9.1 Linear Programs in canonical form

LP in standard form:

$$(LP)\begin{cases}\max & z=\sum_j c_j x_j\\ s.t. & \sum_j a_{ij}x_j \le b_i \quad \forall i=1,\dots,m\\ & x_j \ge 0 \quad \forall j=1,\dots,n\end{cases}$$

where $b_i \in \mathbb{R}, \quad \forall i=1,\dots,m$

But the Simplex method works only on systems of equations!

Introduce nonnegative slack variables s_i for each constraint i and convert the standard form into a system of equations.

9.1 Linear Programs in canonical form

New LP formulation:

$$
(LP)\begin{cases}
max & z & (1a)\\
s.t. & z - \sum_j c_j x_j = 0 & (1b)\\
 & \sum_j a_{ij} x_j + s_i = b_i \quad \forall i = 1, \ldots, m & (1c)\\
 & x_j \geq 0 \quad \forall j = 1, \ldots, n & (1d)\\
 & s_i \geq 0 \quad \forall i = 1, \ldots, m & (1e)
\end{cases}
$$

where $b_i \in \mathbb{R}, \quad \forall i = 1, \ldots, m$. This is also called *canonical form*.

Solving a LP may be viewed as performing the following three tasks

1. Find solutions to the augumented system of linear equations in 1b and 1c.
2. Use the nonnegative conditions (1d and 1e) to indicate and maintain the feasibility of a solution.
3. Maximize the objective function, which is rewritten as equation 1a.

9.2 Basic feasible solutions and reduced costs

Definitions

Given that a system $\mathbf{Ax} = \mathbf{b}$, where the numbers of solutions are infinite, and $rank(\mathbf{A}) = m$ $(m < n)$, a unique solution can be obtained by setting any $n - m$ variables to 0 and solving for the remaining system of m variables in m equations. Such a solution, if it exists, is called a *basic solution*. The variables that are set to 0 are called *nonbasic variables*, denoted by $\mathbf{x_N}$. The variables that are solved are called *basic variables*, denoted by $\mathbf{x_B}$. A basic solution that contains all nonnegative values is called a *basic feasible solution*. A basic solution that contains any negative component is called a *basic infeasible solution*. The $m \times n$ coefficient matrix associated with a give set of basic variables is called a *basis*, or a *basis matrix*, and is denoted as $\mathbf{B}$. The number of basic solutions possible in a system of m equations in n variables is calculated by

$$C_m^n = \frac{n!}{m!(n-m)!}$$

9.2 Basic feasible solutions and reduced costs

$$(LP)\begin{cases} \max & \mathbf{c_B}^T\mathbf{x_B} + \mathbf{c_N}^T\mathbf{x_N} \\ s.t. & \mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b} \\ & \mathbf{x_B}, \mathbf{x_N} \geq \mathbf{0} \end{cases} \tag{2}$$

Example:

Consider

$$x_1 + x_2 + x_3 = 6 \tag{3}$$

$$2x_1 + x_2 + x_4 = 8 \tag{4}$$

The system has six basic solutions displayed below:

	Basic Solution					
	1	2	3	4	5	6
Nonbasic variables $\mathbf{x_N}$	$x_1=0,$ $x_2=0$	$x_1=0,$ $x_3=0$	$x_1=0,$ $x_4=0$	$x_2=0,$ $x_3=0$	$x_2=0,$ $x_4=0$	$x_3=0,$ $x_4=0$
Basic variables $\mathbf{x_B}$	$x_3=6,$ $x_4=8$	$x_2=6,$ $x_4=2$	$x_2=8,$ $x_3=-2$	$x_1=6,$ $x_4=-4$	$x_1=4,$ $x_3=2$	$x_1=2,$ $x_2=4$

Infeasible!

$\exists x_i < 0$

9.2 Basic feasible solutions and reduced costs

Definitions:

A nonbasic variable is called an *entering variable* if it is selected to become basic in the next basis. Its associated coefficient column is called a *pivot column*. A basic variable is called a *leaving variable* if it's selected to become nonbasic in the next basis. Its associated coefficient row is called a *pivot row*. The element that intersects a pivot column and a pivot row is called a *pivot* or *pivot element*. A *pivoting operation* is a sequence of elementary row operations that makes the pivot element 1 and all other elements 0 in the pivot column. Two basic feasible solution is said to be *adjacent* if the set of their basic variables differ by only one basic variable.

