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EXISTENCE AND UNIQUENESS OF AN ENERGY SOLUTION TO THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION IN THE EXTERIOR OF A MULTI-DIMENSIONAL PARABOLOID

V. Maz'ya

Department of Mathematical Sciences, M&O Building
University of Liverpool, Liverpool L69 7ZL, UK

Department of Mathematics, Linköping University
SE-581 83 Linköping, Sweden
vlmaz@mai.liu.se

S. Poborchi*

St. Petersburg State University
28, Universitetskii pr., St. Petersburg, Russia
poborchi@mail.ru

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We study the Dirichlet problem for the Laplace equation in the exterior of a multi-dimensional paraboloid. We describe the boundary traces of functions in the energy space corresponding to the problem and obtain necessary and sufficient conditions for the existence and uniqueness of an energy solution. Bibliography: 10 *titles.*

1 Introduction and Formulation of Results

As is known [1, 2], the study of the solvability of the Dirichlet problem for the Laplace equation is reduced to a description of the boundary traces of functions with finite Dirichlet integral in the domain. By the Gagliardo theorem [3] (cf. also [4, 5]), the boundary traces of functions in $W_p^1(\Omega)$ in a domain with compact closure and Lipschitz boundary belong to the space $W_p^{1-1/p}(\partial\Omega)$.

In the case of an infinite domain with locally Lipschitz boundary, the Gagliardo theorem characterizes the local behavior of the boundary trace, but this information is, generally speaking, unsufficient for obtaining a complete characteristics of trace.

In this paper, we consider functions of class L_2^1 in the exterior of a multi-dimensional paraboloid and characterize the traces of such functions on the infinite boundary of the paraboloid. Thereby we obtain sufficient and necessary conditions for the solvability of the Dirichlet problem (one-lateral or two-lateral) in the class L_2^1.

* To whom the correspondence should be addressed.

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We formulate the main results of this paper. Let F be an increasing locally Lipschitz function on the half-axis $[0,\infty)$ such that $F(0)=0$, $F'(0+0)<\infty$, and $F'(r)\to\infty$ as $r\to\infty$. We write a point in $\mathbb{R}^n$ as $\mathbb{R}^n \ni x=(y,z)$: $y\in\mathbb{R}^{n-1}$, $z\in\mathbb{R}^1$. Introduce the domains $\Omega^\pm\subset\mathbb{R}^n$ by the formula

$$\Omega^+ = \{(y,z)\in\mathbb{R}^n : z > F(|y|)\},$$
$$\Omega^- = \{(y,z)\in\mathbb{R}^n : z < F(|y|)\}.$$

We call the domain Ω^+ an *n-dimensional paraboloid* and Ω^- its *exterior*.

Let Γ be the common boundary of the domains Ω^+ and Ω^-. In each of these domains, we consider the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \ \Omega^+, \qquad u\big|_\Gamma = f, \tag{1.1}$$

$$\Delta u = 0 \quad \text{in } \ \Omega^-, \qquad u\big|_\Gamma = f. \tag{1.2}$$

By the *energy space* $H^1(\Omega^-)$ of the problem (1.2) we mean the closure in the norm $\|\nabla u\|_{L_2(\Omega^-)}$ of the set of smooth functions in Ω^- with bounded support in Ω^-. In a similar way, we introduce the energy space $H^1(\Omega^+)$ of the problem (1.1).

A *solution* to the problem (1.1) (or (1.2)) is a function $u\in H^1(\Omega^\pm)$ satisfying the boundary condition $u|_\Gamma = f$ and such that for all $v\in C_0^\infty(\Omega^\pm)$

$$\int\limits_{\Omega^\pm} \nabla u \nabla v dx = 0.$$

As is known, the Dirichlet problem under consideration admits a variational statement: a solution to the problem (1.1) or (1.2) is a minimizer of the Dirichlet integral on the set of functions satisfying the boundary condition

$$u\big|_\Gamma = f.$$

It is easy to show that the minimizing sequence for the Dirichlet integral converges in the energy space and thereby we justify the solvability of the Dirichlet problem provided that the class of functions satisfying the boundary condition is not empty. The uniqueness of a solution is a consequence of the convexity of the DIrichlet integral.

On the surface Γ, we introduce the space $Tr^\pm(\Gamma)$ of traces $u\big|_\Gamma$ of functions $u\in H^1(\Omega^\pm)$, equipped with the norm

$$\|f\|_{Tr^\pm(\Gamma)} = \inf\{\|\nabla u\|_{L_2(\Omega^\pm)} : u\in H^1(\Omega^\pm), u|_\Gamma = f\}.$$

The space

$$Tr(\Gamma) = Tr^-(\Gamma)\cap Tr^+(\Gamma)$$

can be interpreted as the space of traces on Γ of functions in $H^1(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{Tr(\Gamma)} = \inf\{\|\nabla u\|_{L_2(\mathbb{R}^n)} : u\in H^1(\mathbb{R}^n), u|_\Gamma = f\}.$$

Theorem 1.1. *There exists a linear continuous extension operator*

$$H^1(\Omega^-) \to H^1(\mathbb{R}^n).$$

Consequently, the space $Tr^-(\Gamma)$ *is continuously embedded into* $Tr^+(\Gamma)$.

Remark 1.1. In Theorem 1.1, the domain Ω^- cannot be replaced with Ω^+. We consider the following example. Suppose that $\varphi = F^{-1}$ and $u = u(z)$. Then the inclusion $u \in H^1(\Omega^+)$ is equivalent to the relations

$$\int\limits_1^\infty u'(z)^2 \varphi(z)^{n-1} dz < \infty, \quad \lim_{z\to\infty} u(z) = 0.$$

In particular, if we set $\varphi(z) = z^{1/(n-1)}$ and $u(z) = z^\lambda \eta(z)$, where η is a smooth cut-off function that vanishes in a neighborhood of the point $z = 0$ and is equal to 1 for large z, then for $\lambda \in [(2-n)/n, 0)$ we have $u \in H^1(\Omega^+)$, but $u \notin L_q(\Omega^+)$, $q = 2n(n-2)^{-1}$, which contradicts the possibility to extend u to $\mathbb{R}^n$ with preserving the class H^1.

The following assertion describes the class $Tr^-(\Gamma)$ for $n > 3$.

Theorem 1.2. *Let* $\varphi = F^{-1}$. *For* $n > 3$ *the following equivalence of norms holds*:

$$\|f\|_{Tr^-(\Gamma)} \sim \left(\int\limits_\Gamma f(x)^2 \frac{ds_x}{\max\{\varphi(z), 1\}} + \iint\limits_{\{x,\xi\in\Gamma : |z-\zeta| < M(z,\zeta)\}} |f(x) - f(\xi)|^2 \frac{ds_x ds_\xi}{|x-\xi|^n} \right)^{1/2}, \tag{1.3}$$

where $x = (y, z)$, $\xi = (\eta, \zeta)$, $M(z,\zeta) = \max\{\varphi(z), \varphi(\zeta), 1\}$, *and* ds_x, ds_ξ *are area elements on the surface* Γ. *The finiteness of the norm on the right-hand side* (1.3) *is a necessary and sufficient condition for the existence and uniqueness of an energy solution to the problem* (1.2), *and also a necessary and sufficient condition for the unique solvability of the two-sided Dirichlet problem*

$$\Delta u = 0 \quad \text{in} \ \mathbb{R}^n \setminus \Gamma, \qquad u|_\Gamma = f. \tag{1.4}$$

The assertion below is related to the case $n = 3$, which is specific.

Theorem 1.3. *Assume that* $n = 3$, $\varphi = F^{-1}$, *and*

$$\sup\{\varphi(2z)/\varphi(z) : z \geqslant 1\} < \infty.$$

Then the following equivalence of norms holds:

$$\|f\|_{Tr^-(\Gamma)} \sim \left(\int\limits_\Gamma f(x)^2 \sigma(z) ds_x + \iint\limits_{\{x,\xi\in\Gamma : 2^{-1} < z/\zeta < 2\}} |f(x) - f(\xi)|^2 \, P\left(\frac{|z-\zeta|}{M(z,\zeta}\right) \frac{ds_x ds_\xi}{|x-\xi|^3} \right)^{1/2}, \tag{1.5}$$

where

$$P(t) = 1 + t^2/((\log(1+t)))^2,$$
$$\sigma(z) = (\varphi(z) \log(\max\{2, z/\varphi(z)\})^{-1},$$

and the remaining notation is the same as in the previous theorem. The finiteness of the norm on the right-hand side of (1.5) *is a necessary and sufficient condition for the unique solvability of the problem* (1.2) *in the energy space, and also a necessary and sufficient condition for the unique solvability of the two-sided Dirichlet problem.*

Remark 1.2. Functions with the finite Dirichlet integral in $\Omega^{\pm}$, decreasing at infinity at the rate $O(|x|^{2-n})$, belong to the class $H^1(\Omega^{\pm})$, $n \geqslant 3$. However, solutions to the problem (1.1), (1.2) or (1.4) not necessarily decrease with order $O(|x|^{2-n})$ at infinity. Let, for example, $f(x) = |x|^{\lambda}\eta(x)$, where $\eta = 0$ in a neighborhood of the point $x = 0$ and $\eta(x) = 1$ for large $|x|$. Then for $\lambda \in (2-n, 1-n/2)$ the problem (1.4) is uniquely solvable in the class $H^1(\mathbb{R}^n)$, but for the sequence $\{x_k\} \subset \Gamma$, $|x_k| \to \infty$, and the solution u to the Dirichlet problem we have

$$|u(x_k)||x_k|^{n-2} \to \infty.$$

2 Extension Operator $H^1(\Omega^-) \to H^1(\mathbb{R}^n)$

We denote by B_r (or $B_r^{(n)}$) an open ball in $\mathbb{R}^n$ centered at the origin and by $S_r^{(n-1)}$ the sphere in $\mathbb{R}^n$ of the same radius and center.

