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ON THE REGULARITY OF DOMAINS SATISFYING A UNIFORM HOUR-GLASS CONDITION AND A SHARP VERSION OF THE HOPF—OLEINIK BOUNDARY POINT PRINCIPLE

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We prove that an open, proper, nonempty subset of $\mathbb{R}^n$ *is a locally Lyapunov domain if and only if it satisfies a uniform hour-glass condition. The limiting cases are as follows: Lipschitz domains may be characterized by a uniform double cone condition, and domains of class* $\mathscr{C}^{1,1}$ *may be characterized by a uniform two-sided ball condition. We discuss a sharp generalization of the Hopf–Oleinik boundary point principle for domains satisfying an interior pseudoball condition, for semi-elliptic operators with singular drift and obtain a sharp version of the Hopf strong maximum principle for second order, nondivergence form differential operators with singular drift. Bibliography: 66 titles. Illustrations: 7 figures.*

1 Introduction

This paper has two parts which intertwine closely. One is of a predominantly geometric flavor and is aimed at describing the smoothness of domains (as classically formulated in analytical terms) in a purely geometric language. The other, having a more pronounced analytical nature, studies how the ability of expressing regularity in a geometric fashion is helpful in establishing sharp results in partial differential equations. We begin by motivating the material belonging to the first part just described.

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Over the past few decades, analysis on classes of domains defined in terms of specific geometrical and measure theoretical properties has been a driving force behind many notable advances in partial differential equations and harmonic analysis. Examples of categories of domains with analytic and geometric measure theoretic characteristics are specifically designed to meet the demands and needs of work in the aforementioned fields include the class of nontangentially accessible domains introduced by Jerison and Kenig [1] (nontangentially accessible domains form the most general class of regions where the pointwise nontangential behavior of harmonic functions at boundary points is meaningful), the class of (ε, δ)-domains considered by Jones [2] (these are the most general type of domains known to date for which linear extension operators which preserve regularity measured on Sobolev scales may be constructed), uniformly rectifiable domains introduced by David and Semmes [3] (making up the largest class of domains with the property that singular integral operators of Calderón–Zygmund type defined on their boundaries are continuous on L^p, $1 < p < \infty$), and the class of Semmes–Kenig–Toro domains defined in [4] (Semmes–Kenig–Toro domains make up the most general class of domains for which Fredholm theory for boundary layer potentials, as originally envisioned by Fredholm, can be carried out).

In the process, more progress has been registered in our understanding of more familiar (and widely used) classes of domains such as the family of Lipschitz domains, as well as domains exhibiting low regularity assumptions. For example, the following theorem, which characterizes the smoothness of a domain of locally finite perimeter in terms of the regularity properties of the geometric measure theoretic outward unit normal, has been recently proved in [5]:

Theorem 1.1. *Assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which is of locally finite perimeter and which lies on only one side of its topological boundary, i.e.,*

$$\partial\Omega = \partial(\overline{\Omega}). \tag{1.1}$$

Denote by ν the outward unit normal to Ω, defined in the geometric measure theoretic sense at each point belonging to $\partial^*\Omega$, the reduced boundary of Ω. Finally, fix $\alpha \in (0, 1]$. Then Ω is locally of class $\mathscr{C}^{1,\alpha}$ if and only if ν extends to an S^{n-1}-valued function on $\partial\Omega$ which is locally Hölder of order α. In particular,

$$\Omega \textit{ is a locally } \mathscr{C}^{1,1}\textit{-domain} \iff \textit{the Gauss map } \nu : \partial^*\Omega \to S^{n-1} \textit{is locally Lipschitz.} \tag{1.2}$$

Finally, corresponding to the limiting case $\alpha = 0$, one has that Ω is a locally $\mathscr{C}^1$ domain if and only if the Gauss map $\nu : \partial^*\Omega \to S^{n-1}$ has a continuous extension to $\partial\Omega$.

Open subsets of $\mathbb{R}^n$ (of locally finite perimeter) whose outward unit normal is Hölder are typically called Lyapunov domains (cf., for example, [6] and [7, Chapter I]). Theorem 1.1 shows that, with this definition, Lyapunov domains are precisely those open sets whose boundaries may be locally described by graphs of functions with Hölder first order derivatives (in a suitable system of coordinates). All these considerations are of an analytical or measure theoretical flavor.

By way of contrast, in this paper, we are concerned with finding an intrinsic description of a purely geometrical nature for the class of Lyapunov domains in $\mathbb{R}^n$. In order to be able to elaborate, let us define what we term here to be an hour-glass shape. Concretely, given $a, b > 0$ and $\alpha \in [0, +\infty)$, introduce

$$\mathscr{H}\mathscr{G}^{\alpha}_{a,b} := \big\{x \in \mathbb{R}^n : a|x|^{1+\alpha} < |x_n| < b\big\}. \tag{1.3}$$

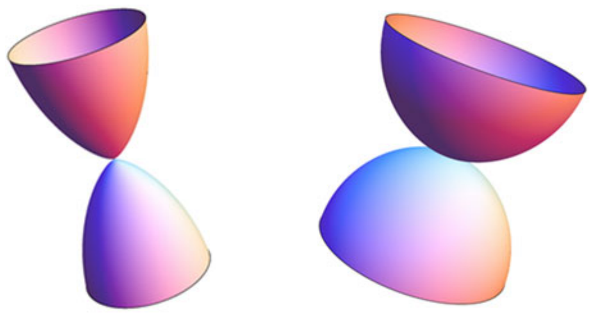

FIGURE 1. The figure on the left is an hour-glass shape with α near 0, while the figure depicted on the right is an hour-glass shape with α near 1.

With this piece of terminology, one of our geometric regularity results may be formulated as follows.

Theorem 1.2. *An open, nonempty set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is Lyapunov if and only if there exist $a, b > 0$ and $\alpha \in (0, 1]$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mathcal{R}(0) = x_0 \quad \textit{and} \quad \partial\Omega \cap \mathcal{R}\big(\mathscr{H}\mathscr{G}^{\alpha}_{a,b}\big) = \varnothing. \tag{1.4}$$

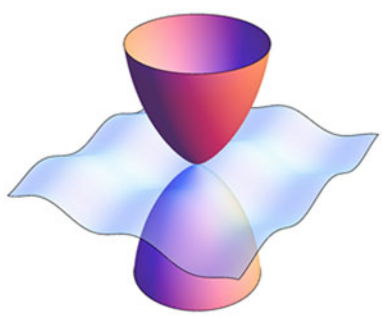

FIGURE 2. Threading the boundary of a domain Ω in between the two rounded components of an hour-glass shape with direction vector along the vertical axis.

The reader is referred to Theorem 3.13 in the body of the paper for a more precise statement, which is stronger than Theorem 1.2 on two accounts: it is local in nature and it allows for more general regions than those considered in (1.3) (cf. (1.9) in this regard). Equally important, Theorem 3.13 makes it clear that the Hölder order of the normal is precisely the exponent $\alpha \in (0, 1]$ used in the definition of the hour-glass region (1.3). As a corollary of Theorem 1.2, we note the following purely geometric characterization of domains of class $\mathscr{C}^{1,1}$: *an open, proper, nonempty subset Ω of $\mathbb{R}^n$, with compact boundary, is a domain of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.* The latter condition amounts to requesting that there exists $r > 0$ along with a function $h : \partial\Omega \to S^{n-1}$ with the property that

$$B(x + rh(x), r) \subseteq \Omega \quad \text{and} \quad B(x - rh(x), r) \subseteq \mathbb{R}^n \setminus \Omega \quad \text{for all } x \in \partial\Omega. \tag{1.5}$$

The idea is that the configuration consisting of two open, disjoint, congruent balls in $\mathbb{R}^n$ sharing a common boundary point may be rigidly transported so that it contains an hour-glass region $\mathscr{H}\mathscr{G}^{\alpha}_{a,b}$ with $\alpha = 1$ and some suitable choice of the parameters a, b (depending only on the radius r appearing in (1.5)).

The limiting case $\alpha = 0$ of Theorem 1.2 is also true, although the nature of the result changes in a natural fashion. Specifically, if $a \in (0, 1)$, then, corresponding to $\alpha = 0$, the hour-glass region $\mathscr{H}\mathscr{G}^{\alpha}_{a,b}$ from (1.3) becomes the two-component, open, circular, upright, truncated cone with vertex at the origin

$$\Gamma_{\theta,b} := \big\{x \in \mathbb{R}^n : \cos(\theta/2)|x| < |x_n| < b\big\}, \tag{1.6}$$

where $\theta := 2\arccos(a) \in (0,\pi)$ is the (total) aperture of the cone. This yields the following characterization of Lipschitzianity: an open, nonempty set $\Omega \subseteq \mathbb{R}^n$ with compact boundary is a Lipschitz domain if and only if there exist $\theta \in (0,\pi)$ and $b > 0$ with the property that for each $x_0 \in \partial\Omega$ there exists an isometry $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{R}(x_0) = 0 \quad \text{and} \quad \mathcal{R}(\partial\Omega) \cap \Gamma_{\theta,b} = \varnothing. \tag{1.7}$$

Our characterizations of Lipschitz domains in terms of uniform cone conditions are of independent interest and, in fact, the result just mentioned is the starting point in the proof of Theorem 1.2. Concretely, the strategy for proving the aforementioned geometric characterization of Lyapunov domains in terms of a uniform hour-glass condition with exponent $\alpha \in (0,1]$ consists of three steps: (1) show that the domain in question is Lipschitz, (2) show that the unit normal satisfies a Hölder condition of order $\alpha/(\alpha+1)$, and (3) show that the boundary of the original domain may be locally described as a piece of the graph of a function whose first order derivatives are Hölder of order α.

In fact, we prove a more general result than Theorem 1.2 (cf. Theorem 1.3 below), where the (components of the) hour-glass shape (1.3) are replaced by a more general family of subsets of $\mathbb{R}^n$, which we call pseudoballs (for the justification of this piece of terminology see item (iii) in Lemma 2.2). To formally introduce this class of sets, consider

$$\begin{gathered} R \in (0,+\infty) \ \text{ and } \ \omega : [0,R] \to [0,+\infty) \text{ a continuous function} \\ \text{with the properties that } \omega(0) = 0 \text{ and } \omega(t) > 0 \quad \forall\, t \in (0,R]. \end{gathered} \tag{1.8}$$

Then the pseudoball with apex at $x_0 \in \mathbb{R}^n$, axis of symmetry along $h \in S^{n-1}$, height $b > 0$, aperture $a > 0$ and shape function ω as in (1.8), is defined as

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) := \{x \in B(x_0,R) : a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\}. \tag{1.9}$$

For certain geometric considerations, it is convenient to impose the following two additional conditions on the shape function ω:

$$\lim_{\lambda\to 0^+}\Big(\sup_{t\in(0,\min\{R,R/\lambda\}]}\frac{\omega(\lambda t)}{\omega(t)}\Big) = 0, \quad \text{and } \omega \text{ strictly increasing}. \tag{1.10}$$

Also, in the second part of the paper, in relation to problems in partial differential equations, we work with functions $\widetilde{\omega} : [0,R] \to [0,+\infty)$ satisfying the Dini integrability condition

$$\int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty. \tag{1.11}$$

Of significant interest for us in this paper is the class of functions $\omega_{\alpha,\beta}$, indexed by pairs of numbers $\alpha \in [0,1]$, $\beta \in \mathbb{R}$, such that $\beta < 0$ if $\alpha = 0$, defined as follows (convening that $\frac{\beta}{0} := +\infty$ for any $\beta \in \mathbb{R}$):

$$\begin{aligned} &\omega_{\alpha,\beta} : \Big[0, \min\big\{e^{\frac{\beta}{\alpha}}, e^{\frac{\beta}{\alpha-1}}\big\}\Big] \to [0,+\infty), \\ &\omega_{\alpha,\beta}(t) := t^{\alpha}(-\ln t)^{\beta} \text{ if } t > 0, \quad \text{and } \omega_{\alpha,\beta}(0) := 0. \end{aligned} \tag{1.12}$$

Corresponding to $\beta = 0$, abbreviate $\omega_\alpha := \omega_{\alpha,0}$. Note $\omega_{\alpha,\beta}$ satisfies all conditions listed in (1.8), (1.10) and (1.11) given $\alpha \in (0,1]$ and $\beta \in \mathbb{R}$. In addition, we also have that $t \mapsto \omega_{\alpha,\beta}(t)/t$ is decreasing. However, if $\alpha = 0$, then $\omega_{\alpha,\beta}$ satisfies the Dini integrability condition if and only if $\beta < -1$.

If $\alpha \in (0,1]$ and $a, b > 0$, then, corresponding to ω_α as in (1.12), the pseudoball

$$\mathscr{G}^{\alpha}_{a,b}(x_0,h) := \mathscr{G}^{\omega_\alpha}_{a,b}(x_0,h) = \big\{x \in B(x_0,1) \subseteq \mathbb{R}^n : \, a|x-x_0|^{1+\alpha} < h \cdot (x-x_0) < b\big\} \tag{1.13}$$

is designed so that the hour-glass region (1.3) consists precisely of the union between $\mathscr{G}^{\alpha}_{a,b}(0,\mathbf{e}_n)$ and $\mathscr{G}^{\alpha}_{a,b}(0,-\mathbf{e}_n)$, where $\mathbf{e}_n$ is the canonical unit vector along the vertical direction in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. The pseudoballs (1.13) naturally make the transition between cones and genuine balls in $\mathbb{R}^n$ in the sense that, corresponding to $\alpha = 1$, the pseudoball $\mathscr{G}^{1}_{a,b}(x_0,h)$ is a solid spherical cap of an ordinary Euclidean ball, whereas corresponding to the limiting case where one formally takes $\alpha = 0$ in (1.13), the pseudoball $\mathscr{G}^{0}_{a,b}(x_0,h)$ is a one-component, circular, truncated, open cone (cf. Lemma 2.2 in the body of the paper for more details).

In order to state the more general version of Theorem 1.2 alluded to above, we need one more definition. Concretely, call an open, proper, nonempty subset Ω of $\mathbb{R}^n$ a domain of class $\mathscr{C}^{1,\omega}$ if, near boundary points, its interior may be locally described (up to an isometric change of variables) in terms of upper-graphs of $\mathscr{C}^1$ functions whose first order partial derivatives are continuous with modulus of continuity ω. Then a version of Theorem 1.2 capable of dealing with the more general type of pseudoballs introduced in (1.9) reads as follows.

Theorem 1.3. *Let ω be a function as in* (1.8) *and* (1.10)*. Then an open, proper, nonempty subset Ω of $\mathbb{R}^n$, with compact boundary, is of class $\mathscr{C}^{1,\omega}$ if and only if there exist $a > 0$, $b > 0$ and two functions $h_\pm : \partial\Omega \to S^{n-1}$ with the property that*

$$\mathscr{G}^{\omega}_{a,b}(x, h_+(x)) \subseteq \Omega \quad \text{and} \quad \mathscr{G}^{\omega}_{a,b}(x, h_-(x)) \subseteq \mathbb{R}^n \setminus \Omega \ \text{ for each } x \in \partial\Omega. \tag{1.14}$$

Moreover, in the case where $\Omega \subseteq \mathbb{R}^n$ is known to be of class $\mathscr{C}^{1,\omega}$, one necessarily has $h_- = -h_+$.

This more general version of Theorem 1.2 is justified by the applications to partial differential equations we have in mind. Indeed, as we see momentarily, this more general hour-glass shape is important since it permits a desirable degree of flexibility (which happens to be optimal) in constructing certain types of barrier functions, adapted to the operator in question.

More specifically, in the second part of this paper we deal with the maximum principles for second order, nondivergence form differential operators. Traditionally, the three most basic maximum principles are labeled as weak, boundary point, and strong (cf. the discussion in [8, 9]). Among these, it is the boundary point principle which has the most obvious geometrical character, both in its formulation and proof. For example, Zaremba [10], Hopf [11], and Oleinik [12] have proved such boundary point principles[1] in domains satisfying an interior ball condition.

Our goal here is to prove a sharper version of their results with the interior ball condition replaced by an interior pseudoball condition. In fact, it is this goal that has largely motivated the portion of the research in this paper described earlier.

Being able to use pseudoballs as a replacement of standard Euclidean balls allows us to relax both the assumptions on the underlying domain, as well as those on the coefficients of

[1] In [13, p. 36] this is referred to as the Zaremba-Giraud principle, while in [14, p. 312] this is called the Zaremba principle.

the differential operator by considering semi-elliptic operators with singular lower order terms (drift). Besides its own intrinsic merit, relaxing the regularity assumptions on the coefficients is particularly significant in view of applications to nonlinear partial differential equations.

To state a version of our main result in this regard (cf. Theorem 4.4), we make one definition. Given a real-valued function u of class $\mathscr{C}^2$ in an open subset of $\mathbb{R}^n$, denote by $\nabla^2 u$ the Hessian matrix of u, i.e., $\nabla^2 u := \big(\partial_i\partial_j u\big)_{1\leqslant i,j\leqslant n}$. We then have the following boundary point principle, relating the type of degeneracy in the ellipticity, as well as the nature of the singularities in the coefficients of the differential operator, to geometry of the underlying domain.

Theorem 1.4. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Assume that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudoball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) : \, a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\} \subseteq \Omega \tag{1.15}$$

for some parameters $a, b, R \in (0,+\infty)$, a direction vector $h \in S^{n-1}$, and a real-valued shape function $\omega \in \mathscr{C}^0([0,R])$, which is positive and nondecreasing on $(0,R]$, and with the property that the mapping $(0,R] \ni t \mapsto \omega(t)/t \in (0,+\infty)$ is nonincreasing. Also, consider a nondivergence form, second order, differential operator L in Ω acting on functions $u \in \mathscr{C}^2(\Omega)$ according to

$$Lu := -\mathrm{Tr}\big(A\,\nabla^2 u\big) + \vec{b}\cdot\nabla u = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b^i\partial_i u \quad \text{in} \ \ \Omega, \tag{1.16}$$

whose coefficients $A = \big(a^{ij}\big)_{1\leqslant i,j\leqslant n} : \Omega \to \mathbb{R}^{n\times n}$ and $\vec{b} = (b^i)_{1\leqslant i\leqslant n} : \Omega \to \mathbb{R}^n$ satisfy

$$\inf_{x\in\mathscr{G}^{\omega}_{a,b}(x_0,h)} \ \inf_{\xi\in S^{n-1}} (A(x)\xi)\cdot\xi \geqslant 0, \quad (A(x)h)\cdot h > 0 \ \text{ for each } x \in \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.17}$$

In addition, suppose that there exists a real-valued function $\widetilde{\omega} \in \mathscr{C}^0\big([0,R]\big)$, which is positive on $(0,R]$ and satisfying the Dini integrability condition

$$\int_0^R t^{-1}\widetilde{\omega}(t)\,dt < +\infty$$

with the property that

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\big(\mathrm{Tr}\,A(x)\big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\big((A(x)h)\cdot h\big)} < +\infty, \tag{1.18}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\big\{0\,,\,\vec{b}(x)\cdot h\big\} + \Big(\sum\limits_{i=1}^{n}\max\{0,-b^i(x)\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\big((A(x)h)\cdot h\big)} < +\infty. \tag{1.19}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ for which $\vec{\ell}\cdot h > 0$, and suppose that $u \in \mathscr{C}^0(\Omega\cup\{x_0\})\cap\mathscr{C}^2(\Omega)$ is a function satisfying

$$(Lu)(x) \geqslant 0 \quad \text{and} \quad u(x_0) < u(x) \ \text{ for each } x \in \Omega. \tag{1.20}$$

Then

$$\liminf_{t\to 0^+} \frac{u(x_0+t\vec{\ell})-u(x_0)}{t} > 0. \tag{1.21}$$

For example, if $\partial\Omega \in \mathscr{C}^{1,\alpha}$ for some $\alpha \in (0,1)$ and if ν denotes the outward unit normal to $\partial\Omega$, then (1.21) holds provided that $\vec{\ell}\cdot\nu(x_0) < 0$ and the coefficients of the semi-elliptic operator L, as in (4.33), satisfy for some $\varepsilon \in (0,\alpha)$

$$(A(x)\nu(x_0))\cdot\nu(x_0) > 0 \quad \text{for each } x\in\Omega \text{ near } x_0, \text{ and} \tag{1.22}$$

$$\limsup_{\Omega\ni x\to x_0} \frac{|x-x_0|^{\alpha-\varepsilon}\big(\operatorname{Tr} A(x)\big)+|x-x_0|^{1-\varepsilon}|\vec{b}(x)|}{(A(x)\nu(x_0))\cdot\nu(x_0)} < +\infty. \tag{1.23}$$

Also, it can be readily verified that if the coefficients of the operator L are bounded near x_0, then a sufficient condition guaranteeing the validity of (1.18)–(1.19) is the existence of some $c>0$ such that

$$(A(x)h)\cdot h \geqslant c\frac{\big((x-x_0)\cdot h\big)\,\omega(|x-x_0|)}{|x-x_0|\widetilde{\omega}((x-x_0)\cdot h)} \quad \forall\, x\in \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.24}$$

This should be thought of as an admissible degree of degeneracy in the ellipticity uniformity of the operator L (a phenomenon concretely illustrated by considering the case when $\omega(t)=t^\alpha$ and $\widetilde{\omega}(t)=t^\beta$ for some $0<\beta<\alpha<1$).

It is illuminating to note that the geometry of the pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ affects (through its direction vector h and shape function ω) the conditions (1.17)–(1.19) imposed on the coefficients of the differential operator L. This is also the case for the proof of Theorem 1.4 in which we employ a barrier function which is suitably adapted both to the nature of the pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$, as well as to the degree of degeneracy of the ellipticity of the operator L (manifested through $\widetilde{\omega}$ and ω). Concretely, this barrier function is defined at each $x\in\mathscr{G}^{\omega}_{a,b}(x_0,h)$ as

$$v(x) := (x-x_0)\cdot h + C_0 \int_0^{(x-x_0)\cdot h}\int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\,dt\,d\xi - C_1 \int_0^{|x-x_0|}\int_0^{\xi} \frac{\omega(t)}{t}\Big(\frac{t}{\xi}\Big)^{\gamma-1}\,dt\,d\xi, \tag{1.25}$$

where $\gamma>1$ is a fine-tuning parameter, and $C_0, C_1>0$ are suitably chosen constants (depending on Ω and L), whose role is to ensure that v satisfies the properties described below. The linear part on the right-hand side of (1.25) is included in order to guarantee that

$$\vec{\ell}\cdot(\nabla v)(x_0) > 0, \tag{1.26}$$

while the constants C_0, C_1 are chosen such that

$$Lv \leqslant 0 \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0,h), \quad \text{and} \quad \exists\,\varepsilon>0 \text{ so that } \varepsilon v\leqslant u-u(x_0) \text{ on } \partial\mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{1.27}$$

Then (1.21) follows from (1.26)–(1.27) and the weak maximum principle.

Note that no measurability assumptions are made on the coefficients, and that the class of second order, nondivergence form, differential operators considered in Theorem 1.4 is invariant under multiplication by arbitrary positive functions. In addition, the said class contains all uniformly elliptic, second order, nondivergence form differential operators with bounded coefficients, granted that the domain Ω satisfies a pseudoball condition at $x_0 \in \partial\Omega$ whose shape function ω satisfies the Dini integrability condition (in which scenario, one simply takes $\widetilde{\omega} := \omega$).

Although a more refined version of Theorem 1.4 is proved later in the paper (cf. Theorem 4.4), we wish to note here that this result is already quantitatively optimal. To see this, consider the case where $\Omega := \{x \in \mathbb{R}^n_+ : x_n < 1\}$, the point x_0 is the origin in $\mathbb{R}^n$, and

$$L := -\Delta + \frac{\psi(x_n)}{x_n}\frac{\partial}{\partial x_n} \quad \text{in } \Omega, \tag{1.28}$$

where $\psi : (0,1] \to (0,+\infty)$ is a continuous function with the property that

$$\int_0^1 \frac{\psi(t)}{t}\,dt = +\infty. \tag{1.29}$$

Then, if $\vec{\ell} := \mathbf{e}_n := (0,\dots,0,1) \in \mathbb{R}^n$ and

$$u(x_1,\dots,x_n) := \int_0^{x_n} \exp\Big\{-\int_\xi^1 \frac{\psi(t)}{t}\,dt\Big\}\,d\xi \quad \forall\,(x_1,\dots,x_n) \in \Omega, \tag{1.30}$$

it follows that $u \in \mathscr{C}^2(\Omega)$, u may be continuously extended at $0 \in \mathbb{R}^n$ by setting $u(0) := 0$, and $u > 0$ in Ω. Furthermore,

$$\begin{aligned} \frac{\partial u}{\partial x_n} &= \exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\,dt\Big\}, \\ \frac{\partial^2 u}{\partial x_n^2} &= \frac{\psi(x_n)}{x_n}\exp\Big\{-\int_{x_n}^1 \frac{\psi(t)}{t}\,dt\Big\} = \frac{\psi(x_n)}{x_n}\frac{\partial u}{\partial x_n} \quad \text{in } \Omega, \end{aligned} \tag{1.31}$$

from which we deduce that $Lu = 0$ in Ω, and $(\nabla u)(0) = 0$, thanks to (1.29). Hence (1.21), the conclusion of the boundary point principle formulated in Theorem 1.4, fails in this case. The sole cause of this breakdown is the inability to find a shape function $\widetilde{\omega}$ satisfying the Dini integrability condition and such that (1.19) holds.

Indeed, since Ω satisfies an interior pseudoball condition at 0 with shape function, say $\omega(t) := t$, and direction vector $h := \mathbf{e}_n \in S^{n-1}$, the latter condition reduces, in the current setting, to (for some fixed $a, b > 0$)

$$\begin{aligned} &\limsup_{\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)\ni x\to 0} \Big(\frac{\max\{0, \vec{b}(x)\cdot\mathbf{e}_n\}}{x_n^{-1}\widetilde{\omega}(x_n)}\Big) < +\infty, \quad \text{where} \\ &\vec{b}(x) := \big(0,\dots,0,\psi(x_n)/x_n\big) \ \text{ for } x = (x_1,\dots,x_n) \in \Omega, \end{aligned} \tag{1.32}$$

which, if true, would force $\widetilde{\omega}(t) \geqslant c\,\psi(t)$ for all $t > 0$ small (for some fixed constant $c > 0$). However, in light of (1.29), this would prevent $\widetilde{\omega}$ from satisfying the Dini integrability condition. This proves the optimality of the condition (1.19) in Theorem 1.4. A variant of this counterexample also shows the optimality of the condition (1.18). Specifically, let Ω, $\vec{\ell}$, x_0, u be as before and, this time, consider

$$L := -\Big(\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{x_n}{\psi(x_n)}\frac{\partial^2}{\partial x_n^2}\Big) + \frac{\partial}{\partial x_n} \quad \text{in } \ \Omega. \tag{1.33}$$

Obviously, $Lu = 0$ in Ω and, as pointed out before, Ω satisfies an interior pseudoball condition at the origin with shape function $\omega(t) = t$ and direction vector $h = \mathbf{e}_n \in S^{n-1}$. As such, the condition (1.18) would entail (for this choice of ω, after some simple algebra), $\widetilde{\omega}(t) \geqslant c\psi(t)$ for all $t > 0$ small. In concert with (1.29) this would, of course, prevents $\widetilde{\omega}$ from satisfying the Dini integrability condition. Other aspects of the sharpness of Theorem 1.4 are discussed later, in Subsection 4.3.

As a consequence of our boundary point principle, we obtain a strong maximum principle for a class of nonuniformly elliptic operators with singular (and possibly nonmeasurable) drift terms. More specifically, we have the following theorem.

Theorem 1.5. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Suppose that L, written as in* (1.16)*, is a* (*possibly, nonuniformly*) *elliptic second order differential operator in nondivergence form* (*without a zero order term*) *in Ω. Also, assume that for each $x_0 \in \Omega$ and each $\xi \in S^{n-1}$ there exists a real-valued function $\widetilde{\omega} = \widetilde{\omega}_{x_0,\xi}$ which is continuous on* $[0,1]$*, positive on* $(0,1]$*, satisfies*

$$\int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty$$

and with the property that

$$\limsup_{(x-x_0)\cdot\xi>0,\ x\to x_0} \frac{\big(\operatorname{Tr} A(x)\big) + \big|\vec{b}(x)\cdot\xi\big| + \big|\vec{b}(x)\big||x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}\big((A(x)\xi)\cdot\xi\big)} < +\infty. \tag{1.34}$$

Then ,if $u \in \mathscr{C}^2(\Omega)$ satisfies $(Lu)(x) \geqslant 0$ for all $x \in \Omega$ and assumes a global minimum value at some point in Ω, it follows that u is constant in Ω.

See Theorem 4.17 for a slightly more refined version, though such a result is already quantitatively sharp. The following example sheds light in this regard. Concretely, in the n-dimensional Euclidean unit ball centered at the origin, consider

$$L := -\tfrac{1}{n+2}\Delta + \vec{b}(x)\cdot\nabla, \quad \text{where} \quad \vec{b}(x) := \begin{cases} |x|^{-2}x \ \text{if} \ \ x \in B(0,1)\setminus\{0\}, \\ 0 \qquad\ \ \text{if} \ \ x = 0. \end{cases} \tag{1.35}$$

and the function $u : \overline{B(0,1)} \longrightarrow \mathbb{R}$ given by $u(x) := |x|^4$ for each $x \in \overline{B(0,1)}$.

It follows that

$$u \in \mathscr{C}^2\big(\overline{B(0,1)}\big), \quad (\nabla u)(x) = 4|x|^2x \ \text{ and } \ (\Delta u)(x) = 4(n+2)|x|^2 \quad \forall\, x \in \overline{B(0,1)}. \tag{1.36}$$

Consequently,

$$(Lu)(x) = 0 \ \text{ for each } x \in B(0,1), \quad u \geqslant 0 \ \text{ in } B(0,1), \quad u(0) = 0 \quad \text{and} \quad u\big|_{\partial B(0,1)} = 1, \tag{1.37}$$

which shows that the strong maximum principle fails in this case. To understand the nature of this failure, observe that given a function $\widetilde{\omega} : (0,1) \to (0,+\infty)$ and a vector $\xi \in S^{n-1}$, the condition (1.34) entails

$$\limsup_{x\to 0,\, x\cdot\xi>0} \frac{|x|^{-2}\, x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}} < +\infty \tag{1.38}$$

which, when specialized to the case where x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0,+\infty)$ such that $\widetilde{\omega}(t) \geqslant c$ for all small $t > 0$. Of course, this would prevent $\widetilde{\omega}$ from satisfying the Dini integrability condition.

In the last part of this section, we briefly review some of the most common notational conventions used in the sequel. Throughout the paper, we assume that $n \geqslant 2$ is a fixed integer, $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$, and '$\cdot$' denotes the canonical dot product of vectors in $\mathbb{R}^n$. Also, as usual, S^{n-1} is the unit sphere centered at the origin in $\mathbb{R}^n$ and by $B(x,r)$ we denote the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, i.e., $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. Whenever necessary to stress the dependence of a ball on the dimension of the ambient Euclidean space we write $B_n(x,r)$ in place of $\{y \in \mathbb{R}^n : |x-y| < r\}$. We let $\{\mathbf{e}_j\}_{1\leqslant j\leqslant n}$ denote the canonical orthonormal basis in $\mathbb{R}^n$. In particular, $\mathbf{e}_n = (0,\dots,0,1) \in \mathbb{R}^n$, and we use the abbreviation (x', x_n) in place of $(x_1,\dots,x_n) \in \mathbb{R}^n$. By $0'$ we typically denote the origin in $\mathbb{R}^{n-1}$, often regarded as a subspace of $\mathbb{R}^n$ under the canonical identification $\mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1} \times \{0\}$. Next, given $E \subseteq \mathbb{R}^n$, we use E^c, E°, $\overline{E}$ and ∂E to denote, respectively, the complement of E (relative to $\mathbb{R}^n$, i.e., $E^c := \mathbb{R}^n \setminus E$), the interior, the closure and the boundary of E. One other useful piece of terminology is as follows. Let $E \subseteq \mathbb{R}^n$ be a set of cardinality $\geqslant 2$. Assume that $(X, \|\cdot\|)$ is a normed vector space. Then $\mathscr{C}^\alpha(E,X)$ denotes the vector space of functions $f : E \to X$ which are Hölder of order $\alpha > 0$, i.e., for which

$$\|f\|_{\mathscr{C}^\alpha(E,X)} := \sup_{x,y\in E,\, x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha} < +\infty. \tag{1.39}$$

As is customary, functions which are Hölder of order $\alpha = 1$ are referred to as Lipschitz functions. Also, corresponding to the limiting case $\alpha = 0$, we agree that $\mathscr{C}^0$ stands for the class of continuous functions (in the given context).

More generally, given a modulus of continuity ω, a real-valued function f is said to be of class $\mathscr{C}^\omega$ provided that there exists $C \in (0,+\infty)$ such that $|f(x)-f(y)| \leqslant C\,\omega(|x-y|)$ for $|x-y|$ small. Functions of class $\mathscr{C}^{1,\omega}$ are then defined by requiring that their first order partial derivatives exist and are in $\mathscr{C}^\omega$.

Finally, we denote by $\operatorname{Tr} A$ and $A^\top$ the trace and transpose of the matrix A respectively.

2 Geometrical Preliminaries

2.1 The geometry of pseudoballs

In this section, we introduce a category of sets which contains both the cones and balls in $\mathbb{R}^n$, and which we call pseudoballs. This concept is going to play a basic role for the entire subsequent discussion. As a preamble, we describe the class of cones in the Euclidean space. Concretely, by an open, truncated, one-component circular cone in $\mathbb{R}^n$ we understand any set of the form

$$\Gamma_{\theta,b}(x_0,h) := \{x \in \mathbb{R}^n : \cos(\theta/2)\,|x-x_0| < (x-x_0)\cdot h < b\}, \tag{2.1}$$

where $x_0 \in \mathbb{R}^n$ is the vertex of the cone, $h \in S^{n-1}$ is the direction of the axis, $\theta \in (0, \pi)$ is the (full) aperture of the cone, and $b \in (0, +\infty)$ is the height of the cone.

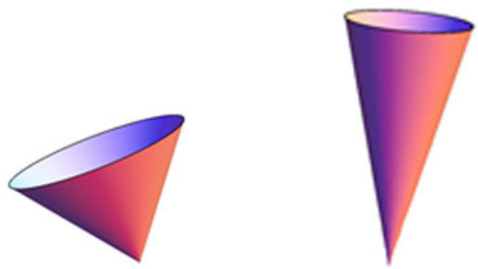

FIGURE 3. One-component circular cones. The aperture of the cone on the left is larger than that of the cone on the right.

Definition 2.1. Assume (1.8) and suppose that the point $x_0 \in \mathbb{R}^n$, vector $h \in S^{n-1}$ and numbers $a, b \in (0, +\infty)$ are given. Then the *pseudoball* with apex at x_0, axis of symmetry along h, height b, amplitude a, and shape function ω is defined by

$$\mathscr{G}^{\omega}_{a,b}(x_0, h) := \{x \in B(x_0, R) \subseteq \mathbb{R}^n : a|x - x_0|\,\omega(|x - x_0|) < h \cdot (x - x_0) < b\}. \tag{2.2}$$

Collectively, a, b, and ω constitute the geometrical characteristics of the named pseudoball.

In the sequel, given a, b, and α positive numbers, abbreviate $\mathscr{G}^{\alpha}_{a,b}(x_0, h) := \mathscr{G}^{\omega_\alpha}_{a,b}(x_0, h)$ with ω_α as in (1.12), i.e., define

$$\mathscr{G}^{\alpha}_{a,b}(x_0, h) := \big\{x \in B(x_0, 1) \subseteq \mathbb{R}^n : a|x - x_0|^{1+\alpha} < h \cdot (x - x_0) < b\big\}. \tag{2.3}$$

FIGURE 4. A pseudoball with shape function $\omega(t) = t^{1/2}$.

Some basic, elementary properties of pseudoballs are collected in the lemma below. In particular, item (iii) justifies the terminology employed in Definition 2.1.

