be solved in new variables $x_1 \pm x_2$ and $m_1 \pm m_2$.

The integrable cases $b = 2$ (CH peakons) and $b = 3$ (DP peakons) have been solved for arbitrary n using inverse spectral methods.

Typical two-peakon interaction (from the CH $n = 2$ solution formulas that we will see very soon):



Asymptotically (as $t \to \pm\infty$) the peakons separate and behave like free particles (travelling waves).

Differences compared to KdV

- The wave profile $u(x,t)$ is not smooth, so the solution must be interpreted in a suitable weak sense (more about that later).
- One knows exactly where all solitons are all the time.
- The problem is reduced to solving a set of ODEs instead of a PDE.
- Since this makes everything finite-dimensional, the inverse spectral techniques become much more elementary. (Ideal for teaching!)

Camassa–Holm peakons

The solution for $n = 2$ is

$$\begin{aligned} x_1(t) &= \log \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2} \\ x_2(t) &= \log(b_1 + b_2) \\ m_1(t) &= \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)} \\ m_2(t) &= \frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2} \end{aligned}$$

where $b_k(t) = b_k(0)e^{t/\lambda_k}$. The constants λ_1, λ_2, $b_1(0)$, $b_2(0)$ are uniquely determined by initial conditions.

(This is the form that the solution takes when one uses inverse spectral methods. Camassa & Holm wrote it a little differently.)

- The *eigenvalues* λ_k are real, simple, nonzero. The number of positive eigenvalues equals the number of positive m_k's.
- The quantities b_k (*residues of the Weyl function*) are always positive.

The terminology comes from the inverse spectral solution method, and will be explained a little later.

Reference:

Richard Beals, David Sattinger, and Jacek Szmigielski
Multipeakons and the classical moment problem
Advances in Mathematics (2000)

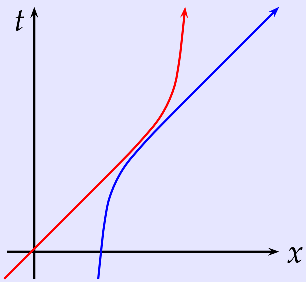
The CH solution for $n = 3$ is

$$
\begin{aligned}
x_1(t) &= \log \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k} \\
x_2(t) &= \log \frac{\sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3} \\
x_3(t) &= \log(b_1 + b_2 + b_3) \\
m_1(t) &= \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1 \lambda_2 \lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k} \\
m_2(t) &= \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k} \\
m_3(t) &= \frac{b_1 + b_2 + b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}
\end{aligned}
$$

The solution for general n looks similar, but to write it down one needs a bit of notation for symmetric functions.

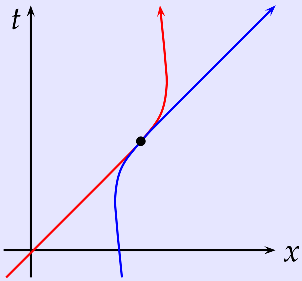
(Camassa–Holm $n = 2$ continued)

Typical plots of $x_1(t)$ and $x_2(t)$ in the (x, t) plane:



$\lambda_1 = 1$ and $\lambda_2 = 10$
(two peakons)

$x_1(t) < x_2(t)$ for all t.

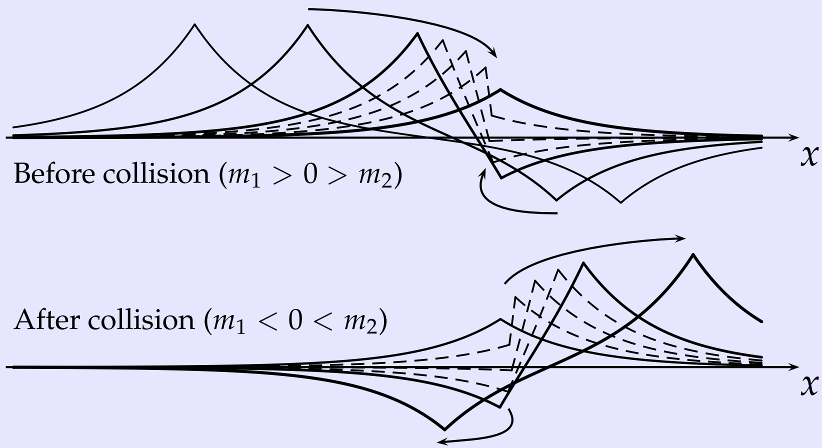


$\lambda_1 = 1$ and $\lambda_2 = -10$
(peakon and antipeakon)

$x_1(t) < x_2(t)$ except at the instant of collision.