If G is a domain in $\mathbb{R}^n$, then $C_0^{\infty}(G)$ is the set of infinitely differentiable functions compactly supported in G. Denote by ∇u the gradient of u. Further, $L_2^1(G)$ is the space of functions in $L_{2,\mathrm{loc}}(G)$ such that their gradients belong to $L_2(G)$. This space is equipped with the norm

$$\|u\|_{L_2^1(G)} = \left(\|u\|^2_{L_2(g)} + \|\nabla u\|^2_{L_2(G)}\right)^{1/2},$$

where g is an interior subdomain of G, i.e., $\overline{g}$ is a compact set containing in G. Varying g, we obtain an equivalent norm [6]. The space $H^1(G)$ is the closure in $L_2^1(G)$ of the set of smooth functions in G with bounded support in G. In accordance with [7], the space $H^1(\mathbb{R}^n)$ for $n > 2$ is the intersection $L_2^1(\mathbb{R}^n) \cap L_q(\mathbb{R}^n)$, $q = 2n(n-2)^{-1}$, and the $H^1(\mathbb{R}^n)$-norm is equivalent to any of the norms

$$\|\nabla u\|_{L_2(\mathbb{R}^n)} \quad \text{or} \quad \|u\|_{L_q(\mathbb{R}^n)} + \|\nabla u\|_{L_2(\mathbb{R}^n)}, \quad q = 2n(n-2)^{-1}.$$

The symbol $a \sim b$ means the ratio of positive numbers a and b is bounded from above and from below by positive constants independent of a and b. Further c denotes positive constants (their values can change within the same chain of inequalities) depending only on n, and Γ denotes the common boundary of the domains Ω^+ and $\Omega-$.

Proof of Theorem 1.1. We set $\varphi = F^{-1}$. Since $F'(r) \to \infty$ as $r \to \infty$,

$$\lim_{z\to\infty} \varphi'(z) = \lim_{z\to\infty} \varphi(z)/z = 0.$$

We fix $z_0 > 0$ such that

$$\varphi(z_0) > 1, \quad \varphi(z)/z \leqslant 1/2, \quad \varphi'(z) \leqslant 1/2, \quad z \in [z_0, \infty). \tag{2.1}$$

We construct a sequence $\{z_k\}$ by the rule

$$z_{k+1} = z_k + \varphi(z_{k+1}), \quad k = 0, 1, \dots$$

It is clear that $\{z_k\}$ increases and $z_k \to \infty$. Furthermore,

$$* 1 - \frac{z_k}{z_{k+1}} = \frac{\varphi(z_{k+1})}{z_{k+1}} \to 0,$$

$$1 - \frac{\varphi(z_k)}{\varphi(z_{k+1})} = \frac{1}{\varphi(z_{k+1})} \int\limits_{z_k}^{z_{k+1}} \varphi'(t)dt \to 0$$

and, in particular,

$$\lim \frac{z_{k+1}}{z_k} = \lim \frac{\varphi(z_{k+1})}{\varphi(z_k)} = 1.$$

We consider the cells

$$\Omega_k^- = \{(y, z) : z \in (z_{k-1}, z_{k+1}), 1 < |y|/\varphi(z) < 2\}$$

for $k \geqslant 1$. The transformation

$$x = (y, z) \ni\mapsto \varkappa_k(x) = X = (Y, Z), \quad Y = \frac{y}{\varphi(z)}, \quad Z = \frac{z - z_{k-1}}{(z_{k+1} - z_{k-1})}, \tag{2.2}$$

sends the cell Ω_k^- to the cylinder

$$G = \{(Y, Z) : Z \in (0, 1), 1 < |Y| < 2\}$$

and the domain

$$\Omega_k = \{x = (y, z) \in \mathbb{R}^n : z \in (0, 1), |y| < 2\varphi(z)\}$$

to the cylinder

$$D = \{(Y, Z) : Z \in (0, 1), |Y| < 2\}.$$

Setting $\varphi(z_k) = \varphi_k$ for the sake of brevity, we find

$$dx \sim \varphi_k^n dX, \quad |\nabla_x u| \sim \varphi_k |\nabla_X (u \circ \varkappa_k^{-1})|. \tag{2.3}$$

We denote by E the linear continuous extension operator $W_2^1(G) \to W_2^1(D)$, i.e., $Eu\big|_G = u$ for all $u \in W_2^1(G)$. Then the operator

$$W_2^1(\Omega_k^-) \ni u \mapsto E_k u = (E(u \circ \varkappa_k^{-1})) \circ \varkappa_k \in W_2^1(\Omega_k)$$

is an extension operator and, in view of (2.3), the following estimates hold:

$$c\,\|E_k u\|_{L_2(\Omega_k)} \leqslant \|u\|_{L_2(\Omega_k^-)} + \varphi_k \|\nabla u\|_{L_2(\Omega_k^-)}, \tag{2.4}$$

$$c\,\|\nabla(E_k u)\|_{L_2(\Omega_k)} \leqslant \varphi_k^{-1} \|u\|_{L_2(\Omega_k^-)} + \|\nabla u\|_{L_2(\Omega_k^-)}. \tag{2.5}$$

We proceed by constructing the bounded extension operator $H^1(\Omega^-) \to H^1(\mathbb{R}^n)$. Let $\{\mu_k\}_{k\geqslant 1}$ be a partition of unity in $[z_1, \infty)$ subject to the covering by intervals (z_{k-1}, z_{k+1}). We can assume that $0 \leqslant \mu_k \leqslant 1$, $\mu_k \in C_0^\infty(z_{k-1}, z_{k+1})$, and

$$\operatorname{dist}\big(\operatorname{supp}\,\mu_k, \mathbb{R}^1 \setminus (z_{k-1}, z_{k+1})\big) \sim \varphi_k, \quad |\mu_k'| \leqslant c\,\varphi_k^{-1}, \quad k = 1, 2, \dots, \tag{2.6}$$

where $\varphi_k = \varphi(z_k)$. We complete this partition of unity by one more element $\mu_0 \in C^\infty[0, \infty)$ by setting $\mu_0(z) = 0$ for $z > z_1$, $\mu_0(z) = 1$ for $z < z_0$, and $\mu_0 = 1 - \mu_1$ in $[z_0, z_1]$. It is obvious that

$$\sum_{k=0}^{\infty} \mu_k(z) = 1, \quad z \in [0, \infty).$$

We also introduce the "cell"

$$\Omega_0^- = \{x \in \Omega_1^- : z \in (z_0, z_1)\} \cup \{x : z \in (-1, z_1), |y| < 2\,\varphi(z_0)\} \setminus \overline{\Omega}^+.$$

Since Ω_0^- is a domain with Lipschitz boundary, there exists [8, Chapter VI] a bounded extension operator $E_0 : W_2^1(\Omega_0^-) \to W_2^1(\mathbb{R}^n)$.

Finally, we consider one more collection of functions $\{\lambda_k\}$ of type of partition of unity with the following properties:

$$\lambda_k \in C_0^\infty(z_{k-1}, z_{k+1}), \quad |\lambda_k'| \leqslant c\,\varphi_k^{-1}, \quad k \geqslant 1, \quad \lambda_i \mu_i = \mu_i, \quad i \geqslant 0.$$

For a given function $u \in H^1(\Omega^-)$, $x \in \mathbb{R}^n \setminus \Omega^-$, we set

$$\begin{aligned} v(x) &= \sum_{k=0}^{\infty} \mu_k(z)\overline{u}_k, \\ w(x) &= \sum_{k=0}^{\infty} \mu_k(z)(E_k(\lambda_k(u - \overline{u}_k)))(x), \end{aligned} \tag{2.7}$$

where $\overline{u}_k$ is the average of u on Ω_k^- and the generic term of the second sum in (2.7) vanishes by definition if $z \notin (z_{k-1}, z_{k+1})$. Now, we can write the required extension operator

$$H^1(\Omega^-) \ni u \mapsto \mathscr{E}u \in H^1(\mathbb{R}^n)$$

in the form

$$(\mathscr{E}u)(x) = \begin{cases} u(x), & x \in \Omega^-, \\ v(x) + w(x), & x \in \mathbb{R}^n \setminus \Omega^-. \end{cases}$$

To justify the last formula, we need to check the following three assertions.