Lemma 2.2. *Assume (1.8) and, in addition, suppose that ω is strictly increasing. Also, fix two parameters $a, b \in (0, +\infty)$, a point $x_0 \in \mathbb{R}^n$, and a vector $h \in S^{n-1}$. Then the following assertions hold.*

(i) *The pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0, h)$ is an open, nonempty subset of $\mathbb{R}^n$ (in fact, it contains a line segment of the form $\{x_0 + th : 0 < t < \varepsilon\}$ for some small $\varepsilon > 0$), which is included in the ball $B(x_0, R)$, and with the property that $x_0 \in \partial\mathscr{G}^{\omega}_{a,b}(x_0, h)$. Corresponding to the choice $x_0 := 0 \in \mathbb{R}^n$ and $h := \mathbf{e}_n \in S^{n-1}$, one has*

$$\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n) = \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n : |x| < R \ \text{ and } \ a|x|\,\omega(|x|) < x_n < b\big\}. \tag{2.4}$$

Furthermore,

$$\begin{aligned}&\textit{if } b\in(0,R\,\omega(R)) \textit{ and } t_b\in(0,R) \textit{ satisfies } t_b\,\omega(t_b)=b,\\ &\textit{then } \mathscr{G}^{\omega}_{a,b}(x_0,h)\subseteq B(x_0,t_b).\end{aligned} \tag{2.5}$$

(ii) *Assume that* $a\in(0,1)$. *Then, corresponding to the limiting case* $\alpha=0$, *the pseudoball introduced in* (2.3) *coincides with the one-component, circular, open cone with vertex at* x_0, *unit axis* h, *aperture* $\theta:=2\arccos a\in(0,\pi)$, *and which is truncated at height* b, *i.e.,*

$$\mathscr{G}^{0}_{a,b}(x_0,h)=\Gamma_{\theta,b}(x_0,h) \quad \textit{for } \theta:=2\arccos a\in(0,\pi). \tag{2.6}$$

(iii) *In the case* $\alpha=1$, *for each* $a>0$ *the pseudoball defined in* (2.3) *coincides with the solid spherical cap obtained by intersecting the open ball in* $\mathbb{R}^n$ *with center at* $x_0+h/(2a)$ *and radius* $r:=1/(2a)$ *with the half-space* $H(x_0,h,b):=\{x\in\mathbb{R}^n:\,(x-x_0)\cdot h<b\}$. *In other words,*[2]

$$\mathscr{G}^{1}_{a,b}(x_0,h)=B\big(x_0+h/(2a),1/(2a)\big)\cap H(x_0,h,b). \tag{2.7}$$

Furthermore, when and $b\geqslant 1/a$, *one actually has*

$$\mathscr{G}^{1}_{a,b}(x_0,h)=B\big(x_0+h/(2a),1/(2a)\big).$$

(iv) *Let* $\mathcal{R}:\mathbb{R}^n\to\mathbb{R}^n$ *be an isometry; hence* $\mathcal{R}=T\circ\mathscr{R}$, *where* $\mathscr{R}$ *is a rotation about the origin in* $\mathbb{R}^n$ *and* T *is a translation in* $\mathbb{R}^n$. *Then*

$$\mathcal{R}\big(\mathscr{G}^{\omega}_{a,b}(x_0,h)\big)=\mathscr{G}^{\omega}_{a,b}(\mathcal{R}(x_0),\mathscr{R}h). \tag{2.8}$$

In particular, $x_1+\mathscr{G}^{\omega}_{a,b}(x_0,h)=\mathscr{G}^{\omega}_{a,b}(x_0+x_1,h)$ *for every* $x_1\in\mathbb{R}^n$.

(v) *Pick* $t_*\in(0,R)$ *with the property that* $\omega(t_*)<1$. *Then whenever the number* b_0 *and the angle* θ *satisfy*

$$0<b_0<\min\{b,t_*\},\quad 2\max\big\{\arccos\big(\omega(t_*)\big)\,,\,\arccos\big(b_0/t_*\big)\big\}\leqslant\theta<\pi, \tag{2.9}$$

one has

$$\Gamma_{\theta,b_0}(x_0,h)\subseteq\mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{2.10}$$

As a consequence, the pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ *contains truncated circular cones* (*with vertex at* x_0 *and axis* h) *of apertures arbitrarily close to* π.

(vi) *Assume that* ω *and* ω' *are two increasing functions satisfying the properties listed in* (1.8). *If* $x_0\in\mathbb{R}^n$, $h,h'\in S^{n-1}$, *and* $a,a',b,b'>0$ *are such that* $\mathscr{G}^{\omega}_{a,b}(x_0,h)\subseteq\mathscr{G}^{\omega'}_{a',b'}(x_0,h')$, *then necessarily* $h=h'$ *and* $a\omega(t)\leqslant a'\omega'(t)$ *for all* $t>0$ *small.*

Proof. These are all straightforward consequences of definitions. □

For the remainder of this section we assume that the shape function ω as in (1.8) also satisfies the conditions listed in (1.10), i.e. (after a slight rephrasing of the first condition in (1.10)),

> $R\in(0,+\infty)$ and $\omega:[0,R]\to[0,+\infty)$ is a continuous, (strictly) increasing function, with the property that $\omega(0)=0$ and there exists a function $\eta:(0,+\infty)\to(0,+\infty)$ which satisfies $\lim\limits_{\lambda\to0^+}\eta(\lambda)=0$ and $\omega(\lambda t)\leqslant\eta(\lambda)\,\omega(t)$ for all $\lambda>0$ and $\in\big[0,\min\{R,R/\lambda\}\big]$. (2.11)

[2]The term "pseudoball" has been chosen, *faute de mieux*, primarily because of this observation.

For future reference, let us note here that the conditions (2.11) entail that

$$\begin{aligned}&\omega : [0, R] \to [0, \omega(R)] \text{ is invertible and its inverse } \omega^{-1} : [0, \omega(R)] \to [0, R] \text{ is}\\&\text{a continuous function which is (strictly) increasing and satisfies } \omega^{-1}(0) = 0.\end{aligned} \tag{2.12}$$

In order to facilitate the presentation of the proof of the main geometric result in this section, we now present a series of technical, preliminary lemmas pertaining to the geometry of pseudoballs. The key ingredient is the fact that a pseudoball has positive, finite curvature near the apex. A concrete manifestation of this property is the fact that two pseudoballs with apex at the origin and whose direction vectors do not point in opposite ways necessarily have a substantial overlap.

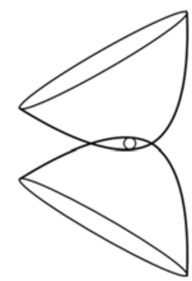

FIGURE 5. Any two pseudoballs with a common apex and whose direction vectors are not opposite contain a ball in their overlap (with quantitative control of its size).

A precise, quantitative aspect of this phenomenon is discussed in Lemma 2.3 below.

Lemma 2.3. *Assume* (2.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Then there exists* $\varepsilon > 0$*, which depends only on* η*,* ω*,* R*,* a*, and* b*, such that for any* $h_0, h_1 \in S^{n-1}$ *the following implication holds*:

$$\begin{aligned}&x \in \mathbb{R}^n \text{ and } \Big| x - \varepsilon\,\omega^{-1}\Big(\omega(R)\frac{|h_0+h_1|}{2}\Big)\frac{h_0+h_1}{|h_0+h_1|}\Big| < \frac{\varepsilon}{2}\,\omega^{-1}\Big(\omega(R)\frac{|h_0+h_1|}{2}\Big)\Big|\frac{h_0+h_1}{2}\Big|\\&\Longrightarrow |x| < R, \quad a|x|\omega(|x|) < \min\big\{x\cdot h_0, x\cdot h_1\big\}, \text{ and } \max\big\{x\cdot h_0, x\cdot h_1\big\} < b,\end{aligned} \tag{2.13}$$

with the convention that $\dfrac{h_0+h_1}{|h_0+h_1|} := 0$ *if* $h_0 + h_1 = 0$*. In other words, for each vectors* $h_0, h_1 \in S^{n-1}$ *the first line in* (2.13) *implies that* $x \in \mathscr{G}^{\omega}_{a,b}(0, h_0) \cap \mathscr{G}^{\omega}_{a,b}(0, h_1)$*.*

Proof. Fix a real number ε such that

$$0 < \varepsilon < \min\Big\{\frac{2b}{3R}, \frac{2}{3}\Big\} \quad \text{and} \quad \eta\Big(\frac{3\varepsilon}{2}\Big) < \big[3a\,\omega(R)\big]^{-1}. \tag{2.14}$$

That this is possible is ensured by the last line in (2.11). Next, pick two arbitrary vectors $h_0, h_1 \in S^{n-1}$ and introduce $v := (h_0 + h_1)/2$. Then, if x is as in the first line in (2.13), we may estimate (keeping in mind that $|v| \leqslant 1$):

$$\begin{aligned}|x| &\leqslant \Big|x - \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big| + \Big|\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big|\\&< \frac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| + \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big) \leqslant \frac{3\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big).\end{aligned} \tag{2.15}$$

Granted the first condition in (2.14), this further implies (recall that ω^{-1} is increasing and $|v| \leqslant 1$) that

$$|x| \leqslant \frac{3\varepsilon}{2}R < \min\{R, b\}, \tag{2.16}$$

so the first estimate in the second line of (2.13) is taken care of. In order to prove the remaining estimates in the second line of (2.13), it is enough to show that

$$\begin{aligned}&a|x|\omega(|x|) < x\cdot h_0 < b \ \text{ if } x\in\mathbb{R}^n \text{ is such that}\\&\Big|x-\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big| < \frac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v|\end{aligned} \tag{2.17}$$

since the roles of h_0 and h_1 in (2.13) are interchangeable. To this end, assume that x is as in the last part of (2.17), write

$$x\cdot h_0 = \Big(x-\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big)\cdot h_0 + \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\cdot h_0, \tag{2.18}$$

and observe that

$$v\cdot h_0 = \frac{1}{2}(1+h_0\cdot h_1) = |v|^2.$$

Thus, on the one hand, we have

$$\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\cdot h_0 = \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{2.19}$$

On the other hand, based on the Cauchy–Schwarz inequality and the assumption on x, we obtain

$$\Big|\Big(x-\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big)\cdot h_0\Big| \leqslant \Big|x-\varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)\frac{v}{|v|}\Big| < \frac{\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{2.20}$$

From this it follows that

$$x\cdot h_0 > \varepsilon\,\omega^{-1}\big(\omega(R)|v|\big)|v| - \frac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| = \frac{\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)|v|. \tag{2.21}$$

At this stage, we make the claim that

$$\frac{\varepsilon}{2}\,\omega^{-1}\big(\omega(R)|v|\big)|v| > a|x|\omega(|x|)$$

which, when used in concert with the estimate just derived, yields $x\cdot h_0 > a|x|\omega(|x|)$. To justify this claim, based on (2.15) and (2.11), we may then write

$$\begin{aligned}a|x|\omega(|x|) &\leqslant a\frac{3\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)\,\omega\Big(\frac{3\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)\Big)\\&\leqslant a\frac{3\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)\,\eta\Big(\frac{3\varepsilon}{2}\Big)\,\omega(R)|v| < \frac{\varepsilon}{2}\omega^{-1}\big(\omega(R)|v|\big)|v|,\end{aligned} \tag{2.22}$$

where the third inequality is a consequence of (2.14). This finishes the proof of the claim. It remains to observe that, thanks to (2.16), $x\cdot h_0 \leqslant |x| < b$, completing the proof of Lemma 2.3. □

The main application of Lemma 2.3 is the following result asserting, in a quantitative manner, that two pseudoballs necessarily overlap if their apexes are sufficiently close to one another relative to the degree of proximity of their axes.

Lemma 2.4. *Assume* (2.11) *and suppose that* $a,b\in(0,+\infty)$ *are given. Also, suppose that the parameter* $\varepsilon=\varepsilon(\omega,\eta,R,a,b)>0$ *is as in Lemma* 2.3. *Then for every* $x_0,x_1\in\mathbb{R}^n$ *and every* $h_0,h_1\in S^{n-1}$ *one has*

$$|x_0-x_1| < \frac{\varepsilon}{2}\omega^{-1}\Big(\omega(R)\frac{|h_0+h_1|}{2}\Big)\Big|\frac{h_0+h_1}{2}\Big| \Longrightarrow \mathscr{G}^{\omega}_{a,b}(x_0,h_0)\cap\mathscr{G}^{\omega}_{a,b}(x_1,h_1)\neq\varnothing. \tag{2.23}$$

Proof. To set the stage, let $\varepsilon > 0$ be as in Lemma 2.3 and assume that $x_0, x_1 \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that the estimate on the left-hand side of (2.23) holds. In particular, $|h_0 + h_1| > 0$ and

$$B\Big(\varepsilon\,\omega^{-1}\Big(\omega(R)\frac{|h_0+h_1|}{2}\Big)\frac{h_0+h_1}{|h_0+h_1|}, \frac{\varepsilon}{2}\omega^{-1}\Big(\omega(R)\frac{|h_0+h_1|}{2}\Big)\Big|\frac{h_0+h_1}{2}\Big|\Big) \subseteq \mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(0,h_1). \tag{2.24}$$

Indeed, this is simply a rephrasing of the conclusion in Lemma 2.3. Henceforth, we denote by $\mathcal{B}_{h_0,h_1}$ the ball on the left-hand side of (2.24) and by c_{h_0,h_1} and r_{h_0,h_1} its center and radius respectively. To proceed, we consider $y := c_{h_0,h_1} + x_0 - x_1 \in \mathbb{R}^n$ and note that $|y - c_{h_0,h_1}| = |x_0 - x_1| < r_{h_0,h_1}$. This implies that $y \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0,h_1)$. Thus, ultimately, $c_{h_0,h_1} = (x_1 - x_0) + y \in x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1)$. Since we also have $c_{h_0,h_1} \in \mathcal{B}_{h_0,h_1} \subseteq \mathscr{G}^{\omega}_{a,b}(0,h_0)$, this analysis shows that $c_{h_0,h_1} \in \mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \big(x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1)\big)$. Upon recalling from item (iv) in Lemma 2.2 that $x_1 - x_0 + \mathscr{G}^{\omega}_{a,b}(0,h_1) = \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1)$, we deduce that $\mathscr{G}^{\omega}_{a,b}(0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1 - x_0, h_1) \neq \varnothing$. Finally, translating by x_0 yields $\mathscr{G}^{\omega}_{a,b}(x_0,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x_1,h_1) \neq \varnothing$. This completes the proof of Lemma 2.4. $\square$

We conclude this section by presenting a consequence of Lemma 2.4 to the effect that two pseudoballs sharing a common apex are disjoint if and only if their axes point in opposite directions.

Corollary 2.5. *Assume* (2.11) *and suppose that* $a, b \in (0, +\infty)$ *are given. Then for each point* $x \in \mathbb{R}^n$ *and any pair of vectors* $h_0, h_1 \in S^{n-1}$ *one has*

$$\mathscr{G}^{\omega}_{a,b}(x,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x,h_1) = \varnothing \Longleftrightarrow h_0 + h_1 = 0. \tag{2.25}$$

Proof. If $x \in \mathbb{R}^n$ and $h_0, h_1 \in S^{n-1}$ are such that $|h_0 + h_1| > 0$ and yet $\mathscr{G}^{\omega}_{a,b}(x,h_0) \cap \mathscr{G}^{\omega}_{a,b}(x,h_1) = \varnothing$, then Lemma 2.4 (used with $x_0 := x =: x_1$) yields a contradiction. This proves the left-to-right implication in (2.25). For the converse implication, observe that if $y \in \mathscr{G}^{\omega}_{a,b}(x,h) \cap \mathscr{G}^{\omega}_{a,b}(x,-h)$ for some $x \in \mathbb{R}^n$ and $h \in S^{n-1}$, then $a|y-x|\,\omega(|y-x|) < h \cdot (y-x)$ and $a|y-x|\,\omega(|y-x|) < (-h) \cdot (y-x)$. Hence $h \cdot (y-x) > 0$ and $(-h) \cdot (y-x) > 0$, a contradiction which concludes the proof of the corollary. $\square$

2.2 Sets of locally finite perimeter

Given $E \subseteq \mathbb{R}^n$, denote by $\mathbf{1}_E$ the characteristic function of E. A Lebesgue measurable set $E \subseteq \mathbb{R}^n$ is said to be of locally finite perimeter provided that

$$\mu := \nabla \mathbf{1}_E \tag{2.26}$$

is a locally finite $\mathbb{R}^n$-valued measure. For a set of locally finite perimeter which has a compact boundary we agree to drop the adverb “locally.” Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter we denote by σ the total variation measure of μ; σ is then a locally finite positive measure supported on ∂E, the topological boundary of E. Also it is clear that each component of μ is absolutely continuous with respect to σ. It follows from the Radon–Nikodym theorem that

$$\mu = \nabla \mathbf{1}_E = -\nu\sigma, \tag{2.27}$$

where

$$\begin{aligned}&\nu \in L^\infty(\partial E, d\sigma) \text{ is an } \mathbb{R}^n\text{-valued function}\\ &\text{satisfying } |\nu(x)| = 1 \text{ for } \sigma\text{-a.e. } x \in \partial E.\end{aligned} \tag{2.28}$$

It is customary to identify σ with its restriction to ∂E with no special mention. We refer to ν and σ respectively as the (geometric measure theoretic) outward unit normal and the surface measure on ∂E. Note that ν defined by (2.27) can only be specified up to a set of σ-measure zero. To eliminate this ambiguity, we redefine $\nu(x)$ for every x as being

$$\lim_{r\to 0} \fint_{B(x,r)} \nu \, d\sigma \tag{2.29}$$

whenever the above limit exists, and zero otherwise. In doing so, we make the convention that

$$\fint_{B(x,r)} \nu \, d\sigma := (\sigma(B(x,r)))^{-1} \int_{B(x,r)} \nu \, d\sigma \tag{2.30}$$

if $\sigma(B(x,r)) > 0$, and zero otherwise. The Besicovitch differentiation theorem (cf., for example, [15]) ensures that ν in (2.27) agrees with (2.29) for σ-a.e. x.

The reduced boundary of E is then defined as

$$\partial^* E := \big\{x \in \partial E : \, |\nu(x)| = 1\big\}. \tag{2.31}$$

This is essentially the point of view adopted in [16, Definition 5.5.1, p. 233]. Let us remark that this definition is slightly different from that given in [15, p. 194]. The reduced boundary introduced there depends on the choice of the unit normal in the class of functions agreeing with it σ-a.e. and, consequently, can be pointwise specified only up to a certain set of zero surface measure. Nonetheless, any such representative is a subset of $\partial^* E$ defined above and differs from it by a set of σ-measure zero.

Moving on, it follows from (2.31) and the Besicovitch differentiation theorem that σ is supported on $\partial^* E$ in the sense that $\sigma(\mathbb{R}^n \setminus \partial^* E) = 0$. From the work of Federer and De Giorgi it is also known that, if $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^* E. \tag{2.32}$$

Recall that, generally speaking, given a Radon measure μ in $\mathbb{R}^n$ and a set $A \subseteq \mathbb{R}^n$, the restriction of μ to A is defined as $\mu \lfloor A := \mathbf{1}_A \, \mu$. In particular, $\mu \lfloor A \ll \mu$ and $d(\mu \lfloor A)/d\mu = \mathbf{1}_A$. Thus,

$$\sigma \ll \mathcal{H}^{n-1} \quad \text{and} \quad \frac{d\sigma}{d\mathcal{H}^{n-1}} = \mathbf{1}_{\partial^* E}. \tag{2.33}$$

Furthermore (cf. [16, Lemma 5.9.5, p. 252] and [15, p. 208]), one has

$$\partial^* E \subseteq \partial_* E \subseteq \partial E, \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0, \tag{2.34}$$

where $\partial_* E$, the *measure–theoretic boundary* of E, is defined by

$$\partial_* E := \big\{x \in \partial E : \, \limsup_{r \to 0^+} r^{-n} \mathcal{H}^n\big(B(x,r) \cap E^{\pm}\big) > 0\big\}. \tag{2.35}$$

Above, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (i.e., up to normalization, the Lebesgue measure) in $\mathbb{R}^n$, and we set $E^+ := E$ and $E^- := \mathbb{R}^n \setminus E$.

Let us also record here a useful criterion for deciding whether a Lebesgue measurable subset E of $\mathbb{R}^n$ is of locally finite perimeter in $\mathbb{R}^n$ (cf. [15, p. 222]):

$$E \text{ has locally finite perimeter} \iff \mathcal{H}^{n-1}(\partial_* E \cap K) < +\infty \quad \forall K \subseteq \mathbb{R}^n \text{ compact.} \tag{2.36}$$

We conclude this section by proving the following result of geometric measure theoretic flavor (which is a slight extension of Proposition 2.9 in [5]), establishing a link between the cone property and the direction of the geometric measure theoretic unit normal.

Lemma 2.6. *Let E be a subset of $\mathbb{R}^n$ of locally finite perimeter. Fix a point x_0 belonging to $\partial^* E$ (the reduced boundary of E) with the property that there exist $b > 0$, $\theta \in (0, \pi)$, and $h \in S^{n-1}$ such that*

$$\Gamma_{\theta,b}(x_0, h) \subseteq E. \tag{2.37}$$

If $\nu(x_0)$ denotes the geometric measure theoretic outward unit normal to E at x_0, then

$$\nu(x_0) \in \Gamma_{\pi-\theta,1}(0, -h). \tag{2.38}$$

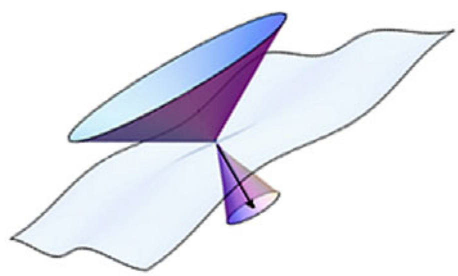

Figure 6.

Proof. The idea is to use a blow-up argument. Specifically, consider the half-space

$$H(x_0) := \{x \in \mathbb{R}^n : \nu(x_0) \cdot (x - x_0) < 0\} \subseteq \mathbb{R}^n \tag{2.39}$$

and for each $r > 0$ and $A \subseteq \mathbb{R}^n$ set

$$A_r := \{x \in \mathbb{R}^n : r(x - x_0) + x_0 \in A\}. \tag{2.40}$$

Also, abbreviate $\Gamma := \Gamma_{\theta,b}(x_0, h)$ and denote by $\widetilde{\Gamma}$ the circular, open, infinite cone which coincides with $\Gamma_{\theta,b}(x_0, h)$ near its vertex. The theorem concerning the blow-up of the reduced boundary of a set of locally finite perimeter (cf., for example, [15, p. 199]) gives that

$$\mathbf{1}_{E_r} \longrightarrow \mathbf{1}_{H(x_0)} \text{ in } L^1_{loc}(\mathbb{R}^n) \text{ as } r \to 0^+. \tag{2.41}$$

On the other hand, it is clear that $\Gamma_r \subseteq E_r$ and $\mathbf{1}_{\Gamma_r} \longrightarrow \mathbf{1}_{\widetilde{\Gamma}}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0^+$. This and (2.40) then allow us to write

$$\mathbf{1}_{\widetilde{\Gamma}} = \lim_{r\to 0^+} \mathbf{1}_{\Gamma_r} = \lim_{r\to 0^+} \big(\mathbf{1}_{\Gamma_r} \cdot \mathbf{1}_{E_r}\big) = \big(\lim_{r\to 0^+} \mathbf{1}_{\Gamma_r}\big) \cdot \big(\lim_{r\to 0^+} \mathbf{1}_{E_r}\big) = \mathbf{1}_{\widetilde{\Gamma}} \cdot \mathbf{1}_{H(x_0)} = \mathbf{1}_{\widetilde{\Gamma} \cap H(x_0)} \tag{2.42}$$

in the pointwise $\mathcal{H}^n$-a.e. sense in $\mathbb{R}^n$. In turn, this implies

$$\widetilde{\Gamma} \subseteq \overline{H(x_0)}. \tag{2.43}$$

Now, (2.38) readily follows from this, (2.39), and simple geometrical considerations. □

2.3 Measuring the smoothness of Euclidean domains in analytical terms

We begin by giving the formal definition of the category of Lipschitz domains and domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1]$. The reader is reminded that the superscript c is the operation of taking the complement of a set, relative to $\mathbb{R}^n$.

Definition 2.7. Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Also, fix $x_0 \in \partial\Omega$. Call Ω a *Lipschitz domain near* x_0 if there exist $r, c > 0$ with the following significance. There exists an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, a choice N of the unit normal to H, and an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$ (called coordinate cylinder near x_0) such that

$$\mathcal{C}_{r,c} \cap \Omega = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\} \tag{2.44}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$, called the defining function for $\partial\Omega$ near x_0, satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < c \ \text{ if } \ |x' - x_0| \leqslant r. \tag{2.45}$$

Collectively, the pair $(\mathcal{C}_{r,c}, \varphi)$ is referred to as a local chart near x_0, whose geometrical characteristics consist of r, c, and the Lipschitz constant of φ.

Moreover, call Ω a *locally Lipschitz domain* if it is a Lipschitz domain near every point $x \in \partial\Omega$. Finally, Ω is simply called a *Lipschitz domain* if it is locally Lipschitz and such that the geometrical characteristics of the local charts associated with each boundary point are independent of the point in question.

The categories of $\mathscr{C}^{1,\alpha}$ domains with $\alpha \in (0,1]$, as well as their local versions, are defined analogously, requiring that the defining functions φ have first order directional derivatives (along vectors parallel to the hyperplane H) which are of class $\mathscr{C}^{\alpha}$ (the Hölder space of order α).

A few useful observations related to the property of an open set $\Omega \subseteq \mathbb{R}^n$ of being a Lipschitz domain near a point $x_0 \in \partial\Omega$ are collected below.

Proposition 2.8. *Assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$, and fix $x_0 \in \partial\Omega$. The following assertions hold.*

(i) *If Ω is a Lipschitz domain near x_0 and $(\mathcal{C}_{r,c}, \varphi)$ is a local chart near x_0 (in the sense of Definition* 2.7*), then, in addition to* (2.44)*, one also has*

$$\mathcal{C}_{r,c} \cap \partial\Omega = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t = \varphi(x')\}, \tag{2.46}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^c = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t < \varphi(x')\}. \tag{2.47}$$

Furthermore,

$$\mathcal{C}_{r,c} \cap \overline{\Omega} = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t \geqslant \varphi(x')\}, \tag{2.48}$$

$$\mathcal{C}_{r,c} \cap (\overline{\Omega})^\circ = \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}, \tag{2.49}$$

and, consequently,

$$E \cap \partial\Omega = E \cap \partial(\overline{\Omega}) \quad \forall\, E \subseteq \mathcal{C}_{r,c}. \tag{2.50}$$

(ii) *Assume that there exists an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$ and a Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying* (2.45) *such that* (2.46) *holds. Then, if $x_0 \notin (\overline{\Omega})^\circ$, it follows that Ω is a Lipschitz domain near x_0.*

Proof. The fact that (2.44) implies (2.46) is a consequence of the general fact

$$\mathcal{O}, \Omega_1, \Omega_2 \subseteq \mathbb{R}^n \text{ open sets such that } \mathcal{O} \cap \Omega_1 = \mathcal{O} \cap \Omega_2 \Longrightarrow \mathcal{O} \cap \partial\Omega_1 = \mathcal{O} \cap \partial\Omega_2, \tag{2.51}$$

used with $\mathcal{O} := \mathcal{C}_{r,c}$, $\Omega_1 := \Omega$, and Ω_2 the upper-graph of φ. In order to justify (2.51), we make the elementary observation that

$$E \subseteq \mathbb{R}^n \text{ arbitrary set and } \mathcal{O} \subseteq \mathbb{R}^n \text{ open set } \Longrightarrow \overline{E} \cap \mathcal{O} \subseteq \overline{E \cap \Omega}. \tag{2.52}$$

Then, in the context of (2.51), based on assumptions and (2.52) we may write

$$\begin{aligned} \mathcal{O} \cap \partial\Omega_1 &\subseteq (\mathcal{O} \cap \overline{\Omega_1}) \setminus (\mathcal{O} \cap \Omega_1) \subseteq \overline{\mathcal{O} \cap \Omega_1} \setminus (\mathcal{O} \cap \Omega_1) \\ &= \overline{\mathcal{O} \cap \Omega_2} \setminus (\mathcal{O} \cap \Omega_2) \subseteq \overline{\Omega_2} \setminus \Omega_2 = \partial\Omega_2. \end{aligned} \tag{2.53}$$

This further entails $\mathcal{O} \cap \partial\Omega_1 \subseteq \mathcal{O} \cap \partial\Omega_2$ from which (2.51) follows by interchanging the roles of Ω_1 and Ω_2. As mentioned earlier, this establishes (2.46). Thus, in order to prove (2.47),

$$(2.44) \text{ and } (2.46) \Longrightarrow (2.47), \tag{2.54}$$

In turn, (2.54) follows by writing

$$\begin{aligned} \mathcal{C}_{r,c} \cap (\overline{\Omega})^c &= \mathcal{C}_{r,c} \setminus (\mathcal{C}_{r,c} \cap \overline{\Omega}) = \mathcal{C}_{r,c} \setminus \big((\mathcal{C}_{r,c} \cap \Omega) \cup (\mathcal{C}_{r,c} \cap \partial\Omega)\big) \\ &= \mathcal{C}_{r,c} \setminus \big((\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}) \\ &\quad \cup (\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t = \varphi(x')\})\big) \\ &= \mathcal{C}_{r,c} \setminus \big(\mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t \geqslant \varphi(x')\}\big) \\ &= \mathcal{C}_{r,c} \cap \{x' + tN : x' \in H,\ t < \varphi(x')\}, \end{aligned} \tag{2.55}$$

as desired. Next, (2.48) is a consequence of (2.44) and (2.46), while (2.49) follows from (2.48) by passing to interiors. In concert, (2.48), (2.49), and (2.46) give that

$$\mathcal{C}_{r,c} \cap \partial(\overline{\Omega}) = \mathcal{C}_{r,c} \cap \partial\Omega, \tag{2.56}$$

which further implies (2.50) by taking the intersection of both sides with a given set $E \subseteq \mathcal{C}_{r,c}$. This completes the proof of assertion (i). As far as (ii) is concerned, it suffices to show that, up to reversing the sense on the vertical axis in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$x_0 \notin (\overline{\Omega})^\circ \Longrightarrow (2.44) \text{ and } (2.47). \tag{2.57}$$

In turn, (2.57) follows from Lemma 2.9, stated and proved below. □

Here is the topological result which was invoked earlier, in the proof of the implication (2.57).

Lemma 2.9. *Assume that $\Omega \subseteq \mathbb{R}^n$ is an open, proper, nonempty set and fix $x_0 \in \partial\Omega$. Also, assume that $B' \subseteq \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional open ball, $I \subseteq \mathbb{R}$ is an open interval, and $\varphi : B' \to I$ is a continuous function. Denote by $\Sigma := \{(x', \varphi(x')) : x' \in B'\}$ the graph of φ. Assume that the open cylinder $\mathcal{C} := B' \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ contains x_0 and satisfies $\Sigma = \mathcal{C} \cap \partial\Omega$. Finally, set*

$$\begin{aligned} D^+ &:= \{(x', x_n) \in \mathcal{C} : \varphi(x') < x_n\}, \\ D^- &:= \{(x', x_n) \in \mathcal{C} : \varphi(x') > x_n\}. \end{aligned} \tag{2.58}$$

Then one of the following three alternatives holds:

$$\Omega \cap \mathcal{C} = D^+ \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^-, \tag{2.59}$$

$$\Omega \cap \mathcal{C} = D^- \quad \textit{and} \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \tag{2.60}$$

$$x_0 \in (\overline{\Omega})^\circ. \tag{2.61}$$

Proof. We begin by noting that $D^\pm$ are connected sets. To see this, consider D^+, as the argument for D^- is similar. It suffices to show that the set in question is pathwise connected and a continuous curve γ contained in D^+ joining any two given points $x, y \in D^+$ may be taken to consist of three line segments, L_1, L_2, L_3, defined as follows. Take L_1 and L_2 to be the vertical line segments contained in D^+ which emerge from x and y respectively and then choose L_3 to be a horizontal line segment making the transition between L_1 and L_2 near the very top of $\mathcal{C}$. Moving on, we claim that one of the following situations necessarily happens:

$$\begin{array}{ll} \text{(i)} & D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq (\overline{\Omega})^c, \quad \text{or} \\ \text{(ii)} & D^- \subseteq \Omega \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c, \quad \text{or} \\ \text{(iii)} & D^+ \subseteq \Omega \quad \text{and} \quad D^- \subseteq \Omega, \quad \text{or} \\ \text{(iv)} & D^- \subseteq (\overline{\Omega})^c \quad \text{and} \quad D^+ \subseteq (\overline{\Omega})^c. \end{array} \tag{2.62}$$

To prove this, note that $D^\pm$ are disjoint from Σ and hence from $\partial\Omega \cap \mathcal{C}$. In turn, this further entails that $D^\pm$ are disjoint from $\partial\Omega$. Based on this and the fact that $\mathbb{R}^n = \Omega \cup (\overline{\Omega})^c \cup \partial\Omega$, we conclude that

$$D^\pm \subseteq \Omega \cup (\overline{\Omega})^c. \tag{2.63}$$

Now, recall that $D^\pm$ are connected sets and observe that Ω are $(\overline{\Omega})^c$ open, disjoint sets. In concert with the definition of connectivity, (2.63) then implies that each of the two sets D^+ and D^- is contained in either Ω or $(\overline{\Omega})^c$. Unraveling the various possibilities now proves that one of the four scenarios in (2.62) must hold. This concludes the proof of the claim made about (2.62). The next step is to show that if conditions (i) in (2.62) happen, then the conditions (2.59) happen as well. To see this, assume that (i) holds, i.e., $D^+ \subseteq \Omega$, $D^- \subseteq (\overline{\Omega})^c$ and recall that $\Sigma = \partial\Omega \cap \mathcal{C}$. Then $D^+ = \Omega \cap \mathcal{C}$. Indeed, the left-to-right inclusion is clear from what we assume. For the opposite inclusion, we reason by contradiction a nd assume that there exists $x \in \Omega \cap \mathcal{C}$ such that $x \notin D^+$, thus $x \in \Omega$, $x \in \mathcal{C}$, $x \notin D^+$. Since

$$\mathcal{C} = D^+ \cup D^- \cup \Sigma \quad \text{disjoint unions,} \tag{2.64}$$

we obtain

$$\mathcal{C} = D^+ \cup D^- \cup (\mathcal{C} \cap \partial\Omega) \quad \text{disjoint unions.} \tag{2.65}$$

From the assumptions on x we have $x \notin D^+$, $x \notin \mathcal{C} \cap \partial\Omega$ (since $x \in \Omega$ and $\Omega \cap \partial\Omega = \varnothing$) and, consequently, using also (2.65), $x \in D^- \subseteq (\overline{\Omega})^c$. This yields $x \notin \Omega$, contradicting the assumption that $x \in \Omega$. This completes the proof of the fact that $D^+ = \Omega \cap \mathcal{C}$. In a similar fashion, we also obtain $D^- = (\overline{\Omega})^c \cap \mathcal{C}$.

We next propose to show that, if condition (ii) in (2.62) holds, then the condition (2.60) holds as well. In particular, if (ii) holds, then

$$\Omega \cap \mathcal{C} = D^-, \quad (\overline{\Omega})^c \cap \mathcal{C} = D^+, \quad \partial\Omega \cap \mathcal{C} = \Sigma. \tag{2.66}$$

To see this, assume that (ii) holds. The first observation is that $D^+ = (\overline{\Omega})^c \cap \mathcal{C}$. The left-to-right inclusion is clear. Assume that there exists $x \in \mathcal{C}$ such that $x \notin \overline{\Omega}$ (hence $x \notin \partial\Omega$) and $x \notin D^+$. Together with (2.65), these imply $x \in D^-$, so $x \in D^- \subseteq \Omega \subseteq \overline{\Omega}$, which contradicts our assumptions. Moving on, the second observation is that $D^- = \Omega \cap \mathcal{C}$. The left-to-right inclusion is obvious. In the opposite direction, assume that there exists $x \in \mathcal{C}$ such that $x \in \Omega$ (hence $x \notin \partial\Omega$) yet $x \notin D^-$. Invoking (2.65), we obtain $x \in D^+ \subseteq (\overline{\Omega})^c$, hence $x \notin \overline{\Omega}$ contradicting the assumption on x.

Next, we show that if condition (iii) in (2.62) happens, then $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, i.e., (2.61) happens. To this end, let $x_* \in \partial\Omega \cap \mathcal{C}$. Then there exists $r > 0$ such that $B(x_*, r) \subseteq \mathcal{C}$. We claim that

$$B(x_*, r) \subseteq \overline{\Omega}. \tag{2.67}$$

Indeed, by (2.64), we have

$$B(x_*, r) = \big(B(x_*, r) \cap D^+\big) \cup \big(B(x_*, r) \cap D^-\big) \cup \big(B(x_*, r) \cap \Sigma\big). \tag{2.68}$$

Making use of the inclusion $B(x_*, r) \subseteq \mathcal{C}$, we then obtain

$$\begin{aligned} &B(x_*, r) \cap D^+ \subseteq D^+ \subseteq \Omega, \\ &B(x_*, r) \cap D^- \subseteq D^- \subseteq \Omega, \\ &B(x_*, r) \cap \Sigma \subseteq \Sigma = \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega. \end{aligned} \tag{2.69}$$

Combining all these with (2.68), it follows that $B(x_*, r) \subseteq \Omega \cap \partial\Omega = \overline{\Omega}$, proving (2.67). In turn, (2.67) implies that $x_* \in (\overline{\Omega})^\circ \cap \partial\Omega$ so that, ultimately,

$$\mathcal{C} \cap \partial\Omega \subseteq (\overline{\Omega})^\circ \cap \partial\Omega. \tag{2.70}$$

Since $x_0 \in \mathcal{C} \cap \partial\Omega$, this forces $x_0 \in (\overline{\Omega})^\circ \cap \partial\Omega$, as claimed.

At this stage in the proof, it remains to show that condition (vi) in (2.62) never happens. Reasoning by contradiction, assume (iv) actually does happen, i.e.,

$$D^- \subseteq (\overline{\Omega})^c, \quad D^+ \subseteq (\overline{\Omega})^c, \quad \Sigma = \partial\Omega \cap \mathcal{C}. \tag{2.71}$$

Taking the union of the first two inclusions above yields

$$D^+ \cup D^- \subseteq (\overline{\Omega})^c \Longrightarrow \mathcal{C} \setminus \Sigma \subseteq (\overline{\Omega})^c \Longrightarrow \mathcal{C} \cap (\Sigma^c) \subseteq (\overline{\Omega})^c \Longrightarrow \overline{\Omega} \subseteq \Sigma \cup \mathcal{C}^c, \tag{2.72}$$

where the last implication follows by taking complements. Taking the intersection with $\mathcal{C}$, this yields $\mathcal{C} \cap \overline{\Omega} \subseteq \Sigma = \mathcal{C} \cap \partial\Omega$, thanks to the fact that $\Sigma = \mathcal{C} \cap \partial\Omega$. Thus, $\mathcal{C} \cap \Omega \subseteq \mathcal{C} \cap \overline{\Omega} \subseteq \mathcal{C} \cap \partial\Omega \subseteq \partial\Omega$, i.e.,

$$\mathcal{C} \cap \Omega \subseteq \partial\Omega. \tag{2.73}$$

Since, by assumption, $\mathcal{C}$ is an open neighborhood of the point $x_0 \in \partial\Omega$, the definition of the boundary implies $\mathcal{C} \cap \Omega \neq \varnothing$. Therefore, there exists $x_* \in \mathcal{C} \cap \Omega$. From (2.73) it follows that $x_* \in \partial\Omega$, which forces us to conclude that the open set Ω contains some of its own boundary points. This is a contradiction which shows that (iv) in (2.62) never happens. The proof of the lemma is therefore complete. $\square$

The proposition below formalizes the idea that an open, proper, connected subset of $\mathbb{R}^n$ whose boundary is a compact Lipschitz surface is a Lipschitz domain. Before stating this, we wish to note that the connectivity assumption is necessary since, otherwise, $\Omega := \{x \in \mathbb{R}^n : |x| < 2 \text{ and } |x| \neq 1\}$ would serve as a counterexample.