The asymptotic speeds as $t \to \pm\infty$ are $1/\lambda_1$ and $1/\lambda_2$.

CH peakon-antipeakon collision: $(\frac{1}{\lambda_1} \approx 1.5, \frac{1}{\lambda_2} \approx -0.86)$



The individual peakon amplitudes $m_1(t)$ and $m_2(t)$ both blow up at the instant of collision, one to $+\infty$ and the other to $-\infty$, but in such a way that the infinities cancel and $u = \sum m_i e^{-|x-x_i|}$ remains continuous. (However, u_x blows up.)

How does one find the n-peakon solution?

Consider a vibrating string whose deflection $U(y,t)$ is governed by the usual linear wave equation $g(y)U_{tt} = U_{yy}$, where $g(y)$ is the mass density distribution. Assume ends fixed at $y = \pm 1$.

Separation of variables $U(y,t) = \phi(y)\psi(t)$ yields the string's vibrational modes via the spectral problem

$$
\begin{aligned}
-\phi''(y) &= z\, g(y)\, \phi(y) \qquad \text{for } -1 < y < 1 \\
\phi(-1) &= 0 \qquad \phi(1) = 0
\end{aligned}
$$

The usual case that we teach our students is when the density $g(y)$ is **constant**; then there is an infinite sequence of eigenvalues, and the eigenfunctions are sinusoidal. (Think of the harmonics of a guitar string.)

Here we'll consider the opposite extreme: isolated **point masses**.

To a given peakon configuration $\{x_k, m_k\}$ we associate a **discrete** measure $g(y) = \sum_1^n g_i\, \delta_{y_i}$ on the interval $(-1, 1)$ with

$$y_i = \tanh(x_i/2) \qquad\qquad g_i = 2m_i/(1 - y_i^2)$$

(**Point masses** g_i at positions y_i connected by massless string.)

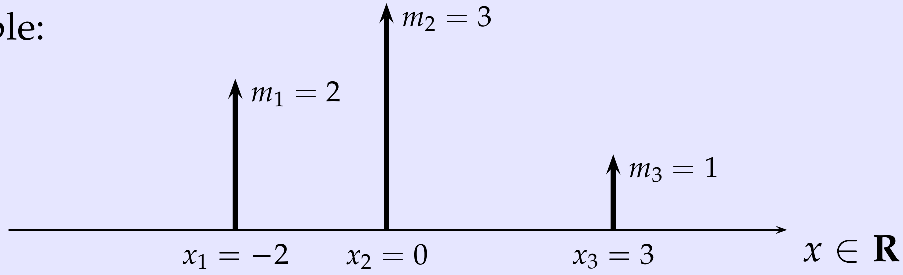
Such a ***discrete string*** has exactly n eigenvalues $z = \lambda_1, \ldots, \lambda_n$ and the corresponding eigenfunctions $\phi_k(y)$ are piecewise linear.

The quantities b_k in the peakon solution formulas are the residues of the (modified) Weyl function:

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)} = \frac{1}{2z} + \sum_{k=1}^{n} \frac{b_k}{z - \lambda_k}$$

In other words, b_k is the *coupling coefficient* $\phi_k'(1)/\phi_k'(-1)$ of the kth eigenfunction, divided by the factor $-2\prod_{j\neq k}(1 - \lambda_k/\lambda_j)$. So the b_k's encode some information about the shape of the eigenfunctions.

