APPLIED INTEGER PROGRAMMING, MODELING AND SOLUTION

CHAPTERS 14.4-14.15

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CHAPTER 14.5: LAGRANGIAN DUAL

LAGRANGIAN RELAXATION

$$\min \quad \mathbf{cx}$$

$$s.t. \quad \mathbf{A_1 x} \geq \mathbf{b_1}$$ Complicated constraints.

$$\mathbf{A_2 x} \geq \mathbf{b_2}$$ Easy constraints.

$$\mathbf{x} \in Z_+^n$$

(IP)

LAGRANGIAN RELAXATION

$$\begin{aligned} \min \quad & \mathbf{cx} \\ s.t. \quad & \mathbf{A_1 x} \geq \mathbf{b_1} \\ & \mathbf{A_2 x} \geq \mathbf{b_2} \\ & \mathbf{x} \in Z_+^n \end{aligned}$$

(IP)

$$\begin{aligned} \min \quad & \mathbf{cx} + \mathbf{u}(\mathbf{b_1} - \mathbf{A_1 x}) \\ s.t. \quad & \mathbf{A_2 x} \geq \mathbf{b_2} \\ & \mathbf{x} \in Z_+^n \end{aligned}$$

(LRIP)

LAGRANGIAN RELAXATION

NOTE: Less than or equal to zero if original IP feasible!

$$\begin{aligned} \min \quad & \mathbf{cx} \\ s.t. \quad & \mathbf{A}_1\mathbf{x} \geq \mathbf{b}_1 \\ & \mathbf{A}_2\mathbf{x} \geq \mathbf{b}_2 \\ & \mathbf{x} \in Z_+^n \end{aligned} \quad \geq \quad \begin{aligned} \min \quad & \mathbf{cx} + \mathbf{u}(\mathbf{b}_1 - \mathbf{A}_1\mathbf{x}) \\ s.t. \quad & \mathbf{A}_2\mathbf{x} \geq \mathbf{b}_2 \\ & \mathbf{x} \in Z_+^n \end{aligned}$$

(IP) (LRIP)

LAGRANGIAN RELAXATION

NOTE: Less than or equal to zero if original IP feasible!

$$\begin{aligned} \min \quad & \mathbf{cx} \\ s.t. \quad & \mathbf{A_1 x} \geq \mathbf{b_1} \\ & \mathbf{A_2 x} \geq \mathbf{b_2} \\ & \mathbf{x} \in Z_+^n \end{aligned} \quad \geq \quad \begin{aligned} \min \quad & \mathbf{cx} + \mathbf{u}(\mathbf{b_1} - \mathbf{A_1 x}) \\ s.t. \quad & \mathbf{A_2 x} \geq \mathbf{b_2} \\ & \mathbf{x} \in Z_+^n \end{aligned}$$

So the Lagrangian relaxation optimal value can be used as a **lower bound** to the original IP. But we want a tight bound…

LAGRANGIAN DUAL PROBLEM

…and to find the largest lower bound we solve the *Lagrangian dual problem*.

$$\max_{\mathbf{u}\geq 0} \min_{\mathbf{x}\in \mathbf{Z}_+^n} \mathbf{cx} + \mathbf{u}(\mathbf{b_1} - \mathbf{A_1 x})$$

$$s.t. \quad \mathbf{A_2 x} \geq \mathbf{b_2}$$

LAGRANGIAN DUAL PROBLEM

…and to find the largest lower bound we solve the *Lagrangian dual problem*.

$$\max_{\mathbf{u}\geq 0} \min_{\mathbf{x}\in \mathbf{Z}_+^n} \mathbf{cx} + \mathbf{u}(\mathbf{b_1} - \mathbf{A_1 x})$$

$$s.t. \quad \mathbf{A_2 x} \geq \mathbf{b_2}$$

Dual variable aka Lagrange multiplier

LAGRANGIAN DUAL PROBLEM

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$$\max_{\mathbf{u} \geq 0} \min_{\mathbf{x} \in \mathbf{Z}_+^n} \mathbf{cx} + \mathbf{u}(\mathbf{b_1} - \mathbf{A_1 x})$$

$$s.t. \quad \mathbf{A_2 x} \geq \mathbf{b_2}$$

Dual variable aka Lagrange multiplier

$$\begin{pmatrix} \max\limits_{u \geq 0} \quad \mathbf{ub_1} + \min\limits_{x \in \mathbf{Z}_+^n} (\mathbf{c} - \mathbf{uA_1})\mathbf{x} \\ s.t. \quad \mathbf{A_2 x} \geq \mathbf{b_2} \end{pmatrix}$$

