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SOLVABILITY OF BOUNDARY SINGULAR INTEGRAL OPERATORS OF ELASTICITY ON SURFACES WITH CONIC POINTS

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A system of boundary singular integral equations of the linear isotropic elasticity is studied in the case where the double layer potential is generated by the stress operator. These equations are considered on a bounded two-dimensional surface without boundary which is smooth outside a finite number of conical points. The solvability of the system is proved in various weighted spaces of differentiable functions. We obtain a representation of the inverse operator of the system in terms of the inverse operators of some boundary value problems. We also obtain pointwise estimates for the kernel of this operator and "quasilocal" estimates for solutions of the integral equations in question. Bibliography: 12 *titles.*

1 Introduction

The paper is closely connected with our previous works [1, 2]. It is devoted to the study of boundary integral equations of the linear isotropic elasticity in the case where the double layer potential is generated by the stress operator. The integral equations are considered on a closed bounded two-dimensional surface which is smooth outside finitely many conical points. In this paper, we apply the same scheme as in [1, 2], where the case of the double layer potential generated by the pseudostress operator was studied. Therefore, we restrict ourselves to more condensed exposition. A preliminary version of this paper appeared as Preprint LiTH-MAT-R-91-06 in 1991.

We describe the main results of the paper. Let G^+ be a simply connected region in $\mathbb{R}^3$ with compact closure. We put $\Gamma = \partial G^+$ and assume that $0 \in \Gamma$. Suppose that $\Gamma\backslash\{0\}$ is a smooth surface and, near the origin, G^+ coincides with a cone K^+ excising the open set Ω^+ on the unit

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sphere S^2. We set $G^{-}\mathbb{R}^3\backslash\overline{G^+}$, $K^- = \mathbb{R}^3\backslash\overline{K^+}$, and $\Omega^- = S^2\backslash\overline{\Omega^+}$.

In what follows, we denote by $\varkappa$ a real number depending on the shape of the cone and the boundary conditions. This number can be expressed in terms of eigenvalues of some boundary value problems with spectral parameter in domains Ω^+ and Ω^-. From [3] it follows that $\varkappa$ is positive if the cone K^+ can be explicitly prescribed in a Cartesian coordinate system.

We consider the system of integral equations associated with the first boundary value problem of linear isotropic elasticity in G^+

$$(1+T)\varphi = f. \tag{1.1}$$

Here, 1 is the identity matrix and T is the operator defined by the equation

$$T\varphi(x) = 2(W_0\varphi)(x) \quad \text{for a.e. } x \in \Gamma,$$

where $W_0\varphi$ is the value of the double layer potential on Γ generated by the stress operator (cf. [4]). If φ is Hölder continuous on Γ, then $T\varphi$ can be extended by continuity to Γ.

Assuming that K^+ can be explicitly prescribed in a Cartesian coordinate system, we show that the operator $1+T$ *isomorphically maps the space* $C^{0,\alpha}(\Gamma)$ *of Hölder continuous functions onto itself for all* $\alpha \in (0,\varkappa)$.

Now, we formulate results on the solvability of the system (1.1) in the spaces $V^l_{p,\beta}(\Gamma)$ and $N^{l,\alpha}_\beta(\Gamma)$ $(1<p<\infty,\ \alpha\in(0,1),\ \beta\in\mathbb{R},\ l=0,1,\dots)$ which are defined as follows.

A norm in the space $V^l_{p,\beta}(\Gamma)$ for a function u with support in an arbitrary coordinate neighborhood on $\Gamma\backslash 0$ (cf. Section 2) has the form

$$\sum_{0\leqslant j\leqslant l} \|r^{\beta+j-l}\nabla_j u\|_{L_p(\Gamma)},$$

where $r(x) = |x|$ and ∇_j is the vector of all derivatives of order j. Similarly, in the space $N^{l,\alpha}_\beta(\Gamma)$, it is possible to introduces the norm

$$\sup_{x\in\Gamma\backslash 0} r^{\beta-l-\alpha}(x)|u(x)| + \sup_{x,y\in\Gamma\backslash 0} \frac{|r^\beta(x)\nabla_l u(x) - r^\beta(y)\nabla_l u(y)|}{|x-y|^\alpha}.$$

We obtain the following result.