1. If the support of u is bounded in Ω^-, then the support of $\mathscr{E}u$ is bounded in $\mathbb{R}^n$.
2. For the same u the function $\mathscr{E}u$ has locally summable gradient in $\mathbb{R}^n$.
3. For the same u the following estimate holds:

$$\|\nabla(\mathscr{E}u)\|_{L_2(\mathbb{R}^n)} \leqslant c\,\|\nabla u\|_{L_2(\Omega^-)}. \tag{2.8}$$

1. If supp u is bounded in Ω^-, then each of the sums in (2.7) has a bounded number of nonzero terms. Hence supp $\mathscr{E}u$ is bounded in Ω^+ and, consequently, in $\mathbb{R}^n$ since

$$\text{supp } \mathscr{E}u \subset (\text{supp } u) \cup (\text{supp } \mathscr{E}u \cap \Omega^+).$$

2. Each term of the sums in (2.7) has locally summable gradient in $\mathbb{R}^n$; moreover, (2.7) implies that for $x \in \bigcup_{k=0}^{\infty} \Omega_k^-$

$$v(x) + w(x) = \sum_{k=0}^{\infty} (\mu_k \overline{u}_k + \mu_k \lambda_k (u - \overline{u}_k) = \sum_{k \geqslant 0} \mu_k u = u.$$

Thus, there exists $\nabla(\mathscr{E}u) \in L_{2,\text{loc}}(\mathbb{R}^n)$.

3. The estimate (2.8) follows from the inequalities proved below:

$$\|\nabla v\|_{L_2(\Omega^+)} \leqslant c\,\|\nabla u\|_{L_2(\Omega^-)}, \tag{2.9}$$

$$\|\nabla w\|_{L_2(\Omega^+)} \leqslant c\,\|\nabla u\|_{L_2(\Omega^-)}. \tag{2.10}$$

Consider (2.9). If $x = (y, z) \in \Omega^+$, then either $z \in (0, z_0]$ or $z \in (z_k, z_{k+1}]$ for some $k \geqslant 0$. In the first case, $v(x) = \overline{u}_0$, so that $\nabla v = 0$. In the second case, $x \in \text{supp } \mu_i$ only if $i = k, k+1$, and for these x

$$v(x) = \mu_k(z)\overline{u}_k + \mu_{k+1}(z)\overline{u}_{k+1} = \overline{u}_k + \mu_{k+1}(\overline{u}_{k+1} - \overline{u}_k).$$

Setting $\delta_k = (z_k, z_{k+1})$, we obtain

$$\|\nabla v\|^2_{L_2(\{x \in \Omega^+ : z \in \delta_k\})} \leqslant c\,\varphi_k^{n-2} |\overline{u}_{k+1} - \overline{u}_k|^2.$$

The right-hand side of the last inequality is not greater than

$$c\,\varphi_k^{-2} \|\overline{u}_{k+1} - \overline{u}_k\|^2_{L_2(\{x : z \in \delta_k 1 < |y|/\varphi(z) < 2\})}.$$

Hence

$$\|\nabla v\|^2_{L_2(\{x \in \Omega^+ : z \in \delta_k\})} \leqslant c\,\varphi_k^{-2} \sum_{i=k}^{k+1} \|u - \overline{u}_i\|^2_{L_2(\Omega_i^-)}.$$

Using the Poincaré inequality in the cell Ω_i^-, we arrive at the estimate

$$\|\nabla v\|^2_{L_2(\{x \in \Omega^+ : z \in \delta_k\})} \leqslant c\,\|\nabla u\|^2_{L_2(\Omega_k^- \cup \Omega_{k+1}^-)}.$$

Taking the sum with respect to k, we obtain (2.9).

Let us prove the inequality (2.10). From (2.7) it follows that

$$\|\nabla w\|^2_{L_2(\Omega^+)} \leqslant c \sum_{k \geqslant 0} \varphi_k^{-2} \|E_k(\lambda_k(u - \overline{u}_k)\|^2_{L_2(\Omega^+)} + c \sum_{k \geqslant 0} \|\nabla E_k(\lambda_k(u - \overline{u}_k)\|^2_{L_2(\Omega^+)}$$

By (2.4), (2.5) and the Poincaré inequality in the cell Ω_k^-, we can majorize the right-hand side of the last inequality by

$$c \sum_{k \geqslant 0} \|\nabla u\|^2_{L_2(\Omega_k^-)}.$$

Hence we obtain (2.10).

We have shown that the linear operator $u \mapsto \mathscr{E}u$ defined on a dense set in $H^1(\Omega^-)$ of smooth functions with bounded support in Ω^- is a continuous extension operator in the space $H^1(\mathbb{R}^n)$. This operator can be uniquely extended to the continuous extension operator $H^1(\Omega^-) \to H^1(\mathbb{R}^n)$. The theorem is proved. □

By the above theorem, the following assertion holds.

Corollary 2.1. *The spaces $Tr(\Gamma)$ and $Tr^-(\Gamma)$ coincide, and, consequently, the problem* (1.2) *is solvable if and only if the two-sided Dirichlet problem* (1.4) is solvable.

3 Space $Tr^-(\Gamma)$ for $n > 3$

For $r < R < \infty$ we set $A_{r,R} = B_R \setminus \overline{B}_r$. By definition, $A_{r,\infty} = \mathbb{R}^n \setminus \overline{B}_r$.

Lemma 3.1. *For $n > 2$ the following estimate holds*:

$$\|u\|^2_{L_2(S_r^{(n-1)})} \leqslant c_1(r,n)\|\nabla u\|^2_{L_2(A_{r,\infty})}, \quad u \in H^1(A_{r,\infty}) \tag{3.1}$$

with the exact constant $c_1 = (n-2)^{-1}r$.

Proof. Since smooth functions with bounded support are dense in $H^1(A_{r,\infty})$, it suffices to verify (3.1) for such functions. For these functions we have

$$u(r,\theta) = -\int\limits_r^\infty u_\varrho(\varrho,\theta)d\varrho, \quad \theta \in S_1^{(n-1)}.$$

Using the Cauchy–Bunyakovskii inequality and integrating with respect to θ, we find

$$\int\limits_{S_r^{(n-1)}} u(r,\theta)^2 dS_r \leqslant r^{n-1} \int\limits_{S_1^{(n-1)}} d\theta \int\limits_r^\infty u_\varrho(\varrho,\theta)^2 \varrho^{n-1} d\varrho \int\limits_r^\infty \varrho^{1-n} d\varrho.$$

Thus,

$$\|u\|^2_{L_2(S_r^{(n-1)})} \leqslant (n-2)^{-1} r \, \|\nabla u\|^2_{L_2(A_{r,\infty})}.$$

It remains to note that the constant $c_1(r,n)$ is attained on $u(x) = |x|^{2-n}$. □

Owing to Lemma 3.1, we can obtain useful estimates for the trace of functions. In the assertion below, we establish the embedding $H^1(\Omega^-) \subset L_{2,\sigma}(\Gamma)$ into some weight space on the surface Γ.

Lemma 3.2. *Suppose that $n > 3$ and $u \in H^1(\Omega^-)$. Then*

$$\int\limits_\Gamma u(x)^2 \frac{ds_x}{\max\{\varphi(z),1\}} \leqslant c \, \|\nabla u\|^2_{L_2(\Omega^-)}, \tag{3.2}$$

where $\varphi = F^{-1}$.