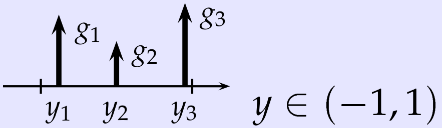
Example:



$y_1 \approx -0.762$ $\quad g_1 \approx 9.52$

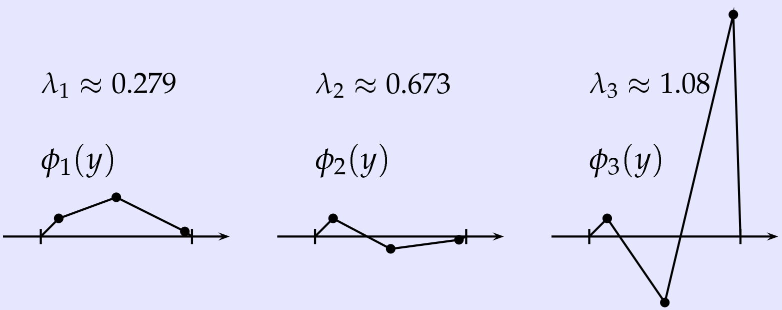
$y_2 = 0$ $\quad g_2 = 6$

$y_3 \approx 0.905$ $\quad g_3 \approx 11.07$



$\lambda_1 \approx 0.279$ $\quad \lambda_2 \approx 0.673$ $\quad \lambda_3 \approx 1.08$

$\phi_1(y)$ $\quad \phi_2(y)$ $\quad \phi_3(y)$



Crucial fact (thanks to the Lax pair associated with the CH eqn):

> The CH peakons move in such a way that the spectral data of the corresponding discrete string satisfy

$$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k / \lambda_k$$

So we determine the spectral data of the string corresponding to the initial peakon configuration (at time $t = 0$), and let it evolve in time as above. Now if we can recover the string data from the spectral data at a later time $t > 0$, then we also obtain the peakon configuration at that time, which is what we seek.

This inverse problem of determining the mass distribution of a discrete string given the spectral data was solved long ago (*analytic continued fractions* T. Stieltjes 1895, *string interpretation* M. Krein 1951).

Here is how Stieltjes continued fractions enter:

Let $l_k = y_{k+1} - y_k$. Propagate $\phi(y;z)$ from the left endpoint $y = -1$ using $-\phi''(y) = z\,g(y)\,\phi(y)$. Then ϕ is continuous and piecewise linear, with jumps in the slope ϕ' where the point masses are. Keeping track of ϕ and ϕ', one finds at the right endpoint $y = +1$ that

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)} = \cfrac{1}{zl_n + \cfrac{1}{-g_n + \cfrac{1}{zl_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{-g_2 + \cfrac{1}{zl_1 + \cfrac{1}{-g_1 + \cfrac{1}{zl_0}}}}}}}}$$

Stieltjes gave formulas for the coefficients in such a continued fraction expansion of a meromorphic function $f(z)$, in terms of the coefficients in its expansion $f(z) = \sum_{j=0}^{\infty}(-1)^j A_j z^{-(j+1)}$ around $z = \infty$. Since we know λ_k and b_k in

$$\frac{W(z)}{z} = \frac{1}{2z} + \sum_{k=1}^{n} \frac{b_k}{z - \lambda_k}$$

we can expand each term in a geometric series to get the expansion around $z = \infty$. Then Stieltjes' formulas give us the coefficients l_k and g_k in the continued fraction for $W(z)/z$.

Using this, one obtains explicit formulas for the general peakon solution $\{x_k(t), m_k(t)\}$ for any n.

J. Moser (1975) showed (in the case $n = 3$) how Stieltjes' results give the solution of the n-particle nonperiodic Toda lattice. The Toda lattice and the CH peakons are special cases of a more general construction due to Beals–Sattinger–Szmigielski (2001).

Degasperis–Procesi peakons

The solution for $n = 2$ is

$$
\begin{aligned}
x_1(t) &= \log \frac{\frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2} \\
x_2(t) &= \log(b_1 + b_2) \\
m_1(t) &= \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1 \lambda_2 \left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2 \right)} \\
m_2(t) &= \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2}
\end{aligned}
$$

with $b_k(t) = b_k(0) e^{t/\lambda_k}$ as before, but with the spectral data now coming from a "discrete cubic string" instead of an ordinary string.

