Journal of Mathematical Sciences, Vol. 191, No. 2, May, 2013

SECOND ORDER DIFFERENTIABILITY FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN THE PLANE

R. McOwen

Northeastern University
360 Huntington Ave, Boston, MA 02115, USA

r.mcowen@neu.edu

V. G. Maz'ya *

Department of Mathematical Sciences, M&O Building
University of Liverpool, Liverpool L69 7ZL, UK

Department of Mathematics, Linköping University
SE-581 83 Linköping, Sweden

vlmaz@mai.liu.se

UDC 517.95

For a second order elliptic equation of nondivergence form in the plane, we investigate conditions on the coefficients which imply that all strong solutions have first order derivatives that are Lipschitz continuous or differentiable at a given point. We assume the coefficients have modulus of continuity satisfying the square-Dini condition, and obtain additional conditions associated with a dynamical system that is derived from the coefficients of the elliptic equation. Our results extend those of previous authors who assume the modulus of continuity satisfies the Dini condition. Bibliography: 6 titles.

1 Introduction

We consider an elliptic equation in nondivergence form

$$a(x,y)\, u_{xx} + b(x,y)\, u_{xy} + c(x,y)\, u_{yy} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^2$. Suppose that $u \in W^{2,2}(\Omega)$ is a strong solution of (1.1). We want to know how regular u is in Ω. This, of course, depends upon the smoothness of the coefficient functions a, b, and c. If we only assume the coefficients are bounded, then u has first order derivatives that are Hölder continuous in Ω, i.e., $u \in C^{1,\alpha}(\Omega)$ where $\alpha \in (0,1)$ depends on the coefficient bounds and the ellipticity constant (cf. [1]). If the coefficients are continuous in Ω, then $u \in C^{1,\alpha}(\Omega)$ for all $\alpha \in (0,1)$ (cf. [2]). On the other hand, if the coefficients are Hölder continuous in Ω, or more generally if the coefficients are Dini-continuous in Ω, then it is

* To whom the correspondence should be addressed.

Translated from *Problems in Mathematical Analysis* **70**, May 2013, pp. 00–00.

well-known that $u \in C^2(\Omega)$ (cf. [3]). In this paper, we want to find conditions on the coefficients, weaker than Dini continuity, under which u will be second order differentiable.

Let us assume that Ω contains the origin $\mathbf{0} = (0,0)$ and focus on the differentiability at $\mathbf{0}$. Using ellipticity and a change of independent variables, we may arrange that $a(\mathbf{0}) = 1 = c(\mathbf{0})$ and $b(\mathbf{0}) = 0$. Consequently, we assume that the coefficients a, b, c satisfy

$$\sup_{|\mathbf{x}|=r} \left(|a(\mathbf{x}) - 1| + |b(\mathbf{x})| + |c(\mathbf{x}) - 1|\right) \leqslant \omega(r) \quad \text{as } r \to 0, \tag{1.2}$$

where $\mathbf{x} = (x, y)$ and the modulus of continuity ω is a continuous, nondecreasing function for $0 \leqslant r < 1$ satisfying $\omega(0) = 0$. The coefficients being Dini continuous means that (1.2) holds with $\omega(r)$ satisfying the Dini condition

$$\int_0^1 \omega(r)\, r^{-1}\, dr < \infty.$$

Hölder continuity, of course, corresponds to the special case $\omega(r) = C\, r^{\alpha}$ where $\alpha \in (0,1)$ and C is a positive constant. But we assume that $\omega(r)$ satisfies the more general square-Dini condition:

$$\int_0^1 \omega^2(r)\, \frac{dr}{r} < \infty. \tag{1.3}$$

Given a solution $u \in W^{2,2}(\Omega)$ of (1.1), let us introduce the vector $U = (U_1, U_2) = (u_x, u_y)$. Using $u_{xx} = (U_1)_x$, $u_{xy} = (U_1)_y$, and $u_{yy} = (U_2)_y$, we can write (1.1) as

$$a(x,y)\,(U_1)_x + b(x,y)\,(U_1)_y + (c(x,y) - 1)(U_2)_y + (U_2)_y = 0.$$

If we differentiate this with respect to x and use $(U_2)_{yx} = u_{yyx} = u_{xyy} = (U_1)_{yy}$ (where third order derivatives are interpreted weakly), we obtain

$$(a(x,y)\,(U_1)_x)_x + (b(x,y)\,(U_1)_y)_x + ((c(x,y) - 1)(U_2)_y)_x + (U_1)_{yy} = 0. \tag{1.4}$$

Now, we perform a similar calculation using $u_{xy} = (U_2)_x$ instead of $(U_1)_y$ and differentiating with respect to y instead of x to obtain

$$(U_2)_{xx} + ((a(x,y) - 1)(U_1)_x)_y + (b(x,y)\,(U_2)_x)_y + (c(x,y)\,(U_2)_y)_y = 0. \tag{1.5}$$

Putting (1.4) and (1.5) together as a second order system, we obtain

$$\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} U_x\right)_x + \left(\begin{pmatrix} 0 & 0 \\ (a-1) & b \end{pmatrix} U_x\right)_y + \left(\begin{pmatrix} b & c-1 \\ 0 & 0 \end{pmatrix} U_y\right)_x + \left(\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} U_y\right)_y = \mathbf{0}. \tag{1.6}$$