Proof. Let z_0 satisfy (2.1), and let $r_0 = \varphi(z_0)$. It is clear that the denominator of the integrand on the left-hand side of (3.2) can be replaced with $\max\{|y|, r_0\}$. By the Gagliardo theorem [3] and Theorem 1.1, we have

$$\int\limits_{\{x\in\Gamma:|y|<r_0\}} u(x)^2 ds_x \leqslant c\,\|\nabla u\|^2_{L_2(\Omega^-)}.$$

Let us prove the inequality

$$\int\limits_{\{x\in\Gamma:|y|>r_0\}} u(x)^2 \frac{ds_x}{|y|} \leqslant c \int\limits_{\Omega^-} |\nabla u(x)|^2 dx. \tag{3.3}$$

The unit vector of normal to Γ directed towards Ω^- is equal almost everywhere to

$$(1 + F'(|y|)^2)^{-1/2} \left(\frac{F'(|y|)y}{|y|}, -1 \right).$$

We introduce the spherical coordinates $y = (r, \theta)$. Then

$$ds_x = (1 + F'(r)^2)^{1/2} r^{n-2} dr d\theta$$

and the left-hand side of (3.3) is equal to

$$\int\limits_{r_0}^{\infty} r^{n-3}(F'(r)^2 + 1)^{1/2} dr \int\limits_{S_1^{(n-2)}} u(r, \theta, F(r))^2 d\theta.$$

By Lemma 3.1, the last integral over $S_1^{(n-2)}$ does not exceed

$$c\, r^{3-n} \int\limits_{|y|>r} |\nabla_y u(y, F(r))|^2 dy.$$

Therefore,

$$\int\limits_{\{x\in\Gamma:|y|>r_0\}} u(x)^2 \frac{ds_x}{|y|} \leqslant c \int\limits_{z_0}^{\infty} (1 + \varphi'(z)^2)^{1/2} dz \int\limits_{|y|>\varphi(z)} |(\nabla u)(y, z)|^2 dy,$$

and we arrive at (3.3) in view of (2.1). □

We proceed by describing the family of equivalent norms on the surface Γ. For the sake of convenience, we first formulate some consequences of the Gagliardo theorem and the theorem about equivalent norms in W_2^1.

Lemma 3.3. *Let G be a domain in $\mathbb{R}^n$, $n > 2$, with compact closure and Lipschitz boundary $S = \partial G$.*

(i) *The following seminorms are equivalent*:

$$[f]_S = \left(\iint\limits_{S\times S} |f(x) - f(\xi)|^2 \frac{ds_x ds_\xi}{|x - \xi|^n} \right)^{1/2}, \tag{3.4}$$

$$(f)_S = \inf\{\|\nabla u\|_{L_2(G)} : u \in W_2^1(G), u|_S = f\},$$

$$\{f\}_S = \inf\{\|\nabla u\|_{L_2(\mathbb{R}^n)} : u \in L_2^1(\mathbb{R}^n), u|_S = f\}.$$

The constants in the relation

$$[f]_S \sim (f)_S \sim \{f\}_S \tag{3.5}$$

depend only on S and the dimension n.

(ii) *If S' is a measurable subset of S of positive area and the average f' of f on S' vanishes, then*

$$[f]_S \sim \inf\{\|\nabla u\|_{L_2(\mathbb{R}^n)} : u \in H^1(\mathbb{R}^n), u|_S = f\}, \tag{3.6}$$

where the constants depend on S and S'.

Proof. We note that $[f]_S < \infty$ implies

$$\int\limits_S |f(x) - f(\xi)|^2 ds_\xi < \infty$$

for a.e. $x \in S$. Hence $f \in L_2(S)$. Thus, the seminorm (3.4) is finite if and only if the norm

$$\|f\|_{W_2^{1/2}(S)} = \|f\|_{L_2(S)} + [f]_S$$

is finite.

Let $u \in W_2^1(G)$, $u\big|_S = f$, and let $\overline{u}$ be the average of u in the domain G. Combining the Gagliardo theorem and the Poincaré inequality in the domain, we obtain

$$[f]_S = [f - \overline{u}]_S \leqslant c\,\|u - \overline{u}\|_{W_2^1(G)} \leqslant c\,\|\nabla u\|_{L_2(G)}.$$

Thereby we establish the estimate

$$[f]_S \leqslant c\,(f)_S.$$

The inequality

$$(f)_S \leqslant \{f\}_S$$

is obvious. To complete the proof of (3.5), it remains to verify the estimate

$$\{f\}_S \leqslant c\,[f]_S.$$

By the Gagliardo theorem and the theorem about extension of functions of the same class W_2^1, there exists a linear continuous extension operator $E : W_2^{1/2}(S) \to W_2^1(\mathbb{R}^n)$. We denote by $\overline{f}$ the average of f on S. We set

$$u = \overline{f} + E(f - \overline{f}).$$

Then $u \in L_2^1(\mathbb{R}^n)$, $u\big|_S = f$, and

$$\|\nabla u\|_{L_2(\mathbb{R}^n)} \leqslant c\,[f - \overline{f}]_S + c\,\|f - \overline{f}\|_{L_2(S)} \leqslant [f]_S.$$

The relation (3.5) is proved. Consider assertion (ii). If $f' = 0$, then

$$\|f\|_{L_2(S)} = \|f - f'\|_{L_2(S)} \leqslant c\,[f]_S,$$

and the above-mentioned bounded extension operator

$$E : W_2^{1/2}(S) \to W_2^1(\mathbb{R}^n)$$

satisfies the estimate

$$\|\nabla(Ef)\|_{L_2(\mathbb{R}^n)} \leqslant c\,[f]_S.$$

We complete the proof of the lemma by the reference to (3.5), which implies, in particular, the inequality $[f]_S \leqslant c\{f\}_S$. □

Remark 3.1. If we omit the condition $f' = 0$, then assertion (ii) of the lemma fails. For example, in the case $S = S_1^{(n-1)}$ and $f = 1$, the left-hand side of (3.6) vanishes and the infimum on the right-hand side is attained at the solution to the two-sided Dirichlet problem

$$\Delta u = 0 \quad \text{in} \;\; \mathbb{R}^n \setminus S, \qquad u\big|_S = 1,$$

i.e., at the functions

$$u(x) = \begin{cases} 1, & |x| \leqslant 1, \\ |x|^{2-n}, & |x| > 1, \end{cases}$$

and, consequently, is positive.

We proceed by considering the norms for functions on the surface Γ, which characterize the membership of functions to the class $Tr(\Gamma)$.

Lemma 3.4. *Suppose that* $n > 3$, $\{z_k\}_{k\geqslant 0}$ *is a sequence of points, and* $\{\mu_k\}_{k\geqslant 1}$ *is the partition of unity described in the proof of Theorem* 1.1. *We set*

$$\Gamma_k = \{x \in \Gamma : z \in (z_{k-1}, z_{k+1})\}, \quad k = 1, 2, \ldots$$

For functions $f \in L_{2,\mathrm{loc}}(\Gamma)$ *such that* $f(y,z) = 0$ *for* $z \leqslant z_1$ *we introduce the norms* (*or seminorms*)

$$|f|_\Gamma = \Bigg(\iint\limits_{x,\xi\in\Gamma:|\zeta-z|<M(z,\zeta)} |f(x) - f(\xi)|^2 \frac{ds_x ds_\xi}{|x-\xi|^n} \Bigg)^{1/2},$$

$$\|f\|_\Gamma = \Bigg(\int\limits_\Gamma f(x)^2 \frac{ds_x}{\max\{\varphi(z), 1\}} \Bigg)^{1/2},$$

where

$$x = (y,z), \quad \xi = (\eta, \zeta), \quad M(z,\zeta) = \max\{\varphi(z), \varphi(\zeta), 1\}.$$

The seminorms $[\cdot]_\Gamma$ *and* $[\cdot]_{\Gamma_k}$ *are defined by formula* (3.4), *where* S *should be replaced with* Γ *or* Γ_k *respectively. The following equivalence relations hold*:

$$\|f\|_\Gamma + [f]_\Gamma \sim \|f\|_\Gamma + |f|_\Gamma \sim \|f\|_\Gamma + \Bigg(\sum_{k\geqslant 1} [f]^2_{\Gamma_k} \Bigg)^{1/2} \sim \|f\|_\Gamma + \Bigg(\sum_{k\geqslant 1} [\mu_k f]^2_{\Gamma_k} \Bigg)^{1/2}. \tag{3.7}$$

Proof. We denote by $\|\cdot\|_1, \ldots, \|\cdot\|_4$ the norms in the order as indicated in (3.7). We verify that

$$\|f\|_i \leqslant c\,\|f\|_{i+1}, \quad i = 1, 2, 3,$$

and

$$\|f\|_4 \leqslant c\,\|f\|_1.$$

The first inequality follows from the estimate

$$\iint\limits_{x,\xi\in\Gamma:|\zeta-z|>M(z,\zeta)} |f(x)-f(\xi)|^2\frac{ds_x ds_\xi}{|x-\xi|^n} \leqslant c\,\|f\|_\Gamma^2. \tag{3.8}$$

To prove (3.8), we note that

$$|z-\zeta| \sim |x-\xi|$$

if $|z-\zeta| > M(z,\zeta)$. Therefore, the left-hand side of (3.8) does not exceed

$$c\int\limits_\Gamma ds_x \int\limits_{\xi\in\Gamma:|\zeta-z|>M(z,\zeta)} (f(x)^2+f(\xi)^2)\frac{ds_\xi}{|z-\zeta|^n},$$

which, in turn, does not exceed

$$c\int\limits_\Gamma f(x)^2 ds_x \int\limits_{|\zeta-z|>\varphi(z)} |\zeta-z|^{-2}d\zeta \leqslant c\int\limits_\Gamma \frac{f(x)^2 ds_x}{\varphi(z)}.$$

Let us prove the inequality

$$\|f\|_2 \leqslant c\,\|f\|_3.$$

We set

$$\Gamma'_k = \{x\in\Gamma : z\in(z_k, z_{k+1})\},$$
$$g(x,\xi) = |f(x)-f(\xi)|^2/|x-\xi|^n.$$

Then

$$|f|_\Gamma^2 = 2\int\limits_\Gamma d\xi \int\limits_{x\in\Gamma:0<\zeta-z<\varphi(\zeta)} g(x,\xi)ds_x \leqslant c\sum_{k\geqslant 1}\int\limits_{\Gamma'_k} ds_\xi \int\limits_{\Gamma_k} g(x,\xi)ds_x \leqslant c\sum_{k\geqslant 1}\iint\limits_{\Gamma_k\times\Gamma_k} g(x,\xi)ds_x ds_\xi.$$

Hence we obtain the required result.