References:

Hans Lundmark and Jacek Szmigielski
Multi-peakon solutions of the Degasperis–Procesi equation
Inverse Problems (2003)

Hans Lundmark and Jacek Szmigielski
Degasperis–Procesi peakons and the discrete cubic string
International Mathematics Research Papers (2005)

Jennifer Kohlenberg, Hans Lundmark, and Jacek Szmigielski
The inverse spectral problem for the discrete cubic string
Inverse Problems (2007)

The DP solution for $n = 3$ is

$$x_1(t) = \log \frac{U_3}{V_2} \quad x_2(t) = \log \frac{U_2}{V_1} \quad x_3(t) = \log U_1$$

$$m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1} \quad m_3(t) = \frac{(U_1)^2}{W_1}$$

where

$$U_1 = b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$U_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3$$

$$V_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3$$

$$U_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3 \qquad V_3 = \lambda_1 \lambda_2 \lambda_3 U_3$$

$$W_1 = U_1 V_1 - U_2 = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4\lambda_2 \lambda_3}{\lambda_2 + \lambda_3} b_2 b_3$$

$$W_2 = U_2 V_2 - U_3 V_1 = \frac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2} \lambda_1 \lambda_2 (b_1 b_2)^2 + \cdots + \frac{4\lambda_1 \lambda_2 \lambda_3 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 b_1^2 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} + \dots$$

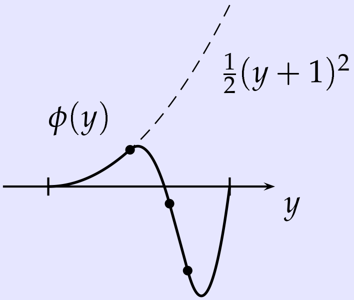
By the *cubic string* we mean the following spectral problem:

$$-\phi'''(y) = z\, g(y)\, \phi(y) \qquad \text{for } -1 < y < 1$$
$$\phi(-1) = \phi'(-1) = 0 \qquad \phi(1) = 0$$

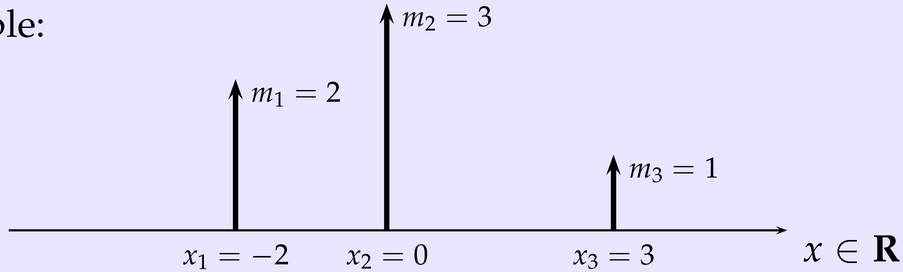
The *discrete cubic string* associated to a DP peakon configuration $\{x_k, m_k\}$ has $g(y) = \sum_1^n g_i\, \delta_{y_i}$ with

$$y_i = \tanh \frac{x_i}{2} \qquad g_i = \frac{8 m_i}{(1 - y_i^2)^2}$$

The eigenfunctions are now piecewise quadratic polynomials in y, since $\phi''' = 0$ away from the support of g.



Example:

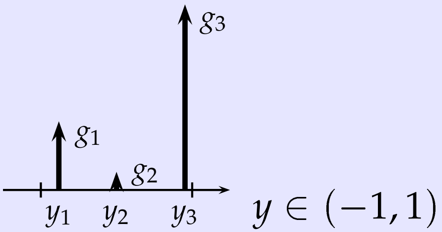


$y_1 \approx -0.762$ $\quad g_1 \approx 90.7$

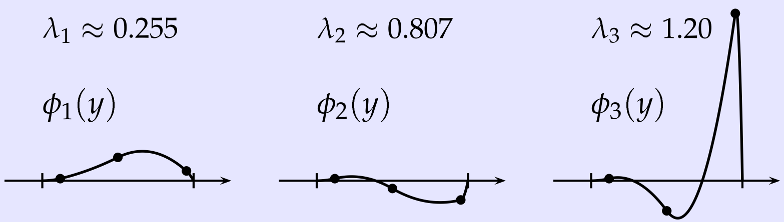
$y_2 = 0$ $\quad g_2 = 24$

$y_3 \approx 0.905$ $\quad g_3 \approx 245$

(as before) (different)



$\lambda_1 \approx 0.255$ $\quad \lambda_2 \approx 0.807$ $\quad \lambda_3 \approx 1.20$



The DP peakons move such that the spectral data of the corresponding discrete cubic string satisfy

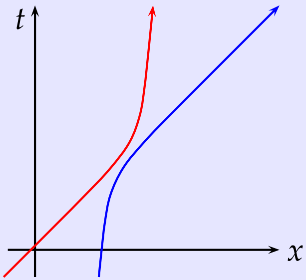
$$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k / \lambda_k$$

(Here b_k equals the relevant coupling coefficient $\phi_k'(1)/\phi_k''(-1)$ divided by the factor $-2\prod_{j \neq k}(1 - \lambda_k/\lambda_j)$.)

