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Discreteness of Spectrum and Strict Positivity Criteria for Magnetic Schrödinger Operators

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ABSTRACT

We establish necessary and sufficient conditions for the discreteness of spectrum and strict positivity of magnetic Schrödinger operators with a positive scalar potential. They are expressed in terms of Wiener's capacity and the local energy of the magnetic field. The conditiions for the discreteness of spectrum depend, in particular, on a functional parameter which is a decreasing function of one variable whose argument is the normalized local energy of the magnetic field. This function enters the negligibility condition of sets for the scalar potential. We give a description for the range of all admissible functions which is precise in a certain sense. In case when there is no magnetic field, our results extend the discreteness of spectrum and positivity criteria by Molchanov [Molchanov, A. M. (1953). On the discreteness of the spectrum conditions for self-adjoint differential equations of the second order (Russian). *Trudy Mosk. Matem. Obshchestva (Proc. Moscow Math. Society)* 2:169–199] and Maz'ya [Maz'ya, V. G. (1973). On (p, l)-capacity, imbedding theorems and the spectrum of a self-adjoint elliptic operator. *Math. USSR Izv.* 7:357–387].

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1. INTRODUCTION AND MAIN RESULTS

The main object of this paper is the magnetic Schrödinger operator in $\mathbb{R}^n$ which has the form

$$H_{a,V} = \sum_{j=1}^{n} P_j^2 + V, \tag{1.1}$$

where

$$P_j = \frac{1}{i}\frac{\partial}{\partial x^j} + a_j,$$

and $a_j = a_j(x)$, $V = V(x)$, $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We assume that a_j and V are real-valued functions. Denote also

$$\nabla_a u = \nabla u + iau = \left(\frac{\partial u}{\partial x^1} + ia_1 u, \dots, \frac{\partial u}{\partial x^n} + ia_n u\right).$$

We will assume a priori that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $a \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ (which will be a shorthand for saying that $a_j \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for all $j = 1, \dots, n$). This allows to define the quadratic form

$$h_{a,V}(u,u) = \int_{\mathbb{R}^n} (|\nabla_a u|^2 + V|u|^2)dx \tag{1.2}$$

on functions $u \in C_c^\infty(\mathbb{R}^n)$. A stronger local requirement on a will be imposed for the discreteness of spectrum results. (For example, it will be sufficient to require that $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.) We will also assume that $V \geq 0$ (the case when V is semi-bounded below by another constant is easily reduced to the case when $V \geq 0$ for the discreteness of spectrum results). Then we can define $H_{a,V}$ as the operator defined by the closure of this quadratic form. This closure is well defined (Leinfelder and Simader, 1981).

We will say that $H_{a,V}$ has a *discrete spectrum* if its spectrum consists of isolated eigenvalues of finite multiplicities. It follows that the only accumulation point of these eigenvalues can be $+\infty$. Equivalently, we may say that $H_{a,V}$ has a compact resolvent.

Our first goal is to provide necessary and sufficient conditions for the discreteness of the spectrum of $H_{a,V}$. We will write $\sigma = \sigma_d$ instead of the statement that the spectrum of $H_{a,V}$ is discrete.

Let us recall some facts concerning the Schrödinger operator $H_{0,V} = -\Delta + V$ without magnetic field (i.e., the operator (1.1) with $a = 0$).

It is a classical result of Friedrichs (1934) (see also e.g., Reed and Simon, 1978, Theorem XIII. 67, or Berezin and Shubin, 1991, Theorem 3.1) that the condition

$$V(x) \to +\infty \quad \text{as } x \to \infty$$

implies $\sigma = \sigma_d$ (for $H_{0,V}$).

Molchanov (1953) found a necessary and sufficient condition for the discreteness of spectrum. It is formulated in terms of the Wiener capacity. The capacity of a compact set F will be denoted $\operatorname{cap}(F)$ (see Sec. 2 for the definition and Edmunds and Evans, 1987, Kondratiev and Shubin, 1999, Maz'ya, 1985 for necessary properties of the capacity, expositions of Molchanov's work and more general results).

Let $B(x, r)$ denote the open ball in $\mathbb{R}^n$ with the radius $r > 0$ and the center at x, $\overline{B}(x, r)$ denote the corresponding closed ball.

In case $n = 2$ the capacity of a set $F \subset \overline{B}(x, r)$ is always taken relative to a ball $B(x, 2r)$. The value of r is usually clear from the context. In case $n \geq 3$ such a definition would be equivalent to the usual Wiener capacity (relative to $\mathbb{R}^n$).

In case $n = 2$ we can also use capacities of sets $F \subset \overline{B}(x, r)$ with respect to the ball $B(x, R)$ where $r \in (0, R/2)$ and $R > 0$ is fixed, but this complicates some formulations.

Similarly we can use closed cubes (squares if $n = 2$) Q_d, where $d > 0$ means the length of the edge and the edges are assumed to be parallel to the coordinate axes. The interior of Q_d will be denoted $\overset{\circ}{Q}_d$. In this paper we prefer to use cubes instead of balls, but balls are more convenient in case of manifolds. In case $n = 2$ the capacity of a compact set $F \subset Q_d$ will be always defined relative to $\overset{\circ}{Q}_{2d}$, where Q_d and Q_{2d} have the same center.

Let us define the Molchanov functional

$$M_c(Q_d; V) = \inf_F \left\{ \int_{Q_d \setminus F} V(x)dx \,\middle|\, \operatorname{cap}(F) \leq c \operatorname{cap}(Q_d) \right\}. \tag{1.3}$$

Here we will always assume that $0 < c < 1$. Due to the standard properties of the capacity, the infimum in (1.3) will not change if we only restrict it to the sets F which are closures of open subsets of Q_d with a smooth boundary.

Molchanov proved that there exists $c = c_n > 0$ such that $H_{0,V}$ has a discrete spectrum if and only if for every $d > 0$

$$M_c(Q_d; V) \to +\infty \quad \text{as } Q_d \to \infty, \tag{M_c}$$

where $Q_d \to \infty$ means that the center of the cube Q_d goes to infinity (with d fixed). He actually established this result with a specific constant c_n (see also Kondratiev and Shubin, 1999), namely, $c_n = (4n)^{-4n}(\operatorname{cap}(Q_1))^{-1}$ for $n \geq 3$, but it is by no means precise and we will not be interested in the precise value of this constant (it seems beyond the reach of the existing technique).

The case $n = 2$ was not discussed in Molchanov (1953), though it can be covered by the same methods with minor modifications.

Note that (M_c) implies $(M_{c'})$ for every $c' < c$. The arguments in Molchanov (1953) actually show that it suffices to assume that (M_c) is satisfied for all sufficiently small $c > 0$. Hence we can equivalently formulate a necessary and sufficient condition of the discreteness of spectrum for $H_{0,V}$ by writing that (M_c) is satisfied for all $c \in (0, c_0)$ with a positive c_0.

Note also that $\operatorname{cap}(\overline{B}(x, r))$ can be explicitly calculated. It equals $c_n r^{n-2}$ (with a different $c_n > 0$). The capacity of a cube Q_d is $c_n d^{n-2}$ (with yet another $c_n > 0$).

Hence in the formulation of the Molchanov condition (M_c) we can replace cap(Q_d) by d^{n-2}.

A simple argument given in Avron et al. (1978) (see also Corollary 1.4 in Kondratiev and Shubin, 2002) shows that if $H_{0,V}$ has a discrete spectrum, then the same is true for $H_{a,V}$ whatever the vector potential a. Therefore the condition (M_c) together with $V \geq 0$ is sufficient for the discreteness of spectrum of $H_{a,V}$. This means that a magnetic field can only improve the situation from our point of view. Papers by Avron et al. (1978), Colin de Verdière (1986), Dufresnoy (1983) and Iwatsuka (1986) provide some quantitative results which show that even in case $V = 0$ the magnetic field can make the spectrum discrete. (This situation is called *magnetic bottle*.)

The results of Avron et al. (1978), Dufresnoy (1983) and Iwatsuka (1986), were improved in Kondratiev and Shubin (2002). In particular, some sufficient conditions for the spectrum of $H_{a,V}$ to be discrete were given. The capacity was added into the picture, so in most cases these conditions become necessary and sufficient in case when there is no magnetic field, i.e., when $a = 0$. Also both electric and magnetic fields were made to work together to achieve the discreteness of spectrum.

However no necessary and sufficient conditions of the discreteness of the spectrum with both fields present were provided in Kondratiev and Shubin (2002). Here we will give such conditions which actually separate the influence of the electric and magnetic fields. If the magnetic field is absent then our conditions turn into the Molchanov condition (M_c) or into some weaker conditions, improving Molchanov's sufficiency result.

We will need the bottoms $\lambda(G; H_{a,V})$ and $\mu(G; H_{a,V})$ of Dirichlet and Neumann spectra for the operator $H_{a,V}$ in an open set $G \subset \mathbb{R}^n$. They are defined in terms of its quadratic form $h_{a,V}$ as follows (see e.g., Courant and Hilbert, 1953; Kato, 1966):

$$\lambda(G; H_{a,V}) = \inf_u \left\{ \frac{h_{a,V}(u,u)_G}{(u,u)_G},\ u \in C_c^\infty(G) \setminus \{0\} \right\}, \tag{1.4}$$

$$\mu(G; H_{a,V}) = \inf_u \left\{ \frac{h_{a,V}(u,u)_G}{(u,u)_G},\ u \in (C^\infty(G) \setminus \{0\}) \cap L^2(G) \right\}, \tag{1.5}$$

where in both cases $h_{a,V}(u,u)_G$ is given by the formula (1.2) with the integrals over G (instead of $\mathbb{R}^n$) i.e.,

$$h_{a,V}(u,u)_G = \int_G (|\nabla_a u|^2 + V|u|^2)\,dx,$$

and $(u,u)_G$ means square of the L^2-norm of u in G. However in the future we will often skip the subscript G since it will be clear from the context which G is used.

We will also use these notations for $G = Q_d$ in which case $\lambda(Q_d; H_{a,V})$ is understood as $\lambda(\overset{\circ}{Q}_d; H_{a,V})$, whereas $\mu(Q_d; H_{a,V})$ can be understood as $\mu(\overset{\circ}{Q}_d; H_{a,V})$ as well as directly by the formula (1.5) (i.e., with the use of functions u which are C^∞ on the closed cube) which gives the same result.

In both (1.4) and (1.5) we can also use locally Lipschitz test functions instead of C^∞ functions u, which does not change the result. (Of course we should take functions with compact support in G in case of $\lambda(G; H_{a,V})$).

We will also need the quantity

$$\mu_0 = \mu_0(Q_d) = \mu_0(Q_d; a) = \mu(Q_d; H_{a,0}), \tag{1.6}$$

which we will call the *local energy of the magnetic field* (in Q_d). Here the first three terms are defined by the last one, but we will use the shorter notations when the choice of Q_d and a is clear from the context. Obviously $\mu_0 \geq 0$. Also, μ_0 is *gauge invariant*, i.e.,

$$\mu_0(Q_d; a) = \mu_0(Q_d; a + d\phi),$$

as soon as $a, a + d\phi \in L^\infty_{\mathrm{loc}}(Q_d)$, ϕ is a locally Lipschitz function, and a is identified with the 1-form

$$a = \sum_{j=1}^{n} a_j \, dx^j.$$

Therefore $\mu_0(Q_d; a)$ depends only on the magnetic field $B = da$ which is understood as a 2-form with distributional coefficients. It is easy to see that $\mu_0(Q_d; a)$ vanishes if and only if B vanishes on $\overset{\circ}{Q}_d$. This justifies calling μ_0 local energy of the magnetic field.

We will also use a *normalized local energy of the magnetic field* in Q_d defined as

$$\tilde{\mu}_0 = \tilde{\mu}_0(Q_d) = \tilde{\mu}_0(Q_d; a) = \mu_0 d^2. \tag{1.7}$$

Definition 1.1. A class $\mathscr{F}$ consists of functions f: $[0, +\infty) \to (0, +\infty)$ which are continuous and decreasing on $[0, +\infty)$.

A class $\mathscr{G}$ consists of functions g: $(0, d_0) \to (0, +\infty)$ such that $g(\tau) \to 0$ as $\tau \to 0$ and $(g(d))^{-1}d^2 \leq 1$ for all $d \in (0, d_0)$.