Now, we consider the inequality

$$\|f\|_3 \leqslant c\,\|f\|_4.$$

If $x,\xi\in\Gamma_k$, then

$$f(x)-f(\xi) = \sum_{|i-k|\leqslant 1}((\mu_i f)(x)-(\mu_i f)(\xi)).$$

Hence

$$\sum_{k\geqslant 1}[f]_{\Gamma_k}^2 \leqslant c\sum_{k\geqslant 1}([\mu_k f]_{\Gamma_k}^2 + \sum_{|i-k|=1}[\mu_i f]_{\Gamma_k}^2). \tag{3.9}$$

Let, for example, $i = k-1$. By (2.6), the support μ_{k-1} is located at a distance equivalent to φ_k from (z_k, z_{k+1}), which implies

$$[\mu_{k-1}f]^2_{\Gamma_k} \leqslant 2 \int\limits_{\Gamma'_{k-1}} (\mu_{k-1}f)(x)^2 ds_x \int\limits_{\Gamma'_k} \frac{ds_\xi}{|x-\xi|^n} + [\mu_{k-1}f]^2_{\Gamma'_{k-1}}$$

$$\leqslant [\mu_{k-1}f]^2_{\Gamma_{k-1}} + c \int\limits_{\Gamma_k} f(x)^2 \frac{ds_x}{\varphi(z)}. \tag{3.10}$$

The following estimate is proved in a similar way:

$$[\mu_{k+1}f]^2_{\Gamma_k} \leqslant [\mu_{k+1}f]^2_{\Gamma_{k+1}} + c \int\limits_{\Gamma_k} f(x)^2 \frac{ds_x}{\varphi(z)}. \tag{3.11}$$

Combining (3.9)–(3.11), we obtain

$$\sum_{k\geqslant 1} [f]^2_{\Gamma_k} \leqslant c \sum_{k\geqslant 1} [\mu_k f]^2_{\Gamma_k} + c\,\|f\|^2_\Gamma,$$

i.e.,

$$\|f\|_3 \leqslant c\,\|f\|_4.$$

To complete the proof of the lemma, it remains to verify the estimate

$$\|f\|_4 \leqslant c\,\|f\|_1.$$

Indeed,

$$\sum_{k\geqslant 1} [\mu_k f]^2_{\Gamma_k} \leqslant c \int\limits_\Gamma \sum_{k\geqslant 1} \mu_k(z)^2 ds_x \int\limits_\Gamma g(x,\xi) ds_\xi + c \sum_{k\geqslant 1} \int\limits_{\Gamma_k} f(\xi)^2 ds_\xi \int\limits_{\Gamma_k} \frac{(\mu_k(z)-\mu_k(\zeta))^2}{|x-\xi|^n} ds_x. \tag{3.12}$$

Since

$$|\mu_k(z) - \mu_k(\zeta)| \leqslant c\,|z-\zeta|/\varphi_k,$$

the last integral over Γ_k is not greater than

$$c\,\varphi_k^{-2} \int\limits_{\Gamma_k} |x-\xi|^{2-n} ds_x. \tag{3.13}$$

Making the change of variables

$$x = (y,z) \to X = (Y,Z): \quad Y = y/\varphi(z), \quad Z = (z-z_k)/\varphi_k,$$

we verify that for $\xi \in \Gamma_k$ the integral in (3.13) does not exceed $c\,\varphi_k$, which implies that the second sum on the right-hand side of (3.12) is majorized by $c\|f\|^2_\Gamma$.

To estimate the first term on the right-hand side (3.12), we note that

$$\sum_k \mu_k^2 \leqslant 1.$$

Thus, the right-hand side of (3.12) is not greater than $c\,[f]_\Gamma^2 + c\,\|f\|_\Gamma^2$. The lemma is proved. $\square$

Proof of Theorem 1.2. Let u be a function in $H^1(\Omega^-)$, and let $f = u\big|_\Gamma$ be its trace. We show that the norms of functions on Γ described in the previous lemma are equivalent to the $Tr^-(\Gamma)$-norm. Since the surface Γ is locally Lipschitz, the locally boundary trace is characterized by the Gagliardo theorem. Using this theorem and a finite partition of unity, we reduce the description of the boundary trace to the case where the function vanishes in a neighborhood of the point O. Without loss of generality, we assume that $u = f = 0$ for $z < z_1$, where $\{z_k\}_{k\geqslant 0}$ is the sequence constructed in the proof of Theorem 1.1. By Theorem 1.1, we can assume that $u \in H^1(\mathbb{R}^n)$.

Thus, let $u \in H^1(\mathbb{R}^n)$. By Lemma 3.2,

$$\|f\|_\Gamma \leqslant c\,\|\nabla u\|_{L_2(\mathbb{R}^n)}$$

(hereinafter, we use the notation introduced in the previous lemma).

We consider a cut-off function $\psi \in C^\infty([0,\infty))$ such that $\psi(t) = 1$ for $t \in [0,2]$ and $\psi(t) = 0$ for $t \geqslant 3$. We set $\psi_k(y) = \psi(|y|/\varphi_k)$, $\varphi_k = \varphi(z_k)$. Let $\{\mu_k\}_{k\geqslant 1}$ be the partition of unity from Theorem 1.1. The function $x = (y,z) \mapsto \mu_k(z)\psi_k(y)$ is supported in the cylinder

$$D_k = \{x:\ z \in (z_{k-1}, z_{k+1}), |y| < 3\,\varphi_k\}. \tag{3.14}$$

From (2.1) it follows that $\psi_k(y)\mu_k(z) = \mu_k(z)$ if $(y,z) \in \overline{\Omega}^+\}$. We set

$$u_k(x) = \mu_k(z)\psi_k(y)u(x), \quad x \in \mathbb{R}^n.$$

It is clear that

$$\operatorname{supp} u_k \subset D_k$$

and

$$u_k\big|_\Gamma = \mu_k f.$$

Let $\varkappa_k$ be the change of variables (2.2), under which the surface

$$\Gamma_k = \{x \in \Gamma : z \in (z_{k-1}, z_{k+1})\}$$

is transformed to the lateral surface of the cylinder

$$Q = \{(Y,Z):\ |Y| < 1, Z \in (0,1)\}.$$

The function $v_k = u_k \circ \varkappa_k^{-1}$ is defined at least in a neighborhood of $\overline{Q}$, $v_k = 0$ in a neighborhood of the bases of Q, and v_k has trace $g_k = (\mu_k f) \circ \varkappa_k^{-1}$ on the lateral surface of the cylinder Q. We define g_k by zero on the bases of the cylinder. By Lemma 3.3,

$$[g_k]_{\partial Q} \leqslant c\,\|\nabla v_k\|_{L_2(Q)}.$$

Returning to the variable $x = \varkappa_k^{-1}X$, we obtain

$$[\mu_k f]_{\Gamma_k} \leqslant c\,\|\nabla u_k\|_{L_2(\mathbb{R}^n)}.$$

We show that

$$\sum_{k\geqslant 1} \|\nabla u_k\|^2_{L_2(\mathbb{R}^n)} \leqslant c\,\|\nabla u\|^2_{L_2(\mathbb{R}^n)}. \tag{3.15}$$

Then the last two inequalities and Lemma 3.2 imply the estimate

$$\int\limits_{x\in\Gamma:z>z_1} f(x)^2\frac{ds_x}{\varphi(z)} + \sum_{k\geqslant 1}[\mu_k f]^2_{\Gamma_k} \leqslant c\,\|\nabla u\|^2_{L_2(\mathbb{R}^n)}. \tag{3.16}$$

Let us prove (3.15). Since supp $u_k \subset D_k$, we have

$$\sum_{k\geqslant 1}\|\nabla u_k\|^2_{L_2(\mathbb{R}^n)} \leqslant c\sum_k \|\nabla u\|^2_{L_2(D_k)} + c\sum_k \|u/\varphi_k\|^2_{L_2(D_k)}$$

$$\leqslant c\,\|\nabla u\|^2_{L_2(\mathbb{R}^n)} + c\,\|u/|x|\|^2_{L_2(\mathbb{R}^n\setminus B_{z_0})},$$

and the right-hand side of the last inequality does not exceed the right-hand side of (3.15) in view of the Hardy inequality.