The solution formulas for $x_k(t)$ and $m_k(t)$ hence follow from the solution of the inverse problem for the discrete cubic string, which is much more involved than for the ordinary string.

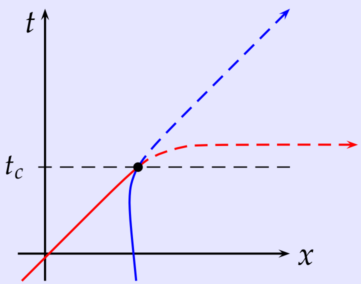
Even the *forward* spectral problem is more complicated, since it is not selfadjoint. (The Gantmacher–Krein theory of oscillatory kernels shows that the spectrum is positive and simple, at least for positive mass distributions.)

(Degasperis–Procesi $n = 2$ continued)



$\lambda_1 = 1$ and $\lambda_2 = 10$
(two peakons)

$x_1(t) < x_2(t)$ for all t.

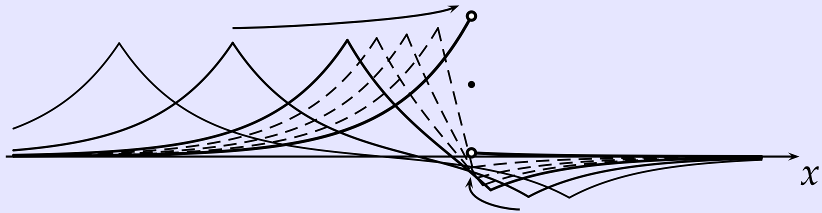


$\lambda_1 = 1$ and $\lambda_2 = -10$
(peakon and antipeakon)

Transversal collision!
(Not tangential as for CH.)

The solution formulas are only valid up to the time of collision t_c since they were derived under the assumption that $|x_1 - x_2|$ can be replaced by $x_2 - x_1$ in the ODEs. Can the solution be continued past the collision?

DP peakon-antipeakon collision:



We see that $u(x,t)$ tends to a discontinuous function as $t \nearrow t_c$. In other words, a *shock* is formed.

- Why is the DP case different from the CH case?
- How does the solution continue?

References:

Giuseppe Coclite and Kenneth Hvistendahl Karlsen
On the well-posedness of the Degasperis–Procesi equation
Journal of Functional Analysis (2006)

Hans Lundmark
Formation and dynamics of shock waves in the Degasperis–Procesi equation
Journal of Nonlinear Science (2007)

Inverting $m = u - u_{xx}$ as $u = \frac{1}{2} G * m$ where $G(x) = e^{-|x|}$, one can formally rewrite the b-equation as a conservation law:

$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(\tfrac{b}{2} u^2 + \tfrac{3-b}{2} u_x^2 \right) \right] = 0$$

After multiplying by a test function and integrating by parts, one obtains a rigorous definition of what weak solutions (including peakons) really mean for this family of equations.

Now a difference between CH and DP emerges:

$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(u^2 + \tfrac{1}{2} u_x^2 \right) \right] = 0 \qquad (\text{CH}, b = 2)$$

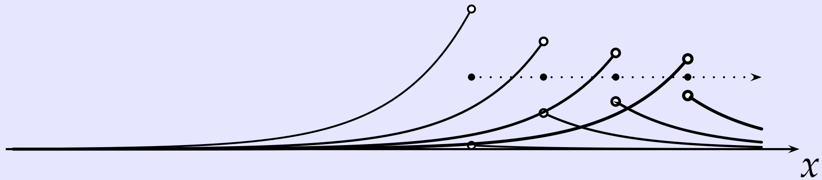
$$u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} G * \left(\tfrac{3}{2} u^2 \right) \right] = 0 \qquad (\text{DP}, b = 3)$$

Since DP does not involve u_x explicitly it is reasonable that it also admits solutions where u (and not just u_x) has jumps.

Coclite and Karlsen: For initial data $u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R})$ there is a unique $u \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}))$ which satisfies DP (in the above weak sense) together with an additional "entropy condition".