Have constraint matrix:

$$\mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b}$$

By performing row operations we obtain:

$$\mathbf{I}\mathbf{x_B} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N} = \mathbf{B}^{-1}\mathbf{b} \;\Rightarrow\; \mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N}$$

Substituting into $z = \mathbf{c_B}^T\mathbf{x_B} + \mathbf{c_N}^T\mathbf{x_N}$ we have

$$z = (\mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T)\mathbf{x_N}$$

Reduced space and reduced costs

The subspace that contains only the nonbasic variables is referred to a *reduced space*. The components of the objective row in a reduced space are called *reduced costs*, denoted by $\bar{\mathbf{c}}$:

$$\bar{\mathbf{c}}^T = (\bar{\mathbf{c}}_\mathbf{B}^T, \bar{\mathbf{c}}_\mathbf{N}^T) = (\mathbf{0}^T, \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T)$$

9.3 The Simplex Method

The Simplex method consists of three steps:

1. *Initialization:* Find an initial basic solution that is feasible.
2. *Iteration:* Find a basic solution that is better, adjacent, and feasible.
3. *Optimality test:* Test if the current solution is optimal. If not, repeat step 2.

2. *Iteration:* Find a basic solution that is better, adjacent, and feasible.

1. determining the entering variable
   A new basic solution will be *better* if an entering variable is properly chosen.
2. determining the leaving variable
   A new basic solution will be *feasible* if a leaving variable is properly chosen.
3. pivoting on the pivot element for exchange of variables and updating the data in the simplex tableau.

How do we chose?

2. *Iteration:* Find a basic solution that is better, adjacent, and feasible.

Basic Variable	$\mathbf{x_B}$						$\mathbf{x_N}$							RHS Solution
	z	x_{B_1}	$\dots$	x_{B_r}	$\dots$	x_{B_m}	$\dots$	x_j	$\dots$	x_k	$\dots$			
z	1	0	$\dots$	0	$\dots$	0	$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_k$	$\dots$			$\bar{b}_o$
x_{B_1}	0	1	$\dots$	0	$\dots$	0	$\dots$	$\bar{a}_{1j}$	$\dots$	$\bar{a}_{1k}$	$\dots$			$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$				$\vdots$
x_{B_r}	0	0	$\dots$	1	$\dots$	0	$\dots$	$\bar{a}_{rj}$	$\dots$	$\bar{a}_{rk}$	$\dots$			$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$				$\vdots$
x_{B_m}	0	0	$\dots$	0	$\dots$	1	$\dots$	$\bar{a}_{mj}$	$\dots$	$\bar{a}_{mk}$	$\dots$			$\bar{b}_m$

Pick an entering variable $x_k = \{x_j \in \mathbf{x_N} : \min_j \bar{c}_j, \bar{c}_j < 0\}$ with most negative reduced cost. $\rightarrow$ will improve the solution most.

$$\bar{\mathbf{c}}^T = (\bar{\mathbf{c}}_\mathbf{B}^T, \bar{\mathbf{c}}_\mathbf{N}^T) = (\mathbf{0}^T, \mathbf{c_B}^T \mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N}^T)$$

2. *Iteration:* Find a basic solution that is better, adjacent, and feasible.

Basic Variable	$\mathbf{x_B}$							$\mathbf{x_N}$						RHS Solution
	z	x_{B_1}	$\dots$	x_{B_r}	$\dots$	x_{B_m}		$\dots$	x_j	$\dots$	x_k	$\dots$		
z	1	0	$\dots$	0	$\dots$	0		$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_k$	$\dots$		$\bar{b}_o$
x_{B_1}	0	1	$\dots$	0	$\dots$	0		$\dots$	$\bar{a}_{1j}$	$\dots$	$\bar{a}_{1k}$	$\dots$		$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			$\vdots$		$\vdots$			$\vdots$
x_{B_r}	0	0	$\dots$	1	$\dots$	0		$\dots$	$\bar{a}_{rj}$	$\dots$	$\bar{a}_{rk}$	$\dots$		$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			$\vdots$		$\vdots$			$\vdots$
x_{B_m}	0	0	$\dots$	0	$\dots$	1		$\dots$	$\bar{a}_{mj}$	$\dots$	$\bar{a}_{mk}$	$\dots$		$\bar{b}_m$

Pick an entering variable $x_k = \{x_j \in \mathbf{x_N} : \min_j \bar{c}_j, \bar{c}_j < 0\}$ with most negative reduced cost. $\rightarrow$ will improve the solution most