Let f be a function on Γ (vanishing for $z < z_1$) with the finite left-hand side of (3.16). We construct a function $u \in H^1(\mathbb{R}^n)$, $u|_\Gamma = f$, such that the inverse of (3.16) holds. Using (2.2), we consider a function $g_k = (\mu_k f)\circ \varkappa_k^{-1}$ on the lateral surface S of the cylinder Q. Since the left-hand side of (3.16) is finite, $[g_k]_S < \infty$ for $k \geqslant 1$. Since $g_k(Y,Z) = 0$ in a neighborhood of $Z = 0$ and $Z = 1$, we define g_k by zero on the bases of the cylinder Q and obtain a function on ∂Q such that

$$[g_k]_{\partial Q} \leqslant c\,[g_k]_S.$$

Using Lemma 3.3, we see that there exists $v_k \in H^1(\mathbb{R}^n)$ such that $v_k\big|_S = g_k$ and

$$\|\nabla v_k\|_{L_2(\mathbb{R}^n)} \leqslant c\,[g_k]_S.$$

Since the Lipschitz functions with bounded support are multiplicators in $H^1(\mathbb{R}^n)$, we can assume that v_k has bounded support. We set

$$u_k = v_k \circ \varkappa_k.$$

From the definition of u_k we obtain the following properties: $u_k \in H^1(\mathbb{R}^n)$, the support of u_k is bounded, $u_k\big|_{\Gamma_k} = \mu_k f$, and

$$\|\nabla u_k\|_{L_2(\mathbb{R}^n)} \leqslant c\,[\mu_k f]_{\Gamma_k}. \tag{3.17}$$

Let $\{\lambda_k\}_{k\geqslant 1}$ be a collection of functions described in the proof of Theorem 1.1. Then the function

$$x = (y,z) \mapsto U_k(x) = \lambda_k(z)\psi_k(y)u_k(x)$$

is defined in $\mathbb{R}^n$, U_k has support in the cylinder (3.14), and $U_k\big|_\Gamma = \mu_k f$. We set

$$u = \sum_{k\geqslant 1} U_k.$$

We have

$$u\big|_\Gamma = \sum_k \lambda_k\mu_k f = \sum_k \mu_k f = f.$$

Thus, to complete the proof, it suffices to verify the estimate

$$\|\nabla u\|_{(L_2\mathbb{R}^n)} \leqslant c\Bigg(\sum_{k\geqslant 1}\|\nabla u_k\|^2_{L_2(\mathbb{R}^n)}\Bigg)^{1/2}.$$

Combining this estimate with (3.17), we obtain the inverse of (3.16). We have

$$\|\nabla u\|^2_{(L_2\mathbb{R}^n)} \leqslant c\sum_k \|\nabla U_k\|^2_{L_2(D_k)} \leqslant c\sum_k \|\nabla u_k\|^2_{L_2(\mathbb{R}^n)} + c\sum_k \Big\|\frac{u_k}{\varphi_k}\Big\|^2_{L_2(D_k)}.$$

Since $\varphi_k \sim |x|$ in the cylinder D_k, the last sum with respect to k is not greater than

$$c\sum_{k\geqslant 1} \|u_k/|x|\|^2_{L_2(\mathbb{R}^n\setminus B_{z_0})} \leqslant c\sum_k \|\nabla u_k\|^2_{L_2(\mathbb{R}^n)}.$$

Here, we used the Hardy inequality. The theorem is proved. □

4 Weight Estimates for Traces in the Case $n = 3$

We begin with the following auxiliary assertion.

Lemma 4.1. *Suppose that $0 < r < R < \infty$ and $A_{r,R} = B_R \setminus \overline{B}_r$ is a planar annulus. If $u \in W^1_2(A_{r,R})$, then*

$$\frac{1}{2}\left(\log\left(\frac{R}{r}\right)\right)^{-1}\int\limits_0^{2\pi} u(r,\theta)^2 d\theta \leqslant \frac{1}{(R-r)^2}\|u\|^2_{L_2(A_{r,R})} + \|\nabla u\|^2_{L_2(A_{r,R})}.$$

Proof. For $v \in C^1[r,R]$ we have

$$\left|\frac{1}{R-r}\int\limits_r^R v(\varrho)d\varrho - v(r)\right| \leqslant \int\limits_r^R |v'(\varrho)|d\varrho,$$

which implies

$$\frac{1}{2}v(r)^2 \leqslant \frac{1}{(R-r)^2}\bigg(\int\limits_r^R v(\varrho)d\varrho\bigg)^2 + \bigg(\int\limits_r^R |v'(\varrho)|d\varrho\bigg)^2.$$

Thus, for $\theta \in (0,2\pi)$ we have

$$\frac{1}{2}u(r,\theta)^2 \leqslant \left(\frac{1}{(R-r)^2}\int\limits_r^R u(\varrho,\theta)^2\varrho d\varrho + \int\limits_r^R |u'_\varrho(\varrho,\theta)|^2\varrho d\varrho\right)\log\frac{R}{r}.$$

Integrating over $\theta \in (0,2\pi)$, we arrive at the required result. □

The assertion below proves the continuity of the trace operator $H^1(\mathbb{R}^3) \to L_{2,\sigma}(\Gamma)$ in some weight space.

Lemma 4.2. *Suppose that $n = 3$, $u \in H^1(\Omega^-)$, and z_0 satisfies the condition* (2.1). *We set*

$$\sigma(z) = \begin{cases} (\varphi(z)\log(z/\varphi(z)))^{-1}, & z \in [z_0,\infty), \\ \sigma(z_0), & z \in (0,z_0). \end{cases}$$

Then

$$\int\limits_{\Gamma} \sigma(z)u(x)^2 ds_x \leqslant c\, \|\nabla u\|^2_{L_2(\Omega^-)}. \tag{4.1}$$

Proof. Let

$$\Gamma_0 = \{x \in \Gamma : z < z_0\}, \quad \Gamma' = \Gamma \setminus \Gamma_0.$$

The estimate

$$\int\limits_{\Gamma_0} u(x)^2 \sigma(z) ds_x \leqslant c\, \|\nabla u\|^2_{L_2(\Omega^-)}$$

follows from the Gagliardo theorem.

Let us estimate the integral over Γ' of the same function. Passing to the cylindrical coordinates, we find

$$\int\limits_{\Gamma'} u(x)^2\sigma(z)ds_x = \int\limits_{r>\varphi(z_0)} (1+F'(r)^2)^{1/2} \frac{dr}{\log(F(r)/r)} \int\limits_0^{2\pi} u(r,\theta,F(r))^2 d\theta.$$

Using the above lemma (with $R = F(r)$), we find

$$\frac{1}{\log(F(r)/r)} \int_0^{2\pi} u(r,\theta,F(r))^2 d\theta \leqslant c\, F(r)^{-2} \|u(\cdot,F(r))\|^2_{L_2(A_{r,F(r)})} + c\, \|\nabla u(\cdot,F(r))\|^2_{L_2(A_{r,F(r)})}.$$

After the change $F(r) = z$ we obtain the estimate

$$\int\limits_{\Gamma'} u(x)^2\sigma(z)ds_x \leqslant c \int\limits_{z_0}^{\infty} \psi(z)dz \int\limits_{\varphi(z)<|y|<z} \left(\frac{|u(y,z)|^2}{z^2} + |\nabla u(y,z)|^2 \right) dy,$$

where $\psi(z) = (1+\varphi'(z)^2)^{1/2}$. By (2.1), the right-hand side of the inequality obtained is not greater than

$$c \int\limits_{z_0}^{\infty} \int\limits_{\varphi(z)<|y|<z} (|u(x)/|x|)^2 dx + c\, \|\nabla u\|^2_{L_2(\Omega^-)}.$$

By Theorem 1.1, we can assume that $u \in H^1(\mathbb{R}^3)$. Then, by the Hardy inequality, the double integral does not exceed $c\, \|\nabla u\|^2_{L_2(\mathbb{R}^3)}$. The lemma is proved. □