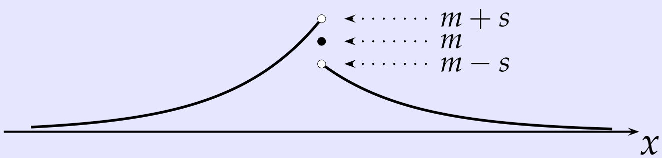
DP shockpeakons

Here is the unique entropy solution with the shock formed at the DP peakon-antipeakon collision as initial data:



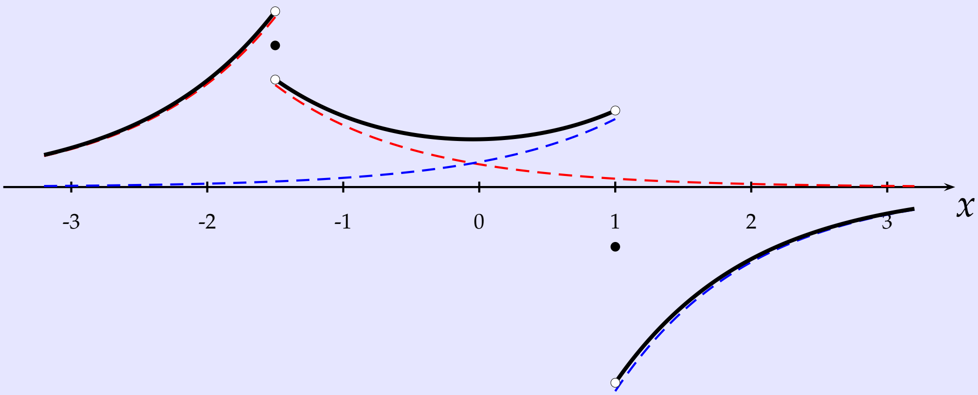
It's a single *shockpeakon*, with shape given by

$$m\,G(x) + s\,G'(x) = m\,e^{-|x|} + s\,\mathrm{sgn}(-x)\,e^{-|x|} = \begin{cases} (m+s)\,e^{x} & (x<0) \\ m & (x=0) \\ (m-s)\,e^{-x} & (x>0) \end{cases}$$



Natural idea: try superposition!

Superposition (solid curve) of two shockpeakons (dashed curves) with $x_1 = -\frac{3}{2}$, $m_1 = 1$, $s_1 = \frac{1}{4}$ and $x_2 = 1$, $m_2 = -\frac{1}{2}$, $s_2 = 1$ looks like this:



Plug a shockpeakon superposition Ansatz into the DP eqn and compute, and you will get…

Theorem: The n-shockpeakon superposition

$$u(x,t) = \sum_{i=1}^{n} m_k(t)\, G(x - x_k(t)) + \sum_{i=1}^{n} s_k(t)\, G'(x - x_k(t))$$

satisfies the DP equation iff

$$\begin{aligned} \dot{x}_k &= u(x_k) \\ \dot{m}_k &= 2\big(s_k\{u_{xx}(x_k)\} - m_k\{u_x(x_k)\}\big) \\ \dot{s}_k &= -s_k\{u_x(x_k)\} \end{aligned}$$

(The entropy condition holds iff $s_k \geq 0$ for all k.)

Here $G(x) = e^{-|x|}$ with $G'(0) := 0$, and curly brackets denote the nonsingular part:

$$u(x_k) = \{u_{xx}(x_k)\} = \sum_{i=1}^{n} m_i\, G(x_k - x_i) + \sum_{i=1}^{n} s_i\, G'(x_k - x_i)$$

$$\{u_x(x_k)\} = \sum_{i=1}^{n} m_i\, G'(x_k - x_i) + \sum_{i=1}^{n} s_i\, G(x_k - x_i)$$

For $n = 1$ we get

$$\dot{x}_1 = m_1 \qquad \dot{m}_1 = 0 \qquad \dot{s}_1 = -s_1^2$$

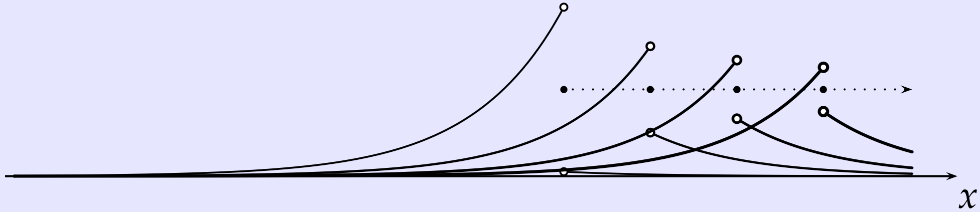
which is a shock wave with constant speed (equal to the average height m_1 at the jump; cf. Rankine–Hugoniot condition).