LAGRANGIAN DUAL PROBLEM

Proposition 14.8: Given a specific $\mathbf{u} \in R_+^{m_1}$, if the following two conditions are satisfied, then the optimal vector $\mathbf{x}^*(\mathbf{u})$ of LRIP($\mathbf{u}$) is optimal for IP:

1. $\mathbf{A}_1\mathbf{x}^*(u) \geq \mathbf{b}_1$

2. $\left(\mathbf{A}_1\mathbf{x}^*(u) \geq \mathbf{b}_1\right)_i = \mathbf{b}_{1i}$ whenever $\mathbf{u_i} > 0$

EQUALITY INSTEAD OF INEQUALITY

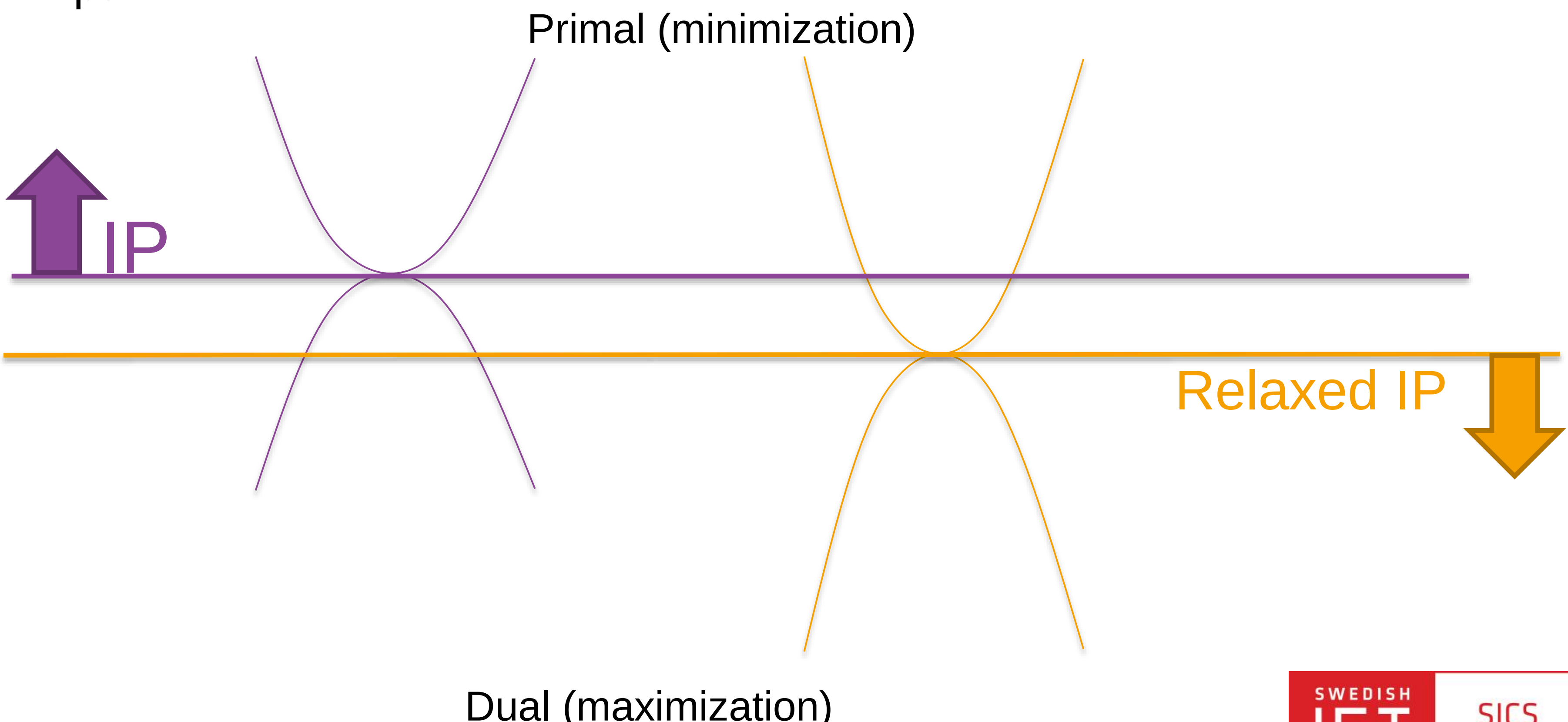
No sign restriction on u!