5 Equivalent Norms for Functions on the Surface Γ

We will describe some norms of functions on the surface Γ that characterize the class of boundary traces of functions in $H^1(\mathbb{R}^3)$. We begin with necessary remarks. Hereinafter, we assume that the function φ describing the paraboloid satisfies the additional condition

$$\sup\{\varphi(2z)/\varphi(z) : z \geqslant 1\} < \infty.$$

We set $t_k = 2^k z_0$, $k = 0, 1, \ldots$, where z_0 satisfies the condition (2.1). Let $\{\mu_k\}_{k\geqslant 1}$ be a smooth partition of unity on the set $[t_1, \infty)$ subject to the covering by intervals (t_{k-1}, t_{k+1}). We can assume that $0 \leqslant \mu_k \leqslant 1$, $\mu_k \in C_0^\infty(t_{k-1}, t_{k+1})$, and

$$\operatorname{dist}\big(\operatorname{supp} \mu_k, \mathbb{R}^1 \setminus (t_{k-1}, t_{k+1})\big) \sim t_k, \quad |\mu_k'| \leqslant c\, t_k^{-1}, \quad k = 1, 2, \ldots$$

For $f \in L_{2,\mathrm{loc}}(\Gamma)$ we introduce the following norms and seminorms:

$$\|\!|f|\!\|_\Gamma = \left(\int\limits_\Gamma f(x)^2 \sigma(z) ds_x \right)^{1/2},$$

$$\{f\}_\Gamma = \left(\iint\limits_{\Gamma\times\Gamma} |f(x) - f(\xi)|^2 \, P\left(\frac{|z-\zeta|}{M(z,\zeta} \right) \frac{ds_x ds_\xi}{|x-\xi|^3} \right)^{1/2}, \tag{5.1}$$

$$\langle f \rangle_\Gamma = \left(\iint\limits_{\{x,\xi\in\Gamma: 2^{-1}<z/\zeta<2\}} |f(x) - f(\xi)|^2 \, P\left(\frac{|z-\zeta|}{M(z,\zeta} \right) \frac{ds_x ds_\xi}{|x-\xi|^3} \right)^{1/2},$$

where σ is the weight function defined in Lemma 4.1, $x = (y, z)$, $\xi = (\eta, \zeta)$,

$$M(z,\zeta) = \max\{\varphi(z), \varphi(\zeta), 1\},$$

ds_x and ds_ξ are surface area elements on Γm and

$$P(t) = 1 + t^2/((\log(1+t)))^2.$$

Lemma 5.1. *Let $f \in L_{2,\mathrm{loc}}(\Gamma)$, and let $f(y,z) = 0$ for $z < 2z_0$. Then*

$$\|\!|f|\!\|_\Gamma + \{f\}_\Gamma \sim \|\!|f|\!\|_\Gamma + \langle f \rangle_\Gamma \sim \|\!|f|\!\|_\Gamma + \left(\sum_{k\geqslant 1} \langle \mu_k f \rangle^2 \right)^{1/2}. \tag{5.2}$$

Proof. The following inequalities hold:

$$\int\limits_{\{\xi\in\Gamma:\zeta>2z\}} \frac{ds_\xi}{|x-\xi|^3} \leqslant c \int\limits_{2z}^{1} \varphi(\zeta) \frac{d\zeta}{\zeta^3} \leqslant \frac{c}{z} \leqslant c\,\sigma(z), \quad z > z_0,$$

$$\int\limits_{\{x\in\Gamma:z_0<z<\zeta/2\}} \frac{ds_x}{|x-\xi|^3} \leqslant c \int\limits_{z_0}^{\zeta/2} \varphi(z) \frac{dz}{\zeta^3} \leqslant \frac{c}{\zeta} \leqslant c\,\sigma(\zeta), \quad \zeta > 2\,z_0,$$

$$\int\limits_{\{\xi\in\Gamma:\zeta>2z\}} \frac{M(z,\zeta)^{-2} ds_\xi}{|x-\xi|[\log(1+|z-\zeta|/M(z,\zeta))]^2} \leqslant c \int\limits_{2z}^{1} \frac{(\zeta/\varphi(\zeta))}{[\log(\zeta/\varphi(\zeta))]^2} \frac{d\zeta}{\zeta^2}$$

$$\leqslant c \int_{2z}^{1} \frac{(\zeta/\varphi(z))}{[\log(\zeta/\varphi(z))]^2} \frac{d\zeta}{\zeta^p} \leqslant c\,\sigma(z),$$

where $z > z_0$, and

$$\int\limits_{\{x\in\Gamma:z<\zeta/2\}} \frac{M(z,\zeta)^{-2} ds_x}{|x-\xi|[\log(1+|\zeta-z|/M(z,\zeta))]^2} \leqslant c\,\varphi(\zeta)^{-1} \int\limits_{\zeta/2}^{\zeta} \frac{dt}{t(\log(t/\varphi(\zeta))^2} \leqslant c\,\sigma(\zeta),$$

where $\zeta > 2\,z_0$. Thus,

$$\iint\limits_{\{x,\xi\in\Gamma:\zeta>2z\}} \big(f(x)^2+f(\xi)^2\big)P\left(\frac{|z-\zeta|}{M(z,\zeta)}\right)\frac{ds_x ds_\xi}{|x-\xi|^3} \leqslant c\int\limits_\Gamma f(x)^2\sigma(z)ds_x,$$

which implies the first relation in (5.2).

Let us establish the second relation. We set

$$E=\{(x,\xi): x,\xi\in\Gamma: 2^{-1}<z/\zeta<2\},$$

where $x=(y,z)$ and $\xi=(\eta,\zeta)$. It is clear that

$$f(x)-f(\xi)=\sum_{k\geqslant 1}\big(\mu_k(z)f(x)-\mu_k(\zeta)f(\xi)\big),\quad (x,\xi)\in E,$$

and the number of nonzero terms in the sum is uniformly bounded with respect to x, ξ. Therefore,

$$|f(x)-f(\xi)|^2\leqslant c\sum_{k\geqslant 1}|\mu_k(z)f(x)-\mu_k(\zeta)f(\xi)|^2,$$

which implies

$$\langle f\rangle_\Gamma^2\leqslant c\sum_{k\geqslant 1}\langle\mu_k f\rangle_\Gamma^2.$$

To prove the second relation in (5.2), it suffices to verify the estimate

$$c\sum_{k\geqslant 1}\iint\limits_E |(\mu_k f)(x)-(\mu_k f)(\xi)|^2 P\bigg(\frac{|z-\zeta|}{M(z,\zeta)}\bigg)\frac{ds_x ds_\xi}{|x-\xi|^3}\leqslant \langle f\rangle_\Gamma^2+|\!|\!| f|\!|\!|_\Gamma^2. \tag{5.3}$$

We note that

$$c\sum_{k\geqslant 1}|(\mu_k f)(x)-(\mu_k f)(\xi)|^2\leqslant |f(x)-f(\xi)|^2\sum_{k\geqslant 1}\mu_k(z)^2+|f(\xi)|^2\sum_{k\geqslant 1}|\mu_k(z)-\mu_k(\zeta)|^2;$$

moreover, the first sum on the right-hand side is not greater than 1, and the last sum does not exceed $c\,\zeta^{-2}|z-\zeta|^2$ if $(x,\xi)\in E$. Furthermore,

$$P\left(\frac{|z-\zeta|}{M}\right)\sim\begin{cases}1, & |\zeta-z|<M,\\ \left(\dfrac{|\zeta-z|}{M}\right)^2\left(\log\left(1+\dfrac{|\zeta-z|}{M}\right)\right)^{-2}, & |\zeta-z|>M,\end{cases}$$

where $M=M(z,\zeta)$. Since $M(z,\zeta)\sim\varphi(\zeta)$, $(x,\xi)\in E$, the left-hand side of (5.3) is majorized by $c\big(\langle f\rangle_\Gamma^2+I_1+I_2\big)$, where

$$I_1=\int\limits_\Gamma f(\xi)^2\frac{ds_\xi}{\zeta^2}\int\limits_{\{x\in\Gamma:|\zeta-z|<M(z,\zeta)\}}|\zeta-z|^2\frac{ds_x}{|x-\xi|^3},$$

$$I_2=\int\limits_\Gamma f(\xi)^2\frac{ds_\xi}{\zeta^2}\int\limits_{\ell(\zeta)}\frac{(|\zeta-z|/\varphi(\zeta))dz}{[\log(1+|\zeta-z|/\varphi(\zeta))]^2}$$

and

$$\ell(\zeta) = \{z : \varphi(\zeta) < |\zeta - z| < \zeta\}.$$

We denote by $J(\xi)$ the inner integral over I_1. Since the integrand in $J(\xi)$ does not exceed $|x-\xi|^{-1}$ and $\varphi(z) \sim \varphi(\zeta)$ for $|\zeta - z| < M(z,\zeta)$, we have

$$J(\xi) \leqslant \int\limits_{\{x\in\Gamma : |z-\zeta| < c\,\varphi(\zeta)\}} |x-\xi|^{-1} ds_x \leqslant c\,\varphi(\zeta).$$