Pick a leaving variable $x_{B_r} = \{x_{B_i} \in \mathbf{x_B} : \min_i \frac{\bar{b}_i}{\bar{a}_{ik}}, \bar{a}_{ik} > 0\}$ Why? Explained in the next slide

Basic Variable	$\mathbf{x_B}$							$\mathbf{x_N}$						RHS Solution
	z	x_{B_1}	$\dots$	x_{B_r}	$\dots$	x_{B_m}		$\dots$	x_j	$\dots$	x_k	$\dots$		
z	1	0	$\dots$	0	$\dots$	0		$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_k$	$\dots$		$\bar{b}_o$
x_{B_1}	0	1	$\dots$	0	$\dots$	0		$\dots$	$\bar{a}_{1j}$	$\dots$	$\bar{a}_{1k}$	$\dots$		$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			$\vdots$		$\vdots$			$\vdots$
x_{B_r}	0	0	$\dots$	1	$\dots$	0		$\dots$	$\bar{a}_{rj}$	$\dots$	$\bar{a}_{rk}$	$\dots$		$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$			$\vdots$		$\vdots$			$\vdots$
x_{B_m}	0	0	$\dots$	0	$\dots$	1		$\dots$	$\bar{a}_{mj}$	$\dots$	$\bar{a}_{mk}$	$\dots$		$\bar{b}_m$

When increasing x_k from 0:

$$z + \bar{c}_k x_k = \bar{b}_0 \;\Rightarrow\; z = \bar{b}_0 - \bar{c}_k x_k$$

$$\text{and } x_{B_i} + \bar{a}_{ik} x_k = \bar{b}_i \text{ or } x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \;\forall i$$

Want new solution to remain feasible.

$$x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \geq 0 \;\forall i$$

Basic Variable	$\mathbf{x_B}$						$\mathbf{x_N}$						RHS Solution
	z	x_{B_1}	$\ldots$	x_{B_r}	$\ldots$	x_{B_m}	$\ldots$	x_j	$\ldots$	x_k	$\ldots$		
z	1	0	$\ldots$	0	$\ldots$	0	$\ldots$	$\bar{c}_j$	$\ldots$	$\bar{c}_k$	$\ldots$		$\bar{b}_o$
x_{B_1}	0	1	$\ldots$	0	$\ldots$	0	$\ldots$	$\bar{a}_{1j}$	$\ldots$	$\bar{a}_{1k}$	$\ldots$		$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$			$\vdots$
x_{B_r}	0	0	$\ldots$	1	$\ldots$	0	$\ldots$	$\bar{a}_{rj}$	$\ldots$	$\bar{a}_{rk}$	$\ldots$		$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$			$\vdots$
x_{B_m}	0	0	$\ldots$	0	$\ldots$	1	$\ldots$	$\bar{a}_{mj}$	$\ldots$	$\bar{a}_{mk}$	$\ldots$		$\bar{b}_m$

$$x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k \geq 0 \ \forall i$$

$\underline{\bar{a}_{ik} < 0}$: then x_{B_i} increases as x_k increases
$\underline{\bar{a}_{ik} > 0}$: then x_{B_i} decreases as x_k increases

To satisfy nonnegativity x_k is increased until x_{B_i} drops to zero. The first basic variable dropping to zero is

$$x_{B_r} = \{x_{B_i} \in \mathbf{x_B} : \min_i \frac{\bar{b}_i}{\bar{a}_{ik}}, \bar{a}_{ik} > 0\}$$

Updating the Simplex Tableau

Basic Variable	$\mathbf{x_B}$						$\mathbf{x_N}$						RHS Solution
	z	x_{B_1}	$\dots$	x_{B_r}	$\dots$	x_{B_m}	$\dots$	x_j	$\dots$	x_k	$\dots$		
z	1	0	$\dots$	0	$\dots$	0	$\dots$	$\bar{c}_j$	$\dots$	$\bar{c}_k$	$\dots$		$\bar{b}_o$
x_{B_1}	0	1	$\dots$	0	$\dots$	0	$\dots$	$\bar{a}_{1j}$	$\dots$	$\bar{a}_{1k}$	$\dots$		$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$			$\vdots$
x_{B_r}	0	0	$\dots$	1	$\dots$	0	$\dots$	$\bar{a}_{rj}$	$\dots$	$\bar{a}_{rk}$	$\dots$		$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$			$\vdots$
x_{B_m}	0	0	$\dots$	0	$\dots$	1	$\dots$	$\bar{a}_{mj}$	$\dots$	$\bar{a}_{mk}$	$\dots$		$\bar{b}_m$

1. Divide row r by $\bar{a}_{rk}$.
2. $\forall i \neq r$, update the ith row by adding to it $(-\bar{a}_{ik})$ times the new rth row.
3. Update row 0 by adding to it $\bar{c}_k$ times the new rth row.