Now, (1.6) may look more complicated than (1.1), but at least it is in divergence form:

$$(A_{11} U_x)_x + (A_{21} U_x)_y + (A_{12} U_y)_x + (A_{22} U_y)_y = \mathbf{0}, \tag{1.7}$$

where the A_{ij} are (2×2)-matrices and $U \in W^{1,2}(\Omega, \mathbb{R}^2)$ is a weak solution. Moreover, the matrices A_{ij} are perturbations of $\delta_{ij} I$, where I is the 2×2 identity matrix, in that

$$\sup_{|\mathbf{x}| = r} |A_{ij}(\mathbf{x}) - \delta_{ij} I| \leqslant \omega(r) \quad \text{as } r \to 0. \tag{1.8}$$

In this way, (1.7) is reminiscent of our work [4] which considered the first order differentiability of weak solutions to an elliptic equation in divergence form. Moreover, the first order differentiability of U corresponds to the second order differentiability of u, so the conclusions of [4] are just what we need here. However, the formulas of [4] pertain to equations and not systems (whose coefficients are matrices so that products do not commute). Consequently, they cannot be used directly in the present situation. Nevertheless, we can apply the methods of [4] to (1.7).

The method of [4] suggests that we find a first order dynamical system on $0 < t < \infty$ whose stability properties as $t \to \infty$ control the differentiability of the solutions of (1.7). To derive the dynamical system, we first write $\mathbf{x} = r\,\theta$ where $r = |\mathbf{x}|$ and $\theta = (\theta_1, \theta_2) = (\cos\varphi, \sin\varphi)$ for $0 \leqslant \varphi < 2\pi$. Then we write

$$U(x,y) = U_0(r) + V_1(r)x + V_2(r)y + W(x,y), \tag{1.9}$$

where U_0, V_1, and V_2 are given by mean integrals

$$U_0(r) = \fint_{S^1} U(r\,\theta)\,d\varphi, \quad V_1(r) = \fint_{S^1} U(r\,\theta)\,\theta_1\,d\varphi, \quad V_2(r) = \fint_{S^1} U(r\,\theta)\,\theta_2\,d\varphi, \tag{1.10}$$

and $W(x,y)$ has zero spherical mean and first spherical moments:

$$\fint_{S^1} W(r\,\theta)\,d\varphi = 0 = \fint_{S^1} W(r\,\theta)\,\theta_1\,d\varphi = \fint_{S^1} W(r\,\theta)\,\theta_2\,d\varphi. \tag{1.11}$$

Similar to [4], we show that the 4-vector function $\vec{V}(r) = (V_1(r), V_2(r))$ satisfies a dynamical system that depends on W, and W satisfies a partial differential equation that depends upon $\vec{V}$. Ultimately, we show that the behavior of U_0, $\vec{V}$, and W are all controlled by the asymptotic behavior of solutions to the following first order system:

$$\frac{d\varphi}{dt} + R\,\varphi = 0 \quad \text{on } 0 < t < \infty, \tag{1.12a}$$

where $r = e^{-t}$ and $R(e^{-t})$ is the (4×4)-matrix function defined on $0 < t < \infty$ by

$$R(r) := \begin{pmatrix} \overline{a}_1(r) & 0 & \overline{b}_1(r) & \overline{c}_1(r) \\ \overline{a}_2(r) & \overline{b}_2(r) & 0 & \overline{c}_2(r) \\ \overline{a}_2(r) & 0 & \overline{b}_2(r) & \overline{c}_2(r) \\ -\overline{a}_1(r) & -\overline{b}_1(r) & 0 & -\overline{c}_1(r) \end{pmatrix}, \tag{1.12b}$$

with coefficients given by certain second spherical moments of the original coefficients:

$$\begin{aligned} \overline{a}_1(r) &:= \fint_{S^1} a(r\theta)(\theta_2^2 - \theta_1^2)\,d\varphi, & \overline{a}_2(r) &:= -2\fint_{S^1} a(r\theta)\theta_1\theta_2\,d\varphi, \\ \overline{b}_1(r) &:= \fint_{S^1} b(r\theta)(\theta_2^2 - \theta_1^2)\,d\varphi, & \overline{b}_2(r) &:= -2\fint_{S^1} b(r\theta)\theta_1\theta_2\,d\varphi, \\ \overline{c}_1(r) &:= \fint_{S^1} c(r\theta)(\theta_2^2 - \theta_1^2)\,d\varphi, & \overline{c}_2(r) &:= -2\fint_{S^1} c(r\theta)\theta_1\theta_2\,d\varphi. \end{aligned} \tag{1.12c}$$

As in [4], the first order regularity of solutions of (1.7) is determined by the stability properties of (1.12). In particular, we say that (1.12) is *uniformly stable* as $t \to \infty$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that any solution φ of (1.12a) satisfying $|\varphi(t_1)| < \delta$ for some $t_1 > 0$ satisfies $|\varphi(t)| < \varepsilon$ for all $t \geqslant t_1$. We show that the following holds.