Theorem 2.10. *Let Ω be an open, proper, nonempty, connected subset of $\mathbb{R}^n$ with $\partial\Omega$ bounded. In addition, suppose that for each $x_0 \in \partial\Omega$ there exists an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, an open cylinder $\mathcal{C}_{r,c} := \{x' + tN : x' \in H, |x' - x_0| < r, |t| < c\}$, and a Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying* (2.45) *such that* (2.46) *holds. Then Ω is a Lipschitz domain.*

In the proof of the above result, the following generalization of the Jordan–Brouwer separation theorem for arbitrary compact topological hypersurfaces in $\mathbb{R}^n$, established in [17, Theorem 1, p. 284], plays a key role.

Proposition 2.11. *Let Σ be a compact, connected, topological $(n-1)$-dimensional submanifold (without boundary) of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus \Sigma$ consists of two connected components, each with boundary Σ.*

Proof of Theorem 2.10. Let Σ be a connected component of $\partial\Omega$ (in the relative topology induced by $\mathbb{R}^n$ on $\partial\Omega$). We claim that

$$\Sigma \subseteq \partial(\overline{\Omega}). \tag{2.74}$$

To justify this claim, we first observe that, granted the current assumptions, it follows that $\partial\Omega$ is a compact, $(n-1)$-dimensional Lipschitz submanifold (without boundary) of $\mathbb{R}^n$. Hence, in particular, Σ is a compact, connected, $(n-1)$-dimensional topological manifold (without boundary in $\mathbb{R}^n$). Invoking Proposition 2.11, we may then conclude that there exist $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathbb{R}^n$ such that

$$\begin{gathered} \mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2 \quad \text{and} \quad \mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing, \\ \mathcal{O}_j \text{ open, connected, } \partial\mathcal{O}_j = \Sigma \text{ for } j = 1, 2. \end{gathered} \tag{2.75}$$

Let us also observe that these conditions further entail

$$\partial(\overline{\mathcal{O}_j}) = \Sigma \text{ for } j = 1, 2. \tag{2.76}$$

Indeed, $\overline{\mathcal{O}_1} = \mathcal{O}_1 \cup \partial\mathcal{O}_1 = \mathcal{O}_1 \cup \Sigma = (\mathcal{O}_2)^c$, which forces $\partial(\overline{\mathcal{O}_1}) = \partial[(\mathcal{O}_2)^c] = \partial\mathcal{O}_2 = \Sigma$, from which (2.76) follows. Moreover, since Ω is a connected set contained in $\mathbb{R}^n \setminus \partial\Omega \subseteq \mathbb{R}^n \setminus \Sigma = \mathcal{O}_1 \cup \mathcal{O}_2$, it follows that Ω is contained in one of the sets $\mathcal{O}_1, \mathcal{O}_2$. To fix ideas, assume that $\Omega \subseteq \mathcal{O}_1$. Then $\overline{\Omega} \subseteq \overline{\mathcal{O}_1}$ and hence

$$(\overline{\Omega})^\circ \subseteq (\overline{\mathcal{O}_1})^\circ = \overline{\mathcal{O}_1} \setminus \partial(\overline{\mathcal{O}_1}) = (\mathcal{O}_1 \cup \partial\mathcal{O}_1) \setminus \partial\mathcal{O}_1 = \mathcal{O}_1, \tag{2.77}$$

where the next-to-last equality is a consequence of (2.76) (and (2.75)). The relevant observation for us here is that, in concert with the second line in (2.75), the inclusion in (2.77) forces

$$\Sigma \cap (\overline{\Omega})^\circ = \varnothing. \tag{2.78}$$

To proceed, note that since $\Sigma \subseteq \partial\Omega \subseteq \overline{\Omega}$, we also have

$$\Sigma \cap (\overline{\Omega})^c = \varnothing. \tag{2.79}$$

Thus, since
$$\Sigma \subseteq \mathbb{R}^n = (\overline{\Omega})^\circ \cup \partial(\overline{\Omega}) \cup (\overline{\Omega})^c, \tag{2.80}$$
we may ultimately deduce from (2.78)–(2.80) that (2.74) holds. The end-game in the proof of the proposition is then as follows. Taking the union of all connected components of $\partial\Omega$, we see from (2.74) that $\partial\Omega \subseteq \partial(\overline{\Omega})$. Consequently, since the opposite inclusion is always true, we arrive at the conclusion that
$$\partial\Omega = \partial(\overline{\Omega}). \tag{2.81}$$
Therefore, $\partial\Omega \cap (\overline{\Omega})^\circ = \partial(\overline{\Omega}) \cap (\overline{\Omega})^\circ = \varnothing$ and, as such, given any $x_0 \in \partial\Omega$ it follows that necessarily $x_0 \notin (\overline{\Omega})^\circ$. With this in hand, the fact that Ω is a Lipschitz domain now follows from part (ii) of Proposition 2.8. □

Definition 2.7 and (i) in Proposition 2.8 show that if $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain near a boundary point x_0 then, in a neighborhood of x_0, $\partial\Omega$ agrees with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, considered in a suitably chosen system of coordinates (which is isometric with the original one). Then the outward unit normal has an explicit formula in terms of $\nabla\varphi$, namely, in the new system of coordinates,
$$\nu(x', \varphi(x')) = \frac{(\nabla'\varphi(x'), -1)}{\sqrt{1 + |\nabla'\varphi(x')|^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \text{ near } x'_0, \tag{2.82}$$
where the gradient $\nabla\varphi(x')$ of φ exists by the classical Rademacher theorem for $\mathcal{H}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. This readily implies that if $\Omega \subseteq \mathbb{R}^n$ is a $\mathscr{C}^{1,\alpha}$ domain for some $\alpha \in (0, 1]$, then the outward unit normal $\nu : \partial\Omega \to S^{n-1}$ is Hölder of order α.

We next discuss a cone property enjoyed by Lipschitz domains whose significance will become more apparent later.

Lemma 2.12. *Assume that $\Omega \subseteq \mathbb{R}^n$ is Lipschitz near $x_0 \in \partial\Omega$. More specifically, suppose that the $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, the unit normal N to H, the Lipschitz function $\varphi : H \to \mathbb{R}$, and the cylinder $\mathcal{C}_{r,c}$ are such that* (2.44) *and* (2.45) *hold. Denote by M the Lipschitz constant of φ and fix $\theta \in (0, 2\arctan(1/M)]$. Finally, select $\lambda \in (0, 1)$. Then there exists $b > 0$ such that*
$$\Gamma_{\theta,b}(x, N) \subseteq \Omega \quad \textit{and} \quad \Gamma_{\theta,b}(x, -N) \subseteq \mathbb{R}^n \setminus \Omega \quad \textit{for each } x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega. \tag{2.83}$$

Proof. Let $\theta \in \big(0, 2\arctan(1/M)\big]$, where $M > 0$ is the Lipschitz constant of φ. Pick $b > 0$ such that
$$b < \min\Big\{c, \frac{(1-\lambda)r}{\tan(\theta/2)}\Big\}. \tag{2.84}$$
These conditions guarantee that $\Gamma_{\theta,b}(x, \pm N) \subseteq \mathcal{C}_{r,c}$ for each $x \in \mathcal{C}_{\lambda r,c} \cap \partial\Omega$. So, as far as the first inclusion in (2.83) is concerned, it suffices to show that
$$x', y' \in H, \quad s \in \mathbb{R} \text{ so that } \; y' + sN \in \Gamma_{\theta,b}\big(x' + \varphi(x')N, N\big) \implies s > \varphi(y'). \tag{2.85}$$
Fix x', y', and s as on the left-hand side of (2.85). Then
$$\cos(\theta/2)\sqrt{|y' - x'|^2 + (s - \varphi(x'))^2} < s - \varphi(x'). \tag{2.86}$$
Consequently, $s = s - \varphi(x') + \varphi(x') > \cos(\theta/2)(|y' - x'|^2 + (s - \varphi(x'))^2)^{\frac{1}{2}} + (\varphi(x') - \varphi(y')) + \varphi(y')$. So, to prove that $s > \varphi(y')$, it is enough to show that $\cos(\theta/2)(|y' - x'|^2 + (s - \varphi(x'))^2)^{\frac{1}{2}} +$

$(\varphi(x')-\varphi(y')) \geqslant 0$. This is trivially true if $y'=x'$, so it remains to consider the situation where $x' \neq y'$. Assuming that this is the case, define

$$A := \frac{|s-\varphi(x')|^2}{|y'-x'|^2} \quad \text{and} \quad B := \frac{\varphi(x')-\varphi(y')}{|x'-y'|},$$

in which scenario we must show that $\cos(\theta/2)(1+A)^{1/2}+B \geqslant 0$. By construction, $A \geqslant 0$ and $B \in [-M, M]$, so it suffices to prove that

$$\cos(\theta/2)(1+A)^{1/2} \geqslant M. \tag{2.87}$$

As a preamble, observe that $\cos(\theta/2)(|y'-x'|^2+(s-\varphi(x'))^2)^{1/2} < s-\varphi(x')$ entails $\cos(\theta/2)(1+A)^{1/2} < A^{1/2}$ or $\cos^2(\theta/2)(1+A) < A$. Thus,

$$\frac{\cos^2(\theta/2)}{1-\cos^2(\theta/2)} < A$$

and, further, $A > \cot^2(\theta/2)$. Using this lower bound on A in (2.87) yields $\cos(\theta/2)(1+A)^{1/2} > \cos(\theta/2)(1+\cot^2(\theta/2))^{1/2} = \cot^2(\theta/2)$. Now, $\cot^2(\theta/2) \geqslant M$ if and only if $\tan^2(\theta/2) \leqslant 1/M$, which is true by our original choice of θ. This completes the proof of (2.85) and finishes the proof of the first inclusion in (2.83). The second inclusion in (2.83) is established in a similar fashion, completing the proof of the lemma. □

Our next result shows that suitable rotations of graphs of differentiable functions continue to be graphs of functions (enjoying the same degree of regularity as the original ones). This is going to be useful later, in the proof of Theorem 3.13.

Lemma 2.13. *Assume that $\mathcal{O} \subseteq \mathbb{R}^{n-1}$ is an open neighborhood of the origin and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function satisfying $\varphi(0')=0$, which is differentiable and whose derivative is continuous at $0' \in \mathbb{R}^{n-1}$. Let $\mathcal{R}$ be a rotation about the origin in $\mathbb{R}^n$ with the property that*

$$\mathcal{R} \text{ maps the vector } \frac{(\nabla\varphi(0'),-1)}{\sqrt{1+|\nabla\varphi(0')|^2}} \text{ into } -\mathbf{e}_n \in \mathbb{R}^n. \tag{2.88}$$

Then there exists a continuous, real-valued function ψ defined in a small neighborhood of $0' \in \mathbb{R}^{n-1}$ with the property that $\psi(0')=0$ and whose graph coincides, in a small neighborhood of $0 \in \mathbb{R}^n$, with the graph of φ rotated by $\mathcal{R}$. Furthermore, φ is of class $\mathscr{C}^{1,\alpha}$, for some $\alpha \in (0,1]$, if and only if so is ψ.

Proof. Matching the graph of φ, after being rotated by $\mathcal{R}$, by that of a function ψ comes down to ensuring that ψ is such that $\mathcal{R}(x',\varphi(x'))=(y',\psi(y'))$ can be solved both for x' in terms y', as well as for y' in terms x', near the origin in $\mathbb{R}^{n-1}$ in each instance. Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection map of $\mathbb{R}^n$ onto the first $n-1$ coordinates. Denote by $\pi_n : \mathbb{R}^n \to \mathbb{R}$ the coordinate projection map of $\mathbb{R}^n$ onto the last coordinate. Then

$$\begin{aligned} (y',y_n)=\mathcal{R}(x',\varphi(x')) &\Leftrightarrow \mathcal{R}^{-1}(y',y_n)=(x',\varphi(x')) \\ &\Leftrightarrow \pi'\mathcal{R}^{-1}(y',y_n)=x' \quad \text{and} \quad \pi_n\mathcal{R}^{-1}(y',y_n)=\varphi(x') \\ &\Leftrightarrow F(y',y_n)=0 \quad \text{and} \quad x'=\pi'\mathcal{R}^{-1}(y',y_n), \end{aligned} \tag{2.89}$$

where F is the real-valued function defined in a neighborhood of the origin in $\mathbb{R}^n$ by

$$F(y', y_n) := \varphi(\pi'\mathcal{R}^{-1}(y', y_n)) - \pi_n\mathcal{R}^{-1}(y', y_n). \tag{2.90}$$

Then a direct calculation shows that $F(0', 0) = 0$ and

$$\begin{aligned}
\partial_n F(y', y_n) &= \sum_{j=1}^{n-1}(\partial_j\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))(\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_j - (\mathcal{R}^{-1}\mathbf{e}_n)\cdot\mathbf{e}_n \\
&= (\mathcal{R}^{-1}\mathbf{e}_n)\cdot((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\,-1) \\
&= \mathbf{e}_n\cdot\mathcal{R}((\nabla\varphi)(\pi'\mathcal{R}^{-1}(y', y_n))\,,\,-1).
\end{aligned} \tag{2.91}$$

In particular, by (2.88),

$$\partial_n F(0', 0) = -\sqrt{1 + |\nabla\varphi(0')|^2} \neq 0. \tag{2.92}$$

Thus, by the implicit function theorem, there exists a continuous real-valued function ψ defined in a small neighborhood of $0' \in \mathbb{R}^{n-1}$ such that $\psi(0') = 0$ and for which

$$F(y', y_n) = 0 \Longleftrightarrow y_n = \psi(y') \quad \text{whenever } (y', y_n) \text{ is near } 0. \tag{2.93}$$

From this and (2.89), all desired conclusions follow. □

3 Geometric Smoothness

This section is divided into two parts dealing, respectively, with geometric characterizations of Lipschitz domains in terms of cones and geometric characterizations of Lyapunov domains in terms of pseudoballs.

3.1 Characterization of Lipschitz domains in terms of cones

The main goal in this subsection is to discuss several types of cones conditions which fully characterize the class of Lipschitz domains in $\mathbb{R}^n$. The results presented here build on and generalize those from [5, Section 2]. To help put matters in the proper perspective, it is worth recalling that an open set $\Omega \subseteq \mathbb{R}^n$ with compact boundary and the property that there exists an open, circular, truncated, one-component cone Γ with vertex at $0 \in \mathbb{R}^n$ such that for every $x_0 \in \partial\Omega$ there exist $r > 0$ and a rotation $\mathcal{R}$ about the origin such that

$$x + \mathcal{R}(\Gamma) \subseteq \Omega \quad \forall\, x \in B(x_0, r) \cap \overline{\Omega} \tag{3.1}$$

is necessarily Lipschitz (the converse is also true). The proof can be found in [18, Theorem 1.2.2.2, p. 12].

A different type of condition which characterizes Lipschitzianity has been recently discovered in [5]. This involves the notion of a transversal vector field to the boundary of a domain $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter which we now record. As a preamble, we remind the reader that $\partial^*\Omega$ denotes the reduced boundary of Ω and that $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff (outer-)measure in $\mathbb{R}^n$.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set of locally finite perimeter, with outward unit normal ν. We fix a point $x_0 \in \partial\Omega$. Then it is said that Ω has a *continuous transversal vector field* *near* x_0 provided that there exists a continuous vector field h which is uniformly (outwardly) transverse to $\partial\Omega$ near x_0 in the sense that there exist $r>0$, $\varkappa>0$ so that $h:\partial\Omega\cap B(x_0,r)\to\mathbb{R}^n$ is continuous and

$$\nu\cdot h\geqslant\varkappa\quad\mathcal{H}^{n-1}\text{-a.e. on }\ B(x_0,r)\cap\partial^*\Omega. \tag{3.2}$$

Here is the statement of the result proved in [5] alluded to above.

Theorem 3.2. *Assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which has locally finite perimeter, and fix $x_0\in\partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if it has a continuous transversal vector field near x_0 and there exists $r>0$ such that*

$$\partial(\Omega\cap B(x_0,r))=\partial(\overline{\Omega\cap B(x_0,r)}). \tag{3.3}$$

We momentarily digress for the purpose of discussing an elementary result of topological nature which is going to be used shortly.

Lemma 3.3. *Let E_1,E_2 be two subsets of $\mathbb{R}^n$ with the property that*

$$\big(\partial E_1\setminus\partial(\overline{E_1})\big)\cap\overline{E_2}=\varnothing\quad\textit{and}\quad\big(\partial E_2\setminus\partial(\overline{E_2})\big)\cap\overline{E_1}=\varnothing. \tag{3.4}$$

Then

$$\partial(E_1\cap E_2)=\partial(\overline{E_1\cap E_2}). \tag{3.5}$$

Proof. Since $\partial(\overline{E})\subseteq\partial E$ for any set $E\subseteq\mathbb{R}^n$, the right-to-left inclusion in (3.5) always holds, so it remains to show that, granted (3.4), one has

$$\partial(E_1\cap E_2)\subseteq\partial(\overline{E_1\cap E_2}). \tag{3.6}$$

To this end, recall that

$$\partial(A\cap B)\subseteq(\overline{A}\cap\partial B)\cup(\partial A\cap\overline{B})\quad\forall\,A,B\subseteq\mathbb{R}^n, \tag{3.7}$$

which further implies

$$\partial(A\cap B)=\big(\partial(A\cap B)\cap\overline{A}\cap\partial B\big)\cup\big(\partial(A\cap B)\cap\overline{B}\cap\partial A\big). \tag{3.8}$$

From this and simple symmetry considerations we see that (3.6) follows as soon as we check the validity of the inclusion

$$\partial(E_1\cap E_2)\cap(\overline{E_1}\cap\partial E_2)\subseteq\partial(\overline{E_1\cap E_2}). \tag{3.9}$$

To this end, we reason by contradiction and assume that there exist a point x and a number $r>0$ satisfying

$$\begin{aligned}&x\in\partial(E_1\cap E_2),\quad x\in\partial E_2,\quad\text{and}\\&\text{either }\ B(x,r)\cap(\overline{E_1\cap E_2})=\varnothing\ \text{ or }\ B(x,r)\subseteq\overline{E_1\cap E_2}.\end{aligned} \tag{3.10}$$

Note that if $B(x,r)\cap(\overline{E_1\cap E_2})=\varnothing$ then also $B(x,r)\cap(E_1\cap E_2)=\varnothing$, contradicting the fact that $x\in\partial(E_1\cap E_2)$. Thus, necessarily, $B(x,r)\subseteq\overline{E_1\cap E_2}$. However, this entails

$$x\in(\overline{E_1\cap E_2})^\circ\cap\partial E_2\subseteq\overline{E_1}\cap(\overline{E_2})^\circ\cap\partial E_2=\overline{E_1}\cap(\partial E_2\setminus\partial(\overline{E_2}))=\varnothing \tag{3.11}$$

by (3.4). This shows that the conditions in (3.10) are contradictory and hence proves (3.9). $\square$

Definition 3.4. An open, proper, nonempty subset Ω of $\mathbb{R}^n$ is said to satisfy an *exterior, uniform, continuously varying cone condition near a point* $x_0 \in \partial\Omega$ provided that there exist two numbers $r, b > 0$, an angle $\theta \in (0, \pi)$, and a function $h : B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is continuous at x_0 and such that

$$\Gamma_{\theta,b}(x, h(x)) \subseteq \mathbb{R}^n \setminus \Omega \quad \forall\, x \in B(x_0, r) \cap \partial\Omega. \tag{3.12}$$

Also, an open, nonempty set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy a *global, exterior, uniform, continuously varying cone condition* if Ω satisfies an interior uniform continuously varying cone condition near each point on $\partial\Omega$.

Finally, define an *interior uniform continuously varying cone condition* (near a boundary point, or globally) in an analogous manner, replacing $\mathbb{R}^n \setminus \Omega$ by Ω in (3.12).

The global, interior, uniform, continuously varying cone condition earlier appeared in the paper by Nadirashvili [19], where he used them as the main background geometrical hypothesis for the class of domains in which he proves a uniqueness theorem for the oblique derivative boundary value problem (cf. [19, Theorem 1, p. 327]). We revisit the latter topic in Section 4. For now, our goal is to establish the following proposition, refining a result of similar flavor proved in [5][3].

Proposition 3.5. *Assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ and suppose that $x_0 \in \partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if Ω satisfies an exterior, uniform, continuously varying cone condition near x_0.*

Proof. In one direction, if Ω is a Lipschitz domain near x_0, then the existence of $r, b > 0$, $\theta \in (0, \pi)$ and a function $h : B(x_0, r) \cap \partial\Omega \to S^{n-1}$ which is actually constant and such that (3.12) holds, follows from Lemma 2.12. The crux of the matter is, of course, dealing with the converse implication. In doing so, we employ the notation introduced in Definition 3.4. We begin by observing that the condition (3.12) forces $B(x_0, r) \cap \partial\Omega \subseteq \overline{[(\Omega^c)^\circ]}$. In concert with the readily verified formula $(\Omega^c)^\circ = (\overline{\Omega})^c$, this yields $B(x_0, r) \cap \partial\Omega \subseteq \overline{[(\overline{\Omega})^c]}$. Hence $B(x_0, r) \cap \partial\Omega \subseteq \overline{\Omega} \cap \overline{[(\overline{\Omega})^c]} = \partial(\overline{\Omega})$ and further $B(x_0, r) \cap \partial\Omega \subseteq B(x_0, r) \cap \partial(\overline{\Omega})$. Since the opposite inclusion is always true, we may ultimately deduce that

$$B(x_0, r) \cap \partial\Omega = B(x_0, r) \cap \partial(\overline{\Omega}). \tag{3.13}$$

As a consequence of (3.13), we obtain

$$\begin{aligned} B(x_0, r) \cap (\overline{\Omega})^\circ &= B(x_0, r) \cap \big(\overline{\Omega} \setminus \partial(\overline{\Omega})\big) = \big(B(x_0, r) \cap \overline{\Omega}\big) \setminus \big(B(x_0, r) \cap \partial(\overline{\Omega})\big) \\ &= \big(B(x_0, r) \cap \overline{\Omega}\big) \setminus \big(B(x_0, r) \cap \partial\Omega\big) = B(x_0, r) \cap \big(\overline{\Omega} \setminus \partial\Omega\big) \\ &= B(x_0, r) \cap \Omega. \end{aligned} \tag{3.14}$$

Hence

$$\Omega \cap B(x_0, r) = (\overline{\Omega})^\circ \cap B(x_0, r). \tag{3.15}$$

Next, fix $b_0 \in (0, b)$ along with $\varepsilon \in (0, 1 - \cos(\theta/2))$. Then there exists $\theta_0 \in (0, \theta)$ with the property that

$$\cos(\theta_0/2) - \varepsilon > \cos(\theta/2) \quad \text{and} \quad \frac{b_0}{\cos(\theta_0/2)} < b. \tag{3.16}$$

[3] In the process, we also use the opportunity to correct a minor gap in the treatment in [5].

Next, with $\varepsilon > 0$ as above, select $r_0 \in (0, r)$ such that

$$|h(x) - h(x_0)| < \varepsilon \quad \text{whenever } x \in B(x_0, r_0) \cap \partial\Omega. \tag{3.17}$$

That this is possible is ensured by the continuity of the function h at x_0. We then claim that

$$\Gamma_{\theta_0,b_0}(x, h(x_0)) \subseteq \Gamma_{\theta,b}(x, h(x)) \quad \forall\, x \in B(x_0, r_0) \cap \partial\Omega. \tag{3.18}$$

Indeed, if $x \in B(x_0, r_0) \cap \partial\Omega$ and $y \in \Gamma_{\theta_0,b_0}(x, h(x_0))$, then

$$\begin{aligned}(y - x) \cdot h(x) &= (y - x) \cdot h(x_0) + (y - x) \cdot (h(x) - h(x_0)) \\ &> \cos(\theta_0/2)|y - x| - \varepsilon|y - x| = (\cos(\theta_0/2) - \varepsilon)|y - x| > \cos(\theta/2)|y - x|\end{aligned}$$

by the Cauchy–Schwarz inequality, the first inequality in (3.16), and the condition (3.17). In addition, since $y \in \Gamma_{\theta_0,b_0}(x, h(x_0))$ forces $|y - x| < (\cos(\theta_0/2))^{-1} b_0$, it follows that

$$(y - x) \cdot h(x) \leqslant |y - x| < \frac{b_0}{\cos(\theta_0/2)} < b \tag{3.19}$$

by the Cauchy–Schwarz inequality and the second inequality in (3.16). All together, this analysis proves (3.18). With this in hand, we deduce from (3.12) that

$$\Gamma_{\theta_0,b_0}(x, h(x_0)) \subseteq \mathbb{R}^n \setminus \Omega \quad \forall\, x \in B(x_0, r_0) \cap \partial\Omega. \tag{3.20}$$

Moving on, consider the open, proper subset of $\mathbb{R}^n$ given by

$$D := (\Omega^c)^\circ \cap B(x_0, r_0). \tag{3.21}$$

Since, by (3.12), $\Gamma_{\theta,b}(x_0, h(x_0)) \subseteq (\Omega^c)^\circ$, it follows that D is also nonempty. The first claim we make about the set D is that

$$\partial D = \partial(\overline{D}). \tag{3.22}$$

To justify this, observe that $D = (\overline{\Omega})^c \cap B(x_0, r_0)$ and note that, since

$$\partial E \setminus \partial(\overline{E}) = \partial E \cap (\overline{E})^\circ \quad \forall\, E \subseteq \mathbb{R}^n, \tag{3.23}$$

we have

$$\partial((\overline{\Omega})^c) \setminus \partial(\overline{(\overline{\Omega})^c}) = \partial(\overline{\Omega}) \cap \big(\overline{(\overline{\Omega})^c}\big)^\circ = \partial(\overline{\Omega}) \cap \big(((\overline{\Omega})^\circ)\big)^c \subseteq \partial(\overline{\Omega}) \cap \big(\overline{\Omega}\big)^c = \varnothing. \tag{3.24}$$

Having established this, (3.22) follows from Lemma 3.3.

Going further, the second claim we make about the set D introduced in (3.21) is that

$$\partial D \subseteq \big(\partial\Omega \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0). \tag{3.25}$$

To see this, with the help of (3.7) we write

$$\begin{aligned}\partial D &\subseteq \big(\partial\big((\Omega^c)^\circ\big) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0) = \big(\partial\big((\overline{\Omega})^c\big) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0) \\ &= \big(\partial(\overline{\Omega}) \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0) = \big(\partial\Omega \cap \overline{B(x_0, r_0)}\big) \cup \partial B(x_0, r_0),\end{aligned} \tag{3.26}$$

where the last equality is a consequence of (3.13). This proves (3.25). Let us note here that, as a consequence of this, (3.20) and elementary geometrical considerations, we have

$$\eta := \min\{b_0,\, \cos(\theta_0/2)\, r_0/2\} \Longrightarrow \Gamma_{\theta_0,\eta}(x, h(x_0)) \subseteq D \quad \forall\, x \in B(x_0, r_0/2) \cap \partial D. \tag{3.27}$$

The third claim we make about the set D from (3.21) is that

$$\mathcal{H}^{n-1}(\partial D) < +\infty. \tag{3.28}$$

Of course, given (3.25), it suffices to show that there exists a finite constant $C = C(\theta, b) > 0$ with the property that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B(x_0, r_0)) \leqslant C r_0^{n-1}. \tag{3.29}$$

With this goal in mind, recall first that, in general, $\mathcal{H}^{n-1}(E) \leqslant C_n \lim_{\delta\to 0^+} \mathcal{H}^{n-1}_\delta(E)$, where $\mathcal{H}^{n-1}_\delta(E)$ denotes the infimum of all sums $\sum_{B\in\mathcal{B}} (\text{radius}\, B)^{n-1}$, associated with all covers $\mathcal{B}$ of E with open balls B of radii $\leqslant \delta$. Next, abbreviate $\Gamma := \Gamma_{\theta_0,b_0}(0, h(x_0))$ so that (3.20) reads $x + \Gamma \subseteq \Omega^c$ for every $x \in B(x_0, r_0) \cap \partial\Omega$. Denote by L the one-dimensional space spanned by the vector $h(x_0)$ in $\mathbb{R}^n$. For some fixed $\lambda \in (0, 1)$, to be specified later, consider $\Gamma_\lambda \subseteq \Gamma$ to be the open, truncated, circular, one-component cone of aperture $\lambda\,\theta_0$ with vertex at $0 \in \mathbb{R}^n$ and having the same height b_0 and symmetry axis L as Γ. Elementary geometry gives

$$|x - y| < h,\ x \notin y + \Gamma,\ y \notin x + \Gamma \Longrightarrow |x - y| \leqslant \frac{\text{dist}\,(x + L\,,\, y + L)}{\sin(\theta_0/2)}. \tag{3.30}$$

In subsequent considerations, it can be assumed that r_0 is smaller than a fixed fraction of b_0. To fix ideas, suppose henceforth that $0 < r_0 \leqslant b_0/10$.

In order to continue, select a small number $\delta \in (0, r_0)$ and cover $\partial\Omega \cap B(x_0, r_0)$ by a family of balls $\{B(x_j, r_j)\}_{j\in J}$ with $x_j \in \partial\Omega$, $0 < r_j \leqslant \delta$, for each $j \in J$. By the Vitali lemma, there is no loss of generality in assuming that $\{B(x_j, r_j/5)\}_{j\in J}$ are mutually disjoint. Then

$$\mathcal{H}^{n-1}_\delta(\partial\Omega \cap B(x_0, r_0)) \leqslant C_n \sum_{j\in J} r_j^{n-1}.$$

Let π be a fixed $(n-1)$-plane perpendicular to the axis of Γ. Denote by A_j the projection of $(x_j + \Gamma_\lambda) \cap B(x_j, r_j/5)$ onto π. It is clear that $\mathcal{H}^{n-1}(A_j) \approx r_j^{n-1}$ for every $j \in J$ and there exists an $(n-1)$-dimensional ball of radius $3r$ in π containing all A_j's.

We now claim that $\lambda > 0$ can be chosen sufficiently small as to ensure that the A_j's are mutually disjoint. Indeed, if $A_{j_1} \cap A_{j_2} \neq \varnothing$ for some $j_1, j_2 \in J$, then $\text{dist}\,(x_{j_1} + L\,,\, x_{j_2} + L) \leqslant (r_{j_1} + r_{j_2}) \sin(\lambda\,\theta_0/2)$. Also, $|x_{j_1} - x_{j_2}| \geqslant (r_{j_1} + r_{j_2})/5$, as $B(x_{j_1}, r_{j_1}/5) \cap B(x_{j_2}, r_{j_2}/5) = \varnothing$. Note that $|x_{j_1} - x_{j_2}| \leqslant 4r < b_0$. Since also $\partial\Omega \ni x_{j_1} \notin x_{j_2} + \Gamma \subseteq (\Omega^c)^\circ$ plus a similar condition with the roles of j_1 and j_2 reversed, it follows from (3.30) that $(r_{j_1} + r_{j_2})/5 \leqslant (r_{j_1} + r_{j_2}) \sin(\lambda\,\theta_0/2)/\sin(\theta_0/2)$ or $\sin(\theta_0/2) < 5\,\sin(\lambda\,\theta_0/2)$. Taking $\lambda \in (0, 1)$ sufficiently small, this leads to a contradiction. This finishes the proof of the claim that the A_j's are mutually disjoint if λ is small enough. Assuming that this is the case, we obtain

$$\sum_{j\in J} r_j^{n-1} \leqslant C \sum_{j\in J} \mathcal{H}^{n-1}(A_j) \leqslant C\mathcal{H}^{n-1}(\cup_{j\in J} A_j) \leqslant C r_0^{n-1},$$

given the containment condition on the A_j's. As a consequence, $\mathcal{H}^{n-1}_{\delta}(\partial\Omega\cap B(x_0,r_0))\leqslant Cr_0^{n-1}$, so, taking the supremum over $\delta>0$, we arrive at $\mathcal{H}^{n-1}(\partial\Omega\cap B(x_0,r_0))\leqslant Cr_0^{n-1}$. This finishes the proof of (3.29) and hence (3.28) holds.

In summary, the above analysis shows that D is an open, proper, nonempty subset of $\mathbb{R}^n$, of finite perimeter and such that (3.27) holds. Granted this, if follows from Lemma 2.6 that if ν_D is the geometric measure theoretic outer unit normal to D, then

$$\nu_D(x)\in\Gamma_{\pi-\theta_0,\eta}(0,h(x_0))\ \text{ for each } x\in B(x_0,r_0/2)\cap\partial^*D. \tag{3.31}$$

Hence the vector $h(x_0)\in S^{n-1}$ is transversal to ∂D near x_0 in the precise sense that

$$\nu_D(x)\cdot h(x_0)\geqslant\cos((\pi-\theta_0)/2)>0\ \text{ for each } x\in B(x_0,r_0/2)\cap\partial^*D. \tag{3.32}$$

From (3.22) (cf. also Lemma 3.3) and (3.32) we deduce that D is a Lipschitz domain near x_0.

The end-game in the proof of the proposition is as follows. Since D is a Lipschitz domain near x_0, it follows that $(\overline{D})^c$ is also a Lipschitz domain near x_0. In turn, this and the fact that, thanks to (3.15), we have $\Omega\cap B(x_0,r_0/2)=(\overline{D})^c\cap B(x_0,r_0/2)$, we may finally conclude that Ω is a Lipschitz domain near the point x_0. □

Proposition 3.6. *Assume that $\Omega\subseteq\mathbb{R}^n$ is an open, nonempty set which is not dense in $\mathbb{R}^n$, and suppose that $x_0\in\partial\Omega$. Then Ω is a Lipschitz domain near x_0 if and only if there exist two numbers $r,b>0$, an angle $\theta\in(0,\pi)$, and a function $h:B(x_0,r)\cap\partial\Omega\to S^{n-1}$ which is continuous at x_0 and such that*

$$B(x_0,r)\cap\partial\Omega=B(x_0,r)\cap\partial(\overline{\Omega}), \tag{3.33}$$

$$\Gamma_{\theta,b}(x,h(x))\subseteq\Omega\quad\forall\,x\in B(x_0,r)\cap\partial\Omega. \tag{3.34}$$

Proof. This follows from applying Proposition 3.5 to the open, proper, nonempty subset $(\overline{\Omega})^c$ of $\mathbb{R}^n$ and keeping in mind (2.50). □

It is instructive to observe that there is a weaker version of Propositions 3.5-3.6 (same conclusion, yet stronger hypotheses), but whose proof makes no use of results or tools from geometric measure theory. This is presented next.

Proposition 3.7. *Assume that $\Omega\subseteq\mathbb{R}^n$ is an open, proper, nonempty set and $x_*\in\partial\Omega$. Then Ω is Lipschitz near x_* if and only if there exist $b,r>0$, $\theta\in(0,\pi)$ and a function $h:B(x_*,r)\cap\partial\Omega\to S^{n-1}$ which is continuous at x_* and with the property that*

$$\Gamma_{\theta,b}(x,h(x))\subseteq\Omega\quad\text{and}\quad\Gamma_{\theta,b}(x,-h(x))\subseteq\mathbb{R}^n\setminus\Omega\quad\forall\,x\in B(x_*,r)\cap\partial\Omega. \tag{3.35}$$

Proof. Assume first that an open, proper, nonempty set $\Omega\subseteq\mathbb{R}^n$ and a point $x_*\in\partial\Omega$ are such that (3.35) holds. Thanks to the analysis in (3.16)–(3.18), there is no loss of generality in assuming that the function $h:B(x_*,r)\cap\partial\Omega\to S^{n-1}$ is constant, say $h(x)\equiv v\in S^{n-1}$ for each $x\in B(x_*,r)\cap\partial\Omega$. Furthermore, since for any rotation $\mathcal{R}:\mathbb{R}^n\to\mathbb{R}^n$ we have

$$\mathcal{R}(\Gamma_{\theta,b}(x,\pm v))=\Gamma_{\theta,b}(\mathcal{R}(x),\pm\mathcal{R}(v)), \tag{3.36}$$

there is no loss of generality in assuming that $v=\mathbf{e}_n$. Finally, performing a suitable translation, we can assume that $x_*=0\in\mathbb{R}^n$. Granted these, fix some small positive number c, say,

$$0<c<\min\Big\{b\,\cos(\theta/2)\,,\,\frac{r}{\sqrt{1+(\cos(\theta/2))^2}}\Big\}, \tag{3.37}$$

and consider the cylinder

$$\mathcal{C} := B_{n-1}\big(0', c\,\cos(\theta/2)\big) \times (-c, c) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n. \tag{3.38}$$

Then the top lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0,v) \subseteq \Omega$, whereas the bottom lid of $\mathcal{C}$ is contained in $\Gamma_{\theta,b}(0,-v) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ = (\Omega^c)^\circ = (\overline{\Omega})^c$. We now make the claim that for every $x' \in B_{n-1}(0', c\,\cos(\theta/2))$,

$$\text{the (relative) interior of the line segment } L(x') := [(x',c),(x',-c)] \text{ intersects } \partial\Omega. \tag{3.39}$$

Indeed, if $x' \in B_{n-1}(0', c\,\cos(\theta/2))$ is such that the (relative) interior of $L(x')$ is disjoint from $\partial\Omega$, the fact that $\mathbb{R}^n = \Omega \cup \partial\Omega \cup (\overline{\Omega})^c$ with the three sets appearing on the right-hand side mutually disjoint, implies that Ω and $(\overline{\Omega})^c$ form an open cover of $L(x')$. Since $L(x') \cap \Omega$ is nonempty (as it contains (x',c)), $L(x') \cap (\overline{\Omega})^c$ is nonempty (as it contains $(x',-c)$), and $\Omega \cap (\overline{\Omega})^c = \varnothing$, this contradicts the fact that $L(x')$ is connected. This proves that there exists $x_0 \in L(x')$ with the property that $x_0 \in \partial\Omega$. It remains to observe that, necessarily, x_0 is different from the endpoints of $L(x')$ in order to conclude that this point actually belongs to the (relative) interior of $L(x')$. This finishes the proof of (3.39).