Updating the Simplex Tableau

1. Divide row r by $\bar{a}_{rk}$.
2. $\forall i \neq r$, update the ith row by adding to it $(-\bar{a}_{ik})$ times the new rth row.
3. Update row 0 by adding to it $\bar{c}_k$ times the new rth row.

Basic Variable	$\mathbf{x_B}$: z	x_{B_1}	$\cdots$	x_{B_r}	$\cdots$	x_{B_m}	$\mathbf{x_N}$: $\cdots$	x_j	$\cdots$	x_k	$\cdots$	RHS Solution
z	1	0	$\cdots$	$\frac{\bar{b}_r}{\bar{a}_{rk}}$	$\cdots$	0	$\cdots$	$\bar{c}_j - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{c}_k$	$\cdots$	0	$\cdots$	$\bar{b}_0 - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{c}_k$
x_{B_1}	0	1	$\cdots$	$\frac{\bar{a}_{1k}}{\bar{a}_{rk}}$	$\cdots$	0	$\cdots$	$\bar{a}_{1j} - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{a}_{1k}$	$\cdots$	0	$\cdots$	$\bar{b}_1 - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{a}_{1k}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_k	0	0	$\cdots$	$\frac{1}{\bar{a}_{rk}}$	$\cdots$	0	$\cdots$	$\frac{\bar{a}_{rj}}{\bar{a}_{rk}}$	$\cdots$	1	$\cdots$	$\frac{\bar{b}_r}{\bar{a}_{rk}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	$\cdots$	$\frac{\bar{a}_{mk}}{\bar{a}_{rk}}$	$\cdots$	1	$\cdots$	$\bar{a}_{mj} - \frac{\bar{a}_{rj}}{\bar{a}_{rk}} \bar{a}_{mk}$		0	$\cdots$	$\bar{b}_m - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{a}_{mk}$

Optimality Test: An optimal solution is found if there is no adjacent basic feasible solution that can improve the objective value. (That is, all reduced costs for nonbasic variables are positive.

Find initial basic feasible solution

Construct phase-I problem:

Example

$$(LP)\begin{cases} \max & z = 4x_1 + 3x_2 \\ s.t. & x_1 + x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & -2x_1 + x_2 \geq 2 \\ x_1, x_2 \geq 0 & \end{cases} \tag{5}$$

1) Convert each constraint so RHS is nonnegative. Then do the following:

$\leq$-form: add nonnegative slack variable

$=$-form: add nonegative artificial variable (basic variables for a stating basis)

$\geq$-form: add nonnegative slack variable and nonnegative artificial variable

Find initial basic feasible solution

Construct phase-I problem:

Example

$$(LP)\begin{cases} \max & z = 4x_1 + 3x_2 \\ s.t. & x_1 + x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & -2x_1 + x_2 + s_3 + x^a = 2 \\ x_1, x_2, s_1, s_2, s_3, x^a \geq 0 & \end{cases} \quad (6)$$

1) Convert each constraint so RHS is nonnegative. Then do the following:

$\leq$-form: add nonnegative slack variable

$=$-form: add nonegative artificial variable (basic variables for a stating basis)

$\geq$-form: add nonnegative slack variable and nonnegative artificial variable

Find initial basic feasible solution

Construct phase-I problem:

Example

$$(LP)\begin{cases} \max & z = 4x_1 + 3x_2 \\ s.t. & x_1 + x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & -2x_1 + x_2 + s_3 + x^a = 2 \\ x_1, x_2, s_1, s_2, s_3, x^a \geq 0 & \end{cases} \tag{7}$$

2) Solve a phase I problem by minimizing the sum of artificial variables using the same set of constraints.

Find initial basic feasible solution

Construct phase-I problem:

Example

$$(Phase\ I)\begin{cases}\min & z = x^a \\ s.t. & x_1 + x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & -2x_1 + x_2 + s_3 + x^a = 2 \\ x_1, x_2, s_1, s_2, s_3, x^a \geq 0 & \end{cases} \quad (8)$$

2) Solve a phase I problem by minimizing the sum of artificial variables using the same set of constraints.