Theorem 1.1. *If* (1.12) *is uniformly stable, then every weak solution* $U \in W^{1,2}(\Omega, \mathbb{R}^2)$ *of* (1.7) *is Lipschitz continuous at* $\mathbf{x} = \mathbf{0}$.

Another important stability condition is that a solution of (1.12) be *asymptotically constant*, i.e., that $\varphi(t) \to \varphi_\infty$ as $t \to \infty$. As discussed in [4], this is actually independent of uniform stability, so we need to assume both conditions to conclude the differentiability of weak solutions. We show that the following holds.

Theorem 1.2. *If* (1.12) *is uniformly stable and every solution is asymptotically constant, then every weak solution* $U \in W^{1,2}(\Omega, \mathbb{R}^2)$ *of* (1.7) *is differentiable at* $\mathbf{x} = \mathbf{0}$.

Recalling the derivation of (1.7) from (1.1), these results yield the following.

Theorem 1.3. *If* (1.12) *is uniformly stable, then every strong solution* $u \in W^{2,2}(\Omega)$ *of* (1.1) *has first order derivatives that are Lipschitz continuous at* $\mathbf{0}$. *If, in addition, every solution of* (1.12) *is asymptotically constant, then* u *is second order differentiable at* $\mathbf{0}$.

We can obtain analytic conditions on the matrix function R that imply the desired asymptotic properties of (1.12). The simplest condition is

$$r^{-1}R(r) \in L^1(0,\varepsilon) \quad \text{for some } \varepsilon > 0, \tag{1.13}$$

which guarantees that (1.12) is both uniformly stable and asymptotically constant (cf. [5]).

Corollary 1.1. *If R as in* (1.12b) *satisfies* (1.13), *then every strong solution* $u \in W^{2,2}(\Omega)$ *of* (1.1) *is second order differentiable at* $\mathbf{0}$.

Analytic conditions weaker than (1.13) can also be obtained. For example, if we introduce the symmetric matrix $S = -(R + R^t)/2$ and let $\mu(S)$ denote the largest eigenvalue of S, then it is shown in [4] that

$$\int_{r_1}^{r_2} \rho^{-1}\mu(S(\rho))\, d\rho < K \quad \text{for all } \varepsilon > r_2 > r_1 > 0 \tag{1.14}$$

implies that (1.12) is uniformly stable. As a consequence, (1.14) guarantees that every strong solution $u \in W^{2,2}(\Omega)$ of (1.1) has first order derivatives that are Lipschitz continuous at $\mathbf{x} = \mathbf{0}$. In addition, it is shown in [4] that

$$r^{-1}R(r) \int_0^r \rho^{-1}R(\rho)\, d\rho \in L^1(0,\varepsilon) \tag{1.15}$$

implies that (1.12) is uniformly stable and asymptotically constant. As a consequence, (1.15) guarantees that every strong solution $u \in W^{2,2}(\Omega)$ of (1.1) is second order differentiable at $\mathbf{x} = \mathbf{0}$.

One may also consider special cases to better understand the significance of the role of the dynamical system (1.12) in determining the regularity of strong solutions of (1.1). In particular, let us assume that the coefficients b and c in (1.1) satisfy

$$\overline{b}_i(r) = \overline{c}_i(r) = 0 \quad \text{for } i = 1, 2. \tag{1.16}$$

This occurs, for example, when b and c are constant, or more generally if they depend only on r: $b = b(r)$ and $c = c(r)$. In the case (1.16), we see that (1.12a) decouples into three scalar equations

$$\frac{d\varphi}{dt} + \overline{a}_1\,\varphi = 0, \quad \frac{d\varphi}{dt} - \overline{a}_1\,\varphi = 0, \quad \frac{d\varphi}{dt} + \overline{a}_2\,\varphi = 0.$$

But these are all of the form $\varphi' + p(t)\varphi = 0$ which can be solved using the integrating factor $\exp[\int^t p(\tau)d\tau]$. We conclude that the three scalar equations will be uniformly stable provided

$$\left|\int_s^t \overline{a}_1(\tau)\,d\tau\right| < K_1 \quad \text{and} \quad \int_s^t \overline{a}_2(\tau)\,d\tau > -K_2 \quad \text{for } t > s \text{ sufficiently large.} \tag{1.17}$$

Moreover, the three scalar equations will be asymptotically constant provided

$$\begin{aligned} &\int_T^\infty \overline{a}_1(\tau)\,d\tau \text{ converges to a finite real number, and} \\ &\int_T^\infty \overline{a}_2(\tau)\,d\tau \text{ converges to an extended real number} > -\infty. \end{aligned} \tag{1.18}$$

Thus, when the coefficients b, c satisfy (1.16) and the coefficient a satisfies (1.17) and (1.18), then every strong solution of (1.1) will be second order differentiable at $\mathbf{0}$.

2 Derivation of the Dynamical System

A weak solution U of (1.7) satisfies

$$\int_\Omega A_{ij}\,\partial_i U\,\partial_j \eta\,dxdy = \mathbf{0} \tag{2.1}$$

for all $\eta \in C_0^\infty(\Omega)$, where we used the Einstein summation convention of summing over repeated indices. To obtain the dynamical system (1.12), we begin by considering (2.1) with different choices of test functions η.