Our next claim is that, in fact (with $\#E$ denoting the cardinality of the set E)

$$\#(L(x') \cap \partial\Omega) = 1 \quad \forall\, x' \in B_{n-1}(0', c\,\cos(\theta/2)). \tag{3.40}$$

To justify this, let $x = (x', x_n) \in L(x') \cap \partial\Omega$. Then

$$|x| = \sqrt{|x'|^2 + x_n^2} \leqslant \sqrt{c^2(\cos(\theta/2))^2 + c^2} = c\sqrt{1 + (\cos(\theta/2))^2} < r, \tag{3.41}$$

so $x \in B(0,r) \cap \partial\Omega$. Consequently, from (3.35) and conventions,

$$\Gamma_{\theta,b}(x, \mathbf{e}_n) \subseteq \Omega \quad \text{and} \quad \Gamma_{\theta,b}(x, -\mathbf{e}_n) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ. \tag{3.42}$$

This forces (with $\mathcal{I}(y,z)$ denoting the relatively open line segment with endpoints $y, z \in \mathbb{R}^n$)

$$\mathcal{I}(x, x + b\,\mathbf{e}_n) \subseteq \Omega \quad \text{and} \quad \mathcal{I}(x, x - b\,\mathbf{e}_n) \subseteq (\overline{\Omega})^c \tag{3.43}$$

and hence $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n) \cap \partial\Omega = \{x\}$. With this in hand, (3.40) follows after noticing that the (relative) interior of $L(x')$ is contained in $\mathcal{I}(x - b\,\mathbf{e}_n, x + b\,\mathbf{e}_n)$ since, by design, $c < b\,\cos(\theta/2) < b$.

Having established (3.40), it is then possible to define a function

$$\varphi : B_{n-1}(0', c\,\cos(\theta/2)) \longrightarrow (-c, c) \tag{3.44}$$

in an unambiguous fashion by setting, for every $x' \in B_{n-1}(0', c\,\cos(\theta/2))$,

$$\varphi(x') := x_n \ \text{ if } \ (x', x_n) \in L(x') \cap \partial\Omega. \tag{3.45}$$

Then, by design (recall (3.38)), we have

$$\mathcal{C} \cap \partial\Omega = \{x = (x', x_n) \in \mathcal{C} :\, x_n = \varphi(x')\}, \tag{3.46}$$

and we now proceed to show that φ defined in (3.44)–(3.45) is a Lipschitz function. Concretely, if we now select two arbitrary points $x', y' \in B_{n-1}(0', c\,\cos(\theta/2))$, then $(y', \varphi(y'))$ belongs to $\partial\Omega$, therefore $(y', \varphi(y')) \notin \Gamma_{\theta,b}((x', \varphi(x')), \pm\mathbf{e}_n)$. This implies

$$\begin{aligned} \pm((y', \varphi(y')) - (x', \varphi(x'))) \cdot \mathbf{e}_n &\leqslant \cos(\theta/2)\,|(y', \varphi(y')) - (x', \varphi(x'))| \\ &\leqslant \cos(\theta/2)\,|y' - x'|. \end{aligned} \tag{3.47}$$

Thus, ultimately, $|\varphi(y') - \varphi(x')| \leqslant \cos(\theta/2)\,|y' - x'|$, which shows that φ is a Lipschitz function, with Lipschitz constant $\leqslant \cos(\theta/2)$. Based on the classical result of McShane [20] and Whitney [21], the function (3.44) may be extended to the entire Euclidean space $\mathbb{R}^{n-1}$ to a Lipschitz function, with Lipschitz constant $\leqslant \cos(\theta/2)$.

Going further, since the cone condition (3.35) also entails that the point $x_0 \in \partial\Omega$ is the limit of points from $\Gamma_{\theta,b}(x_0, h(x_0)) \subseteq (\mathbb{R}^n \setminus \Omega)^\circ$, we may conclude that $x_0 \in \overline{(\overline{\Omega})^c}$, i.e., $x_0 \notin (\overline{\Omega})^\circ$. With this and (3.46) in hand, we may then invoke Proposition 2.8 in order to conclude that Ω is a Lipschitz domain near 0.

Finally, the converse implication in the statement of the proposition is a direct consequence of Lemma 2.12. □

Definition 3.8. Call a set $\Omega \subseteq \mathbb{R}^n$ *starlike with respect to* $x_0 \in \Omega$ if $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \Omega$, where $\mathcal{I}(x, x_0)$ denotes the open line segment in $\mathbb{R}^n$ with endpoints x and x_0.

Also, call a set $\Omega \subseteq \mathbb{R}^n$ *starlike with respect to a ball* $B \subseteq \Omega$ if $\mathcal{I}(x, y) \subseteq \Omega$ for all $x \in \Omega$ and $y \in B$ (that is, Ω is starlike with respect to any point in B).

Theorem 3.9. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Then Ω is a locally Lipschitz domain if and only if every $x_* \in \partial\Omega$ has an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ with the property that $\Omega \cap \mathcal{O}$ is starlike with respect to some ball.*

In particular, any bounded convex domain is Lipschitz.

Proof. Pick an arbitrary point $x_* \in \partial\Omega$. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open neighborhood of x_* with the property that $\Omega \cap \mathcal{O}$ is starlike with respect to a ball $B(x_0, r) \subseteq \Omega \cap \mathcal{O}$. For each $x \in \mathbb{R}^n \setminus \overline{B(x_0, r/2)}$, consider the circular cone with vertex at x and axis along $x_0 - x$ described as

$$C(x) := \Big\{ y \in \mathbb{R}^n : \sqrt{1 - \frac{r}{2|x - x_0|}}\,|y - x| < (y - x) \cdot \frac{x_0 - x}{|x_0 - x|} < \frac{|x_0 - x|^2 - (r/2)^2}{|x_0 - x|} \Big\}. \tag{3.48}$$

Elementary geometry then shows that

$$C(x) \subseteq \bigcup_{y \in B(x_0, r/2)} \mathcal{I}(x, y) \subseteq \mathcal{O} \cap \Omega \quad \forall\, x \in (\mathcal{O} \cap \Omega) \setminus \overline{B(x_0, r/2)}, \tag{3.49}$$

where the second inclusion is a consequence of the fact that $\mathcal{O} \cap \Omega$ is starlike with respect to $B(x_0, r/2)$. Then, for each $x \in \mathcal{O} \cap \partial\Omega$, there exists a sequence $\{x_j\}_{j \in \mathbb{N}}$ of points in $\mathcal{O} \cap \Omega$ such that $x_j \to x$ as $j \to +\infty$. Hence

$$C(x) \subseteq \bigcup_{j \in \mathbb{N}} C(x_j). \tag{3.50}$$

In concert with (3.49), this implies that

$$C(x) \subseteq \mathcal{O} \cap \Omega \quad \forall\, x \in \mathcal{O} \cap \partial\Omega. \tag{3.51}$$

Next, for each $b > 0$ and $x \in \mathbb{R}^n \setminus \overline{B(x_0, r/2)}$ denote by $\widetilde{C}_b(x)$ the cone with vertex at x, same aperture as $C(x)$, axis pointing in the opposite direction to that of $C(x)$, and height b. We then claim that there exist $b > 0$ and $\rho > 0$ with the property that

$$\widetilde{C}_b(x) \subseteq \mathbb{R}^n \setminus \Omega \quad \forall\, x \in B(x_*, \rho) \cap \partial\Omega. \tag{3.52}$$

To justify this claim, note that since $\mathscr{O}$ is an open neighborhood of x_*, it is possible to select $b, \rho > 0$ sufficiently small so that

$$\widetilde{C}_b(x) \subseteq \mathscr{O} \quad \forall\, x \in B(x_*, \rho). \tag{3.53}$$

Assuming that this is the case, the existence of a point $x \in B(x_*, \rho) \cap \partial\Omega$ for which there exists $\widehat{x} \in \widetilde{C}_b(x) \cap \Omega$ would entail, thanks to (3.53) and (3.49),

$$x \in C(\widehat{x}) \subseteq \mathscr{O} \cap \Omega, \tag{3.54}$$

which contradicts the fact that $x \in \partial\Omega$. This finishes the proof of the claim made in (3.52).

Having established (3.51) and (3.52), Proposition 3.7 applies and yields that Ω is Lipschitz near x_*. Since $x_* \in \partial\Omega$ has been arbitrarily chosen, we may therefore conclude that Ω is locally Lipschitz. This establishes one of the implications in the equivalence formulated in the statement of the theorem.

In the opposite direction, observe that if $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $M > 0$ and if $x_0' \in \mathbb{R}^{n-1}$ and $t > 0$ are given, then

$$\begin{array}{l}\text{the open segment with endpoints } (x_0', t + \varphi(x_0')) \text{ and } (x', \varphi(x')) \text{ belongs} \\ \text{to the (open) upper-graph of } \varphi \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ satisfies } |x'| < t/M.\end{array} \tag{3.55}$$

Then the desired conclusion (i.e., that Ω is locally starlike in the sense explained in the statement of the theorem) follows from this and (2.44). □

3.2 Characterization of Lyapunov domains in terms of pseudoballs

This subsection contains the main result in this paper of geometrical flavor, namely the geometric characterization of Lyapunov domains in terms of a uniform, two-sided pseudoball condition. To set the stage, we first make the following definition.

Definition 3.10. Let E be an arbitrary, proper, nonempty, subset of $\mathbb{R}^n$.

(i) The set E is said to satisfy an *interior pseudoball condition* at $x_0 \in \partial E$ with shape function ω as in (1.8) provided that there exist a, $b > 0$ and $h \in S^{n-1}$ such that $\mathscr{G}^{\omega}_{a,b}(x_0, h) \subseteq E$.

(ii) The set E is said to satisfy an *exterior pseudoball condition* at $x_0 \in \partial E$ with shape function ω as in (1.8) provided that $E^c := \mathbb{R}^n \setminus E$ satisfies an interior pseudoball condition at the point x_0 with shape function ω.

(iii) The set E is said to satisfy a *two-sided pseudoball condition* at $x_0 \in \partial E$ with shape function ω as in (1.8) provided that E satisfies both an interior and an exterior pseudoball condition at $x_0 \in \partial E$ with shape function ω.

(iv) The set E is said to satisfy a *uniform hour-glass condition near* $x_0 \in \partial E$ with shape function ω as in (1.8) provided that there exists $r > 0$ such that E satisfies a two-sided pseudoball condition at each point $x \in B(x_0, r) \cap \partial E$ with shape function ω, amplitude $a > 0$ and truncation height $b > 0$ independent of x.

(v) Finally, the set E is said to satisfy a *uniform hour-glass condition* with shape function ω as in (1.8) provided that both E satisfies a two-sided pseudoball condition at each point $x \in \partial E$ with shape function ω, amplitude $a > 0$, and truncation height $b > 0$ independent of x.

While Definition 3.10 only requires that ω is as in (1.8), for the rest of this section, we also assume that ω satisfies (1.10), i.e., that ω is as in (2.11).

That the terminology "hour-glass condition" employed above is justified is made transparent in the lemma below.

Lemma 3.11. *Let E be a subset of $\mathbb{R}^n$ which satisfies a two-sided pseudoball condition at $x_0 \in \partial E$ with shape function ω as in (2.11), i.e., there exist $a, b > 0$, and $h_\pm \in S^{n-1}$ such that $\mathscr{G}^{\omega}_{a,b}(x_0, h_+) \subseteq E$ and $\mathscr{G}^{\omega}_{a,b}(x_0, h_-) \subseteq E^c := \mathbb{R}^n \setminus E$. Then necessarily $h_+ = -h_-$.*

Proof. This is an immediate consequence of Corollary 2.5. $\square$

Remarkably, if $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition, then the function $h : \partial E \to S^{n-1}$, assigning to each boundary point $x \in \partial E$ the direction $h(x) \in S^{n-1}$ of the pseudoball with apex at x contained in E, turns out to be continuous. A precise, local version of this result is recorded next.

Lemma 3.12. *Assume that the set $E \subseteq \mathbb{R}^n$ satisfies a uniform hour-glass condition near $x_* \in \partial E$ with shape function ω as in (2.11), height $b > 0$, and aperture $a > 0$. Let $\varepsilon = \varepsilon(\omega, \eta, R, a, b) > 0$ be as in Lemma 2.4. Define*

$$\widehat{\omega} : [0,1] \to \Big[0, \frac{\varepsilon R}{2}\Big], \quad \widehat{\omega}(t) := \frac{\varepsilon}{2}\omega^{-1}\big(\omega(R)t\big)t \quad \forall\, t \in [0,1]. \tag{3.56}$$

Since $\widehat{\omega}$ is continuous, increasing, and bijective, it is meaningful to consider its inverse, i.e., the function

$$\widetilde{\omega} : \Big[0, \frac{\varepsilon R}{2}\Big] \to [0,1], \quad \widetilde{\omega}(t) := \widehat{\omega}^{\,-1}(t) \quad \forall\, t \in \Big[0, \frac{\varepsilon R}{2}\Big], \tag{3.57}$$

which is also continuous and increasing. Then there exists a number $r > 0$ such that the function $h : B(x_*, r) \cap \partial E \to S^{n-1}$, defined at each point $x \in B(x_*, r) \cap \partial E$ by the demand that $h(x)$ is the unique vector in S^{n-1} with the property that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, is well defined and continuous. In fact, with $\widetilde{\omega}$ as in (3.57), one has

$$h \in \mathscr{C}^{\,\widetilde{\omega}}\big(B(x_*, r) \cap \partial E\,,\, S^{n-1}\big). \tag{3.58}$$

Proof. Let $r > 0$, ω as in (2.11), and $a, b > 0$ be such that E satisfies a two-sided pseudoball condition at each point $x \in B(x_*, r) \cap \partial E$ with shape function ω, height b and aperture a. The fact that for each $x \in B(x_*, r) \cap \partial E$ there exists a unique vector $h(x) \in S^{n-1}$, which is unequivocally determined by the demand that $\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq E$, follows from our assumption on E and Lemma 3.11. Consequently, we also have $\mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \mathbb{R}^n \setminus E$.

We are left with proving that the mapping $B(x_*, r) \cap \partial E \ni x \mapsto h(x) \in S^{n-1}$ is continuous and, in the process, estimate its modulus of continuity. With this goal in mind, pick two arbitrary points $x_0, x_1 \in B(x_*, r) \cap \partial E$. We then have $\mathscr{G}^{\omega}_{a,b}(x_0, h(x_0)) \cap \mathscr{G}^{\omega}_{a,b}(x_1, -h(x_1)) = \varnothing$ since the former set is contained in E and the latter set is contained in $\mathbb{R}^n \setminus E$. In turn, from this, Lemma 2.4, and (3.56) we infer that

$$|x_0 - x_1| \geqslant \frac{\varepsilon}{2}\omega^{-1}\Big(\omega(R)\frac{|h(x_0) - h(x_1)|}{2}\Big)\Big|\frac{h(x_0) - h(x_1)}{2}\Big| = \widehat{\omega}\Big(\frac{|h(x_0) - h(x_1)|}{2}\Big). \tag{3.59}$$

As a consequence, if $0 < r < \dfrac{\varepsilon R}{4}$ to begin with, from (3.59) and (3.57) we find

$$|h(x_0) - h(x_1)| \leqslant 2\,\widetilde{\omega}(|x_0 - x_1|) \quad \forall\, x_0, x_1 \in B(x_*, r) \cap \partial E. \tag{3.60}$$

This shows that $h \in \mathscr{C}^{\,\widetilde{\omega}}\big(B(x_*, r) \cap \partial E, S^{n-1}\big)$, as desired. $\square$

We are now in a position to formulate the main result in this section.

Theorem 3.13. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Assume that ω is as in* (2.11) *and $x_0 \in \partial\Omega$. Then Ω satisfies a uniform hour-glass condition with shape function ω near x_0 if and only if Ω is of class $\mathscr{C}^{1,\omega}$ near x_0.*

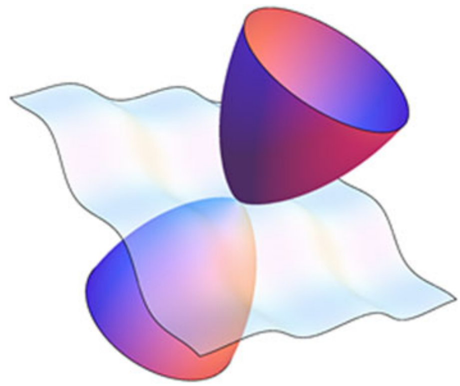

FIGURE 7.

Let us momentarily pause to record an immediate consequence of Theorem 3.13 which is particularly useful in applications.

Corollary 3.14. *Given ω as in* (2.11)*, an open, proper, nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,\omega}$ if and only if Ω satisfies a uniform hour-glass condition with shape function ω. As a corollary, an open, proper, nonempty subset Ω of $\mathbb{R}^n$ with compact boundary is of class $\mathscr{C}^{1,1}$ if and only if it satisfies a uniform two-sided ball condition.*

Proof. The first claim in the statement is a direct consequence of Theorem 3.13, while the last claim follows from the first with the help of part (iii) in Lemma 2.2. □

One useful ingredient in the proof of Theorem 3.13, of independent interest, is the differentiability criterion of geometrical nature presented in the proposition below.

Proposition 3.15. *Assume that $U \subseteq \mathbb{R}^{n-1}$ is an arbitrary set, and that $x_* \in U^\circ$. Given a function $f : U \to \mathbb{R}$, denote by G_f the graph of f, i.e., $G_f := \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^n$. Then f is differentiable at the point x_* if and only if f is continuous at x_* and there exists a nonhorizontal vector $N \in \mathbb{R}^n$ (i.e., satisfying $N \cdot \mathbf{e}_n \neq 0$) with the following significance. For every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ with the property that $G_f \cap B((x_*, f(x_*)), \delta)$ lies in between the cones $\Gamma_{\theta,\delta}((x_*, f(x_*)), N)$ and $\Gamma_{\theta,\delta}((x_*, f(x_*)), -N)$, i.e.,*

$$G_f \cap B((x_*, f(x_*)), \delta) \subseteq \mathbb{R}^n \setminus \big[\Gamma_{\theta,\delta}((x_*, f(x_*)), N) \cup \Gamma_{\theta,\delta}((x_*, f(x_*)), -N)\big]. \tag{3.61}$$

If this happens, then necessarily N is a scalar multiple of $(\nabla f(x_*), -1) \in \mathbb{R}^n$.

Proof. Assume that f is differentiable at x_*. Then f is continuous at x_*. To proceed, take

$$N := \frac{(\nabla f(x_*), -1)}{\sqrt{1 + |\nabla f(x_*)|^2}} \in \mathbb{R}^n. \tag{3.62}$$

It is clear that $|N| = 1$ and $N \cdot \mathbf{e}_n = -(1 + |\nabla f(x_*)|^2)^{-1/2} \neq 0$, so N is nonhorizontal. Then, given $\theta \in (0, \pi)$, the fact that f is differentiable at x_* implies that there exists $\delta > 0$ for which

$$|f(x) - f(x_*) - (\nabla f(x_*)) \cdot (x - x_*)| < \cos(\theta/2)|x - x_*| \quad \forall\, x \in B(x_*, \delta) \cap U. \tag{3.63}$$

For any $x \in B(x_*, \delta) \cap U$ we may then estimate

$$\begin{aligned} \big|\big((x, f(x)) - (x_*, f(x_*))\big) \cdot N\big| &= \frac{|(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)|}{\sqrt{1 + |\nabla f(x_*)|^2}} \\ &\leqslant |(\nabla f(x_*)) \cdot (x - x_*) - f(x) + f(x_*)| \\ &< \cos(\theta/2)|x - x_*| < \cos(\theta/2)|(x, f(x)) - (x_*, f(x_*))|, \end{aligned} \tag{3.64}$$

which (recall that $|N| = 1$) shows that

$$x \in B(x_*, \delta) \cap U \Longrightarrow (x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N). \tag{3.65}$$

Upon observing that any point in $G_f \cap B((x_*, f(x_*)), \delta)$ is of the form $(x, f(x))$ for some $x \in B(x_*, \delta) \cap U$, based on (3.65) we may conclude that (3.61) holds.

For the converse implication, suppose that f is continuous at x_* and assume that there exists a nonhorizontal vector $N \in \mathbb{R}^n$ with the property that for every angle $\theta \in (0, \pi)$ there exists $\delta > 0$ such that (3.61) holds. Dividing N by the nonzero number $-N \cdot \mathbf{e}_n$, we may assume that the nth component of N is -1 to begin with, i.e., $N = (N', -1)$ for some $N' \in \mathbb{R}^{n-1}$.

Fix an arbitrary number $\varepsilon \in (0, 1/2)$ and pick an angle $\theta \in (0, \pi)$ sufficiently close to π so that $0 < \cos(\theta/2) < \varepsilon/\sqrt{1 + |N'|^2}$. By assumption, there exists $\delta_0 > 0$ with the property that if $x \in U$ is such that $|(x, f(x)) - (x_*, f(x_*))| < \delta_0$, then $(x, f(x)) \notin \Gamma_{\theta,\delta}((x_*, f(x_*)), \pm N)$, i.e.,

$$\begin{aligned} \big|\big((x, f(x)) - (x_*, f(x_*))\big) \cdot (N', -1)\big| &\leqslant \cos(\theta/2)|(N', -1)||(x, f(x)) - (x_*, f(x_*))| \\ &\leqslant \varepsilon\sqrt{|x - x_*|^2 + (f(x) - f(x_*))^2} \\ &\leqslant \varepsilon\big[|x - x_*| + |f(x) - f(x_*)|\big]. \end{aligned} \tag{3.66}$$

In turn, this forces (recall that $0 < \varepsilon < 1/2$)

$$\begin{aligned} |f(x) - f(x_*)| &\leqslant \big|\big((x, f(x)) - (x_*, f(x_*))\big) \cdot (N', -1)\big| + |(x - x_*) \cdot N'| \\ &\leqslant \varepsilon\big[|x - x_*| + |f(x) - f(x_*)|\big] + |x - x_*||N'| \\ &\leqslant \Big(\frac{1}{2} + |N'|\Big)|x - x_*| + \frac{1}{2}|f(x) - f(x_*)|. \end{aligned} \tag{3.67}$$

Absorbing the last term above on the left-most side of (3.67) yields

$$\frac{1}{2}|f(x) - f(x_*)| \leqslant \Big(\frac{1}{2} + |N'|\Big)|x - x_*|. \tag{3.68}$$

We have therefore proved that there exists $\delta_0 > 0$ for which

$$\begin{aligned} &x \in U \quad \text{and} \quad |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \\ &\Longrightarrow |f(x) - f(x_*)| \leqslant (1 + 2|N'|)|x - x_*|. \end{aligned} \tag{3.69}$$

Returning with this back in (3.66) then yields

$$\begin{aligned} &x \in U \quad \text{and} \quad |(x, f(x)) - (x_*, f(x_*))| < \delta_0 \\ &\Longrightarrow \big|((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)\big| \leqslant 2\varepsilon(1 + |N'|)|x - x_*|. \end{aligned} \tag{3.70}$$

Since we are assuming that f is continuous at the point x_*, it follows that there exists $\delta_1 > 0$ with the property that

$$x \in U \ \text{ and } \ |x - x_*| < \delta_1 \Longrightarrow |f(x) - f(x_*)| < \delta_0/\sqrt{2}. \tag{3.71}$$

Introducing $\delta := \min\{\delta_1, \delta_0/\sqrt{2}\}$, the implication (3.71) therefore guarantees that

$$x \in U \ \text{ and } \ |x - x_*| < \delta \Longrightarrow |(x, f(x)) - (x_*, f(x_*))| < \delta_0. \tag{3.72}$$

Consequently, from this and (3.70) we deduce that

$$x \in B(x_*, \delta) \cap U \Longrightarrow \left|\frac{((x, f(x)) - (x_*, f(x_*))) \cdot (N', -1)}{|x - x_*|}\right| \leqslant 2\varepsilon(1 + |N'|). \tag{3.73}$$

Since $\varepsilon \in (0, 1/2)$ was arbitrary, this translates into saying that

$$\lim_{x \to x_*,\, x \in U} \frac{f(x) - f(x_*) - N' \cdot (x - x_*)}{|x - x_*|} = 0. \tag{3.74}$$

This proves that f is differentiable at x_* and, in fact, $\nabla f(x_*) = N'$. Hence, in particular, N is a scalar multiple of $(N', -1) = (\nabla f(x_*), -1)$. □

We are now ready to discuss the proof of Theorem 3.13.

Proof of Theorem 3.13. At the first stage, assume that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which satisfies a uniform hour-glass condition with shape function ω (as in (2.11)) near $x_* \in \partial\Omega$. In other words, there exist $b > 0$ and $r_* > 0$, along with a function $h : B(x_*, r_*) \cap \partial\Omega \to S^{n-1}$ such that

$$\mathscr{G}^{\omega}_{a,b}(x, h(x)) \subseteq \Omega \quad \text{and} \quad \mathscr{G}^{\omega}_{a,b}(x, -h(x)) \subseteq \Omega^c \ \text{ for every } x \in B(x_*, r_*) \cap \partial\Omega. \tag{3.75}$$

Note that the uniform hour-glass condition, originally introduced in part (iv) of Definition 3.10, may be written as above thanks to Corollary 2.5. Going further, Lemma 3.12 then guarantees (by eventually decreasing $r_* > 0$ if necessary) that the function $h : B(x_*, r_*) \cap \partial E \to S^{n-1}$ belongs to $\mathscr{C}^{\widetilde{\omega}}$, where $\widetilde{\omega}$ is as in (3.57). Hence, in particular, h is continuous at x_*. Having established this, from part (v) of Lemma 2.2 and Proposition 3.5 we then deduce that Ω is a Lipschitz domain near x_*. Hence there exists an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_*, a choice of the unit normal N to H, a Lipschitz function $\varphi : H \to \mathbb{R}$ and a cylinder $\mathcal{C}_{r,c}$ such that (2.44)–(2.45) hold. Without loss of generality we may assume that x_* is the origin in $\mathbb{R}^n$, that H is the canonical horizontal $(n-1)$-dimensional plane $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ and that $N = \mathbf{e}_n$. In this setting, our goal is to show that

$$\text{the Lipschitz function } \varphi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is actually of class } \mathscr{C}^{1,\omega} \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{3.76}$$

As a preamble, we show that

$$h(x) \cdot \mathbf{e}_n \neq 0 \ \text{ for every } x \in B(0, r_*) \cap \partial\Omega. \tag{3.77}$$

To prove (3.77), assume that there exists $x_0 \in B(0,r_*) \cap \partial\Omega$ with the property that $h(x_0) \cdot \mathbf{e}_n = 0$, with the goal of deriving a contradiction. Then, on the one hand, (3.75) gives that $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \subseteq \Omega^c$, whereas Lemma 2.12 guarantees that $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \subseteq \Omega$ if $\theta_0 := 2\arctan(1/M)$ and $b_0 > 0$ is sufficiently small, where M is the Lipschitz constant of the function φ. Given the locations of the aforementioned pseudoball and cone, the desired contradiction follows as soon as we show that

$$\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0)) \cap \Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n) \neq \varnothing. \tag{3.78}$$

To this end, it suffices to look at the cross-section of $\mathscr{G}^{\omega}_{a,b}(x_0, -h(x_0))$ and $\Gamma_{\theta_0,b_0}(x_0, \mathbf{e}_n)$ with the two-dimensional plane π spanned by the orthogonal unit vectors $h(x_0)$ and $\mathbf{e}_n$. To fix ideas, choose a system of coordinates in π so that $\mathbf{e}_n$ is vertical and $-h(x_0)$ is horizontal, both pointing in the positive directions of these respective axes. In such a setting, it follows that there exists $m \in (0,+\infty)$ with the property that the cross-section of the truncated cone contains all points (x,y) with coordinates satisfying $y > mx$ for $x > 0$ sufficiently small. On the other hand, the portion of the boundary of the cross-section of the pseudoball lying in the first quadrant near the origin is described by the equation $\sqrt{x^2+y^2}\,\omega(\sqrt{x^2+y^2}) = x$. Hence $\omega\big(x\sqrt{1+(y/x)^2}\big) = 1/\sqrt{1+(y/x)^2}$ and, given that $\omega(t) \searrow 0$ as $t \searrow 0$, this forces $y/x \to +\infty$ as $x \searrow 0$. From this, the desired conclusion follows, completing the proof of (3.77).

Moving on, based on (3.77), the fact that φ is continuous, part (v) of Lemma 2.2, and the geometric differentiability criterion presented in Proposition 3.15, we deduce that φ is differentiable at each point near $0' \in \mathbb{R}^{n-1}$ and, in addition,

$$h(x', \varphi(x')) \text{ is parallel to } (\nabla\varphi(x'), -1) \text{ for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{3.79}$$

We now make the claim that for each x' near $0' \in \mathbb{R}^{n-1}$ the vector $(\nabla\varphi(x'), -1)$ points away from Ω, in the sense that

$$(x', \varphi(x')) - t(\nabla\varphi(x'), -1) \in \Omega \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1} \text{ if } t > 0 \text{ is small.} \tag{3.80}$$

This amounts to checking that if x' is near $0' \in \mathbb{R}^{n-1}$ and $t > 0$ is small, then

$$\varphi\big(x' - t\nabla\varphi(x')\big) < \varphi(x') + t$$

which, in turn, follows by observing that (recall that φ is differentiable at points near $0'$)

$$\lim_{t\to 0^+} \frac{\varphi\big(x' - t\nabla\varphi(x')\big) - \varphi(x')}{t} = \frac{d}{dt}\Big[\varphi\big(x' - t\nabla\varphi(x')\big)\Big]\Big|_{t=0} = -|\nabla\varphi(x')|^2 < 1. \tag{3.81}$$

Thus (3.80) holds and, when considered together with the fact that $-h(x', \varphi(x'))$ is a unit vector which also points away from Ω (recall that this is the axis of the pseudoball with apex at $(x', \varphi(x'))$ which is contained in Ω^c) ultimately gives that

$$h(x', \varphi(x')) = \frac{(-\nabla\varphi(x'), 1)}{\sqrt{1+|\nabla\varphi(x')|^2}} \quad \text{for each } x' \text{ near } 0' \in \mathbb{R}^{n-1}. \tag{3.82}$$

Note that since $\mathbb{R}^{n-1} \ni x' \mapsto (x', \varphi(x')) \in \partial\Omega$ is Lipschitz and, since $h \in \mathscr{C}^{\widetilde{\omega}}$, it follows that the mapping $x' \mapsto h(x', \varphi(x'))$ defined for x' near $0' \in \mathbb{R}^{n-1}$ belongs to $\mathscr{C}^{\widetilde{\omega}}$ as well. Moreover, (3.82) also shows that $h_n(x', \varphi(x')) \geqslant (1+M^2)^{-\frac{1}{2}}$, where $M > 0$ is the Lipschitz constant of φ, and

$$\partial_j\varphi(x') = -\frac{h_j(x', \varphi(x'))}{h_n(x', \varphi(x'))}, \quad j = 1, \dots, n-1, \tag{3.83}$$

granted that x' is near $0' \in \mathbb{R}^{n-1}$. Based on this, it follows that $\nabla\varphi$ is of class $\mathscr{C}^{\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$, where $\widetilde{\omega}$ is as in (3.57). Thus, φ is of class $\mathscr{C}^{1,\widetilde{\omega}}$ near $0' \in \mathbb{R}^{n-1}$. While this is a step in the right direction, more work is required in order to justify the stronger claim made in (3.76).

We wish to show that there exists $C > 0$ such that

$$|\nabla\varphi(x_0') - \nabla\varphi(x_1')| \leqslant C\omega(|x_0' - x_1'|) \quad \text{whenever } x_0' \text{ and } x_1' \text{ are near } 0' \in \mathbb{R}^{n-1}. \tag{3.84}$$

Thanks to Lemma 2.13, we may, without loss of generality, assume that $(x_0', \varphi(x_0')) = (0', 0)$ and that $\nabla\varphi(x_0') = 0'$. As such, matters are reduced to proving that

$$|\nabla\varphi(x_1')| \leqslant C\,\omega(|x_1'|) \;\text{ for } x_1' \text{ near } 0'. \tag{3.85}$$

Since this is trivially true when $|\nabla\varphi(x_1')| = 0$, it suffices to focus on the case where $|\nabla\varphi(x_1')| \neq 0$. In this scenario, define

$$x_2' := x_1' + |x_1'|\frac{\nabla\varphi(x_1')}{|\nabla\varphi(x_1')|} \tag{3.86}$$

and note that, by the triangle inequality, $|x_2'| \leqslant 2|x_1'|$. As the point $(x_2', \varphi(x_2'))$ lies on $\partial\Omega$, it does not belong to $\mathscr{G}^{\omega}_{a,b}((x_1', \varphi(x_1')), \pm h((x_1', \varphi(x_1'))))$. As a consequence, we have either

$$\begin{aligned} |(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\,\omega\big(|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\big) \\ \geqslant |h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))| \end{aligned} \tag{3.87}$$

or

$$|h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))| \geqslant b. \tag{3.88}$$

However, given that φ is Lipschitz, the latter eventuality never materializes if we choose x_1' sufficiently close to $0'$. Note that (3.86) forces $|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))| \leqslant \sqrt{1 + M^2}|x_1'|$ where $M > 0$ is the Lipschitz constant of φ. Since ω is increasing and satisfies the condition recorded in the last line of (2.11), we may write

$$\omega\big(|(x_2' - x_1', \varphi(x_2') - \varphi(x_1'))|\big) \leqslant C\,\omega\big(\sqrt{1 + M^2}|x_1'|\big) \leqslant C\,\eta(\sqrt{1 + M^2})\omega(|x_1'|) \tag{3.89}$$

for x_1' near $0'$ and x_2' as in (3.86). The bottom line of this portion of our analysis is that for some finite constant $C > 0$

$$\begin{aligned} \big|h((x_1', \varphi(x_1'))) \cdot (x_2' - x_1', \varphi(x_2') - \varphi(x_1'))\big| \leqslant C\,\omega(|x_1'|) \\ \text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (3.86).} \end{aligned} \tag{3.90}$$

In view of (3.82) and the fact that φ is Lipschitz, from (3.90) we obtain

$$\begin{aligned} \big|-\nabla\varphi(x_1') \cdot (x_2' - x_1') + \varphi(x_2') - \varphi(x_1')\big| \leqslant C|x_1'|\,\omega(|x_1'|) \\ \text{for } x_1' \text{ near } 0' \text{ and } x_2' \text{ as in (3.86).} \end{aligned} \tag{3.91}$$

This estimate further entails

$$|x_1'||\nabla\varphi(x_1')| \leqslant C|x_1'|\,\omega(|x_1'|) + C|\varphi(x_1')| + C|\varphi(x_2')|$$

for some $C > 0$ independent of x_1' near $0'$ (again, x_2' as in (3.86)). Let us now examine $|\varphi(x_1')|$. Given that the point $(x_1', \varphi(x_1'))$ lies on the boundary of Ω, it does not belong to $\mathscr{G}^{\omega}_{a,b}(0, \pm\mathbf{e}_n)$. Much as before, this necessarily implies

$$|(x_1', \varphi(x_1'))|\,\omega\big(|(x_1', \varphi(x_1'))|\big) \geqslant |\varphi(x_1')|.$$

Since it is assumed that $\varphi(0')=0$, we further deduce that

$$\big|(x_1',\varphi(x_1'))\big|=(|x_1'|^2+(\varphi(x_1')-\varphi(0'))^2)^{\frac{1}{2}}\leqslant C|x_1'| \tag{3.92}$$

since φ is Lipschitz. Hence, ultimately, $|\varphi(x_1')|\leqslant C|x_1'|\,\omega(|x_1'|)$, by arguing as before. Likewise, $|\varphi(x_2')|\leqslant C|x_2'|\,\omega(|x_2'|)$ and since $|x_2'|\leqslant 2|x_1'|$, we see that $|\varphi(x_2')|\leqslant C|x_1'|\,\omega(|x_1'|)$. All in all, the above reasoning gives

$$|x_1'||\nabla\varphi(x_1')|\leqslant C|x_1'|\,\omega(|x_1'|)+C|\varphi(x_1')|+C|\varphi(x_2')|\leqslant C|x_1'|\,\omega(|x_1'|).$$

Dividing the most extreme sides of this inequality by $|x_1'|$ then yields $|\nabla\varphi(x_1')|\leqslant C\,\omega(|x_1'|)$, as desired. This concludes the proof of (3.76) and hence Ω is of class $\mathscr{C}^{1,\omega}$ near x_*.