The pair $(f, g) \in \mathscr{F} \times \mathscr{G}$ is called *n-admissible* if f satisfies the inequality $f(t) \leq f_n(t)$ for all $t \geq 0$, where

$$f_n(t) = (1 + t)^{(2-n)/2} \quad \text{if } n \geq 3, \qquad f_2(t) = (1 + \log(1 + t))^{-1}. \tag{1.8}$$

Now we can formulate our main result about the discreteness of spectrum.

Theorem 1.2. *Let us assume that* $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. *There exists* $c_n > 0$ *such that for every n-admissible pair* (f, g) *the following conditions on* $H_{a,V}$ *are equivalent*:

(a) *The spectrum of* $H_{a,V}$ *is discrete.*

($\mathrm{b}_{f,g}$) *There exists* $d_0 > 0$ *such that for every* $d \in (0, d_0)$

$$\mu_0(Q_d) + d^{-n} M_\gamma(Q_d; V) \to +\infty \quad \text{as } Q_d \to \infty, \tag{1.9}$$

where

$$\gamma = \gamma(\mu_0, d) = c_n f(\tilde{\mu}_0) g(d)^{-1} d^2. \tag{1.10}$$

($\mathrm{c}_{f,g}$) *There exists* $d_0 > 0$ *such that for every* $d \in (0, d_0)$

$$\liminf_{Q_d \to \infty} \left(\mu_0(Q_d) + d^{-n} M_\gamma(Q_d; V)\right) \geq g(d)^{-1}, \tag{1.11}$$

where γ *is as in* (1.10).

Note that $f(\tilde{\mu}_0) = f(\mu_0 d^2)$ is decreasing in μ_0 and tends to 0 as $\mu_0 \to \infty$ (with d fixed). So the condition on V is weaker at the places where the local energy of the magnetic field is larger.

Remark 1.3. Assuming that the magnetic field is absent ($a = 0$, $H_{a,V} = H_{0,V} = -\Delta + V$) we obtain $c_n f(\tilde{\mu}_0) = c_n f(0) = c > 0$. Now taking $g(d) = d^2$ we see that the condition (1.9) becomes the Molchanov condition (M_c). So Theorem 1.2 strengthens Molchanov's theorem (Molchanov, 1953) which claims the equivalence of (a) and ($\mathrm{b}_{f,g}$) for this particular case.

Corollary 1.4. *All conditions* ($\mathrm{b}_{f,g}$), ($\mathrm{c}_{f,g}$), *taken for different n-admissible pairs* (f, g) *are equivalent.*

In particular, this Corollary applied in case $a = 0$ (no magnetic field) gives an equivalence of different conditions on the scalar potential $V \geq 0$. This seems to be a new purely function-theoretic property of capacity.

The following corollaries provide examples of more explicit necessary and separately sufficient conditions which easily follow from Theorem 1.2.

Corollary 1.5. *Let us assume that the spectrum of* $H_{a,V}$ *is discrete. Then for every fixed* $d > 0$

$$\mu_0(Q_d) + \frac{1}{d^n} \int_{Q_d} V(x)dx \to +\infty \quad \textit{as } Q_d \to \infty. \tag{1.12}$$

The condition (1.12) corresponds to the case $\gamma \equiv 0$ in ($\mathrm{b}_{f,g}$) in Theorem 1.2. It is known that it is not sufficient for the discreteness of the spectrum, even in the case when there is no magnetic field (Molchanov, 1953).

Corollary 1.6. *Let us assume that there exist* $c > 0$, $d_1 > 0$ *such that for every fixed* $d \in (0, d_1)$

$$\mu_0(Q_d) + d^{-n} M_c(Q_d; V) \to +\infty \quad \textit{as } Q_d \to \infty. \tag{1.13}$$

Then the spectrum of $H_{a,V}$ *is discrete.*

It follows from Theorem 1.7 below that the condition (1.13) is not necessary for the discreteness of spectrum of $H_{a,V}$.

Sufficient conditions (for $\sigma = \sigma_d$) which do not include capacity, can be obtained if the capacity is replaced by the Lebesgue measure in the restriction on F in the definition of $M_c(Q_d; V)$ – see Sec. 6.1 in Kondratiev and Shubin (1999) for a more detailed argument.

Other, more effective sufficient conditions (which do not include μ_0) and related results (in particular, asymptotics of eigenvalues under appropriate conditions) can be found in Colin de Verdière (1986), Dufresnoy (1983), Fefferman (1983), Helffer and Mohamed (1988), Helffer et al. (1989), Ivrii (1998), Iwatsuka (1986, 1990), Kondratiev and Shubin (2002), Levendorskii (1997), Mohamed and Raikov (1994), Shigekawa (1991) and Tamura (1987).

Some necessary and sufficient conditions of discreteness of spectrum for the Schrödinger operators can be obtained by considering them as 1-dimensional Schrödinger operators with operator coefficients (see e.g., Brüning, 1989; Maslov, 1968 and references in Brüning, 1989). An interesting feature of this approach is that it allows to consider operators whose potentials are not necessarily semi-bounded below.

The following theorem shows that the conditions on f in Theorem 1.2 are almost precise.

Theorem 1.7. *There exists an operator $H_{a,V}$ with a discrete spectrum and with the following property. Let $f\colon [0,+\infty) \to (0,1)$ be a decreasing function, such that in case $n \geq 3$*

$$f(t) = (1+t)^{(2-n)/2} h(t), \tag{1.14}$$

and in case $n = 2$

$$f(t) = (1+\log(1+t))^{-1} h(t), \tag{1.15}$$

where in both cases $h(t) \to +\infty$ as $t \to +\infty$. Then, for every fixed $d > 0$, the condition (1.9) *with $\gamma = f(\mu_0 d^2)$ is not satisfied. So the condition* (1.9) *with the function f having the form given above, is not necessary for the discreteness of spectrum, whatever g and c_n. In particular, the exponents in* (1.8) *are the best possible.*

Now we will give a positivity criterion for the operators $H_{a,V}$. We will say that such an operator is *strictly positive* if $H_{a,V} \geq \varepsilon I$ for some $\varepsilon > 0$, or, equivalently, that its spectrum is in $[\varepsilon, \infty)$ for some $\varepsilon > 0$. If $V \geq 0$, then this is equivalent to saying that 0 is not in the spectrum of $H_{a,V}$.

Theorem 1.8. *Let us assume that $V \geq 0$. There exist positive constants c_n, $\tilde{c}_n$ such that the following conditions on $H_{a,V}$ are equivalent*:

(a) *$H_{a,V}$ is strictly positive.*

(b) *There exist positive constants c, d_1, d such that for every cube $Q_d \subset \mathbb{R}^n$*

$$\mu_0(Q_d) + d^{-n} M_c(Q_d; V) \geq \frac{1}{d_1^2}. \tag{1.16}$$

(c) *There exist positive constants d_1, d such that for every cube $Q_d \subset \mathbb{R}^n$*

$$\mu_0(Q_d) + d^{-n} M_{c_n}(Q_d; V) \geq \frac{1}{d_1^2}. \tag{1.17}$$

(d) *There exist positive constants* $c, \tilde{c}, d_2$ *such that for every* $d > d_2$ *and every cube* $Q_d \subset \mathbb{R}^n$

$$\mu_0(Q_d) + d^{-n} M_c(Q_d; V) \geq \frac{\tilde{c}}{d^2}. \tag{1.18}$$

(e) *There exist* $d_2 > 0$ *such that for every* $d > d_2$ *and every cube* $Q_d \subset \mathbb{R}^n$

$$\mu_0(Q_d) + d^{-n} M_{c_n}(Q_d; V) \geq \frac{\tilde{c}_n}{d^2}. \tag{1.19}$$

In case when there is no magnetic field (i.e., $a = 0$, $H_{a,V} = H_{0,V} = -\Delta + V$) this theorem is essentially contained in Maz'ya (1985, Sec. 12.5).

Remark 1.9. The discreteness of spectrum and strict positivity are gauge invariant. More precisely, if we replace $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ by another magnetic potential $a' \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ which has the form $a' = a + d\phi$, then the spectrum does not change, i.e., the spectra of $H_{a,V}$ and $H_{a',V}$ coincide (see Leinfelder, 1983). (Here ϕ is a locally Lipschitz function.) So in fact the spectrum depends not on the magnetic potential a itself but on the magnetic field $B = da$.

Remark 1.10. Theorem 1.2 holds on every manifold of bounded geometry, with cubes replaced by balls in the formulation (see Kondratiev and Shubin, 1999 and Sec. 6 in Kondratiev and Shubin, 2002 for necessary adjustments which should be done to treat the more general case compared with the case of operators on $\mathbb{R}^n$). However it is not at all clear how to extend Theorem 1.8 to this case.

Remark 1.11. In Sec. 7 we will formulate results which extend Theorems 1.2 and 1.8 and their Corollaries to the case when the operator $H_{a,V}$ is considered in $L^2(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary conditions on $\partial\Omega$. Note that the discreteness of spectrum and strict positivity in this case may be influenced or even completely determined by the geometry of Ω. In particular, the results are non-trivial even for the pure Laplacian $H_{0,0} = -\Delta$.

2. PRELIMINARIES

In this section we will list some important technical tools which will be used later. They were actually useful even in case of vanishing magnetic field (see Maz'ya, 1985), when they provide simpler proofs and stronger versions for the Molchanov discreteness of spectrum criterion, as well as for the Maz'ya strict positivity criterion for usual Schrödinger operators with non-negative scalar potentials.

For every subset $\Omega \subset \mathbb{R}^n$ denote by $\mathrm{Lip}(\Omega)$ the space of (complex-valued) functions satisfying the uniform Lipschitz condition in Ω, and by $\mathrm{Lip}_c(\Omega)$ the subspace in $\mathrm{Lip}(\Omega)$ of all functions with compact support in Ω (this will be only used when Ω is open). By $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ we will denote the set of functions on (an open set) Ω which are Lipschitz on any compact subset $K \subset \Omega$.

If F is a compact subset in an open set $\Omega \subset \mathbb{R}^n$, then the Wiener capacity of F relatively to Ω is defined as

$$\operatorname{cap}_\Omega(F) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx \,\middle|\, u \in \operatorname{Lip}_c(\Omega),\, u|_F = 1 \right\}. \tag{2.1}$$

We will also use the notation $\operatorname{cap}(F)$ for $\operatorname{cap}_{\mathbb{R}^n}(F)$ if $F \subset \mathbb{R}^n$, $n \geq 3$, and for $\operatorname{cap}_{\overset{\circ}{Q}_{2d}}(F)$ if $F \subset Q_d \subset \mathbb{R}^2$, where the squares Q_d and Q_{2d} have the same center and the edges parallel to the coordinate axes in $\mathbb{R}^2$.

Note that if we allow only real-valued functions u in (2.1), then the infimum will not change. To see this it suffices to note that $|\nabla|u|| \leq |\nabla u|$ a.e., (almost everywhere) for every complex-valued Lipschitz function. Moreover, the infimum does not change if we restrict ourselves to the Lipschitz functions u such that $0 \leq u \leq 1$ everywhere (see e.g., Maz'ya, 1985, Sec. 2.2.1).

The following Lemmas are particular cases of much more general results from Maz'ya (1985). We supply the simplified formulations for the convenience of the readers.

Lemma 2.1 (Maz'ya, 1985, Theorem 10.1.2, part 1). *There exists $C_n > 0$ such that the following inequality holds for every complex-valued function $u \in \operatorname{Lip}(Q_d)$ which vanishes on a compact set $F \subset Q_d$ (but is not identically zero on Q_d):*

$$\operatorname{cap}(F) \leq \frac{C_n \int_{Q_d} |\nabla u(x)|^2\, dx}{d^{-n} \int_{Q_d} |u(x)|^2\, dx}. \tag{2.2}$$

Lemma 2.2 (Maz'ya, 1985, Lemma 12.1.1). *Let $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, $V \geq 0$. For every $u \in \operatorname{Lip}(Q_d)$ and $\gamma > 0$*

$$\int_{Q_d} |u|^2\, dx \leq \frac{C_n d^2}{\gamma} \int_{Q_d} |\nabla u|^2\, dx + \frac{4d^n}{M_\gamma(Q_d; V)} \int_{Q_d} V|u|^2\, dx, \tag{2.3}$$

(*The last term is declared to be $+\infty$ if its denominator vanishes.*)

Remark 2.3. Both Lemmas 2.1 and 2.2 hold also if we replace ∇ by ∇_a. Indeed, we can first apply the inequalities (2.2) and (2.3) to $|u|$ and then use the diamagnetic inequality $|\nabla|u|| \leq |\nabla_a u|$ (see e.g., Kato, 1972; Lieb and Loss, 2001; Simon, 1976).