$$\min \quad \mathbf{cx}$$
$$s.t. \quad \mathbf{A_1 x} = \mathbf{b_1}$$
$$\mathbf{A_2 x} = \mathbf{b_2}$$
$$\mathbf{x} \in Z_+^n$$

$$\max_u \quad \mathbf{ub_1} + \min_{x \in \mathbf{Z}_+^n} (\mathbf{c} - \mathbf{uA_1})\mathbf{x}$$
$$s.t. \quad \mathbf{A_2 x} \geq \mathbf{b_2}$$

LP DUAL OF IP

Proposition 14.5: The integer program (IP) and the dual of its linear programming relaxation (Relaxed IP) form a weak dual pair.



CHAPTER 14.6: PRIMAL –DUAL SOLUTION VIA BENDERS' PARTITIONING

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

$$\begin{aligned} \min\ & \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y} \\ s.t.\ & \mathbf{A_1}\mathbf{x} + \mathbf{A}_2\mathbf{y} \geq \mathbf{b_1} \\ & \mathbf{D}\mathbf{y} \geq \mathbf{b_2} \\ & \mathbf{x} \geq 0, \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

MIPs are hard. Let's try to partition it.

SWEDISH ICT SICS

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

$$\begin{aligned}\min\ & \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y} \\ s.t.\ & \mathbf{A_1 x} + \mathbf{A}_2\mathbf{y} \geq \mathbf{b_1} \\ & \mathbf{Dy} \geq \mathbf{b_2} \\ & \mathbf{x} \geq 0,\ \mathbf{y} \geq 0 \text{ and integer}\end{aligned}$$

MASTER

$$\begin{aligned}\min\ & \mathbf{c}_2\mathbf{y} + q(\mathbf{y}) \\ s.t.\ & \mathbf{Dy} \geq \mathbf{b_2} \\ & \mathbf{y} \geq 0 \text{ and integer}\end{aligned}$$

SUB-PROBLEM

$$\begin{aligned}q(y) = \min\ & \mathbf{c}_1\mathbf{x} \\ s.t.\ & \mathbf{A_1 x} \geq \mathbf{b}_1 - \mathbf{A}_2\mathbf{y} \\ & \mathbf{x} \geq 0\end{aligned}$$

SWEDISH ICT SICS

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

$$\begin{aligned} \min\ & \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y} \\ s.t.\ & \mathbf{A_1x} + \mathbf{A}_2\mathbf{y} \geq \mathbf{b_1} \\ & \mathbf{Dy} \geq \mathbf{b_2} \\ & \mathbf{x} \geq 0, \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

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It'd be great if the y's weren't in the constraints of q(y).

DUAL!

DUAL OF SUB-PROBLEM

SUB-PROBLEM

$$q(y) = \min \ \mathbf{c}_1\mathbf{x}$$
$$s.t. \quad \mathbf{A}_1\mathbf{x} \geq \mathbf{b}_1 - \mathbf{A}_2\mathbf{y}$$
$$\mathbf{x} \geq 0$$

$$\max \quad \mathbf{u}(\mathbf{b}_1 - \mathbf{A}_2\mathbf{y}')$$
$$s.t. \quad \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1$$
$$\mathbf{u} \geq 0$$

DUAL OF SUB-PROBLEM

SUB-PROBLEM

$$q(y) = \min \ \mathbf{c}_1\mathbf{x}$$
$$s.t. \quad \mathbf{A_1 x} \geq \mathbf{b}_1 - \mathbf{A}_2\mathbf{y}$$
$$\mathbf{x} \geq 0$$

Avoid unboundedness.

$$\max \quad \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}')$$
$$s.t. \quad \mathbf{uA}_1 \leq \mathbf{c}_1$$
$$\mathbf{uE} \leq \mathbf{M}$$
$$\mathbf{u} \geq 0$$

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

1. For every $\mathbf{y}'$ we will pick out a vertex $\mathbf{u_i}$ of the dual feasible solution space. From weak duality:

$$q(\mathbf{y}) \geq \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y})$$

<u>Dual of sub-problem</u>

$$\begin{aligned} \max \quad & \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}') \\ s.t. \quad & \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1 \\ & \mathbf{u}\mathbf{E} \leq \mathbf{M} \\ & \mathbf{u} \geq 0 \end{aligned}$$

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

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$$q(\mathbf{y}) \geq \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y})$$

2. So, we have that the master objective

$$z = \mathbf{c_2 y} + q(\mathbf{y}) \geq \mathbf{c_2 y} + \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y})$$