Consider now the scenario when the open, proper, nonempty set $\Omega\subseteq\mathbb{R}^n$ is of class $\mathscr{C}^{1,\omega}$ near some boundary point $x_*\in\partial\Omega$, where ω is as in (2.11). In particular, $\omega:[0,R]\to[0,+\infty)$ is continuous, strictly increasing and such that $\omega(0)=0$. The goal is to show that Ω satisfies a uniform hour-glass condition near x_* with shape function ω. To this end, based on Definition 2.7 and Lemma 2.13, there is no loss of generality in assuming that x_* is the origin in $\mathbb{R}^n$ and that if $(\mathcal{C}_{r,c},\varphi)$ is the local chart near $0\in\mathbb{R}^n$, then

$$\begin{aligned}&\text{the symmetry axis of the cylinder } \mathcal{C}_{r,c} \text{ is in the vertical direction } \mathbf{e}_n,\\ &\varphi:\mathbb{R}^{n-1}\to\mathbb{R} \text{ is of class } \mathscr{C}^{1,\omega},\quad \varphi(0')=0, \text{ and }\quad \nabla\varphi(0')=0'.\end{aligned} \tag{3.93}$$

Fix a constant $C\in(0,+\infty)$ with the property that

$$C\geqslant\sup_{x',y'\in\mathbb{R}^{n-1},\,x'\neq y'}\frac{|\nabla\varphi(x')-\nabla\varphi(y')|}{\omega(|x'-y'|)}. \tag{3.94}$$

The job at hand is to determine $b>0$, depending only on r,c and φ, with the property that

$$\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)\subseteq\mathcal{C}_{r,c}\cap(\text{upper-graph of }\varphi), \tag{3.95}$$

$$\mathscr{G}^{\omega}_{a,b}(0,-\mathbf{e}_n)\subseteq\mathcal{C}_{r,c}\cap(\text{lower-graph of }\varphi). \tag{3.96}$$

Recall (2.5). Given that the mapping $t\mapsto t\,\omega(t)$ is increasing, it follows that $t_b\searrow 0$ as $b\searrow 0$. Consequently, we may select

$$b\in(0,R\,\omega(R))\quad\text{small enough so that } t_{b/C}<\min\{r,c\}. \tag{3.97}$$

By item (i) in Lemma 2.2, such a choice ensures that $\mathscr{G}^{\omega}_{a,b}(0,\pm\mathbf{e}_n)\subseteq B(0,t_b)\subseteq\mathcal{C}_{r,c}$. Pick now an arbitrary point $x=(x',x_n)\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)$. Then, on the one hand, we have $C|x|\,\omega(|x|)<x_n<b$. On the other hand, (3.93) and the mean value theorem ensure the existence of some $\theta=\theta(x')\in(0,1)$ with the property that $\varphi(x')=x'\cdot(\nabla\varphi(\theta x')-\nabla\varphi(0'))$. This and the fact that φ is of class $\mathscr{C}^{1,\omega}$ then allow us to estimate $\varphi(x')\leqslant|x'||\nabla\varphi(\theta x')-\nabla\varphi(0')|\leqslant C|x'|\,\omega(|x'|)\leqslant C|x|\,\omega(|x|)<x_n$. This estimate shows that the point x belongs to the upper graph of the function φ. In summary, this discussion proves that (3.95) holds in the current setting. The same type of analysis as above (this time, writing $\varphi(x')\geqslant -C|x'|\,\omega(|x'|)\geqslant -C|x|\,\omega(|x|)>x_n$), shows that (3.96) also holds under these conditions. All in all, Ω satisfies a two-sided pseudoball condition at 0 with shape function ω, aperture C and height depending only on the $\mathscr{C}^{1,\omega}$ nature of Ω. This, of course, suffices to complete the proof of the theorem. $\square$

4 A Sharp Version of the Hopf–Oleinik Boundary Point Principle

This section is divided into four parts dealing, respectively, with the history of the topic at hand, our main result (Theorem 4.4), applications to boundary value problems, and a discussion pertaining to the sharpness of our main result.

4.1 A historical perspective

The question of how the geometric properties of the boundary of a domain influence the behavior of a solution to a second order elliptic equation is of fundamental importance and has attracted an enormous amount of attention. A significant topic, with distinguished pedigree, belonging to this line of research is the understanding of the sign of oblique directional derivatives of such a solution at boundary points. A celebrated result in this regard, known as the "boundary point principle," states that an oblique directional derivative of a nonconstant $\mathscr{C}^2$ solution to a second order, uniformly elliptic operator L in nondivergence form[4] with bounded coefficients, at an extremal point located on the boundary of the underlying domain $\Omega \subseteq \mathbb{R}^n$ is necessarily nonzero provided that the domain is sufficiently regular at that point. Part of the importance of this result stems from its role in the development of the strong maximum principle[5], as well as its applications to the issue of regularity near the boundary and uniqueness for a number of basic boundary value problems (such as Neumann, Robin, and mixed).

In the (by now) familiar version in which the regularity demand on the domain in question amounts to an interior ball condition, and when the second order, nondivergence form, differential operator is uniformly elliptic and has bounded coefficients, this principle is due to Hopf and Oleinik who have done basic work on this topic in the early 1950's. However, the history of this problem is surprisingly rich, stretching back for more than a century and involving many contributors. Since the narrative of this endeavor does not appear to be well known[6], below we attempt a brief survey of some of the main stages in the development of this topic.

Special cases of the boundary point principle have been known for a long time since this contains, in particular, the fact that the Green function associated with a uniformly elliptic operator L in a domain Ω has a positive conormal derivative at boundary points provided that $\partial\Omega$ and the coefficients of L are sufficiently regular. Some of the early references on this theme are the works of Neumann [25] and Korn [26] in the case of the Laplacian, and Lichtenstein [27] for more general operators.

In his pioneering 1910 paper [10][7], Zaremba has dealt with the case of the Laplacian in a

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[4]As is well known, the boundary point principle fails in the class of divergence form second order uniformly elliptic operators with bounded coefficients, even when these coefficients are continuous at the boundary point (cf. [22, p. 169], [9, p. 39], [8, Problem 3.9, pp. 49-50], and [23]), though does hold if the coefficients are Hölder continuous at the boundary point (cf. [24]).

[5]This is referred to in [9, p. 1] as a "bedrock result of the theory of second order elliptic partial differential equations."

[6]For example, Zaremba's pioneering work at the beginning of the 20th century is occasionally misrepresented as having been carried out in $\mathscr{C}^2$ domains when, in fact, in 1910 Zaremba has proved a boundary point principle (for the Laplacian) in domains satisfying an interior ball condition at the point in question (a geometrical hypothesis which remains the norm for the next 50 years).

[7]Zaremba's original motivation in this paper is the treatment of Dirichlet–Neumann mixed boundary value problems for the Laplacian. The nowadays familiar name the "Zaremba problem" has been eventually adopted in recognition of his early work in [10] (interestingly, in the preamble of this paper, Zaremba attributes the question

three-dimensional domain Ω satisfying an interior ball condition at a point $x_0 \in \partial\Omega$ (cf. [10, Lemma, pp. 316–317]). His proof makes use of a barrier function, constructed with the help of the Poisson formula for harmonic functions in a ball. Concretely, if, say, $B(0,r) \subseteq \Omega \subseteq \mathbb{R}^3$ and $x_0 \in \partial\Omega \cap \partial B(0,r)$, then Zaremba takes (cf. [10, p. 317])

$$v(x) := \frac{r^2 - |x|^2}{r} \int\limits_{\partial B(0,r)} \frac{\psi(y)}{|x-y|^3}\, d\mathcal{H}^2(y), \quad x \in \overline{B(0,r)}, \tag{4.1}$$

where ψ is a continuous, nonnegative function defined on $\partial B(0,r)$, which is zero near x_0, but otherwise does not vanish identically. As such, the function in (4.1) is harmonic, nonnegative and vanishes at points on $\partial B(0,r)$ near x_0, and satisfies[8]

$$\Big(-\frac{x_0}{r}\Big) \cdot (\nabla v)(x_0) = 2 \int\limits_{\partial B(0,r)} \frac{\psi(y)}{|x_0-y|^3}\, d\mathcal{H}^2(y) > 0. \tag{4.2}$$

These are the key features which virtually all subsequent generalizations based on barrier arguments will emulate in one form or another[9]. This being said, proofs based on other methods have been proposed over the years.

In 1932, Giraud managed to extend the boundary point principle to a larger class of elliptic operators (containing the Laplacian), though this was done at the expense of imposing more restrictive conditions on the domain Ω. Specifically, in [28, Théorème 5, p. 343][10] he requires that Ω is of class $\mathscr{C}^{1,1}$ (cf. Definition 2.44) which, as indicated in the second part of Corollary 3.14, is equivalent to the requirement that Ω satisfies a uniform two-sided ball condition. The strategy adopted by Giraud in the proof of this result (cf. [28, pp. 343-346]) is essentially to reduce matters to the case of the Laplacian by freezing the coefficients and changing variables in a manner in which the Green function associated with the original differential operator may now be regarded as a perturbation of that for the Laplacian. Since the latter has an explicit formula, much as in the work by Zaremba, the desired conclusion follows. Shortly thereafter, in his 1933 paper [29], Giraud was able to sharpen the results he obtained earlier in [28] as to allow second order elliptic operators whose top order coefficients are Hölder while the coefficients of the lower order terms are continuous[11], on domains of class $\mathscr{C}^{1,\alpha}$ where $\alpha \in (0,1)$; cf. [29, p. 50][12]. Giraud's proof of this more general result is a fairly laborious argument based on a change of variables (locally flattening the boundary).

Giraud's progress seems to have created a conundrum at this stage in the early development of the subject, namely there appeared to be two sets of conditions of geometric/analytic nature (which overlap, but are otherwise unrelated) ensuring the validity of the boundary point

of considering such a mixed boundary value problem to Wilhelm Wirtinger).

[8]In essence, this itself is a manifestation of the boundary point principle, but in the very special case of a harmonic function in a ball.

[9]It is worth noting that Zaremba's approach works virtually verbatim for oblique derivative problems for the Laplacian.

[10]In the footnote on page 343 of his 1932 paper, Giraud's acknowledges on this occasion the earlier work done in 1931 by Marcel Brelot in his Thèse, pp. 27-28.

[11]The regularity conditions on the coefficients are not natural since, as is trivially verified, the class of differential operators for which the boundary point principle holds is stable under multiplication by arbitrary (hence, possibly discontinuous) functions.

[12]Giraud's result is restated in [30, Theorem 3, IV, p. 7] for $\mathscr{C}^{1,\alpha}$ domains, though the proof given there is in the spirit of [11] and actually requires smoother boundaries.

principle: on the one hand this holds for the Laplacian in domains satisfying an interior ball condition, while on the other hand this also holds for more general elliptic operators in domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$[13].

A few years later, in 1937, motivated by the question of uniqueness for the Neumann problem for the Laplacian[14], Keldysch and Lavrentiev [31] proved a version of the boundary point principle for the Laplacian in three-dimensional domains satisfying a more flexible property than the interior ball condition. Specifically, if $a, b \in (0, +\infty)$ and $\alpha \in (0, 1]$, consider the three-dimensional, open, truncated paraboloid of revolution (about the z-axis) with apex at $0 \in \mathbb{R}^3$,

$$\mathscr{P}^{\alpha}_{a,b} := \big\{(x, y, z) \in \mathbb{R}^3 : a(x^2 + y^2)^{\frac{1+\alpha}{2}} < z < b\big\}, \tag{4.3}$$

and say that $\Omega \subseteq \mathbb{R}$ satisfies an *interior paraboloid condition* at a boundary point $x_0 \in \partial\Omega$ provided that one can place a congruent version of $\mathscr{P}^{\alpha}_{a,b}$ (for some choice of the exponent $\alpha \in (0, 1]$ and the geometrical parameters $a, b > 0$) inside Ω in such a manner that the apex is repositioned at x_0. With this piece of terminology, Keldysch and Lavrentiev's 1937 result then states that the boundary point principle holds for the Laplacian in any domain satisfying an interior paraboloid condition at the point in question. This extends Zaremba's 1910 work in [10] by allowing considerably more general domains and, at the same time, is more in line with the geometrical context in Giraud's 1933 paper [29] since any domain of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0, 1)$ satisfies a paraboloid condition (for the same α in (4.3); for example, this is implicit in the proof of Theorem 3.13). However, the conundrum described in the previous paragraph continued to persist.

As in Zaremba's approach, Keldysch and Lavrentiev's proof also relies upon the construction of a barrier function, albeit this is now adapted to the nature of the paraboloid (4.3). Specifically, in [31, p. 142], these authors consider the following barrier in $\mathscr{P}^{\alpha}_{a,b}$:

$$v(x, y, z) := z + \lambda\, r^{1+\beta} P_{1+\beta}(z/r) \quad \forall\, (x, y, z) \in \mathscr{P}^{\alpha}_{a,b}, \tag{4.4}$$

where $\beta \in (0, \alpha)$, $\lambda > 0$ i s a normalization constant, $r := \sqrt{x^2 + y^2 + z^2}$, and $P_{1+\beta}$ is the (regular, normalized) solution to the Legendre differential equation[15] of order $1 + \beta$:

$$(1 - t^2)\frac{d^2}{dt^2}P_{1+\beta}(t) - 2t\frac{d}{dt}P_{1+\beta}(t) + (1 + \beta)(2 + \beta)P_{1+\beta}(t) = 0. \tag{4.5}$$

Then β and λ may be chosen so that v in (4.5) has the same key features as in the earlier work of Zaremba. Of course, the case $\alpha = 1$ corresponds to Zaremba's interior ball condition.

As a corollary of their boundary point principle, Keldysch and Lavrentiev then establish the uniqueness for the Neumann problem (classically formulated[16]) for a family of domains

[13] Typically, this is indicative of the fact that a more general phenomenon is at play. Alas, it will take about another 40 years for this issue to be resolved.

[14] This issue of uniqueness for the Neumann problem for the Laplacian has been raised by N. Gunther in his influential 1934 monograph on potential theory; cf. [6, Remark, p. 99]. In this connection, we wish to note that in the 1967 English translation [7] of the original 1934 version of Gunther's book, this particular question has been omitted, and replaced by its solution given by Keldysch and Lavrentiev in [31].

[15] A higher dimensional analogue of the Keldysch-Lavrentiev barrier requires considering Gegenbauer functions in place of solutions of (4.5).

[16] That is, the solution is assumed to be twice continuously differentiable inside the domain and continuous on the closure of the domain, with the normal derivative understood in as a one-sided directional derivative along the unit normal.

which contains all bounded domains of class $\mathscr{C}^{1,\alpha}$ with $\alpha \in (0,1)$. The issue whether this uniqueness result also holds for bounded domains of class $\mathscr{C}^1$ has subsequently become known as the Lavrentiev-Keldysch problem (cf. [32, p. 96]), and it will only be settled later. Momentarily fast-forwarding in time to 1981, it was Nadirashvili who in [19] proved a weaker version[17] of the boundary point principle in bounded domains satisfying a global interior uniform cone condition (as discussed in Definition 3.4) which nonetheless suffices to deduce uniqueness in the Neumann and oblique boundary value problems in such a setting[18] (cf. also [34, p. 307] for further refinements of Nadirashvili's theorem).

The coming of age of the work initiated by Zaremba in the 1910 is marked by the publication in 1952 of the papers [11] and [12], in which Hopf[19] and Oleinik[20] have simultaneously and independently established a version of the boundary point principle for domains satisfying an interior ball condition and for general, nondivergence form, uniformly elliptic operators with bounded coefficients[21]. In fact, Hopf and Oleinik's proofs differ only by their choice of barrier functions. In [11, p. 792], Hopf considered a barrier function in an annulus[22] given by

$$v(x) := e^{a|x|^2} - e^{ar^2} \quad \forall\, x \in B(0,r) \setminus \overline{B(0,r/2)}, \quad r > 0, \tag{4.6}$$

where $a > 0$ is a sufficiently large constant (chosen in terms of the coefficients of L)[23]. Oleinik took a different approach to the construction of a barrier and in [12, p. 696] considered the following function[24] defined in a ball:

$$v(x) := C_1 x_n + x_n^2 - C_2 \sum_{i=1}^{n-1} x_i^2 \quad \forall\, x = (x_1, \dots, x_n) \in B(r\mathbf{e}_n, r), \quad r > 0, \tag{4.7}$$

where $C_1, C_2 > 0$ are suitably chosen constants (depending on the size of the differential operator L).

In this format, the Hopf–Oleinik boundary point principle has become very popular and, even more than half a century later, is still routinely reproduced in basic text-books on partial differential equations (cf., for example, [38, 39, 9] as well as the older monographs [8, 37, 30, 40]). However, the interior ball condition is unnecessarily restrictive and, as such, attempts were made

[17]Indeed, the boundary point principle fails in the general class of Lipschitz domains (cf. [33, p. 4] for a simple counterexample in a two dimensional sector).

[18]The crux of Nadirashvili's paper [19] is that, for domains satisfying a uniform cone condition, while the directional derivative of a supersolution of a uniformly elliptic differential operator in nondivergence form may vanish at an extremal point located on the boundary, it does not, however, vanish identically in any neighborhood of that point.

[19]The crucial observation Hopf makes in 1952 is that the comparison method he employed in his 1927 paper [35, Section I] may be used to establish, similarly yet independently of the strong maximum principle itself, a remarkably versatile version of the boundary point principle.

[20]Oleinik's paper was published two years before she defended her doctoral dissertation, entitled "Boundary-value problems for PDE's with small parameter in the highest derivative and the Cauchy problem in the large for nonlinear equations" in 1954.

[21]Strictly speaking, both Hopf and Oleinik ask in [11] and [12] that the coefficients of the differential operator in question are continuous, but their proofs go trough verbatim under the weaker assumption of boundedness.

[22]The idea of considering this type of region apparently originated with Gilbarg who used it in [36, pp. 312-313].

[23]An elegant alternative to Hopf's barrier function (4.6) in the same annulus is $\widetilde{v}(x) := |x|^{-\lambda} - r^{-\lambda}$ for a sufficiently large constant $\lambda > 0$ (cf. the discussion in [37, Subsection 1.3]).

[24]Interestingly, in the limiting case $\alpha = \beta = 1$, the Keldysch–Lavrentiev barrier (4.4) becomes (given the known formula $P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$ for the second order Lagrange polynomial) precisely $v(x,y,x) = z + z^2 - \frac{1}{2}(x^2 + y^2)$, which strongly resembles Oleinik's barrier (4.7) in the three-dimensional setting.

to generalize Hopf and Oleinik's result (in a conciliatory manner with Giraud's 1933 result valid for domains of class $\mathscr{C}^{1,\alpha}$, $\alpha \in (0,1)$). Motivated by Aleksandrov's basic work in [41]–[46], in a series of papers [47]–[50] beginning in the early 1970's, Kamynin and Khimchenko[25] succeeded[26] in extending the validity range of the boundary point principle for general elliptic operators in nondivergence form with bounded coefficients to the class of domains satisfying an interior paraboloid condition, more general yet reminiscent of that considered by Keldysch and Lavrentiev in [31, p. 141]. More specifically, Kamynin and Khimchenko define in place of (4.3)

$$\mathscr{P}^{\omega}_{a,b} := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, a|x'|\,\omega(|x'|) < x_n < b\big\}\,, \tag{4.8}$$

where $a, b > 0$ and the (modulus of continuity, or) shape function $\omega \in \mathscr{C}^0\big([0,R]\big)$ is nonnegative, vanishes at the origin, and is required to satisfy certain differential/integral properties. For example, in [49], under the assumptions that

$$\omega \in \mathscr{C}^2\big((0,R)\big), \quad \omega'(t) \geqslant 0, \quad \text{and} \quad \omega''(t) \leqslant 0 \;\; \text{for every } t \in (0,R), \tag{4.9}$$

and granted that ω also satisfies a Dini integrability condition

$$\int_0^R \frac{\omega(t)}{t}\, dt < +\infty, \tag{4.10}$$

Kamynin and Khimchenko propose (cf. p. 84 in the English translation of [49]) the following exponential type barrier which involves the above modulus of continuity

$$v(x) := x_n \exp\bigg\{C_1 \int_0^{x_n} \frac{\widehat{\omega}(t)}{t}\, dt\bigg\} - C_2\,|x|\,\omega(|x|) \quad \forall\, x = (x_1, \dots, x_n) \in \mathscr{P}^{\omega}_{a,b}, \tag{4.11}$$

where $C_1, C_2 > 0$ are two suitably chosen constants. Here, $\widehat{\omega}$ is yet another modulus of continuity, satisfying the same type of conditions as in (4.9), and which is related to (in the terminology used in [49]) the nature of the degeneracy of the characteristic part of the differential operator L. A further refinement of this result, which applies to certain classes of differential operators with unbounded coefficients, has subsequently been worked out in [52] (cf. also [34]). Results of similar nature, but for domains satisfying an interior ball condition have been proved earlier by Pucci [53, 54].

While the Dini condition (4.10) may not be omitted (cf. the discussion on pp. 85-88 in the English translation of [49]), the necessity of the differentiability conditions in (4.9) may be called into question. In this regard, see the discussion in p. 6 of [33], a paper in which Safonov proposes another approach to the boundary point principle. His proof of [33, Theorem 1.8, p. 5] does not involve the use of a barrier function and, instead, is based on estimates for quotients u_2/u_1 of positive solutions of $Lu = 0$ in a Lipschitz domain Ω, which vanish on a portion of $\partial\Omega$. The main geometrical hypothesis in [33] is what the author terms interior Q-condition (replacing the earlier interior ball and paraboloid conditions), which essentially states that a region congruent to

$$Q := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |x'| < R, \;\; 0 < x_n - |x'|\omega(|x'|) < R\big\} \tag{4.12}$$

[25] Occasionally also spelled "Himčenko."

[26] Earlier, related results are due to Výborný [51].

may be placed inside Ω so as to make contact with the boundary at a desired point. In this scenario[27], Safonov retains (4.10) and, in place of (4.9), only assumes a monotonicity condition, to the effect that

$$\omega : [0, R] \to [0, 1] \text{ is such that the mapping } [0, R] \ni t \mapsto t\omega(t) \in [0, R] \text{ is nondecreasing.} \quad (4.13)$$

This being said, the method employed by Safonov requires that $u(x) = x_n$ is a solution of the operator L and, as such, he imposes the restriction that L is a differential operator without lower order terms, i.e., $L = \sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j$, which is uniformly elliptic and has bounded coefficients. However, from the perspective of the boundary point principle, a uniformly ellipticity condition is unnecessarily strong (as already noted in [49]) and, in fact, so is the boundedness assumption on the coefficients. Indeed, as is trivially verified, if the boundary point principle is valid for a certain differential operator L, then it remains valid for the operator ψL where ψ is an arbitrary (thus, possibly unbounded) positive function.

The topic of boundary point principles for partial differential equations remains an active area of research, with significant work completed in the recent past (cf., for example, [23, 33] and [55]–[58], among others, and we have already commented on the contents of some of these papers). Here, we only wish to note that in [55, Theorem 4.1, p. 346] Lieberman establishes a version of the boundary point principle which, though weaker than that due to Kamynin and Khimchenko, has a conceptually simpler proof, which works in any $\mathscr{C}^1$ domain whose unit normal has a modulus of continuity satisfying a Dini integrability condition[28].

Finally, it should be mentioned that adaptations of this body of results to parabolic differential operators have been worked out by Nirenberg [59], Kamynin [60], Kamynin and Khimchenko [61, 62], to cite a few, and that a significant portion of the theory continues to hold for nonlinear partial differential equations (cf., for example, [9] and the references therein).

4.2 Boundary point principle for semi-elliptic operators with singular drift

Our main result in this section, formulated in Theorem 4.4 below, is a sharp version of the Hopf–Oleinik boundary point principle. The proof presented here, which is a refinement of work recently completed in [63], is based on a barrier construction in a pseudoball (cf. (2.2)). This is done under less demanding assumptions on the shape function ω than those stipulated by Kamynin and Khimchenko in (4.9) and, at the same time, our pseudoball $\mathscr{G}^{\omega}_{a,b}(0, \mathbf{e}_n)$ (cf. (2.4)) is a smaller set than the paraboloid $\mathscr{P}^{\omega}_{a,b}$ considered by Kamynin and Khimchenko in (4.8). Significantly, the coefficients of the differential operators for which our theorem holds are not necessarily bounded or measurable (in contrast to [11, 12, 47, 49, 50] and others), the matrix of top coefficients is only degenerately elliptic, and the coefficients of the lower order terms are allowed to blow up at a rate related to the geometry of the domain[29]. Furthermore, by means of concrete counterexamples we show that our result is sharp.

To set the stage, we first dispense of a number of preliminary matters.

[27] Our notation is slightly different than that employed in [33], where the author works with $\psi(t) := t\omega(t)$ in place of ω.

[28] The class of domains considered in [55] is, however, not optimal.

[29] This addresses an issue raised in [58, p. 226].

Definition 4.1. Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Fix a point $x_0 \in \partial\Omega$. We say that a vector $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$ *points inside* Ω *at* x_0 provided that there exists $\varepsilon > 0$ with the property that $x_0 + t\vec{\ell} \in \Omega$ whenever $t \in (0, \varepsilon)$. Given a function $u \in \mathscr{C}^0(\Omega \cup \{x_0\})$ and a vector $\vec{\ell} \in \mathbb{R}^n \setminus \{0\}$ pointing inside Ω at x_0, define the *lower and upper directional derivatives of* u *at* x_0 *along* $\vec{\ell}$ as

$$\begin{aligned} D^{(\inf)}_{\vec{\ell}} u(x_0) &:= \liminf_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}, \\ D^{(\sup)}_{\vec{\ell}} u(x_0) &:= \limsup_{t \to 0^+} \frac{u(x_0 + t\vec{\ell}) - u(x_0)}{t}. \end{aligned} \tag{4.14}$$

Of course, in the same geometric setting as above, $D^{(\inf)}_{\vec{\ell}} u(x_0)$, $D^{(\sup)}_{\vec{\ell}} u(x_0)$ are meaningfully defined in $\overline{\mathbb{R}} := [-\infty, +\infty]$, $D^{(\inf)}_{\vec{\ell}} u(x_0) \leqslant D^{(\sup)}_{\vec{\ell}} u(x_0)$ holds, and, as a simple application of the mean value theorem shows,

$$\left.\begin{array}{l} u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^1(\Omega) \text{ and the limit} \\ \nabla u(x_0) := \lim\limits_{t \to 0^+} (\nabla u)(x_0 + t\vec{\ell}) \text{ exists in } \mathbb{R}^n \end{array}\right\} \Rightarrow D^{(\inf)}_{\vec{\ell}} u(x_0) = D^{(\sup)}_{\vec{\ell}} u(x_0) = \vec{\ell} \cdot \nabla u(x_0). \tag{4.15}$$

Shortly, we need a suitable version of the weak minimum principle. In order to facilitate the subsequent discussion, we first make a few definitions. Let $\Omega \subseteq \mathbb{R}^n$ be an open, nonempty set. Consider a second order differential operator L in Ω:

$$L := -\sum_{i,j=1}^{n} a^{ij} \partial_i \partial_j + \sum_{i=1}^{n} b^i \partial_i, \quad \text{where} \quad a^{ij}, b^i : \Omega \to \mathbb{R}, \quad i, j \in \{1, \dots, n\}. \tag{4.16}$$

Hence L is in nondivergence form, without a zero order term, and the reader is alerted to the presence of the minus sign in front of second order part of L. In this context, recall that L is called *semi-elliptic* in Ω provided that the coefficient matrix $A = (a^{ij})_{1 \leqslant i,j \leqslant n}$ is semipositive definite at each point in Ω, i.e.,

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geqslant 0 \ \text{ for every } x \in \Omega \text{ and every } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \tag{4.17}$$

It is clear that the semi-ellipticity condition for L in Ω is equivalent to the requirement that, at each point in Ω, the symmetric part of the coefficient matrix $A := (a^{ij})_{1 \leqslant i,j \leqslant n}$, i.e., $\frac{1}{2}(A + A^\top)$ where $A^\top$ denotes the transpose of A, has only nonnegative eigenvalues. Also, we say that L (as above) is *nondegenerate along* $\xi^* = (\xi_1^*, \dots, \xi_n^*) \in S^{n-1}$ in Ω provided that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i^* \xi_j^* > 0 \ \text{ for every } x \in \Omega. \tag{4.18}$$

For further use, let us also agree to call L *uniformly elliptic near* $x_0 \in \overline{\Omega}$ if there exists $r > 0$ such that

$$\inf_{x \in B(x_0, r) \cap \Omega} \ \inf_{\xi \in S^{n-1}} \sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j > 0, \tag{4.19}$$

and simply *uniformly elliptic* provided that

$$\inf_{x\in\Omega}\inf_{\xi\in S^{n-1}}\sum_{i,j=1}^{n}a^{ij}(x)\xi_i\xi_j>0. \tag{4.20}$$

Here is the variant of the weak minimum principle alluded to above.

Proposition 4.2. *Let $\Omega\subseteq\mathbb{R}^n$ be an open, bounded, nonempty set. Assume that L is a second order differential operator in nondivergence form (without a zero order term) as in (4.16) which is sem-ielliptic and nondegenerate along a vector $\xi^*=(\xi_1^*,\dots,\xi_n^*)\in S^{n-1}$. In addition, suppose that the function*

$$\Omega\ni x\mapsto\frac{\sum\limits_{i=1}^{n}b^i(x)\xi_i^*}{\sum\limits_{i,j=1}^{n}a^{ij}(x)\xi_i^*\xi_j^*}\in\mathbb{R}\quad\textit{is locally bounded from above in }\Omega. \tag{4.21}$$

Then for every real-valued function $u\in\mathscr{C}^2(\Omega)$ with the property that

$$(Lu)(x)\geqslant 0\quad\textit{for every }x\in\Omega, \tag{4.22}$$

it follows that

$$\inf_{x\in\Omega}u(x)=\inf_{x\in\partial\Omega}\Big(\liminf_{\Omega\ni y\to x}u(y)\Big). \tag{4.23}$$

In particular, if u is also continuous on $\overline{\Omega}$, then the minimum of u in $\overline{\Omega}$ is achieved on the topological boundary $\partial\Omega$, i.e.,

$$\min_{x\in\overline{\Omega}}u(x)=\inf_{x\in\Omega}u(x)=\min_{x\in\partial\Omega}u(x). \tag{4.24}$$

Proof. Though the proof of this result follows a well–established pattern, we include it for the sake of completeness. For starters, since $u\in\mathscr{C}^2(\Omega)$, by replacing a^{ij} with $\widetilde{a}^{ij}:=\frac{1}{2}(a^{ij}+a^{ji})$, $1\leqslant i,j\leqslant n$ (a transformation which preserves (4.17) and (4.21)), there is no loss of generality in assuming that the coefficient matrix $A=(a^{ij})_{1\leqslant i,j\leqslant n}$ is symmetric at every point in Ω. Furthermore, observe that (4.23) is implied by the version of (4.24) in which Ω is replaced by any relatively compact subset of Ω, say, of the form $\Omega_k:=\{x\in\Omega:\,\mathrm{dist}\,(x,\partial\Omega)>1/k\}$ where $k\in\mathbb{N}$, by passing to the limit $k\to+\infty$. Hence there is no loss of generality in assuming that the function defined in (4.21) is actually globally bounded in Ω. With these adjustments in mind, the fact that

$$Lu>0\ \text{ in }\ \Omega\Longrightarrow\min_{x\in\overline{\Omega}}u(x)=\min_{x\in\partial\Omega}u(x) \tag{4.25}$$

is then a simple consequence of the semipositive definiteness of the (symmetric) matrix-coefficient (cf. (4.17)), and the S second derivative test for functions of class $\mathscr{C}^2$ (cf., for example, [8, Theorem 3.1, p. 32]). Finally, in the case where the weaker condition (4.22) holds, one makes use of (4.25) with u replaced by $u+\varepsilon v$, where $\varepsilon>0$ is arbitrary, the function $v:\Omega\to\mathbb{R}$ is given by (recall that $\xi^*\in S^{n-1}$ is as in (4.21))

$$v(x):=-e^{\lambda\,x\cdot\xi^*},\quad x\in\Omega, \tag{4.26}$$

and $\lambda \in (0, +\infty)$ is a fixed, sufficiently large constant. Concretely, since for every point $x \in \Omega$ we have

$$(Lv)(x) = \lambda^2 \left(\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^* \right) e^{\lambda\, x\cdot\xi^*} - \lambda \left(\sum_{i=1}^{n} b^i(x)\xi_i^* \right) e^{\lambda\, x\cdot\xi^*}$$

$$= \lambda\, e^{\lambda\, x\cdot\xi^*} \left(\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^* \right) \left(\lambda - \frac{\sum_{i=1}^{n} b^i(x)\xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^*} \right), \tag{4.27}$$

it follows (cf. also (4.18)) that

$$\lambda > \sup_{x\in\Omega} \left(\frac{\sum_{i=1}^{n} b^i(x)\xi_i^*}{\sum_{i,j=1}^{n} a^{ij}(x)\xi_i^*\xi_j^*} \right) \Longrightarrow Lv > 0 \text{ in } \Omega. \tag{4.28}$$

Hence $\min\{(u + \varepsilon v)(x) : x \in \overline{\Omega}\} = \min\{(u + \varepsilon v)(x) : x \in \partial\Omega\}$ for each $\varepsilon > 0$, so (4.24) follows by letting $\varepsilon \to 0^+$. □

Shortly, we also require the following simple algebraic lemma.

Lemma 4.3. *Let A be an $n \times n$ matrix, with real entries, which is semipositive definite, i.e., it satisfies $(A\xi)\cdot\xi \geqslant 0$ for every $\xi \in \mathbb{R}^n$. Then, with $\mathrm{Tr}\,(A)$ denoting the trace of A,*

$$\sup_{\xi\in S^{n-1}} [(A\xi)\cdot\xi] \leqslant \mathrm{Tr}\,(A). \tag{4.29}$$

Proof. Working with $\frac{1}{2}(A + A^\top)$ in place of A, there is no loss of generality in assuming that A is symmetric. Then there exists a unitary $n \times n$ matrix, U, and a diagonal $n \times n$ matrix, D, such that $A = U^{-1}DU$. If $\lambda_1, \dots, \lambda_n$ are the entries on the diagonal of D, then $\lambda_i \geqslant 0$ for each $i \in \{1, \dots, n\}$, and $\mathrm{Tr}\,(A) = \lambda_1 + \cdots + \lambda_n$. On the other hand, $\sup_{\xi\in S^{n-1}}[(A\xi)\cdot\xi] = \max\{\lambda_i : 1 \leqslant i \leqslant n\}$, so the desired conclusion follows. □

As a final preliminary matter to discussing the theorem below, we make a couple of more definitions. Concretely, call a real-valued function f defined on an interval $I \subseteq \mathbb{R}$ *quasi-decreasing* provided that there exists $C \in (0, +\infty)$ with the property that $f(t_1) \leqslant Cf(t_0)$ whenever $t_0, t_1 \in I$ are such that $t_0 \leqslant t_1$. Moreover, call f *quasi-increasing* if $-f$ is quasi-decreasing. Of course, the class of quasi-increasing (respectively, quasi-decreasing) functions contains the class of nondecreasing (respectively, nonincreasing) functions, but the inclusion is strict[30]. In fact, if φ is nondecreasing and $C \geqslant 1$, then any function f with the property that $\varphi \leqslant f \leqslant C\varphi$ is quasi-increasing. Conversely, given a quasi-increasing function f, defining $\varphi(t) := \inf_{s\geqslant t} f(s)$ yields a nondecreasing function for which $\varphi \leqslant f \leqslant C\varphi$ for some $C \geqslant 1$.

We are now prepared to state and prove the main result in this section.

[30]For example, if $\alpha > 0$, then $\omega(t) := (2 + \sin(t^{-1}))t^\alpha$, $t > 0$, is a quasi-increasing function which is not monotone in any interval of the form $(0, \varepsilon)$.