Taking η to be a radial function $\eta(r)$ and using (1.9), we obtain the following first order ordinary differential equation:

$$\mathscr{A}(r)\,U_0' + r\mathscr{B}_1(r)\,V_1' + r\mathscr{B}_2(r)\,V_2' + \Gamma_1(r)\,V \cdots r) = 0, \tag{2.2}$$

should be bold-face 0

where the (2×2)-matrices $\mathscr{A}$, $\mathscr{B}_k$, and Γ_k are

$$\begin{aligned}\mathscr{A}(r) &= \fint_{S^1} A_{ij}(r\theta)\,\theta_i\theta_j\,d\varphi = I_2 + O(\omega(r)) \quad \text{as } r\to 0\\ \mathscr{B}_k(r) &= \fint_{S^1} A_{ij}(r\theta)\,\theta_k\theta_i\theta_j\,d\varphi = O(\omega(r)) \quad \text{as } r\to 0\\ \Gamma_k(r) &= \fint_{S^1} A_{ik}(r\theta)\,\theta_i\,d\varphi = O(\omega(r)) \quad \text{as } r\to 0,\end{aligned} \tag{2.3}$$

and the 2-vector Λ is

$$\Lambda[\nabla W](r) = \fint_{S^1}(A_{i1}\theta_i W_x + A_{i2}\theta_i W_y)\,d\varphi. \tag{2.4}$$

Using Lemma 1 in [4], we can show that

$$|\Lambda[\nabla W](r)| \leqslant \omega(r)\fint_{S^1}|\nabla W|\,d\varphi. \tag{2.5}$$

Note that, although we are thinking of (2.2) as an ordinary differential equation, the coefficients are matrices and so it is really a system of two equations.

Taking $\eta=\eta(r)x$ and then $\eta=\eta(r)y$ in (2.1), we obtain two second order ordinary differential equations which we can put together as a second order system

$$\begin{aligned}&-\left[r^2\left(\begin{pmatrix}\mathscr{B}_1 U_0'\\ \mathscr{B}_2 U_0'\end{pmatrix} + \begin{pmatrix}\mathscr{A}_{11} & \mathscr{A}_{12}\\ \mathscr{A}_{21} & \mathscr{A}_{22}\end{pmatrix}\begin{pmatrix}rV_1'\\ rV_2'\end{pmatrix} + \begin{pmatrix}\mathscr{B}_{11} & \mathscr{B}_{12}\\ \mathscr{B}_{21} & \mathscr{B}_{22}\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix} + \begin{pmatrix}P_1[\nabla W]\\ P_2[\nabla W]\end{pmatrix}\right)\right]'\\ &+ r\left(\begin{pmatrix}\widetilde{\Gamma}_1 U_0'\\ \widetilde{\Gamma}_2 U_0'\end{pmatrix} + \begin{pmatrix}\widetilde{\mathscr{B}}_{11} & \widetilde{\mathscr{B}}_{12}\\ \widetilde{\mathscr{B}}_{21} & \widetilde{\mathscr{B}}_{22}\end{pmatrix}\begin{pmatrix}rV_1'\\ rV_2'\end{pmatrix} + \begin{pmatrix}\mathscr{C}_{11} & \mathscr{C}_{12}\\ \mathscr{C}_{21} & \mathscr{C}_{22}\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix} + \begin{pmatrix}Q_1[\nabla W]\\ Q_2[\nabla W]\end{pmatrix}\right) = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix},\end{aligned} \tag{2.6}$$

where the $\mathscr{B}_j$ are defined above, but the other (2×2)-matrices are as follows:

$$\begin{aligned}\mathscr{A}_{k\ell}(r) &= \fint_{S^1} A_{ij}(r\theta)\,\theta_i\theta_j\theta_k\theta_\ell\,d\varphi = \frac{1}{2}\,\delta_{k\ell}\,I_2 + O(\omega(r)) \text{ as } r\to 0,\\ \mathscr{B}_{k\ell}(r) &= \fint_{S^1} A_{i\ell}(r\theta)\,\theta_i\theta_k\,d\varphi = \frac{1}{2}\,\delta_{k\ell}\,I_2 + O(\omega(r)) \text{ as } r\to 0,\\ \widetilde{\mathscr{B}}_{k\ell}(r) &= \fint_{S^1} A_{ki}(r\theta)\,\theta_i\theta_\ell\,d\varphi = \frac{1}{2}\,\delta_{k\ell}\,I_2 + O(\omega(r)) \text{ as } r\to 0,\\ \mathscr{C}_{k\ell}(r) &= \fint_{S^1} A_{k\ell}(r\theta)\,d\varphi = \delta_{k\ell}I_2 + O(\omega(r)) \text{ as } r\to 0,\\ \widetilde{\Gamma}_k(r) &= \fint_{S^1} A_{ki}(r\theta)\,\theta_i\,d\varphi = O(\omega(r)) \text{ as } r\to 0,\end{aligned} \tag{2.7}$$

and the 2-vectors $P_1,\ P_2,\ Q_1,\ Q_2$ are given by

$$P_k[\nabla W](r) = \fint_{S^1} A_{ij}(r\theta)\,\theta_i\theta_k\frac{\partial W}{\partial x_j}\,d\varphi \quad \text{and} \quad Q_k[\nabla W](r) = \fint_{S^1} A_{ki}(r\theta)\frac{\partial W}{\partial x_j}\,d\varphi. \tag{2.8}$$