Find initial basic feasible solution

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	−2	1	0	0	−1	1	2

Find initial basic feasible solution

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	−2	1	0	0	−1	1	2

x^a basic variable. Reduced cost should be 0.

$\Downarrow$

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	2	−1	0	0	1	0	−2
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	−2	1	0	0	−1	1	2

Find initial basic feasible solution

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	2	−1	0	0	1	0	−2
s_1	0	1	1	1	0	0	0	6
s_2	0	2	1	0	1	0	0	8
x^a	0	−2	1	0	0	−1	1	2

x_2 entering variable, x^a leaving variable

$$\Downarrow$$

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	3	0	1	0	1	−1	4
s_2	0	4	0	0	1	1	−1	6
x_2	0	−2	1	0	0	−1	1	2

Find initial basic feasible solution

Basic Variable	$-z^a$	x_1	x_2	s_1	s_2	s_3	x^a	RHS
$-z^a$	1	0	0	0	0	0	1	0
s_1	0	3	0	1	0	1	−1	4
s_2	0	4	0	0	1	1	−1	6
x_2	0	−2	1	0	0	−1	1	2

x^a no longer in basis. Have basic feasible solution for original problem.

$$\Downarrow$$

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	−4	−3	0	0	0	0
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	−2	1	0	0	−1	2

Find initial basic feasible solution

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	−4	−3	0	0	0	0
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	−2	1	0	0	−1	2

Negative constants in row 0 for basic variable $\Longrightarrow$ not in canonical form (coefficient of basic variable x_2 is negative).

$\Downarrow$

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	−10	0	0	0	−3	6
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	−2	1	0	0	−1	2

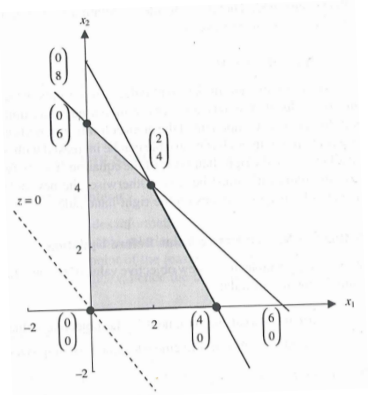
Find initial basic feasible solution

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	−10	0	0	0	−3	6
s_1	0	3	0	1	0	1	4
s_2	0	4	0	0	1	1	6
x_2	0	−2	1	0	0	−1	2

x_1 entering variable and s_1 leaving variable.
No negative coefficients in row 0. Optimal! $\Downarrow$

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	10/3	0	1/3	58/3
x_1	0	1	0	1/3	0	1/3	4/3
s_2	0	0	0	−4/3	1	−1/3	2/3
x_2	0	0	1	2/3	0	−1/3	14/3

9.5 Geometric interpretation of the simplex method

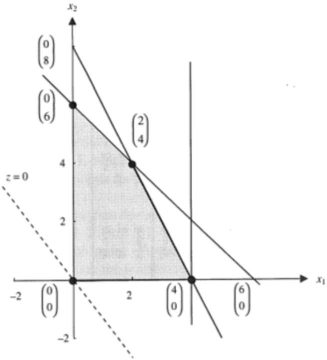


$$x_1 + x_2 \leq 6 \quad (9)$$
$$2x_1 + x_2 \leq 8 \quad (10)$$

The system has six basic solutions displayed below:

	Basic Solution					
	1	2	3	4	5	6
Nonbasic variables $\mathbf{x_N}$	$x_1=0$, $x_2=0$	$x_1=0$, $x_3=0$	$x_1=0$, $x_4=0$	$x_2=0$, $x_3=0$	$x_2=0$, $x_4=0$	$x_3=0$, $x_4=0$
Basic variables $\mathbf{x_B}$	$x_3=6$, $x_4=8$	$x_2=6$, $x_4=2$	$x_2=8$, $x_3=-2$	$x_1=6$, $x_4=-4$	$x_1=4$, $x_3=2$	$x_1=2$, $x_2=4$

Example of a degenerate system



$$x_1 + x_2 \le 6 \quad (11)$$
$$2x_1 + x_2 \le 8 \quad (12)$$
$$x_1 \le 4 \quad (13)$$

Degenerate solutions!