The following lemma is somewhat inverse to Lemma 2.1. It follows from part 2 of Theorem 10.1.2 in Maz'ya (1985).

Lemma 2.4. *There exists positive c_n, c'_n, c''_n such that for every compact subset $F' \subset Q_d$ satisfying*

$$\operatorname{cap}(F') \leq c_n \operatorname{cap}(Q_d), \tag{2.4}$$

there exists $\psi \in \operatorname{Lip}(Q_d)$ with the following properties: $0 \leq \psi \leq 1$, $\psi = 0$ in a neighborhood of F',

$$\operatorname{cap}(F') \geq c'_n \int_{Q_d} |\nabla\psi|^2\, dx \tag{2.5}$$

and

$$d^{-n}\int_{Q_d}\psi^2\,dx \geq \frac{1}{4}, \tag{2.6}$$

hence

$$\operatorname{cap}(F') \geq \frac{c_n''\int_{Q_d}|\nabla\psi|^2\,dx}{d^{-n}\int_{Q_d}\psi^2\,dx}\,. \tag{2.7}$$

For the convenience of the reader we provide self-contained proofs of the lemmas above in Appendix to this paper.

3. DISCRETENESS OF SPECTRUM: SUFFICIENCY

In this section we will consider operators $H_{a,V}$ with $V \in L^1_{\mathrm{loc}}(R^n)$, $V \geq 0$ and $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.

We will start with the following proposition which gives a general (albeit complicated) sufficient condition for the discreteness of spectrum.

Proposition 3.1. *Given an operator $H_{a,V}$, let us assume that the following condition is satisfied*:

$$\begin{gathered}\exists\varepsilon_0 > 0,\quad \forall\varepsilon \in (0,\varepsilon_0),\quad \exists d = d(\varepsilon) > 0,\quad R = R(\varepsilon) > 0,\quad \forall Q_d \ \text{ with}\\ Q_d \cap (\mathbb{R}^n\backslash B(0,R)) \neq \emptyset,\quad \exists\gamma = \gamma(\mu_0, d, \varepsilon) \geq 0,\quad \text{such that}\\ \mu_0 + \frac{\gamma}{C_n d^2} \geq \varepsilon^{-1} \quad \text{and} \quad \mu_0 + d^{-n}M_\gamma(Q_d; V) \geq \varepsilon^{-1},\end{gathered} \tag{3.1}$$

where $\mu_0 = \mu_0(Q_d)$, C_n is the constant from (2.3)*. Then $\sigma = \sigma_d$.*

Proof. We can assume without loss of generality that $V \geq 1$. Define

$$\mathcal{L} = \left\{u \,\middle|\, u \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n}(|\nabla_a u|^2 + V|u|^2)dx \leq 1\right\}. \tag{3.2}$$

By the standard functional analysis argument (see e.g., Lemma 2.3 in Kondratiev and Shubin, 1999) the spectrum of $H_{a,V}$ is discrete if and only if $\mathcal{L}$ is precompact in $L^2(\mathbb{R}^n)$, which in turn holds if and only if $\mathcal{L}$ has "small tails", i.e., for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^n\backslash B(0,R)}|u|^2\,dx \leq \varepsilon \quad \text{for all } u \in \mathcal{L}. \tag{3.3}$$

This will hold if we establish that there exists $d > 0$ such that

$$\int_{Q_d}|u|^2\,dx \leq \varepsilon\int_{Q_d}(|\nabla_a u|^2 + V|u|^2)dx, \tag{3.4}$$

for all cubes Q_d such that $Q_d \cap (\mathbb{R}^n\backslash B(0,R)) \neq \emptyset$.

To prove (3.4) note first that if $\gamma = 0$ then $\mu_0 \geq \varepsilon^{-1}$ due to the first inequality in (3.1), hence (3.4) follows from the definition of μ_0 (even if we skip the term with V in the right-hand side). So from now we will assume that $\gamma > 0$.

Let us look at the inequality

$$\int_{Q_d} |u|^2\, dx \leq \frac{C_n d^2}{\gamma} \int_{Q_d} |\nabla_a u|^2\, dx + \frac{4d^n}{M_\gamma(Q_d; V)} \int_{Q_d} |u|^2 V\, dx \tag{3.5}$$

(see Lemma 2.2 and Remark 2.3). For every fixed $\varepsilon > 0$ we can divide all cubes Q_d into the following two types:

Type I: $\mu_0(Q_d) > (2\varepsilon)^{-1}$;
Type II: $\mu_0(Q_d) \leq (2\varepsilon)^{-1}$.

For a Type I cube Q_d the inequality (3.4) holds with 2ε instead of ε, as was explained above.

For a Type II cube it follows from the conditions (3.1) that

$$\frac{C_n d^2}{\gamma} \leq 2\varepsilon, \qquad \frac{4d^n}{M_\gamma(Q_d; V)} \leq 8\varepsilon,$$

so the inequality (3.4) follows with 8ε instead of ε. □

Instead of requiring that the conditions of Proposition 3.1 satisfied for all $\varepsilon \in (0, \varepsilon_0)$, it suffices to require it for a sequence $\varepsilon_k \to +0$. Keeping this in mind we can replace the dependence $d = d(\varepsilon)$ by the inverse dependence $\varepsilon = g(d)$, so that $g(d) > 0$ and $g(d) \to 0$ as $d \to +0$ (and here we can also restrict to a sequence $d_k \to +0$). This leads to the following:

Proposition 3.2. *Given an operator $H_{a,V}$ with $V \geq 0$, let us assume that the following condition is satisfied*:

$$\begin{gathered} \exists\, d_0 > 0, \quad \forall d \in (0, d_0), \quad \exists R = R(d) > 0, \quad \forall Q_d \ \textit{ with} \\ Q_d \cap (\mathbb{R}^n \backslash B(0, R)) \neq \emptyset, \ \exists \gamma = \gamma(\mu_0, d) \geq 0, \quad \textit{such that} \\ \mu_0 + \frac{\gamma}{C_n d^2} \geq g(d)^{-1} \ \textit{ and } \ \mu_0 + d^{-n} M_\gamma(Q_d; V) \geq g(d)^{-1}, \end{gathered} \tag{3.6}$$

where $\mu_0 = \mu_0(Q_d)$, C_n is the constant from (2.3), *$g(d) > 0$ and $g(d) \to 0$ as $d \to +0$. Then $\sigma = \sigma_d$.*

Proposition 3.3. *Let us assume that $V \geq 0$, $f \in \mathcal{F}$, $g \in \mathcal{G}$ (in the notations of Definition* 1.1) *and one of the conditions* ($\mathrm{b}_{f,g}$), ($\mathrm{c}_{f,g}$) *from Theorem* 1.2 *is satisfied. Then the spectrum of $H_{a,V}$ is discrete.*

Proof. Clearly, ($\mathrm{b}_{f,g}$) implies ($\mathrm{c}_{f,g}$). So it remains to prove that ($\mathrm{c}_{f,g}$) implies that $\sigma = \sigma_d$. To this end it is sufficient to prove that it implies that the conditions of Proposition 3.2 are satisfied.

Note that it suffices to establish that the inequalities (3.6) hold with an additional positive constant factor, independent on d (but possibly dependent on f, g), in the right hand sides.

Clearly, the second inequality in (3.6), with an additional factor 1/2 in the right hand side, is satisfied for distant cubes Q_d due to (1.11). So we need only to take care for the first inequality in (3.6). It obviously holds if $\mu_0 \geq g(d)^{-1}$.

On the other hand, if we assume that $\mu_0 \leq g(d)^{-1}$, then

$$f(\mu_0 d^2) \geq f(g(d)^{-1} d^2),$$

hence

$$\frac{\gamma}{C_n d^2} = \frac{c_n}{C_n} f(\mu_0 d^2) g(d)^{-1} \geq \frac{c_n}{C_n} f(g(d)^{-1} d^2) g(d)^{-1} \geq \frac{c_n}{C_n} f(1) g(d)^{-1},$$

because $g(d)^{-1} d^2 \leq 1$ according to Definition 1.1. Therefore we can apply Proposition 3.2. □

Remark 3.4. No domination requirement (like $f \leq f_n$ in Definition 1.1) is imposed on f in Proposition 3.3.

Remark 3.5. It is clear from the proof that to establish the discreteness of spectrum of an operator $H_{a,V}$, it suffices to check the condition ($\mathrm{b}_{f,g}$) (or ($\mathrm{c}_{f,g}$)) from Theorem 1.2 for every $d \in (0, d_0)$ on the cubes Q_d which form a tiling of $\mathbb{R}^n$ (instead of all cubes Q_d).

Remark 3.6. Let us consider the case of vanishing magnetic field ($a \equiv 0$) and take $g(d) = d^s$ with $0 < s < 2$. Then the conditions ($\mathrm{b}_{f,g}$), ($\mathrm{c}_{f,g}$) provide sufficient conditions for the discreteness of spectrum of the Schrödinger operator $H_{0,V} = -\Delta + V$ which are much better than the Molchanov condition (M_c) which corresponds to the condition ($\mathrm{b}_{f,g}$) with $g(d) = d^2$. The conditions (b_{f,d^s}) in this case impose weaker requirements on the capacity of negligible sets for small d. With the same requirements on the negligible sets the condition (c_{f,d^s}) goes even further: it does not require the functional $M_\gamma(Q_d; V)$ to go to infinity for fixed d, it only requires it to become large for distant cubes and small d.

4. DISCRETENESS OF SPECTRUM: NECESSITY

We will use the notations from Sec. 1. We impose here the same restrictions on $H_{a,V}$ as in Sec. 3, i.e., $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $V \geq 0$, $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Let us fix an arbitrary $d_0 > 0$. We need to prove that the discreteness of spectrum for $H_{a,V}$ implies the condition ($\mathrm{b}_{f,g}$) in Theorem 1.2. This will follow from

Proposition 4.1. *There exist* $c = c_n > 0$, $C = C_n > 0$ *such that for every operator* $H_{a,V}$ *with* $V \geq 0$ *and every cube* Q_d

$$\mu(Q_d; H_{a,V}) \leq CE\left(1 + \frac{1}{f_n(\tilde{\mu}_0) d^{n-2}} M_{cf_n(\tilde{\mu}_0)}(Q_d; V)\right), \tag{4.1}$$

where $E = \mu_0(Q_d) + d^{-2}$, $\tilde{\mu}_0$ *is defined by* (1.7), *and* f_n *is defined by* (1.8).

Proof of Theorem 1.2. Clearly ($b_{f,g}$) implies ($c_{f,g}$). The sufficiency of the condition ($c_{f,g}$) for the discreteness of spectrum was proved in Sec. 3. So we only need to prove that $\sigma = \sigma_d$ implies ($b_{f,g}$) for every n-admissible pair f, g (see Definition 1.1). It is sufficient to consider the special case $f = f_n$, $g(d) = d^2$ because this case corresponds to the maximal allowed value of $\gamma(\mu_0, d)$, therefore to the strongest possible condition ($b_{f,g}$) among all possible n-admissible pairs (f, g).

So let us assume that $H_{a,V}$ has a discrete spectrum. We need to prove that the condition ($b_{f,g}$) holds for $f = f_n$, $g(d) = d^2$. For brevity sake denote this condition by (N).