As with (2.5), we can show

$$|P_k[\nabla W](r)|,\ |Q_k[\nabla W](r)| \leqslant \omega(r) \fint_{S^1} |\nabla W|\, d\varphi. \tag{2.9}$$

We want to use (2.2) to eliminate U_0' from (2.6) and then identify the leading order terms. Since $\mathscr{A}(r) = I_n + O$ [illegible] should be "2" instead of "n" is invertible for small r and we can write

$$\mathscr{B}_i U_0 = -\mathscr{B}_i \mathscr{A} \text{[illegible]} \left(\mathscr{B}_1 r V_1 + \mathscr{B}_2 r V_2' + \Gamma_1 V_1 + \Gamma_2 V_2 + \Lambda[\nabla W]\right).$$

But the coefficients of rV_j' and V_j in this expression are "lower order," i.e.,

$$\mathscr{B}_i \mathscr{A}^{-1} \mathscr{B}_j,\ \mathscr{B}_i \mathscr{A}^{-1} \Gamma_j = O(\omega^2(r)) \quad \text{as } r \to 0.$$

So when we plug this into (2.6), it does not affect the leading order terms in rV_j' and V_j. Similarly for replacing $\widetilde{\Gamma}_i U_0'$ in (2.6).

Let us make the substitution $r = e^{-t}$, so that $r\, d/dr = -d/dt$. Next, let us introduce $\varepsilon(t) = \omega(e^{-t})$ and then write (2.6) (after the elimination of U_0) as

$$\left[e^{-2t}\left(-\mathbf{A}\,\vec{V}_t + \mathbf{B}\,\vec{V} + \vec{P}[\nabla W] + O(\varepsilon^2(t))\right)\right]_t + e^{-2t}\left(-\widetilde{\mathbf{B}}\,\vec{V}_t + \mathbf{C}\,\vec{V} + \vec{Q}[\nabla W] + O(\varepsilon^2(t))\right) = 0,$$

where $\vec{V} = (V_1, V_2)$, $\vec{P} = (P_1, P_2)$, and $\vec{Q} = (Q_1, Q_2)$ are 4-vectors and $\mathbf{A}$, $\mathbf{B}$, $\widetilde{\mathbf{B}}$, and $\mathbf{C}$ are the (4×4)-matrices

$$\mathbf{A} = \begin{pmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} \\ \mathscr{B}_{21} & \mathscr{B}_{22} \end{pmatrix}, \quad \widetilde{\mathbf{B}} = \begin{pmatrix} \widetilde{\mathscr{B}}_{11} & \widetilde{\mathscr{B}}_{12} \\ \widetilde{\mathscr{B}}_{21} & \widetilde{\mathscr{B}}_{22} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} \\ \mathscr{C}_{21} & \mathscr{C}_{22} \end{pmatrix},$$

and we used $O(\varepsilon^2(t))$ to represent terms depending linearly on $\vec{V}_t$, $\vec{V}$, or $\Lambda[\nabla W]$, but with coefficients that are $O(\varepsilon^2(t))$ as $t \to \infty$. We can remove the factor e^{-2t} to obtain

$$\begin{aligned} &\left[-\mathbf{A}\,\vec{V}_t + \mathbf{B}\,\vec{V} + \vec{P}[\nabla W] + O(\varepsilon^2(t))\right]_t + (2\mathbf{A} - \widetilde{\mathbf{B}})\vec{V}_t \\ &\qquad\qquad + (\mathbf{C} - 2\mathbf{B})\vec{V} - 2\vec{P}[\nabla W] + \vec{Q}[\nabla W] + O(\varepsilon^2(t)) = 0. \end{aligned} \tag{2.10}$$

However, this is still a second order system, and we want to avoid differentiating the coefficient matrices, so let us convert it to a first order system by replacing the vector in the brackets in (2.10) by a new 4-vector

$$\vec{U} = -\mathbf{A}\,\vec{V}_t + \mathbf{B}\,\vec{V} + \vec{P}[\nabla W] + O(\varepsilon^2(t)). \tag{2.11}$$