Theorem 4.4. *Suppose that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ and $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudoball condition at x_0. Specifically, assume that*

$$\mathscr{G}^{\omega}_{a,b}(x_0,h) = \{x \in B(x_0,R) :\, a|x-x_0|\,\omega(|x-x_0|) < h\cdot(x-x_0) < b\} \subseteq \Omega \tag{4.30}$$

for some parameters $a, b, R \in (0,+\infty)$, direction vector $h = (h_1,\dots,h_n) \in S^{n-1}$, and a shape function $\omega : [0,R] \to [0,+\infty)$ exhibiting the following features:

$$\omega \text{ is continuous on } [0,R],\ \omega(t) > 0 \text{ for } t \in (0,R], \quad \sup_{0<t\leqslant R}\Big(\frac{\omega(t/2)}{\omega(t)}\Big) < +\infty, \tag{4.31}$$

$$\text{and the mapping } (0,R] \ni t \mapsto \frac{\omega(t)}{t} \in (0,+\infty) \text{ is quasi-decreasing.} \tag{4.32}$$

Consider a nondivergence form, second order, differential operator (without a zero order term)

$$L := -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i, \quad a^{ij}, b^i : \Omega \longrightarrow \mathbb{R}, \quad 1 \leqslant i,j \leqslant n, \tag{4.33}$$

$$L \text{ semi-elliptic in } \Omega \text{ and nondegenerate along } h \in S^{n-1} \text{ in } \mathscr{G}^{\omega}_{a,b}(x_0,h). \tag{4.34}$$

In addition, suppose that there exists a real-valued function

$$\widetilde{\omega} \in \mathscr{C}^0\big([0,R]\big), \quad \widetilde{\omega}(t) > 0 \ \text{ for each } t \in (0,R] \quad \text{and} \quad \int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty, \tag{4.35}$$

with the property that

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < +\infty, \tag{4.36}$$

and

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\, \sum\limits_{i=1}^{n} b^i(x)h_i\Big\} + \Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\, -b^i(x)\big\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < +\infty. \tag{4.37}$$

Finally, suppose that $u : \Omega \cup \{x_0\} \to \mathbb{R}$ is a function satisfying

$$u \in \mathscr{C}^0(\Omega\cup\{x_0\}) \cap \mathscr{C}^2(\Omega), \tag{4.38}$$

$$(Lu)(x) \geqslant 0 \ \text{ for each } x \in \Omega, \tag{4.39}$$

$$u(x_0) < u(x) \ \text{ for each } x \in \Omega, \tag{4.40}$$

and fix a vector $\vec{\ell} \in S^{n-1}$ satisfying the transversality condition

$$\vec{\ell}\cdot h > 0. \tag{4.41}$$

Then $\vec{\ell}$ points inside Ω at x_0 and there exists a compact subset K of Ω which depends only on the geometrical characteristics of $\mathscr{G}^{\omega}_{a,b}(x_0,h)$, and a constant $\varkappa > 0$ which depends only on

$$\begin{aligned}&\textit{the quantities in (4.36)–(4.37)}, \quad (\inf_K u)-u(x_0), \quad \vec{\ell}\cdot h,\\ &\textit{and the pseudoball character of } \Omega \textit{ at } x_0,\end{aligned} \tag{4.42}$$

with the property that

$$(D^{(inf)}_{\vec{\ell}}u)(x_0) \geqslant \varkappa. \tag{4.43}$$

Proof. We debut with a few comments pertaining to the nature of the functions ω, $\widetilde{\omega}$, and also make a suitable (isometric) change of variables in order to facilitate the subsequent discussion. First, the fact that $\widetilde{\omega}$ is continuous on $[0,R]$, positive on $(0,R]$ and satisfies the Dini integrability condition forces $\widetilde{\omega}(0)=0$. Second, for further reference, let us fix a constant $\eta\in(0,+\infty)$ with the property that (cf. (4.32))

$$\frac{\omega(t_1)}{t_1}\leqslant \eta\frac{\omega(t_0)}{t_0} \quad \text{whenever } 0<t_0\leqslant t_1\leqslant R. \tag{4.44}$$

Third, from (4.36) and Lemma 4.3 it follows that there exists $C>0$ with the property that

$$\omega(t)\leqslant C\,\widetilde{\omega}(t) \quad \forall t\in[0,R]. \tag{4.45}$$

As a consequence of this and (4.35), we deduce that ω also satisfies the Dini integrability condition, i.e.,

$$\int_0^R \frac{\omega(t)}{t}\,dt<+\infty. \tag{4.46}$$

Moreover, it is also apparent from (4.31) and the Dini condition satisfied by ω that

$$\omega(0)=0. \tag{4.47}$$

Fourth, we claim that there exist $M\in(0,+\infty)$ and $\gamma\in(1,+\infty)$ such that

$$(\eta\gamma)^{-1}\xi^{\gamma-1}\omega(\xi)\leqslant\int_0^\xi \omega(t)t^{\gamma-2}\,dt\leqslant M\xi^{\gamma-1}\omega(\xi) \quad \forall\,\xi\in(0,R]. \tag{4.48}$$

To justify this claim, observe that if N stands for the supremum in the last condition in (4.31), then $N\in(0,+\infty)$ and

$$\omega(2^{-k}t)\leqslant N^k\omega(t) \quad \forall\,t\in(0,R],\ \forall\,k\in\mathbb{N}. \tag{4.49}$$

Next, fix a number $\gamma\in\mathbb{R}$ such that

$$\gamma>1+\max\{0\,,\,\log_2 N\} \tag{4.50}$$

and recall that the function $(0,R]\ni t\mapsto\omega(t)/t\in(0,+\infty)$ is quasi-increasing. Then, if $\eta\in(0,+\infty)$ is as in (4.44), using the fact that $\gamma>1$ as well as the estimates in (4.49)–(4.50), for every $\xi\in(0,R]$ we may write

$$\begin{aligned}\int_0^\xi \omega(t)t^{\gamma-2}\,dt &= \sum_{k=0}^{+\infty}\int_{2^{-k-1}\xi}^{2^{-k}\xi}\frac{\omega(t)}{t}t^{\gamma-1}\,dt\leqslant\sum_{k=0}^{+\infty}(2^{-k}\xi)^{\gamma-1}\int_{2^{-k-1}\xi}^{2^{-k}\xi}\frac{\omega(t)}{t}\,dt\\ &\leqslant \eta\sum_{k=0}^{+\infty}(2^{-k}\xi)^{\gamma-1}\frac{\omega(2^{-k-1}\xi)}{2^{-k-1}\xi}2^{-k-1}\xi=\eta\xi^{\gamma-1}\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}\omega(2^{-k-1}\xi)\\ &\leqslant \eta\xi^{\gamma-1}\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}N^{k+1}\omega(\xi)=N\eta\,\xi^{\gamma-1}\omega(\xi)\Big(\sum_{k=0}^{+\infty}2^{-k(\gamma-1)}2^{k\log_2 N}\Big)\end{aligned}$$

$$= \eta N\Big(\sum_{k=0}^{+\infty} 2^{-k(\gamma-1-\log_2 N)}\Big)\xi^{\gamma-1}\omega(\xi) = \frac{\eta N}{1-2^{-\gamma+1+\log_2 N}}\xi^{\gamma-1}\omega(\xi). \tag{4.51}$$

Thus, the upper-bound for the integral in (4.48) is proved with

$$M := \frac{\eta N}{1-2^{-\gamma+1+\log_2 N}} \in (0,+\infty). \tag{4.52}$$

Since the lower bound is a direct consequence of (4.44), this completes the proof of (4.48).

Continuing our series of preliminary matters, let U be an $n\times n$ unitary matrix (with real entries) with the property that $Uh = \mathbf{e}_n$ and define an isometry of $\mathbb{R}^n$ by setting $\mathcal{R}x := U(x-x_0)$ for every $x\in\mathbb{R}^n$. Introduce $\widetilde{\Omega} := \mathcal{R}(\Omega)$. Then, if

$$\big(\widetilde{a}^{ij}(y)\big)_{1\leqslant i,j\leqslant n} := U\big[\big(a^{ij}(\mathcal{R}^{-1}y)\big)_{1\leqslant i,j\leqslant n}\big]U^{-1} \quad \forall\, y\in\widetilde{\Omega}, \tag{4.53}$$

$$\big(\widetilde{b}^{i}(y)\big)_{1\leqslant i\leqslant n} := U\big[\big(b^{i}(\mathcal{R}^{-1}y)\big)_{1\leqslant i\leqslant n}\big] \quad \forall\, y\in\widetilde{\Omega} \tag{4.54}$$

and we consider the differential operator in $\widetilde{\Omega}$ given by

$$\widetilde{L} := -\sum_{i,j=1}^{n}\widetilde{a}^{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n}\widetilde{b}^{i}\partial_i, \tag{4.55}$$

then $\widetilde{L}$ satisfies properties analogous to L (relative to the new geometrical context) and

$$\widetilde{L}(u\circ\mathcal{R}^{-1}) = (Lu)\circ\mathcal{R}^{-1}. \tag{4.56}$$

Furthermore, $\mathcal{R}(\mathscr{G}^{\omega}_{a,b}(x_0,h)) = \mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)$ by (2.8). To summarize, given that both the hypotheses and the conclusion in the statement of the theorem transform covariantly under the change of variables $y = \mathcal{R}x$, there is no loss of generality in assuming that, to begin with, x_0 is the origin in $\mathbb{R}^n$ and that $h = \mathbf{e}_n \in S^{n-1}$. In this setting, the transversality condition (4.41) becomes

$$\vec{\ell}\cdot\mathbf{e}_n > 0, \tag{4.57}$$

while the semi-ellipticity condition on L and nondegeneracy condition on L along $h\in S^{n-1}$ read

$$\inf_{x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n)}\ \inf_{\xi\in S^{n-1}}\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geqslant 0 \ \text{ and } \ a^{nn}(x) > 0 \quad \forall\, x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n). \tag{4.58}$$

Going further, for each real number r we set

$$[r]_{\oplus} := \max\{r,0\} \quad \text{ and } \quad [r]_{\ominus} := \max\{-r,0\}. \tag{4.59}$$

Then, as far as how (4.36)–(4.37) transform under the indicated change of variables, we note that after possibly decreasing the value of R, matters may be arranged so that

$$\sum_{i=1}^{n} a^{ii}(x) \leqslant \Lambda_0\frac{|x|\widetilde{\omega}(x_n)}{x_n\omega(|x|)}a^{nn}(x) \quad \forall\, x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n), \tag{4.60}$$

$$\sum_{i=1}^{n}[b^i(x)]_{\ominus} \leqslant \Lambda_1\frac{\widetilde{\omega}(x_n)}{x_n\omega(|x|)}a^{nn}(x) \quad \forall\, x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n), \tag{4.61}$$

$$[b^n(x)]_{\oplus} \leqslant \Lambda_2\frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x) \quad \forall\, x\in\mathscr{G}^{\omega}_{a,b}(0,\mathbf{e}_n), \tag{4.62}$$

for some constants $\Lambda_0, \Lambda_1, \Lambda_2 \in (0, +\infty)$.

We are now ready to begin the proof in earnest. For starters, we note that by eventually increasing the value of $a > 0$ and decreasing the value of $b > 0$ we may assume that

$$\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)} \setminus \{0\} \subseteq \Omega \quad \forall\, b_* \in (0, b]. \tag{4.63}$$

To proceed, fix $b_* \in (0, b]$ and, with $\gamma \in (1, +\infty)$ as in (4.50) and for two finite constants $C_0, C_1 > 0$ to be specified later, consider the barrier function

$$v(x) := x_n + C_0 \int_0^{x_n}\int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\, dt\, d\xi - C_1 \int_0^{|x|}\int_0^{\xi} \frac{\omega(t)}{t}\Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\, d\xi, \tag{4.64}$$

for every $x = (x_1, \dots, x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}$. Since $\omega, \widetilde{\omega}$ are continuous and satisfy the Dini integrability condition, it follows that v is well defined and, in fact,

$$v \in \mathscr{C}^2\big(\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)\big) \cap \mathscr{C}^0\big(\overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)}\big). \tag{4.65}$$

Moreover, a direct computation gives that for each $x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we have

$$\partial_j v(x) = \delta_{jn} + C_0 \delta_{jn} \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \frac{x_j}{|x|} \int_0^{|x|} \frac{\omega(t)}{t}\Big(\frac{t}{|x|}\Big)^{\gamma-1} dt, \quad 1 \leqslant j \leqslant n, \tag{4.66}$$

and, further, for each $i, j \in \{1, \dots, n\}$

$$\partial_i\partial_j v(x) = C_0 \delta_{in}\delta_{jn} \frac{\widetilde{\omega}(x_n)}{x_n} - C_1 \left[\frac{\delta_{ij}}{|x|^{\gamma}} - \gamma \frac{x_i x_j}{|x|^{\gamma+2}}\right] \int_0^{|x|} \omega(t) t^{\gamma-2} dt - C_1 \frac{x_i x_j}{|x|^2} \frac{\omega(|x|)}{|x|}, \tag{4.67}$$

where δ_{ij} is the usual Kronecker symbol. Hence, by combining (4.33) with (4.66)–(4.67), we arrive at the conclusion that

$$(Lv)(x) = I + II + III \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n), \tag{4.68}$$

where, for each $x = (x_1, \dots, x_n) \in \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)$ we have set

$$I := I' + I'' \text{ with } I' := C_1 \Big(\sum_{i=1}^{n} a^{ii}(x)\Big) |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt \text{ and} \tag{4.69}$$

$$I'' := C_1 \Big(\sum_{i,j=1}^{n} a^{ij}(x) \frac{x_i}{|x|} \frac{x_j}{|x|}\Big)\Big(\frac{\omega(|x|)}{|x|} - \gamma |x|^{-\gamma} \int_0^{|x|} \omega(t) t^{\gamma-2}\, dt\Big), \tag{4.70}$$

$$II := -C_0\, a^{nn}(x) \frac{\widetilde{\omega}(x_n)}{x_n}, \tag{4.71}$$

$$III := b^n(x) + C_0 b^n(x) \int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\, dt - C_1 \Big(\sum_{i=1}^{n} b^i(x) \frac{x_i}{|x|}\Big) \int_0^{|x|} \frac{\omega(t)}{t}\Big(\frac{t}{|x|}\Big)^{\gamma-1} dt. \tag{4.72}$$

As a preamble to estimating I, II, III above, we make a couple of preliminary observations. First note that since $C_1 \geqslant 0$, ω is nonnegative, and L is semi-elliptic, we have

$$I'' \leqslant C_1\Big(\sum_{i,j=1}^{n} a^{ij}(x)\frac{x_i}{|x|}\frac{x_j}{|x|}\Big)\frac{\omega(|x|)}{|x|} \leqslant C_1\Big(\sum_{i=1}^{n} a^{ii}(x)\Big)\frac{\omega(|x|)}{|x|} \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n), \tag{4.73}$$

where the last inequality above is based on Lemma 4.3. Second, for every point $x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$, estimate (4.48) used with $\xi := |x| \in (0,R)$ gives that

$$|x|^{-\gamma}\int_0^{|x|} \omega(t)t^{\gamma-2}\,dt \leqslant M\frac{\omega(|x|)}{|x|}, \tag{4.74}$$

where the constant $M \in (0,+\infty)$ is as in (4.52). Consequently,

$$I' \leqslant MC_1\Big(\sum_{i=1}^{n} a^{ii}(x)\Big)\frac{\omega(|x|)}{|x|} \quad \forall\, x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{4.75}$$

In concert with the above observations, formulas (4.69)–(4.72) then allow us to conclude that (recall the notation introduced in (4.59)) for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$

$$I \leqslant C_1(1+M)\Big(\sum_{i=1}^{n} a^{ii}(x)\Big)\frac{\omega(|x|)}{|x|}, \quad II \leqslant -C_0 a^{nn}(x)\frac{\widetilde{\omega}(x_n)}{x_n}, \tag{4.76}$$

$$III \leqslant [b^n(x)]_{\oplus}\left(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\right) + C_1 M\Big(\sum_{i=1}^{n}[b^i(x)]_{\ominus}\Big)\omega(|x|), \tag{4.77}$$

where we have also used (4.74) when deriving the last estimate above. Thus, on account of (4.68), (4.75), (4.76), and (4.44), for every $x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$ we may estimate

$$\begin{aligned}(Lv)(x) \leqslant{}& \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\left\{C_1(1+M)\left(\frac{\sum_{i=1}^{n} a^{ii}(x)}{a^{nn}(x)}\right)\frac{x_n\omega(|x|)}{|x|\widetilde{\omega}(x_n)} - C_0\right\} \\ &+ \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\left\{\frac{x_n[b^n(x)]_{\oplus}}{\widetilde{\omega}(x_n)a^{nn}(x)}\left(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\right) + C_1 M\frac{x_n\omega(|x|)\Big(\sum_{i=1}^{n}[b^i(x)]_{\ominus}\Big)}{\widetilde{\omega}(x_n)a^{nn}(x)}\right\}.\end{aligned} \tag{4.78}$$

In turn, (4.78) and (4.60)–(4.62) permit us to further estimate, for each $x \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$,

$$\begin{aligned}(Lv)(x) &\leqslant \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\Big\{C_1(1+M)\Lambda_0 - C_0\Big\} \\ &\quad + \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\left\{\Lambda_2\left(1 + C_0\int_0^{x_n}\frac{\widetilde{\omega}(t)}{t}\,dt\right) + C_1 M\Lambda_1\right\} \\ &\leqslant \frac{\widetilde{\omega}(x_n)}{x_n}a^{nn}(x)\left\{C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) + \Lambda_2 - C_0\left(1 - \Lambda_2\int_0^{b_*}\frac{\widetilde{\omega}(t)}{t}\,dt\right)\right\}.\end{aligned} \tag{4.79}$$

We return to (4.79) momentarily. For the time being, we wish to estimate the barrier function on the round portion of the boundary of the pseudoball. To this end, let us note from (2.2) that if $x = (x_1, \dots, x_n) \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x \in \mathbb{R}^n : x_n = b_*\}$, then, given that ω is continuous, we have $x_n = a\omega(|x|)|x|$ which further implies

$$x_n + C_0 \int_0^{x_n}\!\!\int_0^{\xi} \frac{\widetilde{\omega}(t)}{t}\,dt\,d\xi \leqslant x_n\left(1 + C_0\int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\,dt\right) = a\omega(|x|)|x|\Big(1 + C_0\int_0^{x_n} \frac{\widetilde{\omega}(t)}{t}\,dt\Big). \tag{4.80}$$

Moreover, since $\omega(t)/t \geqslant \eta^{-1}\omega(|x|)/|x|$ for every $t \in (0, |x|)$ (cf. (4.44)), we may also write

$$\int_0^{|x|}\!\!\int_0^{\xi} \frac{\omega(t)}{t}\Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\,d\xi \geqslant \eta^{-1}\frac{\omega(|x|)}{|x|}\int_0^{|x|}\!\!\int_0^{\xi}\Big(\frac{t}{\xi}\Big)^{\gamma-1} dt\,d\xi = \frac{|x|\omega(|x|)}{2\eta\gamma}. \tag{4.81}$$

Together, (4.64) and (4.80)–(4.81) give that for each $x \in \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = b_*\}$ we have

$$v(x) \leqslant \left(a - \frac{C_1}{2\eta\gamma} + aC_0\int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt\right)|x|\omega(|x|). \tag{4.82}$$

At this stage, we are ready to specify the constants $C_0, C_1 \in (0, +\infty)$ appearing in (4.64), in a manner consistent with the format of (4.79), (4.82) and which suits the goals we have in mind. Turning to details, we start by fixing

$$C_1 > 2a\eta\gamma \quad\text{and}\quad C_0 > 2\big[C_1(\Lambda_0 + M\Lambda_0 + M\Lambda_1) + \Lambda_2\big], \tag{4.83}$$

then, using the Dini integrability condition satisfied by $\widetilde{\omega}$, select $b_* \in (0, b]$ sufficiently small so that

$$\int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt < \frac{1}{2\Lambda_2} \quad\text{and}\quad \int_0^{b_*} \frac{\widetilde{\omega}(t)}{t}\,dt < \frac{C_1 - 2a\eta\gamma}{2a\eta\gamma C_0}. \tag{4.84}$$

Then (4.79) together with the second condition in (4.83) and the first condition in (4.84) ensure that

$$Lv \leqslant 0 \quad\text{in}\quad \mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n). \tag{4.85}$$

Furthermore, the second condition in (4.84) is designed (cf. (4.82)) so that we also have

$$v \leqslant 0 \quad\text{on}\quad \partial\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n) \setminus \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = b_*\}. \tag{4.86}$$

Having specified the constants C_0 and C_1 (in the fashion described above) finishes the process of defining the barrier function v, initiated in (4.64). With this task concluded, we proceed by considering the compact subset of Ω given by

$$K := \{x = (x_1, \dots, x_n) \in \overline{\mathscr{G}^{\omega}_{a,b_*}(0, \mathbf{e}_n)} : x_n = b_*\}, \tag{4.87}$$

and note that (4.40) (and since u is continuous, hence attains its infimum on compact subsets of Ω) entails

$$u(x_0) < \inf_K u. \tag{4.88}$$

Thanks to (4.40), (4.86) and (4.88), we may then choose $\varepsilon > 0$ for which

$$\varepsilon\big(\sup_K |v|\big) < \big(\inf_K u\big) - u(x_0) \tag{4.89}$$

(hence ε depends only on the quantities listed in (4.42)), so that, on the one hand,

$$0 \leqslant u(x) - u(x_0) - \varepsilon v(x) \text{ for every } x \in \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{4.90}$$

On the other hand, from (4.85) and (4.39) we obtain (recall that L annihilates constants)

$$L(u - u(x_0) - \varepsilon v) \geqslant 0 \text{ in } \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n). \tag{4.91}$$

With the estimates (4.90)–(4.91) in hand, and keeping in mind (4.65) plus the fact that the function u belongs to $\mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}) \cap \mathscr{C}^2(\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n))$, bring in the weak minimum principle presented in Proposition 4.2. This is used in the open, bounded, nonempty subset $\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$ of $\mathbb{R}^n$ and with the vector $\mathbf{e}_n$ playing the role of $\xi^* \in S^{n-1}$ from (4.21). Indeed, granted (4.58), it follows that L is nondegenerate along $\mathbf{e}_n \in S^{n-1}$ in $\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$ and, thanks to (4.62), the analogue of the condition (4.21) is valid in the current setting. The bottom line is that Proposition 4.2 applies, and gives

$$u - u(x_0) - \varepsilon v \geqslant 0 \text{ in } \overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}. \tag{4.92}$$

Given that both $u - u(x_0)$ and v vanish at the point $x_0 = 0 \in \partial\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)$, this shows that

$$u - u(x_0) - \varepsilon v \in \mathscr{C}^0(\overline{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)}) \text{ has a global minimum at } x_0 = 0. \tag{4.93}$$

On the other hand, the condition (4.57) and the fact that ω continuously vanishes at the origin (cf. (4.47)) imply the existence of some $t_* \in (0, b_*)$ with the property that $\omega(t) < \vec{\ell}\cdot\mathbf{e}_n/a$ for every $t \in (0, t_*)$. In turn, such a choice of t_* ensures that (cf. (2.2))

$$t\,\vec{\ell} \in \mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n) \text{ for every } t \in (0, t_*). \tag{4.94}$$

In particular, $\vec{\ell}$ points in Ω at x_0 (cf. Definition 4.1), and from (4.14), (4.93)–(4.94) we obtain

$$D^{(\inf)}_{\vec{\ell}}(u - u(x_0) - \varepsilon v)(x_0) \geqslant 0. \tag{4.95}$$

Now (4.66) gives

$$\nabla v(x_0) := \lim_{\mathscr{G}^{\omega}_{a,b_*}(0,\mathbf{e}_n)\ni x\to 0} (\nabla v)(x) = \mathbf{e}_n. \tag{4.96}$$

Hence

$$(D^{(\inf)}_{\vec{\ell}} v)(x_0) = (D^{(\sup)}_{\vec{\ell}} v)(x_0) = \vec{\ell}\cdot\nabla v(x_0) = \vec{\ell}\cdot\mathbf{e}_n \tag{4.97}$$

by (4.65) and the discussion in (4.15). In turn, (4.95)–(4.96) and (4.97) further allow us to conclude that

$$(D^{(\inf)}_{\vec{\ell}} u)(x_0) \geqslant \varepsilon\vec{\ell}\cdot\nabla v(x_0) = \varepsilon\,\vec{\ell}\cdot\mathbf{e}_n > 0, \tag{4.98}$$

where the last inequality is a consequence of (4.57). Choosing $\varkappa := \varepsilon\,\vec{\ell}\cdot\mathbf{e}_n > 0$ then yields (4.43), finishing the proof of the theorem. □

We continue with a series of comments relative to Theorem 4.4 and its proof.

Remark 4.5. (i) As we discuss in detail later, Theorem 4.4 is sharp. A slightly more versatile result is obtained by replacing Ω by $U\cap\Omega$ in (4.38)–(4.40), where $U\subseteq\mathbb{R}^n$ is some open neighborhood of $x_0\in\partial\Omega$. Of course, Theorem 4.4 itself implies such an improvement simply by invoking it with Ω substituted by $U\cap\Omega$ throughout.

(ii) Trivially, the last condition in (4.31) is satisfied if the function $\omega:[0,R]\to[0,+\infty)$ has the property that

$$\text{there exists } m\in\mathbb{R} \text{ such that } (0,R]\ni t\mapsto t^m\omega(t)\in(0,+\infty) \text{ is quasi-increasing,} \tag{4.99}$$

hence, in particular, if ω itself is quasi-increasing. Corresponding to the class of function introduced in (1.12), the shape function $\omega_{\alpha,\beta}$ satisfies all properties displayed in (4.31)–(4.32) for all $\alpha\in(0,1]$ and $\beta\in\mathbb{R}$. However, $\omega_{0,\beta}$ fails to satisfy the Dini integrability condition for $\beta\geqslant -1$ (while still meeting the other conditions).

(iii) It is easy to check that if $\omega:(0,R]\to(0,+\infty)$ is such that the map $(0,R]\ni t\mapsto \omega(t)/t\in(0,+\infty)$ is quasi-decreasing and such that $\sup\limits_{0<t\leqslant R}\Big(\frac{\omega(t/2)}{\omega(t)}\Big)<+\infty$, then for every $c\in(1,+\infty)$ we also have $\sup\limits_{0<t\leqslant R}\Big(\frac{\omega(t/c)}{\omega(t)}\Big)<+\infty$. Based on this observation, one may then verify without difficulty that if $\omega:[0,R]\to[0,+\infty)$ satisfies the conditions in (4.31)–(4.32), then, for each fixed $\theta\in(0,1)$, so does the function $[0,R]\ni t\mapsto\omega(t^\theta)\in[0,+\infty)$. Furthermore, this function satisfies the Dini integrability condition if ω does.

(iv) The amplitude parameter $a>0$ used in defining the pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ plays only a minor role since this may, in principle, be absorbed as a multiplicative factor into the shape function ω (thus, reducing matters to the case where $a=1$). Nonetheless, working with a generic amplitude adds a desirable degree of flexibility in the proof of Theorem 4.4.

(v) It is instructive to note that if $\omega:[0,R]\to[0,+\infty)$ satisfies (4.32) as well as the first two properties listed in (4.31), and is such that (4.48) holds for some $M\in(0,+\infty)$ and $\gamma\in(0,+\infty)$, then actually the last condition in (4.31) is also valid. Indeed, using (4.44), for each $\xi\in(0,R]$ we may estimate

$$\begin{aligned} M\xi^{\gamma-1}\omega(\xi) &\geqslant \int_0^{\xi}\frac{\omega(t)}{t}t^{\gamma-1}\,dt \geqslant \int_0^{\xi/2}\frac{\omega(t)}{t}t^{\gamma-1}\,dt \geqslant \eta^{-1}\frac{\omega(\xi/2)}{\xi/2}\int_0^{\xi/2}t^{\gamma-1}\,dt \\ &= \frac{1}{\gamma\eta}\frac{\omega(\xi/2)}{\xi/2}\Big(\frac{\xi}{2}\Big)^{\gamma} = \frac{1}{\gamma 2^{\gamma-1}\eta}\xi^{\gamma-1}\omega(\xi/2), \end{aligned} \tag{4.100}$$

which entails

$$\sup_{0<\xi\leqslant R}\Big(\frac{\omega(\xi/2)}{\omega(\xi)}\Big)\leqslant \gamma 2^{\gamma-1}M\eta<+\infty. \tag{4.101}$$

We next prove a technical result (refining earlier work in [64]), which is going to be useful in the proof of Theorem 4.7 below.

Proposition 4.6. *Let $R\in(0,+\infty)$ and assume that $\omega:[0,R]\to[0,+\infty)$ is a continuous function with the property that $\omega(t)>0$ for each $t\in(0,R]$. In addition, assume that ω satisfies* *a Dini condition and is quasi-increasing, i.e.,*

$$\int_0^R \frac{\omega(t)}{t}\,dt < +\infty \text{ and } \omega(t_1) \leqslant \eta\,\omega(t_2) \text{ whenever } t_1, t_2 \in [0,R] \text{ are such that } t_1 \leqslant t_2 \tag{4.102}$$

for some fixed constant $\eta \in (0,+\infty)$*. Consider*

$$M := \max\big\{\omega(t) : t \in [0,R]\big\}, \quad t_o := \min\big\{t \in [0,R] : \omega(t) = M\big\} \tag{4.103}$$

and denote by $\theta_* \in (0,1)$ *the unique solution of the equation* $\theta = (\ln\theta)^2$ *in the interval* $(0,+\infty)$*. Then* $t_o > 0$ *and there exists a function* $\widehat{\omega} : [0,t_o] \to [0,+\infty)$ *satisfying the following properties:*

$$\begin{array}{l} \widehat{\omega} \text{ is continuous, concave, and strictly increasing on } [0,t_o],\ \widehat{\omega}(t) \geqslant \omega(t) \text{ for each} \\ t \in [0,t_o],\ \widehat{\omega}(0) = 0,\ \widehat{\omega}(t_o) = M, \text{ the mapping } (0,t_o) \ni t \mapsto \dfrac{\widehat{\omega}(t)}{t} \in [0,+\infty) \text{ is} \\ \text{nonincreasing, and } \displaystyle\int_0^{t_o} \frac{\widehat{\omega}(t)}{t}\,dt \leqslant \eta M + \Big(1 + \eta + \frac{\eta(\theta_* + |\ln\theta_*|)}{\theta_* |\ln\theta_*|}\Big) \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \end{array} \tag{4.104}$$

Proof. We start by noting that since ω is continuous at 0 and satisfies a Dini integrability condition, then necessarily ω vanishes at the origin. In turn, this forces $t_o \in (0,R]$ and $M \in (0,+\infty)$. Given that ω is continuous, we also have that $\omega(t_o) = M$. Next, extend the restriction of ω to the interval $[0,t_o]$ to a function $\overline{\omega} : [0,+\infty) \to [0,+\infty)$ by setting $\overline{\omega}(t) := M$ for every $t \geqslant t_o$, and take $\widetilde{\omega} : [0,+\infty) \to [0,+\infty)$ to be the concave envelope of $\overline{\omega}$, i.e.,

$$\widetilde{\omega}(t) := \sup\Bigg\{\sum_{j=1}^N \lambda_j \overline{\omega}(t_j) : N \in \mathbb{N}, (\lambda_j)_j \in [0,1]^N, \sum_{j=1}^N \lambda_j = 1, (t_j)_j \in [0,+\infty)^N, \sum_{j=1}^N \lambda_j t_j = t\Bigg\} \tag{4.105}$$

for each $t \in [0,+\infty)$. Then (cf., for example, the discussion in [65, pp. 35-57]), $\widetilde{\omega}$ is the smallest concave function which is pointwise $\geqslant \overline{\omega}$, i.e.,

$$\widetilde{\omega} = \inf \psi, \tag{4.106}$$

where the infimum is taken over all ψ such that $\psi \geqslant \overline{\omega}$ on $[0,+\infty)$ and ψ is concave on $\overline{\mathbb{R}}_+$. In particular, $\widetilde{\omega}$ is concave on $[0,+\infty)$, hence continuous on $(0,+\infty)$. Also (as seen from (4.105)),

$$\widetilde{\omega}(0) = \omega(0) = 0 \quad \text{and} \quad \widetilde{\omega}(t) = M \text{ for every } t \geqslant t_o. \tag{4.107}$$

Moreover, since $\widetilde{\omega}, \omega$ are continuous on $(0,R)$, formula (4.106) also entails that

$$\forall\, t \in (0,t_o) \text{ with } \widetilde{\omega}(t) > \omega(t) \Longrightarrow \begin{cases} \exists\, J \text{ open subinterval of } (0,R) \text{ so that } t \in J \\ \text{and such that } \widetilde{\omega} \text{ is an affine function on } J. \end{cases} \tag{4.108}$$

To proceed, from the fact that $\widetilde{\omega}$ and $\overline{\omega}$ are continuous on $(0,+\infty)$ and (4.107) we deduce that

$$W := \big\{t \in (0,+\infty) : \widetilde{\omega}(t) > \overline{\omega}(t)\big\} \text{ is an open subset of } (0,t_o). \tag{4.109}$$

If W is empty, it follows that $\widetilde{\omega}(t) = \overline{\omega}(t)$ for every $t \in (0, +\infty)$, hence ω itself is concave on $(0, t_o)$ As such, we simply take $\widehat{\omega} := \omega\big|_{[0,t_o]}$ and the desired conclusion follows. It remains to study the case where the set W from (4.109) is nonempty. In this scenario, W may be written as the union of an at most countable family of mutually disjoint open intervals (which are precisely the connected components of W), say

$$W = \bigcup_{i \in I} J_i, \quad \text{where } J_i := (\alpha_i, \beta_i), \quad 0 \leqslant \alpha_i < \beta_i \leqslant t_o \text{ for each } i \in I. \tag{4.110}$$

Let us also observe that since both $\widetilde{\omega}$ and ω are continuous on $(0, R)$, from (4.107) and (4.109) we may conclude that $\widetilde{\omega}(t) = \omega(t)$ for each $t \in \partial W$. Given the nature of the decomposition of W in (4.109), this ensures that

$$\begin{aligned} &\widetilde{\omega}(t) > \omega(t) \text{ whenever } i \in I \text{ and } t \in (\alpha_i, \beta_i), \\ &\widetilde{\omega}(\alpha_i) = \omega(\alpha_i) \quad \text{and} \quad \widetilde{\omega}(\beta_i) = \omega(\beta_i) \quad \text{for each } i \in I. \end{aligned} \tag{4.111}$$

Moreover, based on this and (4.108) we arrive at the conclusion that

$$\widetilde{\omega}(t) = \frac{t - \alpha_i}{\beta_i - \alpha_i}\big(\omega(\beta_i) - \omega(\alpha_i)\big) + \omega(\alpha_i) \text{ if } i \in I \text{ and } t \in [\alpha_i, \beta_i]. \tag{4.112}$$

For further use, let us point out that (4.112) readily entails

$$\widetilde{\omega}(t) \leqslant \omega(\alpha_i) + \frac{\omega(\beta_i)}{\beta_i} t \quad \text{if } i \in I \text{ and } t \in [\alpha_i, \beta_i] \tag{4.113}$$

since both functions involved are affine on the interval (α_i, β_i) and the inequality is trivially verified at endpoints. Going further, fix $\theta \in (0, 1)$ and partition the (at most countable) set of indices I (from (4.110)) into the following two subclasses:

$$I_1 := \big\{i \in I : \alpha_i > \theta\beta_i\big\}, \quad I_2 := \big\{i \in I : \alpha_i \leqslant \theta\beta_i\big\}. \tag{4.114}$$

Now, the fact that $\widetilde{\omega}$ is concave entails $\widetilde{\omega}\big(\lambda t_1 + (1-\lambda)t_2\big) \geqslant \lambda\widetilde{\omega}(t_1) + (1-\lambda)\widetilde{\omega}(t_2)$ for all $\lambda \in [0, 1]$ and $t_1, t_2 \in [0, +\infty)$. Pick now two numbers $t'' \geqslant t' > 0$ and specialize the earlier inequality to the case where $\lambda := t'/t''$, $t_1 := t''$ and $t_2 := 0$ (recall that $\widetilde{\omega}$ vanishes at the origin). This yields $\widetilde{\omega}(t') \geqslant (t'/t'')\widetilde{\omega}(t'')$, from which we may ultimately conclude that

$$\text{the mapping } \ (0, +\infty) \ni t \mapsto \frac{\widetilde{\omega}(t)}{t} \in [0, +\infty) \text{ is nonincreasing.} \tag{4.115}$$

For each fixed $i \in I_1$ we necessarily have $\alpha_i > 0$. Keeping this in mind, we may then estimate

$$\int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t} dt \leqslant \frac{\widetilde{\omega}(\alpha_i)}{\alpha_i}(\beta_i - \alpha_i) = \frac{\omega(\alpha_i)}{\alpha_i}(\beta_i - \alpha_i) \leqslant \eta(\beta_i - \alpha_i)\frac{\beta_i}{\alpha_i}\Big(\inf_{t \in (\alpha_i, \beta_i)} \frac{\omega(t)}{t}\Big) \leqslant \eta\theta^{-1}\int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t} dt, \tag{4.116}$$

thanks to (4.115), (4.102), and (4.114). On the other hand, when $i \in I_2$ we may write

$$\int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t} dt = \int_{\alpha_i}^{\beta_i} (\widetilde{\omega}(t) - \omega(\alpha_i)) \frac{dt}{t} + \int_{\alpha_i}^{\beta_i} \omega(\alpha_i) \frac{dt}{t} \leqslant \int_{\alpha_i}^{\beta_i} \frac{\omega(\beta_i)}{\beta_i} dt + \eta \int_{\alpha_i}^{\beta_i} \omega(t) \frac{dt}{t}$$

$$\leqslant \omega(\beta_i) + \eta \int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt \leqslant \frac{\eta}{|\ln\theta|} \int_{\beta_i}^{\beta_i/\theta} \frac{\overline{\omega}(t)}{t}\,dt + \eta \int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt \tag{4.117}$$

by (4.113), (4.102), and the definition of $\overline{\omega}$. At this stage, we proceed to estimate

$$\begin{aligned} \int_0^{t_o} \frac{\widetilde{\omega}(t)}{t}\,dt &= \int_W \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\widetilde{\omega}(t)}{t}\,dt = \sum_{i\in I} \int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_{(0,t_o)\setminus W} \frac{\omega(t)}{t}\,dt \\ &\leqslant \sum_{i\in I_1} \int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \sum_{i\in I_2} \int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt + \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \end{aligned} \tag{4.118}$$

Note that (4.116) gives

$$\sum_{i\in I_1} \int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt \leqslant \eta\theta^{-1} \sum_{i\in I_1} \int_{J_i} \frac{\omega(t)}{t}\,dt \leqslant \eta\theta^{-1} \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \tag{4.119}$$

We continue by observing that

$$\forall\, i,j \in I_2 \text{ with } \ i \neq j \implies (\beta_i, \beta_i/\theta) \cap (\beta_j, \beta_j/\theta) = \varnothing. \tag{4.120}$$

To justify this, fix two different indices $i,j \in I_2$ and, without loss of generality, assume that $\beta_i < \beta_j$. Since (α_i, β_i) and (α_j, β_j) are disjoint connected components of W, it follows that $\beta_i \notin (\alpha_j, \beta_j)$. Hence $\beta_i < \alpha_j \leqslant \theta\beta_j$ given that $j \in I_2$, which shows that $\beta_i/\theta < \beta_j$. With this in hand, (4.120) readily follows. Having established (4.120), we next invoke (4.117) in order to estimate

$$\begin{aligned} \sum_{i\in I_2} \int_{J_i} \frac{\widetilde{\omega}(t)}{t}\,dt &= \sum_{i\in I_2} \int_{\alpha_i}^{\beta_i} \frac{\widetilde{\omega}(t)}{t}\,dt \leqslant \frac{\eta}{|\ln\theta|} \sum_{i\in I_2} \int_{\beta_i}^{\beta_i/\theta} \frac{\overline{\omega}(t)}{t}\,dt + \eta \sum_{i\in I_2} \int_{\alpha_i}^{\beta_i} \frac{\omega(t)}{t}\,dt \\ &\leqslant \frac{\eta}{|\ln\theta|} \int_0^{t_o/\theta} \frac{\overline{\omega}(t)}{t}\,dt + \eta \int_0^{t_o} \frac{\omega(t)}{t}\,dt. \end{aligned} \tag{4.121}$$

In concert, (4.118), (4.119), and (4.121) yield

$$\begin{aligned} \int_0^{t_o} \frac{\widetilde{\omega}(t)}{t}\,dt &\leqslant \big(1 + \eta + \eta\theta^{-1}\big) \int_0^{t_o} \frac{\omega(t)}{t}\,dt + \frac{\eta}{|\ln\theta|} \int_0^{t_o/\theta} \frac{\overline{\omega}(t)}{t}\,dt \\ &= \Big(1 + \eta + \eta\theta^{-1} + \frac{\eta}{|\ln\theta|}\Big) \int_0^{t_o} \frac{\omega(t)}{t}\,dt + \eta M. \end{aligned} \tag{4.122}$$

Finally, minimizing the right-most hand side of (4.122) over all $\theta \in (0,1)$ gives

$$\int_0^{t_o} \frac{\widetilde{\omega}(t)}{t}\,dt \leqslant \Big(1 + \eta + \eta\,\theta_*^{-1} + \frac{\eta}{|\ln\theta_*|}\Big) \int_0^{t_o} \frac{\omega(t)}{t}\,dt + \eta M. \tag{4.123}$$

At this point, much of the ground work ensuring that $\widehat{\omega} := \widetilde{\omega}\big|_{[0,t_o]}$ satisfies the properties listed in (4.104) has been done. Two items which are yet to be settled are as follows. First, formula (4.105) shows that $\widehat{\omega}(t) < M$ for $t \in (0, t_o)$. Hence, if $0 \leqslant t_1 < t_2 \leqslant t_o$ and $\lambda := (t_o - t_2)/(t_o - t_1) \in [0,1)$, then, given that $\widehat{\omega}$ is concave, we obtain $\widehat{\omega}(t_2) \geqslant \lambda\widehat{\omega}(t_1) + (1-\lambda)M > \omega(t_1)$. Consequently, $\widehat{\omega}$ is strictly increasing on $[0, t_o]$. Second, the continuity of $\widehat{\omega}$ at 0 is a consequence of the fact that this function is continuous and increasing on $(0, t_o)$ and satisfies a Dini condition. This concludes the proof of the proposition. □

We are now prepared to present a consequence of Theorem 4.4 in which we impose a more streamlined set of conditions on the shape function (cf. (4.124) with (4.31)–(4.32)). In turn, Theorem 4.7 below readily implies Theorem 1.4.