According to the Localization Theorem 1.2 in Kondratiev and Shubin (2002) it follows from the discreteness of spectrum that

$$\mu(Q_d; H_{a,V}) \to +\infty \quad \text{as } Q_d \to \infty, \tag{4.2}$$

for every fixed $d > 0$. This implies that the right hand side of (4.1) tends to $+\infty$ as $Q_d \to \infty$ with any fixed $d > 0$. This implies that the condition (N) is satisfied. Indeed, if (N) does not hold for some $d > 0$, then there exists a sequence of cubes $Q_d \to \infty$ such that

$$E + d^{-n} M_{cf_n(\tilde{\mu}_0)}(Q_d; V) \le C$$

along this sequence. But then both terms in the left hand side are bounded, hence the right hand side of (4.1) is bounded, which contradicts (4.2). □

Now we will start our proof of Proposition 4.1. Let us choose $u \in \mathrm{Lip}(Q_d)$ such that

$$h_{a,0}(u, u) = \int_{Q_d} |\nabla_a u|^2 \, dx \le E d^n, \tag{4.3}$$

and

$$\|u\|^2_{Q_d} = \int_{Q_d} |u|^2 \, dx = d^n. \tag{4.4}$$

Note that due to the diamagnetic inequality we have

$$\int_{Q_d} |\nabla |u||^2 \, dx \le E d^n. \tag{4.5}$$

For every $k \ge 0$ define a set $E_k \subset Q_d$ by

$$E_k = \{x \mid |u(x)| \ge k\},$$

and estimate the capacity of E_k. This estimate is given in the following Lemma, and it can be also obtained from Theorem 10.1.3 in Maz'ya (1985).

Lemma 4.2. *For every* $k > 0$

$$\mathrm{cap}(E_k) \le C_n E k^{-2} d^n. \tag{4.6}$$

Proof. Let us take $v(x) = \max(k - |u(x)|, 0)$. Then $v \in \mathrm{Lip}(Q_d)$, $0 \leq v \leq k$, and $v|_{E_k} = 0$. Using Lemma 2.1, we get

$$\mathrm{cap}(E_k) \leq \frac{C_n \int_{Q_d} |\nabla v|^2\, dx}{d^{-n} \int_{Q_d} v^2\, dx}. \tag{4.7}$$

Note that $|\nabla v| \leq |\nabla |u||$ almost everywhere, so (4.5) implies that

$$\int_{Q_d} |\nabla v|^2\, dx \leq E d^n. \tag{4.8}$$

Let us estimate the denominator in (4.7) from below. We have

$$\|k\| \leq \|k - |u|\| + \|u\| \leq \|(k - |u|)_+\| + 2\|u\| = \|v\| + 2\|u\|,$$

where $\|\cdot\|$ is the norm in $L^2(Q_d)$. Therefore

$$\|v\| \geq \|k\| - 2\|u\| = (k-2)d^{n/2}.$$

and the desired inequality (4.6) follows from (4.7) and (4.8) provided $k \geq 3$. It also obviously holds for $k < 3$ because $E \geq d^{-2}$. □

To continue the proof of Proposition 4.1 note that the desired inequality (4.1) holds if and only if the estimate

$$\mu(Q_d; H_{a,V}) \leq C_n E \left(1 + \frac{1}{f_n(\tilde{\mu}_0) d^{n-2}} \int_{Q_d \setminus F} V\, dx\right) \tag{4.9}$$

holds for every compact $F \subset Q_d$ such that

$$\mathrm{cap}(F) \leq \beta\, \mathrm{cap}(Q_d), \tag{4.10}$$

where $\beta = c f_n(\tilde{\mu}_0)$. Let us choose such a compact set F and denote $F' = E_k \cup F$. Then

$$\mathrm{cap}(F') \leq \beta\, \mathrm{cap}(Q_d) + C_n E k^{-2} d^n \tag{4.11}$$

due to the subadditivity of capacity and Lemma 4.2.

We would like to apply Lemma 2.4 to the set F'. Using (4.11), we see that it is sufficient to assume that

$$\beta \leq c_n/2 \quad \text{and} \quad k^2 \geq \frac{C_n E d^n}{\beta\, \mathrm{cap}(Q_d)} = \frac{\widetilde{C}_n E d^2}{\beta}, \tag{4.12}$$

where C_n, c_n are the constants from (4.11) and (2.4). We will assume in the future that the relations (4.12) are satisfied. Then

$$\mathrm{cap}(F') \leq 2\beta\, \mathrm{cap}(Q_d). \tag{4.13}$$

Now we can choose a function ψ as in Lemma 2.4 and define

$$u' = \psi u, \tag{4.14}$$

where $u \in \mathrm{Lip}(Q_d)$ satisfies (4.3) and (4.4). Clearly, $u'|_{F'} = 0$ by the definition of ψ.

To see that we do not cut off too much, we need to estimate the capacity of the set

$$R = \left\{ x\colon x \in Q_d, |\psi(x)| \le \frac{1}{4} \right\}. \tag{4.15}$$

Clearly $R \supset F'$, so $\mathrm{cap}(R) \ge \mathrm{cap}(F')$. The following Lemma establishes an opposite estimate.

Lemma 4.3. *There exists $C_n > 0$ such that*

$$\mathrm{cap}(R) \le C_n \,\mathrm{cap}(F'). \tag{4.16}$$

Proof. Take $\tilde{\psi} = \max\{|\psi| - 1/4, 0\}$, where ψ is constructed by Lemma 2.4. Then $\tilde{\psi}|_R = 0$, $\tilde{\psi} \ge 0$ and

$$\int_{Q_d} |\nabla \tilde{\psi}|^2 \, dx \le \int_{Q_d} |\nabla \psi|^2 \, dx \le C_n \,\mathrm{cap}(F'), \tag{4.17}$$

where we used (2.5). On the other hand, using (2.6) we obtain

$$\frac{d^n}{4} \le \int_{Q_d} |\psi|^2 \, dx \le \int_{Q_d} \left(|\tilde{\psi}| + \frac{1}{4} \right)^2 dx \le 2 \int_{Q_d} |\tilde{\psi}|^2 \, dx + \frac{d^n}{8},$$

hence

$$d^{-n} \int_{Q_d} |\tilde{\psi}|^2 dx \ge \frac{1}{16}$$

Together with (4.17) and Lemma 2.1 this implies the desired inequality (4.16). □

Now let us recall the following inequalities which relate the capacity of a compact set $F \subset Q_d$ with its Lebesgue measure $\mathrm{mes}\, F$:

$$\mathrm{cap}(F) \ge c_n [\mathrm{mes}\, F]^{(n-2)/n}, \quad n \ge 3, \tag{4.18}$$

with $c_n = \omega_n^{-2/n} n^{(2-n)/n} (n-2)^{-1}$, ω_n is the $(n-1)$-volume of the unit sphere in $\mathbb{R}^n$;

$$\mathrm{cap}_{\overset{\circ}{Q}_{d_0}}(F) \ge c_2 \left[\log \frac{d_0^2}{\mathrm{mes}\, F} \right]^{-1}, \quad n = 2, \ d_0 \ge 2d, \tag{4.19}$$

with $c_2 = (4\pi)^{-1}$ (see e.g., Maz'ya, 1985, Sec. 2.2.3). They can be rewritten as follows:

$$\mathrm{mes}\, F \le C_n [\mathrm{cap}(F)]^{n/(n-2)}, \quad n \ge 3; \tag{4.20}$$

$$\operatorname{mes} F \le d_0^2 \exp\left(-\frac{1}{C_2 \operatorname{cap}_{\overset{\circ}{Q}_{d_0}}(F)}\right), \quad n=2, \ d_0 \ge 2d. \tag{4.21}$$

If $n=2$, then we only need $d_0 = 2d$, which will be assumed below. Then $\operatorname{cap}_{\overset{\circ}{Q}_{d_0}}(F) = \operatorname{cap}_{\overset{\circ}{Q}_{2d}}(F) = \operatorname{cap}(F)$ according to our conventions.

Lemma 4.4. *Let R be a compact subset in Q_d. If $n \ge 3$, then*

$$\int_R |u|^2\, dx \le C_n (\operatorname{mes} R)^{2/n} \int_{\mathbb{R}^n} |\nabla u|^2\, dx, \quad u \in \operatorname{Lip}_c(\mathbb{R}^n). \tag{4.22}$$

If $n=2$, then

$$\int_R |u|^2\, dx \le C_2 \operatorname{mes} R \log\left(\frac{4d^2}{\operatorname{mes} R}\right) \int_{Q_{2d}} |\nabla u|^2\, dx \tag{4.23}$$

for any $u \in \operatorname{Lip}(Q_{2d})$ with $u|_{\partial Q_{2d}} = 0$. (Here Q_d and Q_{2d} are assumed to have the same center.)

Proof. It is clear from the inequality $|\nabla|u|| \le |\nabla u|$ that without loss of generality we can assume that $u \ge 0$. Denote for any $t \ge 0$

$$N_t = \{x \mid u(x) \ge t\} \cap R.$$

According to Theorem 2.3.1 from Maz'ya (1985), for any open $\Omega \supset Q_d$

$$\int_0^\infty \operatorname{cap}_\Omega(N_t)\, d(t^2) \le 4 \int_\Omega |\nabla u|^2\, dx, \quad u \in C_c^\infty(\Omega). \tag{4.24}$$

Using this for $\Omega = \mathbb{R}^n$ together with (4.18), we obtain for $n \ge 3$:

$$\begin{aligned}\int_R u^2\, dx &= \int_0^\infty \operatorname{mes} N_t\, d(t^2) \le (\operatorname{mes} R)^{2/n} \int_0^\infty (\operatorname{mes} N_t)^{(n-2)/n} d(t^2) \\ &\le c_n^{-1} (\operatorname{mes} R)^{2/n} \int_0^\infty \operatorname{cap}(N_t) d(t^2) \le 4c_n^{-1} (\operatorname{mes} R)^{2/n} \int_{\mathbb{R}^n} |\nabla u|^2\, dx,\end{aligned}$$

where c_n is the constant from (4.18). So (4.22) follows with $C_n = 4c_n^{-1}$.

Let us consider the case $n=2$. We can assume $u=0$ on $\mathbb{R}^2 \backslash Q_{2d}$. Using the inequalities (4.19), (4.24) and the fact that the function $\tau \mapsto \tau \log(b/\tau)$ is increasing on $(0, b/e)$, $b > 0$, we obtain

$$\begin{aligned}\int_R u^2\, dx &= \int_0^\infty \operatorname{mes} N_t\, d(t^2) \\ &\le \operatorname{mes} R \log\left(\frac{4d^2}{\operatorname{mes} R}\right) \int_0^\infty \left(\log \frac{4d^2}{\operatorname{mes} N_t}\right)^{-1} d(t^2) \\ &\le 4\pi \operatorname{mes} R \log\left(\frac{4d^2}{\operatorname{mes} R}\right) \int_0^\infty \operatorname{cap}(N_t)\, d(t^2) \\ &\le 16\pi \operatorname{mes} R \log\left(\frac{4d^2}{\operatorname{mes} R}\right) \int_{Q_{2d}} |\nabla u|^2\, dx,\end{aligned}$$

so we get (4.23) with $C_2 = 16\pi$. □

Corollary 4.5. *There exist positive constants C_n, $n \geq 2$, such that if R is a compact subset in Q_d then for any $u \in \mathrm{Lip}(Q_d)$*

$$\int_R |u|^2\,dx \leq C_n(\operatorname{mes} R)^{2/n}\left(\int_{Q_d}|\nabla u|^2\,dx + d^{-2}\int_{Q_d}|u|^2\,dx\right), \tag{4.25}$$

if $n \geq 3$, and

$$\int_R |u|^2\,dx \leq C_2 \operatorname{mes} R\ \log\left(\frac{4d^2}{\operatorname{mes} R}\right)\left(\int_{Q_d}|\nabla u|^2\,dx + d^{-2}\int_{Q_d}|u|^2 dx\right), \tag{4.26}$$

if $n = 2$.