We now have a first order system in the 8-vector $(\vec{V}, \vec{U})$:

$$\begin{aligned} &\vec{V}_t - \mathbf{A}^{-1}\mathbf{B}\,\vec{V} + \mathbf{A}^{-1}\,\vec{U} = \mathbf{A}^{-1}\vec{P}[\nabla W] + O(\varepsilon^2), \\ &\vec{U}_t + (\mathbf{C} - \widetilde{\mathbf{B}}\mathbf{A}^{-1}\mathbf{B})\vec{V} + (\widetilde{\mathbf{B}}\mathbf{A}^{-1} - 2\mathbf{I})\vec{U} = -\vec{Q}[\nabla W] + O(\varepsilon^2). \end{aligned}$$

where $\mathbf{I}$ is the (4×4) identity matrix. The coefficients of $\vec{V}$ and $\vec{U}$ behave as follows:

$$-\mathbf{A}^{-1}\mathbf{B} \sim -\mathbf{I}, \quad \mathbf{A}^{-1} \sim 2\mathbf{I}, \quad \mathbf{C} - \widetilde{\mathbf{B}}\mathbf{A}^{-1}\mathbf{B} \sim \frac{1}{2}\mathbf{I}, \quad \widetilde{\mathbf{B}}\mathbf{A}^{-1} - 2\mathbf{I} \sim -\mathbf{I}.$$

should be comma instead of period

where $\sim$ means differs by a term that is $O(\varepsilon)$ as $t \to \infty$. Consequently, let us rewrite the first order system as

$$\frac{d}{dt}\begin{pmatrix}\vec{V}\\ \vec{U}\end{pmatrix} + \mathbf{M}(t)\begin{pmatrix}\vec{V}\\ \vec{U}\end{pmatrix} = \begin{pmatrix}F_1(t,\nabla W)\\ F_2(t,\nabla W)\end{pmatrix}, \tag{2.12}$$

where the (8×8)-matrix-valued function $\mathbf{M}(t)$ is of the form

$$\mathbf{M}(t) = \mathbf{M}_\infty + \mathbf{S}_1(t) + \mathbf{S}_2(t),$$

with a constant matrix

$$\mathbf{M}_\infty = \begin{pmatrix}-\mathbf{I} & 2\,\mathbf{I}\\ \frac{1}{2}\mathbf{I} & -\mathbf{I}\end{pmatrix}.$$

The variable coefficient matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ satisfy

$$\mathbf{S}_1(t) = \begin{pmatrix}\mathbf{I}-\mathbf{A}^{-1}\mathbf{B} & \mathbf{A}^{-1}-2\mathbf{I}\\ \mathbf{C}-\widetilde{\mathbf{B}}\mathbf{A}^{-1}\mathbf{B}-\frac{1}{2}\mathbf{I} & \widetilde{\mathbf{B}}\mathbf{A}^{-1}-\mathbf{I}\end{pmatrix} = O(\varepsilon(t)) \quad \text{as } t\to\infty$$

and $\mathbf{S}_2 = O(\varepsilon^2(t))$ as $t \to \infty$. The right-hand side of (2.12) satisfies

$$|F_i(t,\nabla W)| \leqslant \varepsilon(t) \fint_{S^1} |\nabla W|\, d\varphi.$$

In order to analyze (2.12), as in [4] we introduce a change of variables

$$\begin{pmatrix}\vec{V}\\ \vec{U}\end{pmatrix} = \mathbf{J}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}, \tag{2.13}$$

where the matrix

$$\mathbf{J} = \begin{pmatrix}2\,\mathbf{I} & 2\,\mathbf{I}\\ \mathbf{I} & -\mathbf{I}\end{pmatrix}$$

diagonalizes $\mathbf{M}_\infty$, i.e., $\mathbf{J}^{-1}\mathbf{M}_\infty\mathbf{J} = \operatorname{diag}(0,0,0,0,-2,-2,-2,-2)$. We find that (φ,ψ) satisfies a dynamical system of the form

$$\frac{d}{dt}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -2\,\mathbf{I}\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \mathbf{R}(t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = G(t,\nabla W), \tag{2.14}$$

where

$$\mathbf{R}(t) = \begin{pmatrix}R_1(t) & R_2(t)\\ R_3(t) & R_4(t)\end{pmatrix},$$

with

$$R_1(t) \approx \frac{1}{4}\mathbf{A}^{-1} - \frac{1}{2}\mathbf{A}^{-1}\mathbf{B} + \mathbf{C} - \widetilde{\mathbf{B}}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{2}\widetilde{\mathbf{B}}\mathbf{A}^{-1} - \mathbf{I},$$

where $\approx$ means differs by a term that is $O(\varepsilon^2(t))$ as $t \to \infty$. The right-hand side of (2.14) satisfies

$$|G(t,\nabla W)| \leqslant \varepsilon(t) \fint_{S^1} |\nabla W|\, d\varphi. \tag{2.15}$$

Estimates on W that we shall discuss in the next section together with the stability theory presented in Section 2 of [4] show that the stability of (2.14) is determined by that of

$$\frac{d\varphi}{dt} + R_1(t)\varphi = 0, \tag{2.16}$$

so we need to determine the asymptotic behavior of R_1. But to do this, let us write

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} + \mathbf{A}_0), \quad \mathbf{B} = \frac{1}{2}(\mathbf{I} + \mathbf{B}_0), \quad \widetilde{\mathbf{B}} = \frac{1}{2}(\mathbf{I} + \widetilde{\mathbf{B}}_0), \quad \mathbf{C} = \mathbf{I} + \mathbf{C}_0,$$

where $|\mathbf{A}_0|, |\mathbf{B}_0|, |\widetilde{\mathbf{B}}_0|, |\mathbf{C}_0| = O(\varepsilon(t))$ as $t \to \infty$. Also note that $\mathbf{A}^{-1} \approx 2(\mathbf{I} - \mathbf{A}_0)$. Using these, we can simplify R_1 to obtain $R_1 \approx \mathbf{C}_0 - \mathbf{B}_0 = \mathbf{C} - 2\mathbf{B}$, and after a careful calculation we obtain the formula given in (1.12b).