The jump is $[u] = -2s_1$ where

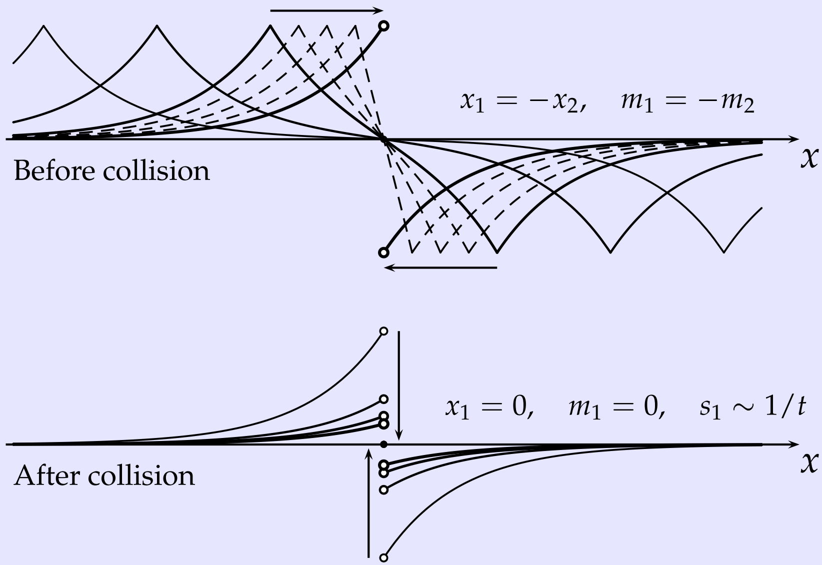
$$s_1(t) = \frac{s_1(t_0)}{1 + (t - t_0)\, s_1(t_0)}$$

so that the shock “dissipates away” like $1/t$ as $t \to \infty$.

This is of course the single shockpeakon shown earlier:



The totally symmetric DP peakon-antipeakon collision results in a stationary shockpeakon (zero momentum):



The only obvious constant of motion is $M = \sum m_k$ so we are still quite far from finding an explicit solution of the shock-peakon ODEs, even in the case $n = 2$:

$$
\begin{aligned}
\dot{x}_1 &= m_1 + (m_2 + s_2)R \\
\dot{x}_2 &= m_2 + (m_1 - s_1)R \\
\dot{m}_1 &= -2(m_1 - s_1)(m_2 + s_2)R \\
\dot{m}_2 &= +2(m_1 - s_1)(m_2 + s_2)R \\
\dot{s}_1 &= -s_1^2 - s_1(m_2 + s_2)R \\
\dot{s}_2 &= -s_2^2 + s_2(m_1 - s_1)R
\end{aligned}
$$

(Assume $x_1 < x_2$ and set $R = e^{x_1 - x_2}$)

Is this system even integrable?

(The DP Lax pair involves $m = u - u_{xx}$ and doesn't seem to make sense in this weak setting.)

Numerical experiments show: small shocks $\Rightarrow$ business as usual, large shocks $\Rightarrow$ new phenomena appear. A little bit more can be said in particular cases:

- Antisymmetric 2-shockpeakon case.

  $0 = x_1 + x_2 = m_1 + m_2 = s_1 - s_2.$

  Found additional constant of motion K. In the subcase $K = 0$ the system can be integrated in terms of the inverse of the function $x \mapsto \int_1^{\exp x} (r^2 - 1) e^{r^2/2} dr$. Moral: Can't hope for solution formulas as simple as in the shockless case.

- Symmetric peakon-antipeakon with stationary shockpeakon in the middle ($\Rightarrow$ triple collision).

  Test case used in Coclite–Karlsen–Risebro: *Numerical schemes for computing discontinuous solutions of the Degasperis–Procesi equation* (preprint 2006).

THE END