Proof. The result will follow if we apply Lemma 4.4 to the function $v = \chi U$, where $U \in \mathrm{Lip}(Q_{3d})$ is an extension of u by reflections, such that

$$\int_{Q_{3d}}|U|^2\,dx \leq 3^n\int_{Q_d}|u|^2\,dx, \quad \int_{Q_{3d}}|\nabla U|^2\,dx \leq 3^n\int_{Q_d}|\nabla u|^2\,dx,$$

and $\chi \in \mathrm{Lip}(Q_{3d})$, $\chi = 1$ on Q_d, $\chi = 0$ on $Q_{3d}\backslash Q_{2d}$, $0 \leq \chi \leq 1$, $|\nabla\chi(x)| \leq 2d^{-1}$ for all x. □

Remark 4.6. In case $n \geq 3$ another proof of the estimate (4.22) can be obtained if we use the Sobolev inequality

$$\left(\int_{\mathbb{R}^n}|u|^{2n/(n-2)}\,dx\right)^{(n-2)/n} \leq C_n\int_{\mathbb{R}^n}|\nabla u|^2\,dx, \quad u \in \mathrm{Lip}_c(\mathbb{R}^n). \tag{4.27}$$

(See e.g., Lieb and Loss, 2001, Sec. 8.3.) By the Hölder inequality

$$\int_R |u|^2\,dx \leq (\operatorname{mes} R)^{2/n}\left(\int_R |u|^{2n/(n-2)}\,dx\right)^{(n-2)/n}.$$

Combining this with (4.27), we obtain (4.22).

Proof of Proposition 4.1. Let us return to the function u satisfying (4.3) (hence (4.5)) and (4.4). We would like to apply Corollary 4.5 to the set R defined by (4.15) and to the function $|u|$ in order to establish that

$$\int_R |u|^2\,dx \leq \frac{1}{4}\int_{Q_d}|u|^2\,dx = \frac{1}{4}d^n. \tag{4.28}$$

The inequalities in Corollary 4.5 (applied to $|u|$) and the diamagnetic inequality imply for this u

$$\begin{aligned}\int_R |u|^2\,dx &\leq C_n(\operatorname{mes} R)^{2/n}\left(\int_{Q_d}|\nabla_a u|^2\,dx + d^{-2}\int_{Q_d}|u|^2\,dx\right)\\ &\leq C_n(\operatorname{mes} R)^{2/n}(E + d^{-2})\int_{Q_d}|u|^2\,dx \leq 2C_nE(\operatorname{mes} R)^{2/n}\int_{Q_d}|u|^2\,dx,\end{aligned} \tag{4.29}$$

if $n \geq 3$, and

$$\int_R |u|^2\,dx \leq 2C_2 E \operatorname{mes} R \,\log\left(\frac{4d^2}{\operatorname{mes} R}\right) \int_{Q_d} |u|^2\,dx, \tag{4.30}$$

if $n = 2$.

Note that Lemma 4.3 and (4.13) imply

$$\operatorname{cap}(R) \leq 2C_n\beta \operatorname{cap}(Q_d). \tag{4.31}$$

Now for $n \geq 3$, using the estimate (4.29), we see that (4.28) will follow if

$$E(\operatorname{mes} R)^{2/n} \leq \frac{c_n}{8}$$

with a sufficiently small $c_n > 0$. Due to (4.20), this will hold if

$$E[\operatorname{cap}(R)]^{2/(n-2)} \leq \frac{c_n}{8}$$

(possibly with a different c_n). Recalling (4.31), we see that it suffices to take

$$\beta \leq c_n(Ed^2)^{(2-n)/2} = c_n f_n(\mu_0 d^2) = c_n f_n(\tilde{\mu}_0).$$

with a small $c_n > 0$.

Now let us assume that $n = 2$ and use the estimates (4.30), (4.31). Taking into account that $\operatorname{cap}(Q_d) = \operatorname{cap}(Q_1)$ does not depend on d, we see that it suffices to have

$$\beta \leq c_2\left(1 + \log(Ed^2)\right)^{-1} = c_2 f_2(\mu_0 d^2) = c_2 f_2(\tilde{\mu}_0)$$

with a sufficiently small $c_2 > 0$.

In both cases we see that the condition

$$\beta \leq c_n f_n(\tilde{\mu}_0) \tag{4.32}$$

with f_n as in Definition 1.1, is sufficient for the estimate (4.28) to hold. Then we conclude that

$$\int_{Q_d\setminus R} |u|^2\,dx \geq \frac{1}{4}d^n.$$

It follows that for $u' = \psi u$, as in (4.14),

$$\int_{Q_d} |u'|^2\,dx \geq \frac{1}{16}\int_{Q_d\setminus R_\varepsilon} |u|^2\,dx \geq \frac{1}{64}d^n,$$

whenever $\varepsilon \in (0, 1/4]$. Let us take $\varepsilon = 1/4$. Then we get

$$\int_{Q_d} |u'|^2\,dx \geq \frac{1}{64}d^n. \tag{4.33}$$

Now we can use u' as a test function to estimate $\mu(Q_d; H_{a,V})$. We obviously have

$$\mu(Q_d; H_{a,V}) \le \frac{h_{a,0}(u', u')_{Q_d} + (Vu', u')_{Q_d}}{\|u'\|^2_{Q_d}} = \frac{\int_{Q_d} |\nabla_a u'|^2\, dx + \int_{Q_d} V|u'|^2\, dx}{\int_{Q_d} |u'|^2\, dx} \tag{4.34}$$

Let us estimate the terms in the right hand side turn by turn. Since $0 \le \psi \le 1$, we obtain

$$h_{a,0}(u', u')_{Q_d} = \int_{Q_d} |\nabla_a u'|^2\, dx = \int_{Q_d} |\psi \nabla_a u + u\nabla\psi|^2\, dx \le 2\int_{Q_d} |\nabla_a u|^2\, dx + 2\int_{Q_d} |u\nabla\psi|^2\, dx.$$

The first term in the right hand side is estimated by $2Ed^n$ by the choice of u (see (4.3) and (4.4)), whereas the second one is estimated, with the use of (2.7), by

$$2k^2 \int_{Q_d} |\nabla\psi|^2\, dx \le C_n k^2 \operatorname{cap}(F') d^{-n} \int_{Q_d} |\psi|^2\, dx \le C_n k^2 \operatorname{cap}(F').$$

Taking into account (4.13), we see that the right hand side here is estimated by $C_n k^2 \beta \operatorname{cap}(Q_d)$. Now we can choose k so that the inequality (4.12) becomes equality, i.e.,

$$k^2 = \frac{\widetilde{C}_n E d^n}{\beta \operatorname{cap}(Q_d)}.$$

With this choice we get $k^2 \operatorname{cap}(F') \le \widetilde{C}_n E d^n$, so we finally get

$$h_{a,0}(u', u')_{Q_d} \le C_n E d^n. \tag{4.35}$$

We also obviously have

$$(Vu', u')_{Q_d} = \int_{Q_d} V|u'|^2\, dx \le k^2 \int_{Q_d \setminus F'} V\, dx \le k^2 \int_{Q_d \setminus F} V\, dx = \frac{\widetilde{C}_n E d^n}{\beta \operatorname{cap}(Q_d)} \int_{Q_d \setminus F} V\, dx, \tag{4.36}$$

where we used that $V \ge 0$, $0 \le \psi \le 1$ and $\psi|_{F'} = 0$.

Substituting the estimates (4.35) and (4.36) into (4.34) and taking into account (4.33), we obtain

$$\mu(Q_d; H_{a,V}) \le C_n E \left(1 + \frac{1}{\beta \operatorname{cap}(Q_d)} \int_{Q_d \setminus F} V\, dx\right).$$

Recalling the restriction (4.32), we see that it is best to take $\beta = c_n f_n(\tilde{\mu}_0)$ with an appropriate (sufficiently small) constant c_n. Thus we arrive at the inequality (4.9) which proves Proposition 4.1, hence Theorem 1.2. □

Remark 4.7. The condition $a \in L^{\infty}_{\text{loc}}(\mathbb{R}^n)$ can be substantially relaxed. Indeed, it was only used to guarantee that the set $\mathcal{L}$ given by (3.2) (we assume that $V \geq 1$) is precompact in $L^2(B(0, R))$ for any $R \in (0, \infty)$. Let us assume that $|a| \in M(H^1(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n))$, the space of pointwise multipliers mapping $H^1(\mathbb{R}^n)$ into $L^2_{\text{loc}}(\mathbb{R}^n)$. (Here $H^1(\mathbb{R}^n)$ is the standard Sobolev space of functions $u \in L^2(\mathbb{R}^n)$ such that $\nabla u \in L^2(\mathbb{R}^n)$.) This means that for any $R \in (0, \infty)$

$$\int_{B(0,R)} |a|^2 |v|^2 \, dx \leq c(R)(\|\nabla v\|^2 + \|v\|^2), \quad v \in C_c^{\infty}(\mathbb{R}^n),$$

where $\|\cdot\|$ is the norm in $L^2(\mathbb{R}^n)$. Applying this to $v = |u|$, we obtain by the diamagnetic inequality

$$\int_{B(0,R)} |a|^2 |u|^2 \, dx \leq c(R) \quad \text{for all } u \in \mathcal{L}.$$

Therefore,

$$\|\nabla u\|^2_{L^2(B(0,R))} \leq 2\|\nabla_a u\|^2_{L^2(B(0,R))} + 2\||a|u\|^2_{L^2(B(0,R))} \leq 2(1 + c(R)), \quad u \in \mathcal{L}.$$

It remains to note that the set

$$\{u \in C_c^{\infty}(\mathbb{R}^n) |\ \|\nabla u\|^2_{L^2(B(0,R))} + \|u\|^2_{L^2(B(0,R))} \leq 3 + 2c(R)\}$$

is precompact in $L^2(B(0, R))$ due to the Rellich Lemma.

The space $M(H^1(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n))$ can be described analytically in various ways (see Maz'ya, 1973; Corollary 2.3.3 in Kerman and Sawyer, 1986; Maz'ya, 1985; Maz'ya and Verbitsky, 1995). For example, $|a| \in M(H^1(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n))$ if and only if for any unit ball $B(x, 1)$

$$\sup_F \frac{\int_F |a|^2 \, dx}{\operatorname{cap}(F)} \leq c(x),$$

where the supremum is taken over all compact subsets $F \subset \overline{B}(x, 1)$, and $c = c(x)$ is continuous on $\mathbb{R}^n$.

Using the inequalities (4.18) and (4.19), we see that it is sufficient to require that a satisfies the condition

$$\int_F |a|^2 \, dx \leq c(x)(\operatorname{mes}(F))^{(n-2)/n}, \quad n > 2,$$

and

$$\int_F |a|^2 \, dx \leq c(x) \left(\log \frac{4}{\operatorname{mes}(F)} \right)^{-1}, \quad n = 2.$$

It is easy to see that that the following condition on a is stronger, hence also sufficient: $a \in L^n_{\text{loc}}(\mathbb{R}^n)$ if $n > 2$ and $|a|^2 \log_+ |a| \in L^1_{\text{loc}}(\mathbb{R}^2)$ if $n = 2$.

Due to the gauge invariance it suffices that one of the conditions above is satisfied for some $a' = a + d\phi$ with a scalar function (or a distribution) ϕ.

5. NECESSITY: PRECISION

In this section we will construct an operator $H_{a,V}$ which will provide a proof of Theorem 1.7, in particular, the precision of the exponents in (1.8).

Let us consider a hyperplane

$$L = \{x \mid x^1 + x^2 + \cdots + x^n = 0\} \subset \mathbb{R}^n. \tag{5.1}$$

It divides its complement in $\mathbb{R}^n$ into two parts

$$L_\pm = \{x \mid \pm (x^1 + x^2 + \cdots + x^n) > 0\}. \tag{5.2}$$

Let us take two operators $H_{\tilde{a},0}$ and $H_{0,\widetilde{V}}$ in $\mathbb{R}^n$, so that each of them has discrete spectrum in $\mathbb{R}^n$, and then define $H_{a,V}$ as follows:

$$H_{a,V} = H_{\tilde{a},0} \quad \text{in } L_-, \qquad H_{a,V} = H_{0,\widetilde{V}} \quad \text{in } L_+. \tag{5.3}$$

So a and V are obtained by restriction of $\tilde{a}$ and $\widetilde{V}$ to L_- and L_+, respectively, with subsequent extensions by 0 to the complementary half-spaces L_+ and L_-.

Theorem 1.7 will immediately follow from

Proposition 5.1. *The operator $H_{a,V}$, defined by* (5.3), *has a discrete spectrum, and satisfies the condition formulated in Theorem* 1.7.