Theorem 4.7. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Assume that $x_0 \in \partial\Omega$ is a point with the property that Ω satisfies an interior pseudoball condition at x_0. Concretely, assume that* (4.30) *holds for some parameters $a, b, R \in (0, +\infty)$, direction vector $h = (h_1, \dots, h_n) \in S^{n-1}$, and a shape function $\omega : [0, R] \to [0, +\infty)$ with the property that*

$$\omega \text{ is continuous, positive, and quasi-increasing on } (0,R] \text{ and } \int_0^R \frac{\omega(t)}{t}\,dt < +\infty. \tag{4.124}$$

Also, consider a nondivergence form, second order, differential operator L which is semi-elliptic and nondegenerate along h (as in (4.33)*–*(4.34)*) and whose coefficients satisfy*

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\sum\limits_{i=1}^{n} a^{ii}(x)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{4.125}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|x-x_0|\Big(\sum\limits_{i=1}^{n} \max\big\{0\,,\,-b^i(x)\big\}\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j} < +\infty, \tag{4.126}$$

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{\max\Big\{0\,,\,\sum\limits_{i=1}^{n} b^i(x)h_i\Big\}}{\frac{\omega(|x-x_0|)}{|x-x_0|}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_i h_j\Big)} < +\infty. \tag{4.127}$$

Finally, fix a vector $\vec{\ell} \in S^{n-1}$ for which $\vec{\ell}\cdot h > 0$, and suppose that $u \in \mathscr{C}^0(\Omega \cup \{x_0\}) \cap \mathscr{C}^2(\Omega)$ is a function satisfying

$$(Lu)(x) \geqslant 0 \quad \text{and} \quad u(x_0) < u(x) \quad \text{for each } x \in \Omega. \tag{4.128}$$

Then $\vec{\ell}$ points inside Ω at x_0, and there exists a constant $\varkappa > 0$ (which depends only on the quantities in (4.42)*) with the property that*

$$(D^{(inf)}_{\vec{\ell}}u)(x_0) \geqslant \varkappa. \tag{4.129}$$

Proof. It may be readily verified that ω continuously vanishes at the origin given that ω satisfies a Dini integrability condition, is continuous and nonnegative on $[0,R]$, as well as quasi-increasing on $(0,R]$. Now, if $\widehat{\omega}$ is associated with the original shape function ω as in Proposition 4.6, properties (4.104) hold. In particular, $\widehat{\omega} \geqslant \omega$ near the origin and hence $\mathscr{G}^{\widehat{\omega}}_{a,b}(x_0,h) \subseteq \mathscr{G}^{\omega}_{a,b}(x_0,h) \subseteq \Omega$. Also, (4.125)–(4.127) imply the versions of (4.36)–(4.37) written with both ω and $\widetilde{\omega}$ replaced by $\widehat{\omega}$. Then Theorem 4.4 applies, with both ω and $\widetilde{\omega}$ in the original statement replaced by $\widehat{\omega}$. From this, the desired conclusion follows. □

4.3 Sharpness of the boundary point principle formulated in Theorem 4.4

As mentioned previously, Theorem 4.4 is sharp, and here the goal is to make this precise through a series of counterexamples presented as remarks.

Remark 4.8. The strict inequality in (4.40) is obviously necessary since otherwise any constant function would serve as a counterexample.

Remark 4.9. In the context of Theorem 4.4, the nondegeneracy of L along the direction vector h of the pseudoball $\mathscr{G}^{\omega}_{a,b}(x_0,h)$ is a necessary condition. A simple counterexample is obtained by taking $n \geqslant 2$, $\Omega := \mathbb{R}^n_+$, $x_0 := (0,\dots,0) \in \mathbb{R}^n$, $\vec{\ell} := \mathbf{e}_n$, $L := -\partial^2/\partial x_1^2$, and $u(x_1,\dots,x_n) := x_n^2$.

Remark 4.10. The discussion in Section 1 pertaining to (1.28)–(1.33) shows that both conditions (4.36) and (4.37) in Theorem 4.4 are necessary.

Remark 4.11. Fix $\alpha \in (1,2)$ and in the two dimensional setting consider

$$\begin{aligned} &\Omega := \{(x,y) \in \mathbb{R}^2 : y > (x^2)^{1/\alpha}\}, \\ &L := -\partial_x^2 - \frac{2}{\alpha(\alpha+1)} y^{2-\alpha}\partial_y^2 \quad \text{in } \Omega, \\ &u(x,y) := y^{1+\alpha} - x^2 y \quad \forall\, (x,y) \in \Omega. \end{aligned} \tag{4.130}$$

Then $u \in \mathscr{C}^2(\overline{\Omega})$ satisfies $u(0) = 0$, $u > 0$ in Ω, $Lu = 0$ in Ω, and $(\nabla u)(0) = 0$. Thus, (4.43) fails in this case, even though Ω satisfies a pseudoball condition at the origin, with shape function $\omega(t) := t^{(2/\alpha)-1}$ satisfying (4.31)–(4.32), and L is (nonuniformly) elliptic in Ω and homogeneous (i.e., L has no lower order terms). Here, the breakdown is caused by the failure of the condition (4.36) for a function $\widetilde{\omega}$ as in (4.37). Indeed, since $x^2 + y^2 \leqslant cy^\alpha$ in Ω, (4.36) would imply $\widetilde{\omega}(y)/y \geqslant c/y$ for all $y > 0$ small, in violation of the Dini integrability condition for $\widetilde{\omega}$. Moreover, varying the parameter $\alpha \in (1,2)$, this counterexample shows that for any fixed $\varepsilon > 0$ the condition (4.36) may not be relaxed to

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h) \ni x \to x_0} \frac{|x-x_0|^{\varepsilon}\Big(\sum\limits_{i=1}^{n} a^{ii}(x)\Big)}{\sum\limits_{i,j=1}^{n} a^{ij}(x) h_i h_j} < +\infty. \tag{4.131}$$

Remark 4.12. Here the goal is to show that the conclusion (4.43) of Theorem 4.4 may be violated if the condition (4.37) fails to be satisfied for some $\widetilde{\omega}$ as in (4.35) (even though (4.30)–(4.36) do hold for some $\widetilde{\omega}$ as in (4.35)). We start by making the general observation that if Ω is an arbitrary open set and if $u \in \mathscr{C}^2(\Omega)$ is any real-valued function without critical points in Ω, then, obviously,

$$-\Delta u + \Big(\frac{\Delta u}{|\nabla u|^2}\nabla u\Big) \cdot \nabla u = 0 \quad \text{in } \Omega. \tag{4.132}$$

This tautology may be interpreted as the statement that u is a null-solution of the second order differential operator

$$L := -\Delta + \vec{b} \cdot \nabla, \quad \text{where} \quad \vec{b} := \frac{\Delta u}{|\nabla u|^2}\nabla u \text{ in } \Omega. \tag{4.133}$$

Let us now specialize these general considerations to the case where (in the two-dimensional setting)

$$\Omega := \mathbb{R}^2_+ \cap B(\mathbf{0}, e^{-1}) \text{ and } u(x,y) := y\big[-\ln\sqrt{x^2+y^2}\,\big]^{-\varepsilon} \text{ for every } (x,y) \in \Omega, \tag{4.134}$$

where $\mathbf{0} := (0,0)$ is the origin in $\mathbb{R}^2$ and $\varepsilon > 0$ is a fixed, small number. It is clear that (4.30)–(4.36) do hold and Ω does satisfy an interior pseudoball condition at $\mathbf{0} \in \partial\Omega$ if we take $\omega(t) := \widetilde{\omega}(t) := t^\alpha$ for some arbitrary, fixed $\alpha \in (0,1)$. Note that such a choice guarantees that both (4.31)–(4.32) and (4.35) are satisfied. Going further, a direct computation in polar coordinates (r,θ) shows that

$$(\nabla u)(r,\theta) = \big(\varepsilon \sin\theta\cos\theta(-\ln r)^{-\varepsilon-1}\,,\,(-\ln r)^{-\varepsilon-1}(\varepsilon\sin^2\theta - \ln r)\big), \tag{4.135}$$

so choosing ε small enough ensures that u does not have critical points in Ω. Assuming that this is the case, the drift coefficients $\vec{b} = (b^1, b^2)$ of the operator L associated with this function may be expressed in polar coordinates (r,θ) as

$$b^1(r,\theta) = \frac{\varepsilon^2\sin^2\theta\cos\theta(2\ln r - 1 - \varepsilon)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}, \tag{4.136}$$

$$b^2(r,\theta) = \frac{\varepsilon\sin\theta(2\ln r - 1 - \varepsilon)(\varepsilon\sin^2\theta - \ln r)}{r(\ln r)\big[\varepsilon\sin^2\theta(\varepsilon - 2\ln r) + (\ln r)^2\big]}. \tag{4.137}$$

It is then clear from (4.134) that $u > 0$ in Ω, $u \in \mathscr{C}^2(\Omega)$, and that u may be continuously extended to $\Omega \cup \{\mathbf{0}\}$ by setting $u(\mathbf{0}) := 0$. Furthermore, as is readily seen from (4.135), the fact that $\varepsilon > 0$ forces $\lim\limits_{y\to 0^+}(\partial_y u)(x,y) = 0$, uniformly in x. As a result, $(D^{(\mathrm{inf})}_{\mathbf{e}_2}u)(\mathbf{0}) = 0$ which shows that the conclusion in Theorem 4.4 fails. The reason for this failure is the fact that the condition (4.37) does not hold in the current situation for any choice of $\widetilde{\omega}$ as in (4.35). Indeed, if (4.37) were to hold, it would then be possible to find a constant $c > 0$ with the property that

$$\frac{\widetilde{\omega}(r)}{r} \geqslant c\max\{0, b^2(r,\pi/2)\} \geqslant \frac{c_\varepsilon}{r(-\ln r)} \quad \text{for all } r > 0 \text{ small}, \tag{4.138}$$

where $c_\varepsilon > 0$ depends only on ε. However, this would then imply that $\widetilde{\omega}$ fails to satisfy the Dini integrability condition since

$$\int_0^{e^{-1}} \frac{1}{r(-\ln r)}\,dr = \int_1^{+\infty} s^{-1}\,ds = +\infty$$

(after making the change of variables $r = e^{-s}$).

The above discussion also shows that the condition (4.37) may not be weakened to

$$\limsup_{\mathscr{G}^{\omega}_{a,b}(x_0,h)\ni x\to x_0} \frac{|\vec{b}(x)|\big|\ln|x-x_0|\big|^{-\delta}}{\frac{\widetilde{\omega}((x-x_0)\cdot h)}{(x-x_0)\cdot h}\Big(\sum\limits_{i,j=1}^{n} a^{ij}(x)h_ih_j\Big)} < +\infty \quad \text{for some } \delta > 0. \tag{4.139}$$

Indeed, in the case of (4.133)–(4.134), such a weakened condition would be satisfied for any given $\delta > 0$ by taking, in the notation introduced in (1.12), $\widetilde{\omega} := \omega_{0,-1-\delta}$ i.e., $\widetilde{\omega}(t) = |\ln t|^{-1-\delta}$. However, as already noted, the conclusion in Theorem 4.4 fails for (4.133)–(4.134).

The same type of counterexample may be easily adapted to the higher-dimensional setting, taking $\Omega := \mathbb{R}^n_+ \cap B(0, e^{-1})$ and $u(x) := x_n(-\ln|x|)^{-\varepsilon}$ in place of (4.134). In this case, the drift coefficients continue to exhibit the same type of singularity at the origin as (4.136)–(4.137). In particular, we have

$$\vec{b} : \Omega \longrightarrow \mathbb{R}^n, \quad |\vec{b}(x)| = O\Big(\frac{1}{|x|\big|\ln|x|\big|}\Big) \text{ as } |x| \to 0, \tag{4.140}$$

which shows that

$$\vec{b} \in L^n(\Omega). \tag{4.141}$$

This should be compared with the classical Aleksandrov–Bakel'man–Pucci theorem which asserts that the weak maximum principle holds for uniformly elliptic operators in open subsets of $\mathbb{R}^n$ whose drift coefficients are locally in L^n. In this light, the significance of (4.141) is that, in contrast with the Aleksandrov–Bakel'man–Pucci weak maximum principle, the boundary point principle may fail even though the drift coefficients are in L^n (cf. also [33, Example 1.12], [33, Example 4.1], and [23, Remark 3] in this regard).

Remark 4.13. Here we present another example for which the same type of conclusions (pertaining the singularity of the drift coefficients) as in Remark 4.12 may be inferred. Specifically, consider the domain $\Omega := \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ and define the function $u : \overline{\Omega} \to \mathbb{R}$ by setting

$$u(x,y) := x\, e^{-\sqrt{-\ln[(x^2+y^2)/4]}} \text{ for each } (x,y) \in \overline{\Omega} \setminus \{\mathbf{0}\} \quad \text{and} \quad u(\mathbf{0}) := 0, \tag{4.142}$$

where, as before, $\mathbf{0}$ denotes the origin in $\mathbb{R}^2$. Then it is not difficult to check that $u \in \mathscr{C}^1\big(\overline{\Omega}\big) \cap \mathscr{C}^\infty(\Omega)$, $u > 0$ in Ω, and $(\nabla u)(\mathbf{0}) = 0$. Furthermore, as noted in [22, p. 169], the function u satisfies the divergence-form, elliptic, second order equation

$$\partial_x\big(a\,\partial_x u + b\,\partial_y u\big) + \partial_y\big(b\,\partial_x u + c\,\partial_y u\big) = 0 \quad \text{in } \Omega, \tag{4.143}$$

where the coefficients $a, b, c \in \mathscr{C}^0\big(\overline{\Omega}\big) \cap \mathscr{C}^\infty(\Omega)$ are defined as follows:

$$\begin{aligned} &a := \frac{1}{\mu} + \frac{y^2(\mu^2-1)}{(x^2+y^2)\mu}, \quad b := \frac{xy(1-\mu^2)}{(x^2+y^2)\mu}, \quad c := \frac{1}{\mu} + \frac{x^2(\mu^2-1)}{(x^2+y^2)\mu} \quad \text{in } \overline{\Omega}\setminus\{\mathbf{0}\}, \\ &\textit{where } \mu := 1 + \big(2\sqrt{-\ln[(x^2+y^2)/4]}\,\big)^{-1} \text{ and } a(\mathbf{0}) := 1, \ \ b(\mathbf{0}) := 0, \ \ c(\mathbf{0}) := 1. \end{aligned} \tag{4.144}$$

Taking advantage of the differentiability of these coefficients, we may convert (4.143) into the uniformly elliptic, nondivergence form, second order equation $Lu = 0$ in Ω, where

$$L := -\sum_{i,j=1}^{2} a^{ij}\partial_i\partial_j + \sum_{i=1}^{2} b^i \partial_i \tag{4.145}$$

with $a^{11} := -a$, $a^{22} := -c$, $a^{12} := a^{21} := -c$, $b^1 := -\partial_x a - \partial_y b$, $b^2 := -\partial_y c - \partial_x b$ in Ω.

Then the top-coefficients of L are bounded in Ω, while the drift coefficients exhibit the following type of behavior near the origin:

$$b^i(x,y) \text{ blows up at } \mathbf{0} \text{ like } \quad \frac{1}{\sqrt{x^2+y^2}(-\ln(x^2+y^2))^{3/2}}, \quad i = 1, 2. \tag{4.146}$$

Then (cf. (4.140)) the same type of conclusions as in Remark 4.12 may be drawn in this case as well.

Remark 4.14. The point of the next example is to show that if the Dini condition on $\widetilde{\omega}$ is allowed to fail (while all the other hypotheses are retained), then (4.43) is no longer expected to hold, even for such simple differential operators as $L := -\Delta$. To see that this is the case, denote by $\mathbf{0}$ the origin of $\mathbb{R}^2$ and consider the two-dimensional domain

$$\Omega := \Big\{(x,y) \in B\big(\mathbf{0}, e^{-1}\big) \setminus \{\mathbf{0}\} : \sqrt{x^2+y^2} + y \ln\sqrt{x^2+y^2} < 0\Big\} \subseteq \mathbb{R}^2. \tag{4.147}$$

Then Ω satisfies an interior pseudoball condition at $\mathbf{0} \in \partial\Omega$ given that, in fact,

$$\Omega = \mathscr{G}^{\omega_{0,-1}}_{1,1}(\mathbf{0}, \mathbf{e}_2), \tag{4.148}$$

where the shape function $\omega_{0,-1}$ is as in (1.12), i.e., $\omega_{0,-1}(t) = \dfrac{-1}{\ln t}$ if $t \in (0, 1/e]$ and $\omega_{0,-1}(0) = 0$.

Next, pick $\varepsilon \in (0, 1/2)$ and define $u : \Omega \cup \{\mathbf{0}\} \to \mathbb{R}$ by setting

$$u(x,y) := \begin{cases} \Big(y + \dfrac{\sqrt{x^2+y^2}}{\ln\sqrt{x^2+y^2}}\Big)\big(-\ln\sqrt{x^2+y^2}\big)^{-\varepsilon} & \text{if } (x,y) \neq \mathbf{0}, \\ 0 & \text{if } (x,y) = \mathbf{0} \end{cases} \qquad \forall\,(x,y) \in \Omega \cup \{\mathbf{0}\}. \tag{4.149}$$

It is clear that $u \in \mathscr{C}^0(\Omega \cup \{\mathbf{0}\}) \cap \mathscr{C}^2(\Omega)$ and $u(\mathbf{0}) < u(x,y)$ for every $(x,y) \in \Omega$. Working in polar coordinates (r,θ), an elementary calculation (recall that here $L := -\Delta$) shows that, in Ω,

$$(Lu)(r,\theta) = \frac{1}{r\,(-\ln r)^{\varepsilon+3}}\Big\{(1-2\varepsilon\sin\theta)(\ln r)^2 + (\varepsilon+1)(\varepsilon\sin\theta - 2)\ln r + (\varepsilon+1)(\varepsilon+2)\Big\}. \tag{4.150}$$

Since the squared logarithm in the curly brackets above has a positive coefficient given that $\varepsilon \in (0, \frac{1}{2})$, we infer that $(Lu)(x,y) \geqslant 0$ at each point (x,y) in Ω. On the other hand, a direct calculation gives that, for each (x,y) in Ω,

$$\begin{aligned}(\partial_y u)(x,y) = {} & \Big\{1 + \frac{2y}{\sqrt{x^2+y^2}}\frac{1}{\ln(x^2+y^2)} - \frac{4y}{\sqrt{x^2+y^2}}\frac{1}{\big(\ln(x^2+y^2)\big)^2}\Big\}\big(-\ln(x^2+y^2)\big)^{-\varepsilon} \\ & + \varepsilon\Big\{\frac{2y^2}{x^2+y^2} + \frac{4y}{\sqrt{x^2+y^2}}\frac{1}{\ln(x^2+y^2)}\Big\}\big(-\ln(x^2+y^2)\big)^{-\varepsilon-1}.\end{aligned} \tag{4.151}$$

Since the two expressions in curly brackets are bounded and $\varepsilon > 0$, it follows that $\lim\limits_{y\to 0^+}(\partial_y u)(x,y) = 0$, uniformly in x. Thus, ultimately, $(D^{(\inf)}_{\mathbf{e}_2} u)(\mathbf{0}) = 0$, i.e., the lower directional derivative of u at $\mathbf{0}$ along $\mathbf{e}_2$ is in fact null. As such, the conclusion in Theorem 4.4 fails. The source of this breakdown is the fact that for any continuous function $\omega : [0,R] \to [0,+\infty)$ and any $a, b > 0$

with the property that $\mathscr{G}^{\omega}_{a,b}(\mathbf{0},\mathbf{e}_2) \subseteq \Omega$, from (4.148) we deduce that $\omega(t) \geqslant a^{-1}\omega_{0,-1}(t)$ for each $t > 0$ sufficiently small. Granted this and given that

$$\int_0^{1/e} \frac{\omega_{0,-1}(t)}{t}\, dt = +\infty,$$

we conclude that ω necessarily fails to satisfy the Dini integrability condition. In concert with (4.45), this ultimately shows that $\widetilde{\omega}$ fails to satisfy the Dini integrability condition.

Remark 4.15. There exists a bounded, convex domain, which is globally of class $\mathscr{C}^1$ as well as of class $\mathscr{C}^\infty$ near all but one of its boundary points, and with the property that the conclusion in the boundary point principle in Theorem 4.4 fails, even for such simple differential operators as $L := -\Delta$.

Indeed, it suffices to show that the two-dimensional domain Ω introduced in (4.147) is convex and of class $\mathscr{C}^1$ near the origin $\mathbf{0}$ of $\mathbb{R}^2$, and of class $\mathscr{C}^\infty$ near each point on $\partial\Omega \setminus \{\mathbf{0}\}$ near the origin. With this goal in mind, we seek a representation of $\partial\Omega$ near $\mathbf{0}$ as the graph of some real-valued function $f \in \mathscr{C}^1\big((-r,r)\big)$, for some small $r > 0$, which is $\mathscr{C}^\infty$ on $(-r,r) \setminus \{0\}$, vanishes at 0, and such that $f''(x) > 0$ for every $x \in (-r,r) \setminus \{0\}$. To get started, define

$$F : \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}, \quad F(x,y) := \sqrt{x^2+y^2} + y \ln\sqrt{x^2+y^2} \quad \forall\,(x,y) \in \mathbb{R}^2 \setminus \{\mathbf{0}\} \tag{4.152}$$

and note that a point (x,y) near the origin in $\mathbb{R}^2$ belongs to $\partial\Omega$ if and only if $F(x,y) = 0$. Also, fix $r \in (0, e^{-10})$ and observe that for each $x \in (-r,r) \setminus \{0\}$ we have

$$F(x,0) = |x| > 0 \text{ and } F(x, \sqrt{4r^2 - x^2}) = 2r + \sqrt{4r^2 - x^2}\ln(2r) < 0.$$

Moreover, since

$$\partial_y F(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \ln\sqrt{x^2+y^2} + \frac{y^2}{x^2+y^2}$$

is negative in $B(\mathbf{0},r)$, it follows that $F(x,\cdot)$ is strictly decreasing near 0. This analysis shows that if for each fixed $x \in (-r,r)\setminus\{0\}$ we define $f(x)$ to be the unique number $y \in (0, \sqrt{4r^2 - x^2})$ such that $F(x,y) = 0$, and also set $f(0) := 0$, then the upper-graph of f coincides with Ω near $\mathbf{0}$ and $F(x, f(x)) = 0$ for every $x \in (-r,r)$. Furthermore, since f is bounded and

$$\sqrt{x^2 + f(x)^2} + f(x)\ln\sqrt{x^2 + f(x)^2} = 0 \quad \forall\, x \in (-r,r) \setminus \{0\}, \tag{4.153}$$

a simple argument shows that $\lim_{x\to 0} f(x) = 0$, so that $f \in \mathscr{C}^0\big((-r,r)\big)$. On the other hand, the fact that $F(x, f(x)) = 0$ for every $x \in (-r,r)$ gives, on account of the implicit function theorem, that $f \in \mathscr{C}^\infty\big((-r,r) \setminus \{0\}\big)$ and, for each $x \in (-r,r) \setminus \{0\}$,

$$f'(x) = -\frac{\frac{x}{\sqrt{x^2+y^2}}\frac{1}{\ln\sqrt{x^2+y^2}} + \frac{xy}{(x^2+y^2)}\frac{1}{\ln\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}}\frac{1}{\ln\sqrt{x^2+y^2}} + 1 + \frac{y^2}{(x^2+y^2)}\frac{1}{\ln\sqrt{x^2+y^2}}} = \frac{xf(x)\big(f(x) + \sqrt{x^2+f(x)^2}\big)}{x^2\sqrt{x^2+f(x)^2} - f(x)^3}. \tag{4.154}$$

The first formula above readily gives that $\lim_{x\to 0} f'(x) = 0$. Based on this and the mean value theorem, we arrive at the conclusion that f is differentiable at 0 and $f'(0) = 0$. Thus, ultimately,

$f \in \mathscr{C}^1\big((-r,r)\big) \cap \mathscr{C}^\infty\big((-r,r)\setminus\{0\}\big)$. Going further, based on the second formula for f' in (4.154) and (4.153), an involved, but elementary calculation shows that for each $x \in (-r,r)\setminus\{0\}$

$$f''(x) = \frac{f(x)^2(x^2+f(x)^2)}{\big(x^2\sqrt{x^2+f(x)^2}-f(x)^3\big)^3}\Big\{f(x)(2x^2+f(x)^2)\sqrt{x^2+f(x)^2}+x^4+x^2f(x)^2+f(x)^4\Big\}. \tag{4.155}$$

In turn, since $x^2\sqrt{x^2+f(x)^2}-f(x)^3 = x^3\big(\sqrt{1+(f(x)/x)^2}-(f(x)/x)^3\big) > 0$ if $r>0$ is small, thanks to the fact that $f'(0)=0$, we may conclude from (4.155) that $f''(x)>0$, as desired.

Remark 4.16. Here we strengthen the counterexample discussed in Remark 4.14 by showing that there exists a bounded, convex domain, which is globally of class $\mathscr{C}^1$ as well as of class $\mathscr{C}^\infty$ near all but one of its boundary points, and with the property that the conclusion in the boundary point principle in Theorem 4.4 fails for $L := -\Delta$ even under the assumption that u is a null-solution in Ω (i.e., u is a harmonic function).

To see that this is the case, we work in the two-dimensional setting and, following a suggestion from [8, p. 35], for every point $(x,y) \in \mathbb{R}^2 \setminus \big\{(x,0) : x \in (-\infty,0]\cup\{1\}\big\}$ define

$$u(x,y) := \mathrm{Re}\Big(\frac{x+iy}{-\ln(x+ix)}\Big) = \frac{-x\ln(\sqrt{x^2+y^2})-y\,\mathrm{Arg}(x,y)}{(\ln(\sqrt{x^2+y^2}))^2+(\mathrm{Arg}(x,y))^2}, \tag{4.156}$$

where $Arg : \mathbb{R}^2 \setminus \big((-\infty,0]\cup\times\{0\}\big) \to (-\pi/2,\pi/2)$, defined as

$$\mathrm{Arg}(x,y) := \begin{cases} \arctan(y/x) & \text{if } x \geqslant 0,\ y\in\mathbb{R},\ (x,y)\neq(0,0),\\ \pi+\arctan(y/x) & \text{if } x<0,\ y>0,\\ -\pi+\arctan(y/x) & \text{if } x<0,\ y<0,\end{cases} \tag{4.157}$$

is the argument of the complex number $z := x+iy \in \mathbb{C}$. In particular, Arg is $\mathscr{C}^\infty$ on its domain, and $\partial_x\mathrm{Arg}(x,y) = -y(x^2+y^2)^{-1}$ and $\partial_y\mathrm{Arg}(x,y) = x(x^2+y^2)^{-1}$ there. Next, consider the open subset of $\mathbb{R}^2$ given by

$$\Omega := \big\{(x,y)\in\mathbb{R}^2\setminus\big\{(x,0) : x\in(-\infty,0]\cup\{1\}\big\} : u(x,y)>0\big\}. \tag{4.158}$$

Then $u \in \mathscr{C}^\infty(\Omega)$ and is harmonic in Ω, since u is the real part of the complex-valued function $\dfrac{z}{-\ln z}$, which is analytic there. Moreover, it is clear that u may be continuously extended to $\mathbf{0}$ by setting $u(\mathbf{0}) := 0$. Also, $u>0$ in Ω by design. To proceed, introduce the continuous function

$$F(x,y) := \begin{cases} \dfrac{x}{2}\ln(x^2+y^2)+y\,\mathrm{Arg}(x,y) & \text{if } (x,y)\in\mathbb{R}^2\setminus((-\infty,0]\times\{0\}),\\ 0 & \text{if } (x,y)=(0,0),\end{cases} \tag{4.159}$$

and note that $F \in \mathscr{C}^\infty\big(\mathbb{R}\setminus\{0\}\big)\times\mathbb{R}\big)$. The significance of this function stems from the fact that

$$\Omega = \big\{(x,y)\in\mathbb{R}^2\setminus\big((-\infty,-1)\times\{0\}\big) : F(x,y)<0\big\}. \tag{4.160}$$

A careful elementary analysis of the nature of the function F shows that there exists $\theta \in (0,\pi/2)$ such that for any $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta)\setminus\{0\}$ the function $F(\cdot,y) : \{x\in\mathbb{R} : (x,y)\in B(\mathbf{0},e^{-2})\} \to \mathbb{R}$ is continuous, strictly decreasing, and satisfies $F(0,y)>0$ and

$F(\sqrt{e^{-4}-y^2},y) < 0$. Consequently, for each $y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \setminus \{0\}$ there exists a unique number

$$f(y) \in \big(0, \sqrt{e^{-4}-y^2}\big) \quad \text{such that} \quad F(f(y),y) = 0. \tag{4.161}$$

The implicit function theorem then shows that the function $f : (-e^{-2}\sin\theta, e^{-2}\sin\theta) \setminus \{0\} \to (0,+\infty)$ just defined is of class $\mathscr{C}^\infty$. Moreover, a simple argument based on (4.161) gives that $\lim_{y\to 0} f(y) = 0$. Therefore, setting $f(0) := 0$ extends f continuously to the entire interval $(-e^{-2}\sin\theta, e^{-2}\sin\theta)$.

We claim that actually $f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$. To justify this claim, we first note that, by the implicit function theorem

$$f'(y) = -\frac{(\partial_y F)(f(y),y)}{(\partial_x F)(f(y),y)} = -\frac{\frac{2f(y)y}{f(y)^2+y^2} + \mathrm{Arg}\,(f(y),y)}{\frac{1}{2}\ln(f(y)^2+y^2) + \frac{f(y)^2-y^2}{f(y)^2+y^2}}, \quad y \neq 0. \tag{4.162}$$

Given that both the numerator of the fraction on the right-hand side of (4.162) and the expression $(f(y)^2-y^2)/(f(y)^2+y^2)$ in the denominator are bounded, while the logarithmic factor converges to $-\infty$ as $y \to 0$, we deduce that $\lim_{y\to 0} f'(y) = 0$. In turn, from this and the mean value theorem we may then conclude that f is differentiable at 0, $f'(0) = 0$ and, moreover, that $f \in \mathscr{C}^1\big((-e^{-2}\sin\theta, e^{-2}\sin\theta)\big)$.

Moving on, if $U := \{(x,y) \in B(\mathbf{0}, e^{-2}) : |y| < e^{-2}\sin\theta\}$, the manner in which the function f has been designed ensures that

$$U \cap \Omega = U \cap \big\{(x,y) \in \mathbb{R}^2 : y \in (-e^{-2}\sin\theta, e^{-2}\sin\theta) \text{ and } x > f(y)\big\}. \tag{4.163}$$

The latter implies that Ω is of class $\mathscr{C}^1$ near $\mathbf{0}$ and of class $\mathscr{C}^\infty$ near any point on $\partial\Omega$ sufficiently close to $\mathbf{0}$. We also claim that Ω is convex near $\mathbf{0}$. To see this, we make use of the fact that $F(f(y),y) = 0$ and re-write (4.162) in the form

$$f'(y) = \frac{2f(y)^2y + f(y)(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y)}{y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y) + f(y)(y^2-f(y)^2)}, \quad y \neq 0. \tag{4.164}$$

Differentiating this and once more making use of the fact that $F(f(y),y) = 0$ then yields (after a lengthy yet elementary calculation)

$$\begin{aligned} f''(y) = {} & \frac{1}{\big(y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y) + f(y)(y^2-f(y)^2)\big)^2}\Big\{(5f(y)^4 y\,\mathrm{Arg}\,(f(y),y) \\ & + 2f(y)^3(f(y)^2+y^2)\,\mathrm{Arg}\,(f(y),y)^2) + 3f(y)^3(y^2-f(y)^2) \\ & - \frac{\big(2f(y)^2y + f(y)(f(y)^2+y^2)\,\mathrm{Arg}\,(f(y),y)\big)^2\big(2f(y)y\,\mathrm{Arg}\,(f(y),y) - 3f(y)^2\big)}{y(f(y)^2+y^2)\mathrm{Arg}\,(f(y),y) + f(y)(y^2-f(y)^2)}\Big\} \end{aligned} \tag{4.165}$$

for $y \neq 0$. Note that $3f(y)^3(y^2-f(y)^2) = 3f(y)^3y^2\big(1-(f(y)/y)^2\big)$ and $\big(1-(f(y)/y)^2\big) \to 1$ as $y \to 0$. Since the last fraction in (4.165) may be written as $f(y)^3y^2(-\pi^2/4 + o(1))$ as $y \to 0$, this analysis shows that $f''(y) > 0$ for all $y \neq 0$ sufficiently close to 0. The bottom line is that Ω is convex near $\mathbf{0}$.

However, as it is easily checked from (4.156), the inner normal derivative of the function u to $\partial\Omega$ vanishes at the origin, so the boundary point principle fails even for harmonic functions in this domain.

A more insightful explanation is offered by the following observation. For any continuous function ω with the property that Ω satisfies an interior pseudoball condition at $\mathbf{0}$ with shape function ω, we necessarily have $\sqrt{f(y)^2+y^2}\,\omega(\sqrt{f(y)^2+y^2}) \geqslant f(y)$ for $y>0$ small. Hence, if ω is slowly growing (say, $\omega(2t) \leqslant c\omega(t)$ for all $t>0$ small), then $\omega(y) \geqslant c\,f(y)/y$ for all $y>0$ small, for some constant $c>0$. As a consequence, if $R>0$ is small, then

$$\int_0^R \frac{\omega(y)}{y}\,dy \geqslant \int_0^R \frac{f(y)}{y^2}\,dy = -R^{-1}f(R) + \int_0^R \frac{f'(y)}{y}\,dy \tag{4.166}$$

after an integration by parts. However, based on formula (4.162) and the fact that $f(y)/y \to 0$, $\operatorname{Arg}(f(y),y) \to \pi/2$ as $y \to 0$, it is not difficult to see that $f'(y)/y \geqslant c/(-y\ln y)$ for all $y>0$ small, where $c>0$ is a fixed constant. Hence

$$\int_0^R \frac{f'(y)}{y}\,dy = +\infty$$

which shows that Ω fails to satisfy an interior pseudoball condition at $\mathbf{0}$ with a shape function for which the Dini integrability condition holds.