Same solution, different basis

	Basic Solution									
	1	2	3	4	5	6	7	8	9	10
$\mathbf{x_N}$	$x_1=0$, $x_2=0$	$x_1=0$, $s_1=0$	$x_1=0$, $s_2=0$	$x_1=0$, $s_3=0$	$x_2=0$, $s_1=0$	$x_2=0$, $s_2=0$	$x_2=0$, $s_3=0$	$s_1=0$, $s_2=0$	$s_1=0$, $s_3=0$	$s_2=0$, $s_3=0$
$\mathbf{x_B}$	$s_1=6$, $s_2=8$, $s_3=4$	$x_2=6$, $s_2=2$, $s_3=4$	$x_2=8$, $s_1=-2$, $s_3=4$	No solution	$x_1=6$, $s_2=-4$, $s_3=-2$	$x_1=2$, $s_1=4$, $s_3=-2$	$x_1=4$, $s_1=2$, $s_2=0$	$x_1=2$, $x_2=4$, $s_3=2$	$x_1=4$, $x_2=2$, $s_2=-2$	$x_1=4$, $x_2=0$, $s_1=2$

Identifying an Extreme Ray in a Simplex Tableau

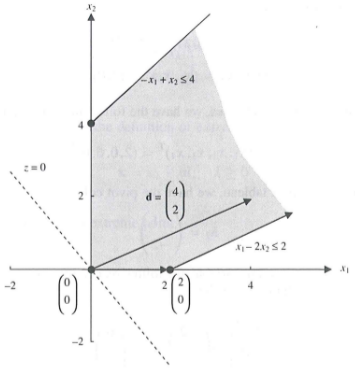
Extreme ray

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{d}\lambda, \ \ \lambda \geq 0,$$

where $\mathbf{x}_0$ is the *root* or *vertex* and $\mathbf{d}$ is the *extreme direction*.

Example

$$
(LP)\begin{cases} \max & z = 4x_1 + 3x_2 \\ s.t. & -x_1 + x_2 \le 4 \\ & x_1 - 2x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{cases}
$$



Extreme ray

$\binom{2}{0} + \lambda\binom{2}{1},\ \lambda \ge 0$

x_1 entering variable, x_4 leaving variable.

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	−4	−3	0	0	0
x_3	0	−1	1	1	0	4
x_4	0	1	−2	0	1	2

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	0	−11	0	4	8
x_3	0	0	−1	1	1	6
x_1	0	1	−2	0	1	2

9.5 Geometric interpretation of the simplex method

x_1 entering variable, x_4 leaving variable.

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	−4	−3	0	0	0
x_3	0	−1	1	1	0	4
x_4	0	1	−2	0	1	2

Unbounded!

Basic Variable	z	x_1	x_2	x_3	x_4	RHS
z	1	0	−11	0	4	8
x_3	0	0	−1	1	1	6
x_1	0	1	−2	0	1	2

9.5 Geometric interpretation of the simplex method

Simplex tableau reveals that current basic feasible solutions is

$$\mathbf{x} = (2, 0, 6, 0)^T = \mathbf{x}_0$$

The pivot column is

$$\bar{a}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

To ensure feasiblility

$$\begin{pmatrix} 2 \\ 0 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \end{pmatrix} x_2 \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 \geq 0$$

The extreme direction is $\mathbf{d} = (2, 0, 1, 0)^T$

General description (maximization problem):

Have basic feasible solution with $\bar{c}_k < 0$ and $\bar{a_{ik}} \leq 0 \forall i$ for some nonbasic variable x_k (i.e. unbounded solution).
Also $\mathbf{x_B} = \bar{\mathbf{b}} - \bar{\mathbf{a}}_k x_k$
Coefficient of entering variable x_k is 1, so $\mathbf{x_N} = \mathbf{e}_k$
This yields

$$\mathbf{x} = \begin{pmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{b}} - \bar{\mathbf{a}}_k x_k \\ \mathbf{e}_k \end{pmatrix} x_k = \begin{pmatrix} \bar{\mathbf{b}} \\ \bar{\mathbf{0}} \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{a}}_k \\ \mathbf{e}_k \end{pmatrix} x_k$$

The extreme ray is now given by:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{d}\lambda, \ \lambda \geq 0$$

where $\mathbf{x}_0 = \binom{\bar{\mathbf{b}}}{\bar{\mathbf{0}}}$, $\mathbf{d} = \binom{\bar{\mathbf{a}}_k}{\mathbf{e}_k}$ and $\lambda = x_k$

9.6 The Simplex Method for upper bounded variables

Have variables with upper and lower bound.

$$x_j \geq l_j, \quad x_j \leq u_j$$

The lower bound can be handled by a simple variable substitution:

$$x_j' = x_j - l_j, \quad x_j' \geq 0$$

Upper bounds are slightly more tricky.