Proof. We will establish the discreteness of spectrum of $H_{a,V}$ by the necessary and sufficient conditions from Theorem 1.2. To this end we can use tiling cubes with one of the faces parallel to L, and with interiors in one of the half-spaces $L_\pm$ (see Remark 3.5). Then the discreteness of the spectrum of $H_{a,V}$ immediately follows from the corresponding properties of the operators $H_{\tilde{a},0}$ and $H_{0,\widetilde{V}}$.

Now let us choose arbitrary $d > 0$, and a decreasing function $f : [0, +\infty) \to (0, 1)$ satisfying (1.14) in case $n \geq 3$ and (1.15) in case $n = 2$. We claim then that the condition (1.9) (with $c_n = 1$) is not satisfied for the cubes Q_d with the edges parallel to the coordinate axes (where the hyperplane L has the form (5.1)).

We will consider only the cubes Q_d which have "small" intersection with L_+, with $x^1 + x^2 + \cdots + x^n = \delta > 0$ at the corner of the cube where the sum $x^1 + x^2 + \cdots + x^n$ is maximal. We will assume that $\delta \leq d$. Then the intersection of Q_d with $\overline{L}_+$ (the closure of L_+) will be a tetrahedron which is isometric to the tetrahedron

$$\left\{ x = (x^1, \ldots, x^n) \mid x^j \geq 0, \sum_{j=1}^{n} x^j \leq \delta \right\}.$$

Clearly

$$\operatorname{cap}(Q_d \cap \overline{L}_+) = c_n^{(1)} \delta^{n-2}, \quad n \geq 3, \tag{5.4}$$

$$C_2^{-1} \left[\log\left(\frac{2d}{\delta} \right) \right]^{-1} \leq \operatorname{cap}(Q_d \cap \overline{L}_+) \leq C_2 \left[\log\left(\frac{2d}{\delta} \right) \right]^{-1}, \quad n = 2. \tag{5.5}$$

Since $Q_d \cap \overline{L}_+$ is free of magnetic field ($a = 0$ there) and contains a ball of diameter $c_n^{(2)}\delta$, then, taking only test functions from $C_c^\infty(\overset{\circ}{Q}_d \cap L_+)$, we obtain

$$\mu_0(Q_d) \le C_n\delta^{-2}, \tag{5.6}$$

if $C_n > 0$ is sufficiently large.

Now we would like the sets $Q_d \cap \overline{L}_+$ to be negligible in the sense of Theorem 1.2 with the use of the function f, i.e.,

$$\operatorname{cap}(Q_d \cap \overline{L}_+) \le f(\mu_0 d^2)\operatorname{cap}(Q_d). \tag{5.7}$$

If this is the case, then we will have $M_\gamma(Q_d; V) = 0$ and

$$\mu_0(Q_d) + d^{-n}M_\gamma(Q_d; V) \le C_n\delta^{-2}. \tag{5.8}$$

with $\gamma = f(\mu_0 d^2)$. The condition (1.9) means that the left hand side of (5.8) tends to $+\infty$ as $Q_d \to \infty$. This will not hold if we are able to provide a sequence of cubes $Q_d \to \infty$ satisfying (5.8) with a fixed $\delta > 0$. This, in turn, will follow if we find $\delta > 0$ (sufficiently small) and a sequence of cubes, constructed by the procedure above, such that the negligibility condition (5.7) holds for these cubes.

Due to the monotonicity of f and the estimate (5.6), the condition (5.7) will follow if we have

$$\operatorname{cap}(Q_d \cap \overline{L}_+) \le c_n f\left(C_n\left(\frac{\delta}{d}\right)^{-2}\right)d^{n-2}, \tag{5.9}$$

where $c_n = \operatorname{cap}(Q_1)$. Now using (5.4) and (1.14) in case $n \ge 3$ we can rewrite this condition in the form

$$c_n^{(1)}\left(\frac{\delta}{d}\right)^{n-2} \le c_n\left(1 + C_n\left(\frac{\delta}{d}\right)^{-2}\right)^{(2-n)/2} h\left(C_n\left(\frac{\delta}{d}\right)^{-2}\right), \tag{5.10}$$

so it obviously holds if δ/d is sufficiently small, because $h(t) \to +\infty$ as $t \to +\infty$.

In case $n = 2$, due to (5.5) and (1.15), the inequality (5.9) will be fulfilled if we require that

$$C_2\left[\log\left(\frac{2d}{\delta}\right)\right]^{-1} \le \left[1 + \log\left(C_2\left(\frac{\delta}{d}\right)^{-2}\right)\right]^{-1} h\left(C_2^{-1}\left(\frac{\delta}{d}\right)^{-2}\right) \tag{5.11}$$

for a sufficiently large $C_2 > 0$. This again holds if δ/d is sufficiently small. □

6. POSITIVITY

In this section we will prove Theorem 1.8. We will consider operators $H_{a,V}$ with $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $V \geq 0$, $a \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

The proof will be essentially based on the same arguments as the proof of Theorem 1.2, except that the large cubes are essential here (instead of small cubes).

We will use the notations from Sec. 1 and start with the following localization result:

Proposition 6.1. *For an operator $H_{a,V}$ the following conditions are equivalent*:

(a) *There exists $d_1 > 0$ such that $H_{a,V} \geq d_1^{-2}I$, or, equivalently, 0 is not in the spectrum of $H_{a,V}$ in $L^2(\mathbb{R}^n)$ (i.e., the spectrum is in $[\varepsilon_0, +\infty)$ for some $\varepsilon_0 > 0$).*

(b) *There exist $d > 0$ and $d_1 > 0$ such that $\mu(Q_d; H_{a,V}) \geq d_1^{-2}$ for every cube $Q_d \subset \mathbb{R}^n$.*

(c) *There exist $d_1 > 0$ and $d_2 > 0$ such that for every $d > d_2$ we have $\mu(Q_d; H_{a,V}) \geq d_1^{-2}$ for every cube $Q_d \subset \mathbb{R}^n$.*

(d) *There exists $d_1 > 0$ such that for every $d > 0$ we have $\lambda(Q_d; H_{a,V}) \geq d_1^{-2}$ for every cube $Q_d \subset \mathbb{R}^n$.*

(e) *There exists $d_1 > 0$, $d_2 > 0$ such that for every $d > d_2$ we have $\lambda(Q_d; H_{a,V}) \geq d_1^{-2}$ for every cube $Q_d \subset \mathbb{R}^n$.*

Proof. The equivalence of (a), (d) and (e) follows from the fact that the quadratic form $h_{a,V}$ of $H_{a,V}$ is obtained as the closure from the original domain $C_c^\infty(\mathbb{R}^n)$.

Using the inequality (Kondratiev and Shubin, 1999, 2002; Molchanov, 1953)

$$\mu(Q_d; H_{a,V}) \leq \lambda(Q_d; H_{a,V}) \leq A_n \mu(Q_d; H_{a,V}) + \frac{B_n}{d^2}, \tag{6.1}$$

where $A_n > 0$, $B_n > 0$, we immediately see that (d) implies that

$$\mu(Q_d; H_{a,V}) \geq A_n^{-1}\left[\lambda(Q_d; H_{a,V}) - \frac{B_n}{d^2}\right] \geq A_n^{-1}\left[\frac{1}{d_1^2} - \frac{B_n}{d^2}\right] \geq \frac{1}{2A_n d_1^2},$$

if $d > d_2 > 0$ with $d_2^2 \geq 2B_n d_1^2$. So (d) implies (c). Obviously (c) implies (b).

Now we see that the proposition will be proved if we establish that (b) implies (a). So let us assume that (b) holds. Then we have

$$\|u\|^2_{Q_d} \leq d_1^2 h_{a,V}(u,u)_{Q_d}, \quad u \in \mathrm{Lip}(Q_d), \tag{6.2}$$

for every cube Q_d with $d > 0$ taken from the condition (b). If we take an arbitrary $u \in \mathrm{Lip}_c(\mathbb{R}^n)$ and sum up the inequalities (6.2) over a tiling of $\mathbb{R}^n$ by cubes Q_d, we will get the inequality $\|u\|^2 \leq d_1^2 h_{a,V}(u,u)$ which proves (a). □

Proof of Theorem 1.8. Clearly the following implications hold:

$$\text{(e)} \Longrightarrow \text{(c)} \Longrightarrow \text{(b)} \quad \text{and} \quad \text{(e)} \Longrightarrow \text{(d)} \Longrightarrow \text{(b)}.$$

So it suffices to prove the following two implications:

(b) $\Longrightarrow$ (a) (sufficiency of (b)) and (a) $\Longrightarrow$ (e) (necessity of (e)).

Proof of the implication (b) $\Longrightarrow$ (a). Let us assume that there exist $c > 0$, $d_1 > 0$ and $d > 0$ such that the inequality (1.16) holds for all cubes Q_d.

The desired strict positivity will follow if we prove the inequality

$$\int_{\mathbb{R}^n} |u|^2\,dx \le d_2^2 \int_{\mathbb{R}^n} \left(|\nabla_a u|^2 + V|u|^2\right)dx, \quad u \in C_c^\infty(\mathbb{R}^n). \tag{6.3}$$

Note first that for every $u \in \operatorname{Lip}(Q_d)$

$$\mu_0(Q_d) \int_{Q_d} |u|^2\,dx \le \int_{Q_d} |\nabla_a u|^2\,dx \le \int_{Q_d} \left(|\nabla_a u|^2 + V|u|^2\right)dx. \tag{6.4}$$

As we did in the proof of Proposition 3.1, let us split the cubes Q_d from a tiling of $\mathbb{R}^n$ into two types:

Type I (*large energy of the magnetic field in* Q_d):

$$\mu_0(Q_d) > \frac{1}{2d_1^2};$$

Type II (*small energy of the magnetic field in* Q_d):

$$\mu_0(Q_d) \le \frac{1}{2d_1^2}.$$

For a type I cube Q_d we obtain from (6.4) that for every $u \in \operatorname{Lip}(Q_d)$ the inequality (6.2) holds with $2d_1^2$ instead of d_1^2.

Now let Q_d be a type II cube. Then we have

$$d^{-n} M_c(Q_d; V) \ge \frac{1}{2d_1^2}.$$

Due to Lemma 2.2 and Remark 2.3 we obtain for every $u \in \operatorname{Lip}(Q_d)$ and $c > 0$

$$\int_{Q_d} |u|^2\,dx \le \frac{C_n d^2}{c} \int_{Q_d} |\nabla_a u|^2\,dx + \frac{4d^n}{M_c(Q_d; V)} \int_{Q_d} |u|^2 V\,dx,$$

and we get

$$\int_{Q_d} |u|^2\,dx \le Cd^2 \int_{Q_d} |\nabla_a u|^2\,dx + 8d_1^2 \int_{Q_d} |u|^2 V\,dx,$$

where $C = C_n/c$. Taking $d_2 > 0$ such that

$$d_2^2 = \max\left(Cd^2, 8d_1^2\right),$$

we obtain (6.2) with d_2^2 instead of d_1^2.

So we obtained the inequalities (6.2) (with d_2^2 instead of d_1^2) for both types of cubes. This means that the condition (b) in Proposition 6.1 is satisfied, hence the spectrum of $H_{a,V}$ is discrete. $\square$

Proof of the implication (a) $\Longrightarrow$ (e). We will use Proposition 4.1 in the same way as in the proof of Theorem 1.2. Recall the notation $E = \mu_0(Q_d) + d^{-2}$ which was introduced in the formulation of Proposition 4.1, and will be used here too, though for large d when the difference between E and $\mu_0(Q_d)$ becomes small.

According to Proposition 6.1 we can assume that its condition (c) is satisfied, i.e., $\mu(Q_d; H_{a,V}) \geq d_3^{-2}$ for every cube Q_d with $d > d_4$, where $d_3, d_4 > 0$ are sufficiently large. Then due to (4.1) we have for such d

$$\mu_0(Q_d) + \frac{E}{f(\tilde{\mu}_0)d^{n-2}} M_{c_n f_n(\tilde{\mu}_0)}(Q_d; V) \geq \frac{1}{C_n d_3^2} - \frac{1}{d^2} \geq \frac{1}{d^2}, \tag{6.5}$$

provided $d^2 \geq 2C_n d_3^2$.