Consequently,

$$I_1 \leqslant c\,|\!|\!|f|\!|\!|^2_\Gamma.$$

To estimate I_2, we represent the integral over the set $\ell(\zeta)$ in the form

$$2\varphi(\zeta) \int\limits_1^{\zeta/\varphi(\zeta)} g(t)dt, \quad g(t) = t\,(\log(1+t))^{-2}.$$

Since the function g attains the maximal value at the right endpoint of the integration interval,

$$I_2 \leqslant c \int\limits_\Gamma f(\xi)^2 g(\zeta/\varphi(\zeta)) \frac{ds_\xi}{\zeta} \leqslant c\,|\!|\!|f|\!|\!|^2_\Gamma.$$

The inequality (5.3) is proved. Thereby the proof of the lemma is complete. □

6 Proof of Theorem 1.3

Let $u \in H^1(\Omega^-)$, $u\big|_\Gamma = f$. Using the Gagliardo theorem and a finite partition of unity, we can reduce the description of the trace f of a function u to the case where $u = f = 0$ in a neighborhood of the origin. For the sake of definiteness, we assume that $u(x) = f(x) = 0$ for $z \in (0, 2\,z_0)$, where the number z_0 satisfies the condition (2.1).

By Theorem 1.1, we can assume that $u \in H^1(\mathbb{R}^3)$. We obtain the estimate

$$|\!|\!|f|\!|\!|_\Gamma + \langle f\rangle_\Gamma \leqslant c\,\|\nabla u\|_{L_2(\mathbb{R}^3)}. \tag{6.1}$$

The first term on the left-hand side is majorized by the right-hand side of (6.1) in view of Lemma 4.2. Let $\{t_k\}$ be a number sequence, and let $\{\mu_k\}$ be the partition of unity described in the previous section before Lemma 5.1. Suppose that $\psi \in C^\infty[0,\infty)$ is such that $\psi(t) = 1$ for $t \in [0,1]$ and $\psi(t) = 0$ for $t \geqslant 2$. We set $\psi_k(y) = \psi(|y|/t_k)$ and $u_k(x) = \mu_k(z)\psi_k(y)u(x)$ for $k \geqslant 1$. The function u_k is defined in $\mathbb{R}^3$ and is supported in the cylinder

$$Q_k = \{x :\ z \in (t_{k-1}, t_{k+1}), |y| < 2\,t_k\}. \tag{6.2}$$

Moreover,

$$u_k\big|_\Gamma = \mu_k f, \quad k = 1, 2, \ldots.$$

The transformation

$$x = (y,z) \mapsto \nu_k(x) = X = (Y,Z), \quad Y = \frac{y}{\varphi(z)}, \quad Z = \frac{z}{\varphi(t_k)},$$

sends the subdomain $\{x \in \Omega^- : z > 0\}$ into the exterior of the cylinder

$$Q = \{(Y, Z) : |Y| < 1, Z \in \mathbb{R}^1\}$$

and the surface $\Gamma \setminus \{O\}$ into the lateral surface S of the cylinder Q. We set

$$v_k = u_k \circ \nu_k^{-1}, \quad g_k = (\mu_k f) \circ \nu_k^{-1}. \tag{6.3}$$

Then $v_k\big|_S = g_k$ and $v_k(Y, Z) = 0$ if $Z \notin (t_{k-1}/\varphi(t_k), t_{k+1}/\varphi(t_k))$. By Theorem 3.7 in [9] (cf. also [10]),

$$\inf\left\{\|\nabla v\|^2_{L_2(\mathbb{R}^3)} : v \in L^1_2(\mathbb{R}^3), v\big|_S = g_k\right\} \sim \iint\limits_{S\times S} |g_k(X) - g_k(X')|^2 P(|Z - Z'|) \frac{ds_X ds_{X'}}{|X - X'|^3}, \tag{6.4}$$

where $P(t) = 1 + t^2/(\log(1 + t))^2$. We return to the variables $x = \nu_k^{-1} X$, $\xi = \nu_k^{-1} X'$. Taking into account that

$$|X - X'|\varphi(t_k) \sim |x - \xi|,$$

we find

$$\|\nabla u_k\|_{L_2(\mathbb{R}^3)} \geqslant c\,\langle \mu_k f\rangle_\Gamma, \quad k = 1, 2, \dots \tag{6.5}$$

where the seminorm $\langle\cdot\rangle_\Gamma$ is defined in (5.1).

Let us check the estimate

$$\sum_{k\geqslant 1} \|\nabla u_k\|^2_{L_2(\mathbb{R}^3)} \leqslant c\,\|\nabla u\|^2_{L_2(\mathbb{R}^3)}. \tag{6.6}$$

Since the support of u_k lies in the cylinder (6.2), we have

$$\sum_{k\geqslant 1} \|\nabla u_k\|^2_{L_2(\mathbb{R}^3)} \leqslant c \sum_{k\geqslant 1} \|\nabla u\|^2_{L_2(Q_k)} + c \int\limits_{Q_k} \frac{u(x)^2}{t_k^2} dx \leqslant c\,\|\nabla u\|^2_{L_2(\mathbb{R}^3)} + c \int\limits_{\mathbb{R}^3\setminus B_{z_0}} \frac{u(x)^2}{|x|^2}$$

and (6.6) follows from the Hardy inequality. Combining (6.5) and (6.6) with Lemma (5.1), we obtain (6.1).

Now, we prove the inverse of (6.1). We assume that

$$|\!|\!| f |\!|\!|_\Gamma + \langle f\rangle_\Gamma < \infty$$

and construct a function u such that $u\big|_\Gamma = f$ and

$$\|\nabla u\|_{L_2(\mathbb{R}^3)} \leqslant c\,|\!|\!| f |\!|\!|_\Gamma + c\,\langle f\rangle_\Gamma. \tag{6.7}$$

For $k \geqslant 1$ we have

$$\operatorname{supp} \mu_k f \subset \{x \in \Gamma : z \in (t_{k-1}, t_{k+1})\}.$$

Let $\mu_k f$ be equal to zero in the remaining part of Γ (recall that we assume that $f(y, z) = 0$ for $z < t_1$). We define v_k and g_k by formula (6.3). Then g_k is the trace of v_k on the lateral surface

S of the cylinder Q. By (6.4), there exists a function $v_k \in L^1_2(\mathbb{R}^3)$ such that $v_k\big|_S = g_k$ and the norm $\|\nabla v_k\|_{L_2(\mathbb{R}^3)}$ is majorized up to a constant by the right-hand side of (6.4). We set

$$U_k = v_k \circ \nu_k.$$

The function U_k has support in the strip $z \in (t_{k-1}, t_{k+1})$; moreover, $U_k\big|_\Gamma = \mu_k f$ and

$$\|\nabla U_k\|_{L_2(\mathbb{R}^3)} \leqslant c\,\langle \mu_k f\rangle_\Gamma, \quad k = 1, 2, \dots \tag{6.8}$$

We also define the function

$$x = (y, z) \mapsto u(x) = \sum_{k \geqslant 1} \lambda_k(z)\psi_k(y)U_k(x),$$

where

$$\lambda_k \in C_0^\infty(t_{k-1}, t_{k+1}), \quad |\lambda_k'| \leqslant c\,t_k^{-1}, \quad k \geqslant 1, \quad \lambda_i\mu_i = \mu_i,\ i \geqslant 1. \tag{6.9}$$

Since $\operatorname{supp} \lambda_i \subset (0, \infty)$, the function u is defined in $\mathbb{R}^3$. It is clear that

$$u\big|_\Gamma = \sum_{k \geqslant 1} \psi_k \lambda_k \mu_k f = \sum_{k \geqslant 1} \mu_k f = f.$$

It remains to verify the estimate (6.7). Taking into account (6.8), (6.9), and the fact that the product of $\lambda_k\psi_k$ is supported in the cylinder (6.2), we obtain

$$\|\nabla u\|^2_{L_2(\mathbb{R}^3)} \leqslant c \sum_{k \geqslant 1} \|\nabla U_k\|^2_{L_2(Q_k)} + c \sum_{k \geqslant 1} \int\limits_{Q_k} \frac{U_k^2}{t_k^2} dx.$$

By the Hardy inequality, the right-hand side is not greater than

$$c \sum_{k \geqslant 1} \|\nabla U_k\|^2_{L_2(\mathbb{R}^3)}.$$

Applying (6.8) and Lemma 5.1, we obtain (6.7). The theorem is proved. □

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