Upper bounded variable: basic concept

Allow an upper bounded variable to be nonbasic if $x_j = 0$(as usual) or $x_j = u_j$.

Using the following strategy.

Change variable to $\bar{x}_j$ defined by the relationship

$$x_j + \bar{x}_j = u_j \quad \Rightarrow \bar{x}_j = u_j - x_j$$

Note! If $x_j = 0$, $\bar{x}_j = u_j$ and vice versa.

9.6 The Simplex Method for upper bounded variables

Suppose solving a maximization problem using the simplex method. An entering variable is chosen as usual. The method for choosing a leaving variable is altered. Have three cases:

Case 1: x_k cannot exceed the minimum ratio $\theta = \min_i\{\frac{\bar{b}_i}{\bar{a}_{ik}}, \ \bar{a}_{ik} > 0\}$ as usual.

Case 2: x_k cannot exceed the amount by which will cause one or more current basic feasible variables to exceed its upper bound. (Denote amount by $\theta' = \min_i\{\frac{u_i - \bar{b}_i}{-\bar{a}_{ik}}, \quad \bar{a}_{ik} < 0\})$

Case 3: x_k cannot exceed its upper bound u_k.

9.6 The Simplex Method for upper bounded variables

Denote $\Delta = \min\{\theta, \theta', u_k\}$

If $\Delta = \theta$: then determina leaving variable x_k and perform ordinary pivoting.
If $\Delta = \theta'$: then replace leaving variable x_{B_r} with $u_{B_r} - \bar{x}_{B_r}$ in row r and the "label" for $x_{\mathbf{B_r}}$ with $\bar{x}_{\mathbf{B_r}}$ and perform ordinary pivoting.
If $\Delta = u_k$: then replace the entering variable x_k with $u_k - \bar{x}_k$ in each row of the tableau, and x_k with $\bar{x}_k$ in the "label" row. Go to step one and do an optimality test.

9.6 The Simplex Method for upper bounded variables - Example

$$(LP)\begin{cases} max & z = 4x_1 + 3x_2 \\ s.t. & x_1 + x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1 \geq 1 \\ & 1 \leq x_2 \leq 3 \end{cases}$$

Using $x_1' = x_1 - 1$ and $x_2' = x_2 - 1$.

$$(LP)\begin{cases} max & z = 4x_1' + 3x_2' + 7 \\ s.t. & x_1' + x_2' + s_1 = 4 \\ & 2x_1' + x_2' + s_2 = 5 \\ & x_2' \leq 2 \\ & x_1', x_2' \geq 0 \end{cases}$$

9.6 The Simplex Method for upper bounded variables - Example

Let $x_2' + \bar{x}_2' = 2$. starting base is s_1, s_2. Initial tableau is:

Basic Variable	z	x_1'	x_2'	s_1	s_2	RHS
z	1	−4	−3	0	0	7
s_1	0	1	1	1	0	4
s_2	0	2	1	0	1	5

Not optimal! x_1' is entering variable. θ' does not exist since $\bar{a}_{11}, \bar{a}_{21} \geq 0$.

$$\theta = min\{\frac{4}{1}, \frac{5}{2}\} = 2.5$$

s_2 leaving variable

9.6 The Simplex Method for upper bounded variables - Example

Basic Variable	z	x'_1	x'_2	s_1	s_2	RHS
z	1	0	−1	0	2	17
s_1	0	0	0.5	1	−0.5	1.5
x'_1	0	1	0.5	0	0.5	2.5

Not optimal! x'_2 entering variable. Still haven't any θ'. Since $x_2 \leq 2$, $\Delta = \min\{\theta = 3, u'_2 = 2\}$. Replace x'_2 with $2 - \bar{x}'_2$.

9.6 The Simplex Method for upper bounded variables - Example

Basic Variable	z	x'_1	x'_2	s_1	s_2	RHS
z	1	0	1	0	2	19
s_1	0	0	−0.5	1	−0.5	0.5
x'_1	0	1	−0.5	0	0.5	1.5

Optimal!