Now note that in the case when

$$\mu_0(Q_d) \geq \frac{1}{d^2},$$

the desired inequality (1.19) becomes obvious (with $\tilde{c}_n = 1$). So from now on we can assume that

$$\mu_0(Q_d) \leq \frac{1}{d^2}.$$

This implies that

$$f_n(\tilde{\mu}_0) = f_n(\mu_0 d^2) \geq f_n(1) > 0, \quad n \geq 2.$$

We also have in this case $E \leq 2d^{-2}$. It follows that the coefficient in front of $M_{c_n f_n(\tilde{\mu}_0)}(Q_d; V)$ in (6.5) is bounded from above by $C_n d^{-n}$. Hence the left hand side in (6.5) is bounded from above by $\widetilde{C}_n[\mu_0(Q_d) + d^{-n} M_{c_n}(Q_d; V)]$ and the desired inequality (1.19) follows with $\tilde{c}_n = \min(\widetilde{C}_n^{-1}, 1)$. This ends the proof of Theorem 1.8. $\square$

7. OPERATORS IN DOMAINS

In this section we will discuss the discreteness of spectrum and strict positivity for the magnetic Schrödinger operators in arbitrary open subsets $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary conditions on $\partial\Omega$. It occurs that the methods developed above can be extended to this case and provide necessary and sufficient conditions so that the results of the previous sections appear as a particular case when $\Omega = \mathbb{R}^n$. Note that the geometry of the domain may contribute to the discreteness of spectrum or strict positivity and even be the only cause of these properties.

Let $H_{a,V}$ be the magnetic Schrödinger operator defined as in Sec. 1 but in $L^2(\Omega)$. We will assume that $V \in L^1_{\mathrm{loc}}(\Omega)$, $V \geq 0$, $a \in L^2_{\mathrm{loc}}(\Omega)$. For the discreteness of spectrum results we will assume that a is bounded in $\Omega \cap B(0, R)$ for every $R > 0$, though this condition may be substantially weakened as explained in Remark 4.7. The operator $H_{a,V}$ is defined by the quadratic form (1.2) on functions $u \in C_c^\infty(\Omega)$.

We will define the Molchanov functional in Ω as follows

$$M_{\gamma,\Omega}(Q_d; V) = \inf_F \left\{ \int_{Q_d \setminus F} V\,dx \,|\, \mathrm{cap}(F) \le \gamma\,\mathrm{cap}(Q_d),\, F \supset Q_d \cap (\mathbb{R}^n \setminus \Omega) \right\},$$

where $0 < \gamma < 1$, F is a closed subset in Q_d. By definition it is $+\infty$ if there is no sets F satisfying the condition in the braces, i.e., if

$$\mathrm{cap}(Q_d \cap (\mathbb{R}^n \setminus \Omega)) > \gamma\,\mathrm{cap}(Q_d). \tag{7.1}$$

The numbers $\lambda(Q_d; H_{a,V})$ and $\mu(Q_d; H_{a,V})$ should be replaced by the numbers $\lambda_\Omega(Q_d; H_{a,V})$ and $\mu_\Omega(Q_d; H_{a,V})$ which are defined by the same formulas (1.4), (1.5) (with $G = Q_d$) but with an additional requirement on u to vanish in a neighborhood of $Q_d \cap (\mathbb{R}^n \setminus \Omega)$. Then the same localization results (see e.g., Theorems 1.1–1.3 in Kondratiev and Shubin, 2002) hold. For example, $H_{a,V}$ has a discrete spectrum in $L^2(\Omega)$ if and only if for any fixed $d > 0$

$$\mu_\Omega(Q_d; H_{a,V}) \to +\infty \quad \text{as } Q_d \to \infty.$$

The appropriate modification of μ_0 (the local energy of the magnetic field) is

$$\mu_{0,\Omega} = \mu_{0,\Omega}(Q_d) = \mu_{0,\Omega}(Q_d; a) = \mu_\Omega(Q_d; H_{a,0}).$$

With these notations the following theorems are obtained by simple repetition of arguments given in the previous sections.

Theorem 7.1. *Theorem* 1.2 *holds for* $H_{a,V}$ *in* $L^2(\Omega)$ *if we replace* μ_0 *by* $\mu_{0,\Omega}$ *and* $M_\gamma(Q_d; V)$ *by* $M_{\gamma,\Omega}(Q_d; V)$.

Theorem 7.2. *Theorem* 1.8 *holds for* $H_{a,V}$ *in* $L^2(\Omega)$ *if we replace* μ_0 *by* $\mu_{0,\Omega}$ *and* $M_\gamma(Q_d; V)$ *by* $M_{\gamma,\Omega}(Q_d; V)$.

The appropriate modifications of Corollaries 1.5 and 1.6 hold as well. The same replacements of μ_0 by $\mu_{0,\Omega}$ and M_c by $M_{c,\Omega}$ should be made in the formulations, and the integral in (1.12) should be replaced by $M_{0,\Omega}(Q_d; V)$ which is equal to this integral if $Q_d \subset \Omega$ and to $+\infty$ otherwise.

Now we will formulate some more specific corollaries of Theorem 7.1, which treat the cases when one or both fields vanish. We will start with the case when $a \equiv 0, V \equiv 0$.

Corollary 7.3. *There exists* $c_n > 0$ *such that for every function* $g \in \mathcal{G}$ (*see Definition* 1.1) *the following conditions are equivalent*:

(a) *The spectrum of the operator* $H_{0,0} = -\Delta$ *in* $L^2(\Omega)$ *with the Dirichlet boundary conditions on* $\partial\Omega$ *is discrete.*

(b_g) *$\exists d_0 > 0, \forall d \in (0, d_0), \exists R = R(d) > 0, \forall Q_d$ such that $Q_d \cap (\mathbb{R}^n \backslash B(0, R)) \neq \emptyset$, the inequality* (7.1) *is satisfied with $\gamma = c_n g(d)^{-1} d^2$.*
In particular, all conditions (b_g) *for different $g \in \mathcal{G}$ are equivalent.*

Instead of (b_g) we can equivalently write that $\exists d_0 > 0, \forall d \in (0, d_0)$

$$\liminf_{Q_d \to \infty} \frac{\mathrm{cap}(Q_d \cap (\mathbb{R}^n \backslash \Omega))}{\mathrm{cap}(Q_d)} > \gamma,$$

with the same γ as above (we can replace c_n by a smaller positive number).

Note that the condition (b_g) is a purely geometric condition on the open set $\Omega \subset \mathbb{R}^n$. The equivalence of these conditions for different functions $g \in \mathcal{G}$ is a non-trivial geometric property of the capacity.

The next corollary treats the case when $a \equiv 0$, i.e., there is no magnetic field.

Corollary 7.4. *There exists $c_n > 0$ such that for every $g \in \mathcal{G}$ the following conditions are equivalent*:

(a) *The spectrum of the operator $H_{0,V} = -\Delta + V$ in $L^2(\Omega)$ with the Dirichlet boundary conditions on $\partial\Omega$ is discrete.*

(b_g) $\exists d_0 > 0, \forall d \in (0, d_0)$

$$M_{\gamma,\Omega}(Q_d; V) \to +\infty \quad \text{as } Q_d \to \infty,$$

where $\gamma = c_n g(d)^{-1} d^2$.

(c_g) $\exists d_0 > 0, \forall d \in (0, d_0)$

$$\liminf_{Q_d \to \infty} d^{-n} M_{\gamma,\Omega}(Q_d; V) \geq g(d)^{-1},$$

with the same γ as in (b_g).
In particular, all conditions (b_g), (c_g) *for different $g \in \mathcal{G}$ are equivalent.*

Finally, we consider the case when $V \equiv 0$. To this end we need the quantity $\mu^{(\gamma)}_{0,\Omega}(Q_d)$ which is defined as $\mu_{0,\Omega}(Q_d)$ if $\mathrm{cap}(Q_d \cap (\mathbb{R}^n \backslash \Omega)) \leq \gamma \mathrm{cap}(Q_d)$ and $+\infty$ otherwise (i.e., if (7.1) is satisfied).

Corollary 7.5. *There exists $c_n > 0$ such that for every n-admissible pair (f, g)* (*see Definition* 1.1) *the following conditions are equivalent*:

($\tilde{a}$) *The spectrum of the operator $H_{a,0}$ in $L^2(\Omega)$ with the Dirichlet boundary conditions on $\partial\Omega$ is discrete.*

($\tilde{b}_g$) $\exists\, d_0 > 0, \forall\, d \in (0, d_0)$

$$\mu^{(\gamma)}_{0,\Omega}(Q_d) \to +\infty \quad \text{as } Q_d \to \infty,$$

where $\gamma = c_n f(\mu_{0,\Omega} d^2) g(d)^{-1} d^2$.

($\tilde{c}_g$) $\exists d_0 > 0,\ \forall d \in (0, d_0)$

$$\liminf_{Q_d \to \infty} \mu_{0,\Omega}^{(\gamma)}(Q_d) \geq g(d)^{-1},$$

with the same γ *as in* ($\tilde{b}_g$).
In particular, all conditions ($\tilde{b}_g$), ($\tilde{c}_g$) *for different* $g \in \mathcal{G}$ *are equivalent.*

We skip formulations of similar corollaries of Theorem 7.2.

APPENDIX: PROOFS OF LEMMAS 2.1, 2.2, AND 2.4

In this appendix, for the convenience of the readers, we will provide proofs of Lemmas 2.1, 2.2, and 2.4. These proofs are simpler compared with the proofs given in Maz'ya (1985) due to the fact that the corresponding results in Maz'ya (1985) have much bigger generality.

Let us recall the following classical *Poincaré inequality* (see e.g., Gilbarg and Trudinger, 1983, Sec. 7.8, or Kondratiev and Shubin, 1999, Lemma 5.1):

$$||u - \bar{u}||_{Q_d}^2 \leq \frac{d^2}{\pi^2} \int_{Q_d} |\nabla u(x)|^2 \, dx, \tag{A.1}$$

where $\|\cdot\|_{Q_d}$ is the norm in $L^2(Q_d)$, $u \in \mathrm{Lip}(Q_d)$, and

$$\bar{u} = d^{-n} \int_{Q_d} u(x)\, dx$$

is the mean value of u on Q_d.

Proof of Lemma 2.1. Let us normalize u by

$$d^{-n} \int_{Q_d} |u(x)|^2 \, dx = 1,$$

i.e., $\overline{|u|^2} = 1$ (we will call it *the standard normalization*). By the Cauchy–Schwarz inequality we obtain

$$\overline{|u|} \leq \left(\overline{|u|^2}\right)^{1/2} = 1 \tag{A.2}$$

Replacing u by $|u|$ does not change the denominator and may only decrease the numerator in (2.2). Therefore we can restrict ourselves to Lipschitz functions $u \geq 0$.