3 Proofs of Theorems 1.1 and 1.2

Since we are only interested in the behavior of our weak solution near $\mathbf{0}$, we may assume $\Omega = B_\varepsilon(\mathbf{0})$ with $\varepsilon > 0$ chosen small enough to make

$$\int_0^\varepsilon r^{-1}\omega(r)\,dr < \delta \quad \text{and} \quad \omega(\varepsilon) < \delta, \tag{3.1}$$

with $\delta > 0$ as small as we like. In fact, for any $p \in (1, \infty)$ we can choose $\delta = \delta(p) > 0$ in (3.1) small enough that the small oscillation condition on the coefficients (1.2) ensures that $\nabla U \in L^p_{\mathrm{loc}}(\Omega)$ (cf. [6, Corollary 6.2]). Henceforth, we pick $p > 2$ and choose ϵ small enough that $\nabla U \in L^p_{\mathrm{loc}}(\Omega)$. But by rescaling the independent variables, we may arrange $\varepsilon > 1$, so we may assume that our weak solution U of (1.7) satisfies

$$\nabla U \in L^p(\Omega), \quad \text{where } p > 2 \text{ and } \Omega = B_1(0). \tag{3.2}$$

Should be bold face 0

In particular, by Sobolev's inequality we know that U is continuous in Ω.

For our analysis, it is useful to consider (1.7) on all of $\mathbb{R}^2$, so we extend the matrices A_{ij} to all of $\mathbb{R}^2$ by $A_{ij} = \delta_{ij} I$ for $|x| > 1$. We also extend our modulus of continuity ω to $(0, \infty)$ by $\omega(1)$ for $r > 1$. It will also be useful to introduce the L^p-mean of a function over the annulus $A_r = \{x : r < |\mathbf{x}| < 2r\}$:

$$M_p(f, r) = \left(\fint_{A_r} |f(x)|^p\,dx \right)^{1/p}.$$

To control growth of the first derivatives of functions, we introduce

$$M_{1,p}(f, r) = r M_p(\nabla f, r) + M_p(f, r).$$

Let us introduce a smooth cut-off function $\chi(r)$ that is 1 for $0 \leqslant r \leqslant 1/4$ and 0 for $r \geqslant 1/2$. We find that $\chi(r)U(x, y)$ satisfies

$$(A_{11}(\chi U)_x)_x + (A_{21}(\chi U)_x)_y + (A_{12}(\chi U)_y)_x + (A_{22}(\chi U)_y)_y = F_0 + (F_1)_x + (F_2)_y$$

where $F_0 = A_{11}\chi'\theta_1 U_x + A_{21}\chi'\theta_2 U_x + A_{12}\chi'\theta_1 U_y + A_{22}\chi'\theta_2 U_y$, $F_1 = \chi'(A_{11}\theta_1 + A_{12}\theta_2)U$, and $F_2 = \chi'(A_{21}\theta_1 U + A_{22}\theta_2)U$. Using (2.1) with $\eta = \chi$, we see that

$$\int_{\mathbb{R}^2} F_0 \, dxdy = 0. \tag{3.3}$$

Since we are interested in the behavior near $x = 0 = y$ where U and χU agree, we can simply assume that U is supported in $r \leqslant 1/2$ and satisfies

$$\partial_i(A_{ij}\partial_j U) = F_0 + \partial_i(F_i), \tag{3.4}$$

where $F_0, F_1, F_2 \in L^p(\mathbb{R}^2)$ are supported in $1/4 \leqslant r \leqslant 1/2$ and F_0 satisfies (3.3). Of course, we now must replace (2.1) by

$$\int_{\mathbb{R}^2} A_{ij}\,\partial_j U\,\partial_i \eta \, dxdy = \int_{\mathbb{R}^2} (F_i \partial_i \eta - F_0)\, dxdy \quad \text{for all } \eta \in C_0^\infty(\Omega). \tag{3.5}$$

At this point, we observe that (3.4) with (3.3) for the vector function U is identical with (51ab) in [4] for the scalar function u. This means that we can repeat the analysis of [4] to connect the stability of the dynamical system (1.12b) with the regularity of our weak solution. We do not want to repeat all of the details here, but let us give an outline of the argument.