In the context of Theorem 4.4, the significance of this failure is that any function $\widetilde{\omega}$ for which (4.36) holds will, thanks to (4.45), necessarily fail to satisfy the Dini integrability condition, thus contradicting the last condition in (4.35).

The harmonic function $u(x,y) := xy$ for $x,y>0$ is a counterexample to the boundary point principle for $L := -\Delta$ when Ω is the first quadrant in the two-dimensional setting. A related counterexample in an arbitrary sector in the plane is presented in [33, Example 1.6]. Compared to these, the counterexamples discussed in Remark 4.14 and Remark 4.16 are considerably stronger since they deal with open sets from the much more smaller class of $\mathscr{C}^1$ domains whose unit normal has a modulus of continuity which fails to satisfy the Dini integrability condition.

We conclude this subsection with a comment pertaining to the nature of the boundary point principle proved by Safonov in [58, Theorem 4.3 and Remark 4.4, p. 18]. Specifically, the demands here are that L is uniformly elliptic and that a truncated circular cylinder Q which touches the boundary at x_0 may be placed inside Ω and that the drift coefficients belong to $L^q(\Omega)$ for some $q>n$. What we wish to note here is that there exist vector fields $\vec{b} = (b^1,\dots,b^n)$ which satisfy (4.126)–(4.127) for some shape function ω as in (4.124), but for which

$$\vec{b} \notin \bigcup_{q>n} L^q(\Omega). \tag{4.167}$$

For example, one may take $\omega : (0,1/e) \to (0,+\infty)$ given by $\omega(t) := (\ln t)^{-2}$ for each $t \in (0,1/e)$, and $\vec{b} : \Omega \to \mathbb{R}$ such that

$$|\vec{b}(x)| \approx \frac{1}{|x-x_0|(\ln|x-x_0|)^2} \quad \text{uniformly for } x \in \Omega. \tag{4.168}$$

4.4 The strong maximum principle for nonuniformly elliptic operators with singular drift

The strong maximum principle is a bedrock result in the theory of second order elliptic partial differential equations since it enables us to derive information about solutions of differential inequalities without any explicit knowledge of the solutions themselves. In reference to the seminal work of Hopf in [35], Serrin wrote in [66, p. 9]: "*It has the beauty and elegance of a Mozart symphony, the light of a Vermeer painting. Only a fraction more than five pages in length, it still contains seminal ideas which are still fresh after 75 years.*" The traditional formulation of a strong maximum principle typically requires the coefficients to be locally bounded (among other things), and here our goal is to prove a version of the strong maximum principle in which this assumption is relaxed to an optimal pointwise blow-up condition. Specifically, we prove the following theorem.

Theorem 4.17. *Let Ω be an open, proper, nonempty subset of $\mathbb{R}^n$. Suppose that*

$$L := -\mathrm{Tr}\big(A\,\nabla^2\big) + \vec{b}\cdot\nabla = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + \sum_{i=1}^{n} b^i\partial_i \tag{4.169}$$

is a (possibly, nonuniformly) elliptic second order differential operator in nondivergence form (without a zero order term) in Ω. Also, assume that for each $x_0 \in \Omega$ and each $\xi \in S^{n-1}$ there exists a real-valued function $\widetilde{\omega} = \widetilde{\omega}_{x_0,\xi}$ satisfying

$$\widetilde{\omega} \in \mathscr{C}^0\big([0,1]\big), \quad \widetilde{\omega}(t) > 0 \ \text{ for each } t \in (0,1], \quad \int_0^1 \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty, \tag{4.170}$$

and with the property that

$$\limsup_{(x-x_0)\cdot\xi>0,\ x\to x_0} \frac{(\mathrm{Tr}\,A(x)) + \max\{0\,, \vec{b}(x)\cdot\xi\} + \Big(\sum_{i=1}^{n}\max\{0\,, -b^i(x)\}\Big)|x-x_0|}{\frac{\widetilde{\omega}((x-x_0)\cdot\xi)}{(x-x_0)\cdot\xi}((A(x)\xi)\cdot\xi)} < +\infty. \tag{4.171}$$

Let $u \in \mathscr{C}^2(\Omega)$ satisfy the differential inequality $(Lu)(x) \geqslant 0$ for all $x \in \Omega$. Then

$$\text{if } u \text{ assumes a global minimum value at some point in } \Omega, \text{ it follows that } u \text{ is constant in } \Omega. \tag{4.172}$$

Remark 4.18. We wish to emphasize that no assumption on the (Lebesgue) measurability of the coefficients a^{ij}, b^i, of the operator L is made in the statement of the above theorem.

Proof of Theorem 4.17. The proof proceeds along the lines of the classical Hopf strong maximum principle (as presented in, for example, [8, Theorem 3.5, p. 35]) with the boundary point principle established in Theorem 4.7 replacing its weaker, more familiar, counterpart. With the goal of arriving at a contradiction, suppose that $u \in \mathscr{C}^2(\Omega)$ is a nonconstant function satisfying $Lu \geqslant 0$ in Ω and which assumes a global minimum value $M \in \mathbb{R}$ at some point $x_* \in \Omega$. Then, if $U := \{x \in \Omega : u(x) = M\}$, it follows that U is a nonempty, relatively closed, proper subset of the connected set Ω hence, in order to reach a contradiction, it suffices to show that U is open, i.e., that $U \setminus U^\circ = \varnothing$. To this end, reason by contradiction and assume that there exists $y \in U \setminus U^\circ$. Since Ω is open and $y \in \Omega$, one may pick $r > 0$ such that $B(y,r) \subseteq \Omega$. On the other hand, the fact that $y \in U \setminus U^\circ$ implies that $B(y, r/2)$ is not contained in U. Hence there exists $z \in B(y,r/2) \setminus U$ and we select $x_0 \in U$ with the property that $\operatorname{dist}(z,U) = |z - x_0| =: R > 0$ (since U is relatively closed). In turn, such a choice forces $\operatorname{dist}(z, \partial\Omega) > r/2 > |y - z| \geqslant \operatorname{dist}(z,U) = R$, hence ultimately

$$B(z,R) \subseteq \Omega \setminus U \quad \text{and} \quad x_0 \in U \cap \partial B(z,R). \tag{4.173}$$

For further use, let us also note here that the fact that $x_0 \in U$ and (4.173) entail, respectively,

$$(\nabla u)(x_0) = 0 \quad \text{and} \quad x_0 \in \partial(\Omega \setminus U). \tag{4.174}$$

To proceed, define $h := R^{-1}(z - x_0) \in S^{n-1}$ and let $\widetilde{\omega} : (0,1) \to (0,+\infty)$ be the function associated with the point $x_0 \in \Omega$ and the vector $h \in S^{n-1}$ as in the statement of the theorem. On account of (4.173), it follows that the open, nonempty set $\Omega \setminus U$ satisfies a pseudoball condition at the point $x_0 \in \partial(\Omega \setminus U)$ with shape function $\omega(t) := t$ and direction vector $h = R^{-1}(z - x_0) \in S^{n-1}$. Also, thanks to (4.170)–(4.171), properties (4.36)–(4.37) are satisfied. Since $u(x_0) = M < u(x)$ for each $x \in \Omega \setminus U$, the conclusion in Theorem 4.4 applies with Ω replaced by $\Omega \setminus U$ and, say, $\vec{\ell} := h \in S^{n-1}$. In the current context, this yields

$$0 < (D^{(\inf)}_{\vec{\ell}} u)(x_0) = \vec{\ell} \cdot (\nabla u)(x_0), \tag{4.175}$$

which contradicts the first condition in (4.174). $\square$

Remark 4.19. In the original formulation of the strong maximum principle in Hopf's 1927 paper [35], the coefficient matrix of the top order part of the differential operator L is assumed to be locally uniformly positive definite in Ω, and the drift coefficients locally bounded in Ω (cf. also [37, pp. 14-15] and [9, p. 14]). The version of the strong maximum principle given in [40, Theorem 5, p. 61 and Remark (i), p. 64] and [8, p. 35] is slightly more general (and natural) in the sense that the conditions on the coefficients of the second and first order terms of L are

$$(A(x)\xi) \cdot \xi > 0 \ \text{ for each } x \in \Omega \text{ and } \xi \in S^{n-1}, \tag{4.176}$$

$$\frac{\operatorname{Tr} A(x)}{\min\limits_{\xi \in S^{n-1}} (A(x)\xi) \cdot \xi} \quad \text{and} \quad \frac{|\vec{b}(x)|}{\min\limits_{\xi \in S^{n-1}} (A(x)\xi) \cdot \xi} \ \text{ are locally bounded in } \Omega. \tag{4.177}$$

Compared with the status-quo, our main contribution in Theorem 4.17 is weakening (4.177) to the blow-up condition for the coefficients formulated in (4.171). Of course, the key factor in this regard, is the more flexible version of the boundary point principle proved in Theorem 4.4.

Remark 4.20. Theorem 4.17 readily implies a weak minimum principle of the following form. Let Ω be an open, bounded, nonempty subset of $\mathbb{R}^n$ and retain the same assumptions on L as in the statement of Theorem 4.17. Then, if $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ is a function which satisfies the differential inequality $(Lu)(x) \geqslant 0$ for all $x \in \Omega$, one has

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{4.178}$$

Theorem 4.17 is sharp, in the sense which we now describe. Fix two numbers $\alpha > 1$, $\beta > 0$ and, for each $i \in \{1, \ldots, n\}$, define the function $b^i : B(0,1) \to \mathbb{R}$ by setting

$$b^i(x) := \begin{cases} (n+\beta)\dfrac{x_i}{|x|^\alpha} & \text{if } x \in B(0,1) \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases} \tag{4.179}$$

Next, consider the differential operator

$$L := -\Delta + \sum_{i=1}^{n} b^i(x)\partial_i \quad \text{in } B(0,1) \tag{4.180}$$

and note that if

$$u : \overline{B(0,1)} \to \mathbb{R}, \quad u(x) := |x|^{2+\beta} \quad \forall\, x \in \overline{B(0,1)}, \tag{4.181}$$

then

$$\begin{aligned} & u \in \mathscr{C}^2\big(\overline{B(0,1)}\big), \quad \nabla u(x) = (\beta+2)|x|^\beta x, \quad \text{and} \\ & \Delta u(x) = (\beta+2)(n+\beta)|x|^\beta \ \text{ for each } x \in \overline{B(0,1)}. \end{aligned} \tag{4.182}$$

Moreover, u is a nonconstant function which attains its global minimum at the origin. More precisely,

$$u \geqslant 0 \text{ in } \ B(0,1), \quad u(0) = 0 \quad \text{and} \quad u\big|_{\partial B(0,1)} = 1. \tag{4.183}$$

Furthermore,

$$\begin{aligned} & (Lu)(0) = 0, \\ & (Lu)(x) = (\beta+2)(n+\beta)|x|^\beta\big[1 - |x|^{2-\alpha}\big] \ \text{ for each } x \in B(0,1) \setminus \{0\}, \end{aligned} \tag{4.184}$$

which shows that

$$\alpha \geqslant 2 \iff (Lu)(x) \geqslant 0 \ \text{ for each } x \in B(0,1). \tag{4.185}$$

On the other hand, given a function $\widetilde{\omega} : (0,1) \to (0,+\infty)$ and a vector $\xi \in S^{n-1}$, the condition (4.171) entails

$$\limsup_{x\cdot\xi>0,\ x\to 0} \frac{|x|^{-\alpha}\, x\cdot\xi}{\frac{\widetilde{\omega}(x\cdot\xi)}{x\cdot\xi}} < +\infty \tag{4.186}$$

which, when specialized to the case where x approaches 0 along the ray $\{t\xi : t > 0\}$, implies the existence of some constant $c \in (0,+\infty)$ such that $\widetilde{\omega}(t) \geqslant c\,t^{2-\alpha}$ for all small $t > 0$. In turn, this readily shows that

$$\exists\, \widetilde{\omega} : (0,1) \to (0,+\infty) \text{ such that (4.171) holds and } \int_0^1 \frac{\widetilde{\omega}(t)}{t}\, dt < +\infty \iff \alpha < 2. \tag{4.187}$$

The bottom line is that, in the context of the situation considered above, the range of α's for which the conclusion in Theorem 4.17 fails is precisely the complement of the range of α's for which the blow-up condition described in (4.171) is violated (cf. (4.185) with (4.187)). Hence Theorem 4.17 is optimal.

4.5 Applications to boundary value problems

Of course, a direct corollary of the strong maximum principle established in Subsection 4.4 is the uniqueness in the Dirichlet problem formulated in the geometrical-analytical context considered in Theorem 4.17. We aim at proving similar results for Neumann and oblique type boundary value problems. In the subsequent discussion, suppose that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu : \partial^*\Omega \to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω (cf. Subsection 2.2). In addition, consider a second order, elliptic, differential operator L, in nondivergence form, as in (4.33). In this context, the goal is to assign a concrete meaning to the *conormal derivative associated with the operator* L, which is originally formally expressed (at boundary points) as

$$\partial_\nu^L := -\sum_{i,j=1}^n a^{ij}\nu_i\partial_j = \Big(-\sum_{i,j=1}^n a^{ij}\nu_i\,\mathbf{e}_j\Big)\cdot\nabla, \tag{4.188}$$

where $(\nu_i)_{1\leqslant i\leqslant n}$ are the components of ν. To this end, fix a point $x_0 \in \partial^*\Omega$ and assume that

$$\begin{array}{l} L \text{ is uniformly elliptic near } x_0 \text{ and its top order} \\ \text{coefficients may be continuously extended at } x_0. \end{array} \tag{4.189}$$

In this setting, define the vector

$$\mathbf{n} := \mathbf{n}(L,\Omega,x_0) := -\sum_{i,j=1}^n a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j \in \mathbb{R}^n \tag{4.190}$$

and note that, since $\nu(x_0)\in S^{n-1}$, we have

$$\mathbf{n}\cdot\nu(x_0) = -\sum_{i,j=1}^n a^{ij}(x_0)\nu_i(x_0)\nu_j(x_0) < 0. \tag{4.191}$$

In particular, this shows that $\mathbf{n}\neq 0$. Finally, assume that, in the sense of Definition 4.1,

$$\mathbf{n} \text{ points in } \Omega \text{ at } x_0. \tag{4.192}$$

Then, given a function $u \in \mathscr{C}^0(\Omega\cup\{x_0\})\cap\mathscr{C}^1(\Omega)$, formula (4.190) and the second equality in (4.188) suggest defining

$$\partial_\nu^L u(x_0) := \big(D_{\mathbf{n}}^{(\mathrm{inf})}u\big)(x_0). \tag{4.193}$$

Let us also agree to drop the dependence on L when writing ∂_ν^L in the special case where $L=-\Delta$, in which scenario $\partial_\nu := -\sum_{i=1}^n \nu_i\partial_i$ is referred to as the *inner normal derivative to* $\partial\Omega$.

Before concluding this preliminary discussion, we wish to note that

$$\begin{array}{l} \text{if } \Omega \text{ is of locally finite perimeter, satisfying an interior pseudoball} \\ \text{condition at } x_0\in\partial^*\Omega, \text{ and if } L \text{ is as in (4.189), then (4.192) holds.} \end{array} \tag{4.194}$$

Indeed, in this scenario Proposition 2.6 shows that $-\nu(x_0)\in S^{n-1}$ is the direction vector for the pseudoball at x_0. Then (4.192) follows from this and (4.191), by Theorem 4.4.

Proposition 4.21. *Suppose that Ω is an open, proper, nonempty subset of $\mathbb{R}^n$ which is of locally finite perimeter. Denote by $\partial^*\Omega$ the reduced boundary of Ω, and by $\nu : \partial^*\Omega\to S^{n-1}$ the geometric measure theoretic outward unit normal to Ω. Assume that $x_0\in\partial^*\Omega$ is a point with the property that Ω satisfies an interior pseudoball condition at x_0 for a shape function $\omega : [0,R]\to[0,+\infty)$ satisfying the properties listed in* (4.31)–(4.32) *as well as the Dini integrability condition. Also, suppose that $\vec{\ell}\in S^{n-1}$ is a vector which is inner transversal to $\partial\Omega$ at x_0, in the sense that*

$$\vec{\ell}\cdot\nu(x_0) < 0. \tag{4.195}$$

Next, consider a second order, differential operator L, in nondivergence form, as in (4.33)*, which* *is uniformly elliptic near x_0 and whose top order coefficients, originally defined in Ω, may be continuously extended at the point $x_0 \in \partial\Omega$. In addition, assume that there exists a real-valued function $\widetilde{\omega} \in \mathscr{C}^0([0,R])$, positive on $(0,R]$, satisfying*

$$\int_0^R \frac{\widetilde{\omega}(t)}{t}\,dt < +\infty,$$

and with the property that

$$\limsup_{\substack{\Omega\ni x\to x_0\\ (x-x_0)\cdot\nu(x_0)>0}} \frac{\max\{0\,,\vec{b}(x)\cdot\nu(x_0)\} + \Big(\sum\limits_{i=1}^{n}\max\{0\,,\,-b^i(x)\}\Big)\omega(|x-x_0|)}{\frac{\widetilde{\omega}((x-x_0)\cdot\nu(x_0))}{(x-x_0)\cdot\nu(x_0)}} < +\infty. \tag{4.196}$$

Finally, suppose that $u \in \mathscr{C}^0(\Omega\cup\{x_0\})\cap\mathscr{C}^2(\Omega)$ is a real-valued subsolution of L in Ω which has a strict global minimum at x_0 (in the sense of (4.39)–(4.40)). Then the vector $\vec{\ell}$ points inside Ω at x_0 and

$$\big(D^{(inf)}_{\vec{\ell}}u\big)(x_0) > 0. \tag{4.197}$$

In particular, with ∂_ν and ∂^L_ν denoting, respectively, the inner normal derivative to $\partial\Omega$, and the conormal derivative associated with L, one has

$$(\partial_\nu u)(x_0) > 0 \quad \textit{and} \quad (\partial^L_\nu u)(x_0) > 0. \tag{4.198}$$

Proof. Proposition 2.6 shows that $-\nu(x_0) \in S^{n-1}$ is the direction vector for the pseudoball at x_0. Granted this, the inequality in (4.197) becomes a consequence of (4.43). Then the two inequalities in (4.198) are obtained by specializing (4.197), respectively, to the case where $\vec{\ell} := -\nu(x_0) \in S^{n-1}$, and to the case where

$$\vec{\ell} := -\frac{\sum\limits_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j}{\Big|\sum\limits_{i,j=1}^{n} a^{ij}(x_0)\nu_i(x_0)\,\mathbf{e}_j\Big|} \in S^{n-1}, \tag{4.199}$$

which is a well–defined unit vector satisfying (4.195) (by the uniform ellipticity of L). □

Corollary 4.22. *With the same background assumptions on the operator L and the function u as in Proposition 4.21, all earlier conclusions hold in domains of class $\mathscr{C}^{1,\omega}$ provided that ω satisfies (2.11), (4.31)–(4.32), as well as the Dini integrability condition. This is sharp, in the sense that there exists a bounded domain of class $\mathscr{C}^1$ (which is even convex and of class $\mathscr{C}^\infty$ near all but one of its boundary points) for which the aforementioned conclusions fail.*

Proof. The claim in the first part of the statement is a direct consequence of Theorem 3.13 and Proposition 4.21. Its sharpness is implied by the counterexamples described earlier, in Remarks 4.16 and 4.14. □

Theorem 4.23. *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an open, bounded, nonempty, connected set and consider a second order, elliptic differential operator L, in nondivergence form in Ω, as in (4.33). Also, suppose that there exists a family of real-valued functions $\widetilde{\omega}_{x,\xi} \in \mathscr{C}^0\big([0,1]\big)$, indexed by $x \in \overline{\Omega}$ and $\xi \in S^{n-1}$, each positive on $(0,1)$ and satisfying the Dini integrability condition, such that the following two properties hold:*

(i) *for each* $x \in \partial\Omega$ *there exists* $h = h_x \in S^{n-1}$ *so that* Ω *satisfies an interior pseudoball condition at* x *with shape function* $\omega = \omega_x$ *satisfying the properties listed in* (4.31)–(4.32), *and direction vector* h, *for which*

$$\limsup_{\substack{\Omega \ni y \to x \\ (y-x)\cdot h > 0}} \frac{\frac{\omega(|y-x|)}{|y-x|} l(\operatorname{Tr} A(y)) + \max\{0, \vec{b}(y) \cdot h\} + \Big(\sum_{i=1}^{n} \max\big\{0, -b^i(y)\big\}\Big)\omega(|y-x|)}{\frac{\widetilde{\omega}_{x,h}((y-x)\cdot h)}{(y-x)\cdot h}((A(y)h) \cdot h)} + \infty, \tag{4.200}$$

(ii) *for each* $x \in \Omega$ *and each* $\xi \in S^{n-1}$

$$\limsup_{\Omega \ni y \to x} \frac{(\operatorname{Tr} A(y)) + \max\{0\,, \vec{b}(y) \cdot \xi\} + \Big(\sum_{i=1}^{n} \max\big\{0\,,\, -b^i(y)\big\}\Big)|y-x|}{\frac{\widetilde{\omega}_{x,\xi}((y-x)\cdot \xi)}{(y-x)\cdot \xi}((A(y)\xi) \cdot \xi)} < +\infty. \tag{4.201}$$

Finally, assume that $\vec{\ell} : \partial\Omega \to S^{n-1}$ *is a vector field with the property that*

$$\vec{\ell}(x) \cdot h_x > 0 \quad \textit{for each } x \in \partial\Omega. \tag{4.202}$$

Then for each $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$

$$u \textit{ is constant in } \overline{\Omega} \iff \begin{cases} (Lu)(x) \geqslant 0 \quad \textit{for each } x \in \Omega, \\ \big(D^{(inf)}_{\vec{\ell}(x)} u\big)(x) \leqslant 0 \quad \textit{for each } x \in \partial\Omega. \end{cases} \tag{4.203}$$

In particular, one has uniqueness for the oblique derivative boundary value problem for L *in* Ω, *i.e., for any given data* $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$, *there is at most one function* u *satisfying*

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ (Lu)(x) = f(x) \quad \textit{for } x \in \Omega, \\ \vec{\ell}(x) \cdot (\nabla u)(x) = g(x) \quad \textit{for } x \in \partial\Omega. \end{cases} \tag{4.204}$$

As a consequence, if Ω *is also of finite perimeter and has the property that* $\partial^*\Omega = \partial\Omega$, *and if* L *is actually uniformly elliptic and its top order coefficients belong to* $\mathscr{C}^0(\overline{\Omega})$, *then*

$$u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \quad Lu \geqslant 0 \textit{ in } \Omega, \quad \partial^L_\nu u \leqslant 0 \textit{ on } \partial\Omega \implies u \textit{ is constant in } \overline{\Omega}. \tag{4.205}$$

Hence, in this setting, one has uniqueness for the Neumann boundary value problem for L *in* Ω, *i.e., for any given data* f, g *there is at most one function* u *satisfying*

$$\begin{cases} u \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^2(\Omega), \\ Lu = f \quad \textit{in } \Omega, \\ \partial^L_\nu u = g \quad \textit{on } \partial\Omega. \end{cases} \tag{4.206}$$

Finally, all these results are sharp in the sense that, even in the class of uniformly elliptic operators with constant top coefficients, the condition (4.201) *may not be relaxed to*

$$\limsup_{\Omega \ni y \to x} \big[|x-y||\vec{b}(y)|\big] < +\infty \quad \forall\, x \in \overline{\Omega}. \tag{4.207}$$

Proof. As a preliminary matter, we note that (4.202) and the fact that, by (i), Ω satisfies an interior pseudoball condition at each $x \in \partial\Omega$ with direction vector $h_x \in S^{n-1}$, imply that $\vec{\ell}(x)$ points inside Ω for each $x \in \partial\Omega$ (cf. the proof of Theorem 4.4). In particular, $\big(D^{(\inf)}_{\vec{\ell}(x)} u\big)(x)$ is well defined for each $x \in \partial\Omega$. To proceed, assume that $u \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ attains a strict global minimum on $\partial\Omega$, i.e., there exists a point $x_0 \in \partial\Omega$ such that $u(x_0) < u(x)$ for all $x \in \Omega$. In this case, granted property (i) in the statement of the theorem, Theorem 4.7 yields $\big(D^{(\inf)}_{\vec{\ell}(x_0)}u\big)(x_0) > 0$, contradicting the second condition on the right-hand side of (4.203). Thus, $u \in \mathscr{C}^0(\overline{\Omega})$ attains its minimum in Ω. In concert with the assumption that Ω is connected, property (ii) in the statement of the theorem, and the fact that $Lu \geqslant 0$ in Ω, the strong maximum principle established in Theorem 4.17 allows us to conclude u is constant in Ω. This proves (4.203) which, in turn, readily yields uniqueness in the oblique boundary value problem (4.204).

As far as (4.205) is concerned, the fact that $\partial^*\Omega = \partial\Omega$ ensures that the geometric measure theoretic outward unit normal ν to Ω is everywhere defined on $\partial\Omega$. Thus, if the top order coefficients of L belong to $\mathscr{C}^0(\overline{\Omega})$, we may define

$$\vec{\ell}: \partial\Omega \longrightarrow S^{n-1}, \quad \vec{\ell}(x) := -\frac{\sum\limits_{i,j=1}^{n} a^{ij}(x)\nu_i(x)\,\mathbf{e}_j}{\Big|\sum\limits_{i,j=1}^{n} a^{ij}(x)\nu_i(x)\,\mathbf{e}_j\Big|} \quad \text{for every } x \in \partial\Omega. \tag{4.208}$$

Now (4.205) follows by specializing (4.203) to this choice of a vector field.

Finally, to see that the above results are sharp, take $\Omega := B(0,1) \subseteq \mathbb{R}^n$ and consider the differential operator L and the function $u \in \mathscr{C}^2\big(\overline{B(0,1)}\big)$ as in (1.35). Then

$$\begin{aligned} &(Lu)(x) = 0 \quad \text{for each } x \in B(0,1),\\ &(\partial^L_\nu u)(x) = -\frac{4}{n+2} \leqslant 0 \quad \text{for each } x \in \partial B(0,1) \end{aligned} \tag{4.209}$$

which shows that (4.205) fails in this case, precisely because the blow-up of the drift at the origin is of order one, i.e., $|\vec{b}(x)| = |x|^{-1}$ for $x \in B(0,1)\setminus\{0\}$. □

Corollary 4.24. *With the same background assumptions on the operator L and the function u as in Theorem* 4.23*, all conclusions in this theorem hold in bounded connected domains of class $\mathscr{C}^{1,\omega}$ in $\mathbb{R}^n$ provided that ω satisfies* (2.11)*,* (4.31)–(4.32)*, as well as the Dini integrability condition.*

Proof. This readily follows from Theorem 3.13 and Theorem 4.23. □

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References

1. D. S. Jerison and C. E. Kenig, "Boundary behavior of harmonic functions in nontangentially accessible domains," *Adv. Math.* **46**, No. 1, 80–147 (1982).

2. P. W. Jones, "Quasiconformal mappings and extendability of functions in Sobolev spaces," *Acta Math.* **47**, 71–88 (1981).

3. G. David and S. Semmes, "Singular Integrals and Rectifiable Sets in $\mathbb{R}^n$: Beyond Lipschitz Graphs," *Astérisque*, No. 193 (1991).

4. S. Hofmann, M. Mitrea, and M. Taylor, "Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains," *Int. Math. Res. Not.* No. 14, 2567–2865 (2010).

5. S. Hofmann, M. Mitrea, and M. Taylor, "Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains," *J. Geom. Anal.* **17**, No. 4, 593–647 (2007).

6. N. M. Gunther, *La Théorie du Potentiel et ses Applications aux Problèmes Fondamentaux de la Physique Mathématique*, Gauthier-Villars, Paris (1934).

7. N. M. Gunther, *Potential Theory*, Ungar, New York (1967).

8. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin etc. (1983).

9. P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel etc. (2007).

10. M. S. Zaremba, "Sur un problème mixte relatif à l'équation de Laplace," *Bull. Int. l'Acad. Sci. Cracovie, Série A, Class Math. Nat.*, 313–344 (1910).

11. E. Hopf, "A remark on linear elliptic differential equations of second order," *Proc. Am. Math. Soc.* **3**, 791–793 (1952).

12. O. A. Oleinik, "On properties of solutions of certain boundary problems for equations of elliptic type" [in Russian], *Mat. Sb., N. Ser.* No. 30(72), 695–702 (1952).

13. A. V. Bitsadze, *Boundary Value Problems for Second Order Elliptic Equations*, North-Holland, Amsterdam (1968).

14. M. E. Taylor, *Partial Differential Equations. 1: Basic Theory*, Springer, Berlin (1996).

15. L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL (1992).

16. W. Ziemer, *Weakly Differentiable Functions*, Springer, New York (1989).

17. S. Alexander, "Local and global convexity in complete Riemannian manifolds," *Pac. J. Math.* **76**, No. 2, 283–289 (1978).

18. P. Grisvard, *Elliptic Problems in Nonsmooth Domains* Pitman, Boston etc. (1985).

19. N. S. Nadirashvili, "Lemma on the interior derivative and uniqueness of the solution of the second boundary value problem for second order elliptic equations" [in Russian], *Dokl. Akad. Nauk SSSR* **261**, No. 4, 804–808 (1981); English transl.: *Sov. Math., Dokl.* **24**, 598–601 (1981).

20. E. J. McShane, "Extension of range of functions," *Bull. Am. Math. Soc.* **40**, 837–842 (1934).

21. H. Whitney, "Analytic extensions of functions defined on closed sets," *Trans. Am. Math. Soc.* **36**, 63–89 (1934).

22. D. Gilbarg, "Some hydrodynamic applications of function theoretic properties of elliptic equations," *Math. Z.* **72**, No. 1, 165–174 (1959).

23. A. I. Nazarov, *A Centennial of the Zaremba–Hopf–Oleinik Lemma*, Preprint (2010); arXiv:1101.0164v1

24. R. Finn and D. Gilbarg, "Asymptotic behavior and uniqueness of plane subsonic flows," *Commun. Pure Appl. Math.* **10**, No. 1, 23–63 (1957).

25. C. Neumann, "Über die Methode des Arithmetischen Mittels," *Abhand. Königl. Sächs. Ges. Wiss.*, Leipzig, **10**, 662–702 (1888).

26. A. Korn, *Lehrbuch der Potentialtheorie. II. Allgemeine Theorie des logarithmischen Potentials und der Potentialfunctionen in der Ebene*, F. Dümmler., Berlin (1901).

27. L. Lichtenstein, "Neue Beiträge zur Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus," *Math. Zeitschr.* **20**, 194–212 (1924).

28. G. Giraud, "Generalisation des problèmes sur les opérations du type elliptique," *Bull. Sci. Math.* **56**, 316–352 (1932).

29. G. Giraud, "Problèmes de valeurs à la frontière rélatifs à certaines données discontinues," *Bull. Soc. Math. Fr.* **61**, 1–54 (1933).

30. C. Miranda, *Partial Differential Equations of Elliptic Type* Springer, New York etc. (1970).

31. M. Keldysch and M. Lavrentiev, "Sur l'unicité de la solution du problème de Neumann," *C. R. (Dokl.) Acad. Sci. URSS* **16**, No. 3, 141–142 (1937).

32. V. A. Kondrat'ev and E. M. Landis, "Qualitative theory of second order linear partial differential equations," *Partial Differential Equations III. Encycl. Math. Sci.* **32**, 87-192 (1991).

33. M. V. Safonov, *Boundary Estimates for Positive Solutions to Second Order Elliptic Equations*, Preprint (2008).

34. L. I. Kamynin, "A theorem on the internal derivative for a weakly degenerate second order elliptic equation" [in Russian], *Mat. Sb., Nov. Ser.* **126**(**168**), No. 3, 307–326 (1985); English transl.: *Math. USSR, Sb.* **54**, No. 2, 297–316 (1986).

35. E. Hopf, "Elementare Bemerkung über die Lösung partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus," *Sitzungsberichte Akad. Berlin*, 147–152 (1927).

36. D. Gilbarg, "Uniqueness of axially symmetric flows with free boundaries," *J. Ration. Mech. Anal.* **1**, 309–320 (1952).

37. E .M. Landis, *Second Order Equations of Elliptic Type and Parabolic Type* Am. Math. Soc., Providence, RI (1998).

38. L. C. Evans, *Partial Differential Equations*, Am. Math. Soc., Providence, RI (1998).

39. L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge Univ. Press, Cambridge (2000).

40. M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., Englewood Cliffs, NJ (1967).

41. A. D. Aleksandrov, "Investigations on the maximum principle. I" [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 5(6), 126–157 (1958).

42. A. D. Aleksandrov, "Investigations on the maximum principle. II" [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 3(10), 3-12 (1959).

43. A. D. Aleksandrov, "Investigations on the maximum principle. III" [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 5(12), 16-32 (1959).

44. A. D. Aleksandrov, “Investigations on the maximum principle. IV” [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 3(16), 3-15 (1960).

45. A. D. Aleksandrov, “Investigations on the maximum principle. V” [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 5(18), 16-26 (1960).

46. A. D. Aleksandrov, “Investigations on the maximum principle. VI” [in Russian], *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 1(20), 3-20 (1961).

47. L. I. Kamynin and B. N. Khimchenko, “On theorems of Giraud type for a second order elliptic operator weakly degenerate near the boundary” [in Russian], *Dokl. Akad. Nauk SSSR* **224**, No. 4, 752–755 (1975); English transl.: *Sov. Math., Dokl.* **16**, No. 5, 1287–1291 (1975).

48. L. I. Kamynin and B. N. Khimchenko, “The maximum principle for an elliptic-parabolic equation of the second order” [in Russian], *Sib. Mat. Zh.* **13**, 773–789 (1972); English transl.: *Sib. Math. J.* **13**, 533–545 (1973).

49. L. I. Kamynin and B. N. Khimchenko, “Theorems of the Giraud type for second order equations with weakly degenerate nonnegative characteristic part” [in Russian], *Sib. Mat. Zh.* **18**, No. 1, 103–121 (1977); English transl.: *Sib. Math. J.* **18**, 76–91 (1977).

50. L. I. Kamynin and B. N. Khimchenko, “Investigations of the maximum principle,” *Dokl. Akad. Nauk SSSR* **240**, No. 4, 774–777 (1978); English transl.: *Sov. Math., Dokl.* **19**, 677–681 (1978).

51. R. Výborný, “On certain extensions of the maximum principle,” In: *Differential Equations and their Applications, Proc. Copnf. Prague Sept. 1962*, pp. 223–228, Academic Press, New York (1963).

52. L. I. Kamynin and B. N. Khimchenko, “Development of Aleksandrov’s theory of the isotropic strict extremum principle” [in Russian], *Differ. Uravn.* **16**, No. 2, 280–292 (1980); English transl.: *Differ. Equations* **16**, No. 2, 181–189 (1980).

53. C. Pucci, “Proprietà di massimo e minimo delle soluzioni di equazioni a derivate parziali del secondo ordine di tipo ellittico e parabolico. I,” *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat., (8)* **23**, No. 6, 370–375 (1957).

54. C. Pucci, “Proprietà di massimo e minimo delle soluzioni di equazioni a derivate parziali del secondo ordine di tipo ellittico e parabolico. II,” *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat., (8)* **24**, No. 1, 3–6 (1958).

55. G. M. Lieberman, “Regularized distance and its applications,” *Pac. J. Math.* **117**, No. 2, 329–352 (1985).

56. A. I. Nazarov and N. N. Ural’tseva, *Qualitative Properties of Solutions to Elliptic and Parabolic Equations with Unbounded Lower Order Coefficients*, Preprint (2009).

57. A. I. Nazarov and N. N. Ural’tseva, *The Harnack Inequality and Related Properties for Solutions to Elliptic and Parabolic Equations with Divergence-Free Lower Order Coefficients* [in Russian], *Algebra Anal.* **23**, No. 1, 136–168 (2011); English transl.: *St. Petersburg Math. J.* **23**, No. 1 (2012). [To appear]

58. M. V. Safonov, “Non-divergence elliptic equations of second order with unbounded drift,” In: *Nonlinear Partial Differential Equations and Related Topic*, pp. 211–232. Am. Math. Soc., Providence, RI (2010).

59. L. Nirenberg, “A strong maximum principle for parabolic equations,” *Commun. Pure Appl. Math. 6*, 167–177 (1953).

60. L. I. Kamynin, A theorem on the internal derivative for a second order uniformly parabolic equation" [in Russian], *Dokl. Akad. Nauk SSSR* **299**, No. 2, 280–283 (1988); English transl.: *Sov. Math., Dokl.* **37**, No. 2, 373–376 (1988).

61. L. I. Kamynin and B. N. Khimchenko, "The analogues of the Giraud theorem for a second order parabolic equation" [in Russian], *Sib. Mat. Zh.* **14**, No. 1, 86–110 (1973); English transl.: *Sib. Math. J.* **14**, 59–77 (1973).

62. L. I. Kamynin and B. N. Khimchenko, "An aspect of the development of the theory of the anisotropic strict A. D. Aleksandrov extremum principle" [in Russian], *Differ. Uravn.* **19**, No. 3, 426–437 (1983); English transl.: *Differ. Equations* **19**, No. 3, 318–327 (1983).

63. A. Carbery, V. Maz'ya, M. Mitrea, and D. J. Rule, *The Integrability of Negative Powers of the Solution of the Saint Venant Problem*, Preprint (2010).

64. A. Ancona, "On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$," *J. London Math. Soc. (2)* **34**, 247–290 (1986).

65. R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, NJ (1970).

66. C. S. Morawetz, J. B. Serrin, and Y. G. Sinai, *Selected Works of Eberhard Hopf with Commentaries*, Am. Math. Soc., Providence, RI (2002).

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