Let us denote $\phi = 1 - u$. Then $\phi = 1$ on F, and $\bar{\phi} = 1 - \bar{u} \geq 0$ due to (A.2). Let us estimate $\bar{\phi}$ from above. Obviously

$$\bar{\phi} = d^{-n/2}(\|u\| - \|\bar{u}\|) \leq d^{-n/2}\|u - \bar{u}\|,$$

where $\|\cdot\|$ means the norm in $L^2(Q_d)$. So the Poincaré inequality gives

$$\bar{\phi} \leq \pi^{-1} d^{-n/2+1} \|\nabla u\| = \pi^{-1} d^{-n/2+1} \|\nabla \phi\|,$$

hence

$$\bar{\phi}^2 \leq \frac{1}{\pi^2} d^{2-n} \int_{Q_d} |\nabla\phi|^2 \, dx.$$

Using the Poincaré inequality again, we obtain

$$\|\phi\|^2 = \|(\phi - \bar{\phi}) + \bar{\phi}\|^2 \leq 2\|\phi - \bar{\phi}\|^2 + 2\|\bar{\phi}\|^2 \leq \frac{4d^2}{\pi^2} \int_{Q_d} |\nabla\phi|^2 \, dx,$$

or

$$\int_{Q_d} \phi^2 \, dx \leq \frac{4d^2}{\pi^2} \int_{Q_d} |\nabla\phi|^2 \, dx. \tag{A.3}$$

Let us extend ϕ outside Q_d by symmetries in the faces of Q_d, so that the extension $\tilde{\phi}$ satisfies

$$\int_{Q_{3d}} |\nabla\tilde{\phi}|^2 \, dx = 3^n \int_{Q_d} |\nabla\phi|^2 \, dx, \ \int_{Q_{3d}} |\tilde{\phi}|^2 \, dx = 3^n \int_{Q_d} |\phi|^2 \, dx.$$

Denote by η a continuous piecewise linear function, such that $\eta = 1$ on Q_d, $\eta = 0$ outside Q_{2d}, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq 2d^{-1}$. Then

$$\operatorname{cap}(F) \leq \int_{Q_{2d}} |\nabla(\tilde{\phi}\eta)|^2 \, dx \leq 2 \cdot 3^n \left(\int_{Q_d} |\nabla\phi|^2 \, dx + 4d^{-2} \int_{Q_d} \phi^2 \, dx \right).$$

Taking into account that $|\nabla\phi| = |\nabla u|$ and using (A.3), we obtain

$$\operatorname{cap}(F) \leq C_n \int_{Q_d} |\nabla u|^2 \, dx,$$

which is equivalent to the desired estimate (2.2). □

Proof of Lemma 2.2. Let $\mathcal{M}_\tau = \{x \in Q_d : |u(x)| > \tau\}$, where $\tau \geq 0$. Since

$$|u|^2 \leq 2\tau^2 + 2(|u| - \tau)^2 \quad \text{on } \mathcal{M}_\tau,$$

we have for all τ

$$\int_{Q_d} |u|^2 \, dx \leq 2\tau^2 d^n + 2 \int_{\mathcal{M}_\tau} (|u| - \tau)^2 \, dx.$$

Let us take

$$\tau^2 = \frac{1}{4d^n} \int_{Q_d} |u|^2 \, dx,$$

i.e., $\tau = \frac{1}{2}\left(\overline{|u|^2}\right)^{1/2}$. Then for this particular value of τ we obtain

$$\int_{Q_d} |u|^2 \, dx \leq 4 \int_{\mathcal{M}_\tau} (|u| - \tau)^2 \, dx. \tag{A.4}$$

Assume first that $\operatorname{cap}(Q_d \backslash \mathscr{M}_\tau) \geq \gamma \operatorname{cap}(Q_d)$. Using (A.4) and applying Lemma 2.1 to the function $(|u| - \tau)_+$, which equals $|u| - \tau$ on $\mathscr{M}_\tau$ and 0 on $Q_d \backslash \mathscr{M}_\tau$, we see that

$$\operatorname{cap}(Q_d \backslash \mathscr{M}_\tau) \leq \frac{C_n \int_{\mathscr{M}_\tau} |\nabla(|u| - \tau)|^2 \, dx}{d^{-n} \int_{Q_d} |u|^2 \, dx} \leq \frac{C_n \int_{Q_d} |\nabla u|^2 \, dx}{d^{-n} \int_{Q_d} |u|^2 \, dx},$$

where C_n is 4 times the one in (2.2). Therefore

$$\int_{Q_d} |u|^2 \, dx \leq \frac{C_n d^n \int_{Q_d} |\nabla u|^2 \, dx}{\operatorname{cap}(Q_d \backslash \mathscr{M}_\tau)} \leq \frac{C_n d^n \int_{Q_d} |\nabla u|^2 \, dx}{\gamma \operatorname{cap}(Q_d)}$$

Taking into account that $\operatorname{cap}(Q_d) = c_n d^{n-2}$ we see that

$$\int_{Q_d} |u|^2 \, dx \leq \frac{C_n d^2}{\gamma} \int_{Q_d} |\nabla u|^2 \, dx \tag{A.5}$$

with yet another constant C_n.

Now consider the opposite case $\operatorname{cap}(Q_d \backslash \mathscr{M}_\tau) \leq \gamma \operatorname{cap}(Q_d)$. Then we can write

$$\begin{aligned} \int_{Q_d} |u|^2 V \, dx &\geq \int_{\mathscr{M}_\tau} |u|^2 V \, dx \geq \tau^2 \int_{\mathscr{M}_\tau} V \, dx = \frac{1}{4d^n} \int_{Q_d} |u|^2 \, dx \cdot \int_{\mathscr{M}_\tau} V \, dx \\ &\geq \frac{1}{4d^n} \int_{Q_d} |u|^2 \, dx \cdot \inf_F \int_{Q_d \backslash F} V \, dx, \end{aligned}$$

where the infimum should be taken over all compact sets $F \subset Q_d$ such that $\operatorname{cap}(F) \leq \gamma \operatorname{cap}(Q_d)$, so it becomes $M_\gamma(Q_d; V)$. Finally we obtain in this case

$$\int_{Q_d} |u|^2 \, dx \leq \frac{4d^n}{M_\gamma(Q_d; V)} \int_{Q_d} V|u|^2 \, dx. \tag{A.6}$$

The resulting inequality (2.3) follows from (A.5) and (A.6). □

Proof of Lemma 2.4. We start with a function $\phi \in \operatorname{Lip}_c(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ in a neighborhood of F', $\phi = 0$ outside Q_{d_0} (where for $n = 2$ we take $d_0 = 2d$), and

$$\operatorname{cap}(F') \geq c_n' \int_{Q_{d_0}} |\nabla \phi|^2 \, dx \tag{A.7}$$

with $c_n' > 0$. It follows that

$$\operatorname{cap}(F') \geq c_n' \int_{Q_d} |\nabla \phi|^2 \, dx.$$

Now take $\psi = 1 - \phi$, so $0 \leq \psi \leq 1$ and $\psi|_{F'} = 0$. Then $|\nabla \psi| = |\nabla \phi|$, hence the condition (2.5) is obviously satisfied. Now our goal will be achieved if we prove that (2.6) holds provided (2.4) is satisfied with a sufficiently small $c_n > 0$.

To prove (2.6), note first that Lemma 4.4 with $R = Q_d$ gives

$$\int_{Q_d} |\phi|^2\,dx \le C_n d^2 \int_{Q_{d_0}} |\nabla\phi|^2\,dx, \tag{A.8}$$

Hence, using (A.7), we obtain

$$\bar{\phi}^2 = d^{-n}\int_{Q_d} \phi^2\,dx \le C_n d^{2-n}\int_{Q_{d_0}} |\nabla\phi|^2\,dx \le \frac{\widetilde{C}_n(c_n')^{-1}\mathrm{cap}(F')}{\mathrm{cap}(Q_d)} \le \widetilde{C}_n(c_n')^{-1}c_n,$$

where c_n is the constant from (2.4). Now we can adjust c_n so that we have $\widetilde{C}_n(c_n')^{-1}c_n \le 1/4$. Then (2.6) follows from the triangle inequality. $\square$

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REFERENCES

Avron, J., Herbst, I., Simon, B. (1978). Schrödinger operators with magnetic fields, I., general interactions. *Duke Math. J.* 45:847–883.

Berezin, F. A., Shubin, M. A. (1991). *The Schrödinger Equation*. Dordrecht e.a.: Kluwer Academic Publishers.

Brüning, J. (1989). On Schrödinger operators with discrete spectrum. *J. Funct. Anal.* 85:117–150.

Colin de Verdière, Y. (1986). L'asymptotique de weyl pour les bouteilles magnétiques. *Comm. Math. Phys.* 105:327–335.

Courant, R., Hilbert, D. (1953). *Methods of Mathematical Physics*. New York: Interscience Publishers.

Dufresnoy, A. (1983). Un exemple de champ magnétique dans $\mathbb{R}^\nu$. *Duke Math. J.* 50:729–734.

Edmunds, D. E., Evans, W. D. (1987). *Spectral Theory and Differential Operators*. New York: Oxford University Press.

Fefferman, C. (1983). The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)* 9:129–206.

Friedrichs, K. (1934). Spektraltheorie halbbeschränkter operatoren und anwendung auf die spektralzerlegung von differentialoperatoren. *Math. Ann.* 109:465–487, 685–713.

Gilbarg, D., Trudinger, N. (1983). *Elliptic Partial Differential Equations of Second Order*. 2nd ed. Berlin: Springer-Verlag.

Helffer, B., Mohamed, A. (1988). Caractérisation du spectre essentiel de l'opérateur de Schrödinger avec un champ magnétique. *Ann. Inst. Fourier (Grenoble)* 38:95–112.

Helffer, B., Nourrigat, J., Wang, X. P. (1989). Sur le spectre de l'équation de dirac (dans $\mathbb{R}^3$ ou $\mathbb{R}^2$) avec champ magnétique. *Ann. Sci. École Norm. Sup.* 22:515–533.

Ivrii, V. (1998). *Microlocal Analysis and Precise Spectral Asymptotics*. Berlin: Springer-Verlag.

Iwatsuka, A. (1986). Magnetic Schrödinger operators with compact resolvent. *J. Math. Kyoto Univ.* 26:357–374.

Iwatsuka, A. (1990). On Schrödinger operators with magnetic fields. In: *Functional-Analytic Methods for Partial Differential Equations*. Lecture Notes in Math., 1450, Berlin: Springer, pp. 157–172, Tokyo, 1989.

Kato, T. (1966). *Perturbation Theory for Linear Operators*. Springer-Verlag.

Kato, T. (1972). Schrödinger operators with singular potentials. *Israel J. Math.* 13:135–148.

Kerman, R., Sawyer, E. (1986). The trace inequality and eigenvalue estimates for Schrödinger operators. *Ann. Inst. Fourier (Grenoble)* 36:207–228.

Kondratiev, V., Shubin, M. (1999). Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry. In: *Operator Theory: Advances and Applications*. Vol. 110. Basel: Birkhäuser Verlag, pp. 185–226, (dedicated to V. G. Maz'ya 60th anniversary).

Kondratiev, V., Shubin, M. (2002). Discreteness of spectrum for the magnetic Schrödinger operators. *Commun. Partial Differential Equations* 27:477–525.

Leinfelder, H. (1983). Gauge invariance of Schrödinger operators and related spectral properties. *J. Operator Theory* 9:163–179.

Leinfelder, H., Simader, C. G. (1981). Schrödinger operators with singular magnetic potentials. *Math. Z.* 176:1–19.

Levendorskii, S. Z. (1997). Spectral properties of Schrödinger operators with irregular magnetic potentials, for a spin $\frac{1}{2}$ particle. *J. Math. Anal. Appl.* 216:48–68.

Lieb, E. H., Loss, M. (2001). *Analysis*. 2nd ed. Amer. Math. Soc.

Maslov, V. P. (1968). A criterion for the discreteness of the spectrum of a Sturm-Liouville equation with operator coefficient (apropos the article of B. M. Levitan and G. A. Suvorchenkova) (Russian). *Funkcional. Anal. i Prilozhen.* 2(2):63–67.

Maz'ya, V. G. (1973). On (p, l)-capacity, imbedding theorems and the spectrum of a self-adjoint elliptic operator. *Math. USSR Izv.* 7:357–387.

Maz'ya, V. G. (1985). *Sobolev spaces*. Berlin: Springer Verlag.

Maz'ya, V. G., Verbitsky, I. E. (1995). Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* 33:81–115.

Mohamed, A., Raikov, G. D. (1994). On the spectral theory of the Schrödinger operator with electromagnetic potential. In: *Pseudo-Differential Calculus and Mathematical Physics*. Math. Top. 5, Berlin: Akademie Verlag, pp. 298–390.

Molchanov, A. M. (1953). On the discreteness of the spectrum conditions for self-adjoint differential equations of the second order (Russian). *Trudy Mosk. Matem. Obshchestva (Proc. Moscow Math. Society)* 2:169–199.

Reed, M., Simon, B. (1978). *Methods of Modern Mathematical Physics, IV: Analysis of Operators*. New York e.a.: Academic Press.

Shigekawa, I. (1991). Spectral properties of Schrödinger operators with magnetic fields for a spin $\frac{1}{2}$ particle. *J. Funct. Anal.* 101:255–285.

Simon, B. (1976). Universal diamagnetism of spinless Bose systems. *Phys. Rev. Lett.* 36:1083–1084.

Tamura, H. (1987). Asymptotic distribution of eigenvalues for Schrödinger operators with magnetic fields. *Nagoya Math. J.* 105:49–69.

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