To begin with, we recall the decomposition $U = U_0 + V_1 x + V_2 y + W$ in (1.9). We have shown that $\vec{V}$ satisfies a dynamical system (2.12) that depends on ∇W, so we need to know ∇W is sufficiently well-behaved in order to obtain estimates for $\vec{V}$. This is done by showing that W satisfies a partial differential equation that depends on $\vec{V}$. To derive the differential equation for W, we introduce

$$\Omega_{ij} = A_{ij} - \delta_{ij} I, \tag{3.6}$$

which satisfies $|\Omega_{ij}| \leqslant \omega(r)$ for $0 < r < 1$ and $\Omega_{ij} = 0$ for $r \geqslant 1$. We also introduce for

should be bold face 0

$f \in L^1(\mathbb{R}^2\setminus\{0\})$

$$f(r\theta)^\perp = f(r\theta) - Pf(r\theta), \tag{3.7}$$

where P is the projection of f onto the functions on S^1 spanned by $1, \theta_1, \theta_2$:

$$Pf(r\theta) = c_0(r) + c_1(r)\theta_1 + c_2(r)\theta_2, \text{ where}$$

$$c_0(r) = \fint_{S^1} f(r\theta)\, d\varphi \quad \text{and} \quad c_i(r) = 2\fint_{S^1} \theta_i\, f(r\theta)\, d\varphi.$$

Note that $P[\Delta(U_0 + V_1 x + V_2 y)] = \Delta(U_0 + V_1 x + V_2 y)$ and $P[\Delta W] = 0$, so W satisfies the following perturbation of Laplace's equation on $\mathbb{R}^2$:

$$\Delta W + [\partial_i(\Omega_{ij}\partial_j U_0)]^\perp + [\partial_i(\Omega_{ij}\partial_j(V_k x_k))]^\perp + [\partial_i(\Omega_{ij}\partial_j W)]^\perp = [F_0 + \partial_i(F_i)]^\perp. \tag{3.8}$$

Now, we simultaneously consider the dynamical system (2.12) for V and Equation (3.8) for W. The analysis in [4] shows the assumptions that $U \in W^{1,2}(\Omega)$ and that (1.12) is uniformly stable together imply that V satisfies

$$\sup_{0<r<1} (|V(r)| + r|V'(r)|) \leqslant C \tag{3.9}$$

and W satisfies

$$M_{1,p}(W, r) \leqslant C\,\omega(r) r, \quad 0 < r < 1. \tag{3.10}$$

(In both (3.9) and (3.10), the constants C depend upon the $W^{1,2}$-norm of U, but not on r.) Since $p > 2$, we can use Sobolev embedding to conclude that $|W(x,y)|r^{-1} \leqslant C\omega(r)$ f[illegible] $|W(x,y)|r^{-1} \to 0$ as $r = |\mathbf{x}| \to 0$. This shows that W is differentiable at 0.

Should be bold face 0

To estimate U_0, we use (2.2) and the estimates that we have obtained on V_i and ∇W to conclude

$$|U_0(r) - U_0(0)| = \left| \int_0^r U_0'(\rho)\, d\rho \right| \leqslant C\,\omega(r) \int_0^r ([illegible] W|)\, d\rho \leqslant C\,\omega(r)\, r.$$

Should be bold face x

But this implies that U_0 is differentiable at $x = 0$ and $U_0'(0)$ [illegible]

Should be bold face 0

We have now shown the assumption that (1.12) is uniform[illegible] show that our weak solution $U \in W^{1,2}(\Omega)$ satisfies

Should be bold face 0

$$|U(x,y) - U(0)| \leqslant |U_0(r) - U_0(0)| + [illegible] y| + |W(x,y)| \leqslant C\, r.$$

But this shows that U is Lipschitz continuous at $x = 0$, completing the proof of Theorem 1.11.

Should be bold face x

Should be bold face 0

For Theorem 1.22, w[illegible] that every [illegible] asymptotically constant. The dynamical systems analysis of [4] applied to (2.14) then shows that $\varphi(t) \to \varphi_\infty$ and $\psi(t) \to 0$ as $t \to \infty$. However, we can use (2.13) to express φ, ψ in terms of V and V_t:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}V - \frac{1}{4}V_t \\ \frac{1}{4}V_t \end{pmatrix} + O(\varepsilon(t)).$$

Hence the conclusion $\psi \to 0$ implies $V_t \to 0$ as $t \to \infty$, in other words,

$$\lim_{r \to 0} rV'(r) = 0. \tag{3.11}$$

But (3.11) implies that $V_1(r)x + V_2(r)y$ is differentiable at 0. Since we have already shown that U_0 and $W(x,y)$ are differentiable at 0, we obtain the conclusion of Theorem 1.22.

References

1. L. Nirenberg, "On nonlinear elliptic partial differential equations and Hölder continuity," *Commun. Pure Appl. Math.* **6**, 103–156 (1953).

2. S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic differential equations satisfying general boundary conditions. 1," *Commun. Pure Appl. Math.* **12**, No. 4, 623–727 (1959).

3. M. Taylor, *Tools for PDE*, Am. Math. Soc., Providence, RI (2000).

4. V. Maz'ya and R. McOwen, "Differentiability of solutions to second order elliptic equations via dynamical systems," *J. Differ. Equ.* **250**, 1137–1168 (2011).

5. W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath & Co., Boston (1965).

6. V. Maz'ya, M. Mitrea, and T. Shaposhnikova, "The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients," *J. Anal. Math.* **110**, 167–239 (2010).

Submitted on March 12, 2013