Dual of sub-problem

$$\max \quad \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}')$$
$$s.t. \quad \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1$$
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SWEDISH ICT SICS

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

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$$z = \mathbf{c_2}\mathbf{y} + q(\mathbf{y}) \geq \mathbf{c_2}\mathbf{y} + \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y})$$

3. ADD CONSTRAINT TO MASTER!

Dual of sub-problem

$$\begin{aligned} \max \quad & \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}') \\ s.t. \quad & \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1 \\ & \mathbf{u}\mathbf{E} \leq \mathbf{M} \\ & \mathbf{u} \geq 0 \end{aligned}$$

Master

$$\begin{aligned} \min \quad & \mathbf{c}_2\mathbf{y} + q(\mathbf{y}) \\ s.t. \quad & \mathbf{D}\mathbf{y} \geq \mathbf{b_2} \\ & \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

PRIMAL-DUAL SOLUTION VIA BENDERS' PARTITIONING

For every **y**' we will pick out a vertex $\mathbf{u_i}$ of the dual feasible solution space. From weak duality:

$$q(\mathbf{y}) \geq \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y})$$

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3. ADD CONSTRAINT TO MASTER!

$$\begin{aligned} &\min\ z \\ &s.t.\ \ z \geq \mathbf{c}_2\mathbf{y} + \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}) \quad \forall i = 1 \ldots T \\ &\qquad \mathbf{Dy} \geq \mathbf{b_2} \\ &\qquad \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

Dual of sub-problem

$$\begin{aligned} &\max\ \ \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}') \\ &s.t.\ \ \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1 \\ &\qquad \mathbf{uE} \leq \mathbf{M} \\ &\qquad \mathbf{u} \geq 0 \end{aligned}$$

Master

$$\begin{aligned} &\min\ \mathbf{c}_2\mathbf{y} + q(\mathbf{y}) \\ &s.t.\ \ \mathbf{Dy} \geq \mathbf{b_2} \\ &\qquad \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

THE MASTER

$$\min \; z$$

$$s.t. \;\; z \geq \mathbf{c}_2\mathbf{y} + \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}) \quad \forall i = 1 \ldots T$$

$$\mathbf{Dy} \geq \mathbf{b_2}$$

$$\mathbf{y} \geq 0 \text{ and integer}$$

The number of vertices (T) can be very large, so instead of solving the Master with all T, start with one and then **generate** more constraints as needed.

BENDERS' ALGORITHM FOR MIP

Step 0: Initialization

Set $t = 1, B_u = +\infty$ and select some ε (convergence criterion). Select some $\mathbf{u}^1$ that is feasible for the dual sub-problem.

Step1: Solve master in iteration t

Solve the relaxed pure integer program:

$$\begin{aligned} &\min \ z \\ &s.t. \ \ z \geq \mathbf{c}_2\mathbf{y} + \mathbf{u}_i(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}) \quad \forall i = 1 \ldots t \\ &\qquad \mathbf{Dy} \geq \mathbf{b_2} \\ &\qquad \mathbf{y} \geq 0 \text{ and integer} \end{aligned}$$

Let z^t and $\mathbf{y}^t$ be the solution. If z is unbounded from below, take $\mathbf{y}^t$ to be some value that gives z^t some arbitrarily large negative value.

BENDERS' ALGORITHM FOR MIP

Step 2: Solve sub-problem in iteration t

Generate the most violated constraint of IP by solving the linear program:

$$\begin{aligned} \max \quad & \mathbf{u}(\mathbf{b_1} - \mathbf{A}_2\mathbf{y}') \\ s.t. \quad & \mathbf{u}\mathbf{A}_1 \leq \mathbf{c}_1 \\ & \mathbf{u}\mathbf{E} \leq \mathbf{M} \\ & \mathbf{u} \geq 0 \end{aligned}$$

Let the solution of this LP be optimal value u_0^{t+1} at $\boldsymbol{u}^{t+1}$.

Step 3: Optimality check

Check convergence criterion. Set $B_u = min\{B_u, u_0^{t+1} + \boldsymbol{c}_1\boldsymbol{y}^t\}$. If $z^t > B_u - \varepsilon$, stop; the optimal solution has been reached. Otherwise, add the constraint $z \geq \boldsymbol{c}_2\boldsymbol{y} + \boldsymbol{u}^{t+1}(\boldsymbol{b}_1 - \boldsymbol{A}_2\boldsymbol{y})$ to the master problem. Set $t = t + 1$. Return to step 1.