The operator $1+T$ *is an isomorphism of the space* $V^l_{p,l+t}(\Gamma)$ *onto itself for all* p, t, l *such that*

$$1<p<\infty, \quad 0<t+2/p<1+\varkappa, \quad l=0,1,\dots. \tag{1.2}$$

A similar assertion holds for the space $N^{l,\alpha}_{\delta+l}(\Gamma)$ *if the inequalities* (1.2) *are replaced by*

$$0<\delta-\alpha<1+\varkappa, \quad \alpha\in(0,1), \quad l=0,1,\dots,.$$

Along with (1.1)*, we consider the formally adjoint system associated with the second boundary value problem of linear isotropic elasticity in* G^-*:*

$$(1+T^*)\psi = g \tag{1.3}$$

and show that the operator $1+T^*$ *is an isomorphism of the space* $V^l_{p,t+l}(\Gamma)$ *onto itself if*

$$1<p<\infty, \quad 1-\varkappa<t+2/p<2, \quad l=0,1,\dots.$$

Similarly, $1+T^*$ *is an isomorphism of* $N^{l,\alpha}_{\delta+l}(\Gamma)$ *onto itself if*

$$\alpha \in (0,1), \quad 1-\varkappa < \delta - a < 2, \quad l = 0,1,2,\dots .$$

In the case of all above-mentioned spaces, we prove that the *inverse operators of the systems* (1.1) *and* (1.3) *can be written as*

$$(1+T)^{-1} = 1 - \frac{1}{1-s^2}T + \frac{1}{1-s^2}T^2 + L,$$

$$(1+T^*)^{-1} = 1 - \frac{1}{1-s^2}T^* + \frac{1}{1-s^2}(T^*)^2 + L^*.$$

Here, $s = \mu(\lambda+2\mu)^{-1}$, λ, and μ are the Lamé constants and L is an integral operator on Γ with the kernel $\mathcal{L}(x,y)$ satisfying the estimates

$$|\partial_x^\sigma \partial_y^\tau \mathcal{L}(x,y)| \leqslant \begin{cases} c|x|^{-|\sigma|}|y|^{-2-|\tau|}, & 2|x| < |y|, \\ c|y|^{-1}||x-y|^{-1-|\sigma|-|\tau|}, & |y| < 2|x| < 4|y|, \\ c|x|^{-1-|\sigma|}|y|^{-1-|\tau|}(\,|y|/|x|)^{\varkappa-\varepsilon}, & |x| > 2|y|, \end{cases}$$

where δ and τ are multiindices of order $|\sigma|$ and $|\tau$— respectively.

2 Representations for Inverse Operators of the Boundary Integral Equations and Pointwise Estimates for Its Kernels

2.1 Function spaces

We use the notation from Section 1. As in [1], we deal with the following function spaces. Let l be an integer, $l \geqslant 0$, and let $\alpha \in (0,1)$, $\beta \in \mathbb{R}^1$. By $N^{l,\alpha}_\beta(G^+)$ we mean the space of functions in $\overline{G^+}\backslash 0$ with the finite norm

$$\|u\|_{N^{l,\alpha}_\beta(G^+)} = \sup_{x\in G^+} |x|^\beta [u]^{l+\alpha}_{B(|x|/2,x)\cap G^+} + \sup_{x\in G^+} |x|^{\beta-l-\alpha}|u(x)|, \tag{2.1}$$

where $B(r,x)$ is the open ball of radius r centered at x,

$$[u]^\rho_\Omega = \sup_{x,y\in\Omega} \sum_{|\sigma|=[\rho]} |x-y|^{[\rho]-\rho}|\partial_x^\sigma u(x) - \partial_y^\sigma u(y)|,$$

$[\rho]$ is the integer part of ρ, $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ is a multiindex of order $|\sigma| = \sigma_1+\sigma_2+\sigma_3$,

$$\partial_x^\sigma = \frac{\partial^{|\sigma|}}{\partial {x_1}^{\sigma_1}\partial {x_2}^{\sigma_2}\partial {x_3}^{\sigma_3}}.$$

For $0<\beta<l+\alpha$ we define the space $C^{l,\alpha}_\beta(G^+)$ of functions u on G^+ with the finite norm

$$\|u\|_{C^{l,\alpha}_\beta(G^+)} = \sup_{x\in G^+} |x|^\beta [u]^{l+\alpha}_{B(|x|/2,x)\cap G^+} + \sup_{x\in G^+} \sum_{l+\alpha-\beta<|\sigma|\leqslant l} |x|^{\beta-l-\alpha+|\sigma|}|\partial_x^\sigma u(x)|$$
$$+ [u]^{l+\alpha-\beta}_{G^+} + \sum_{0\leqslant|\sigma|<l+\alpha-\beta} \sup_{x\in G^+} |\partial_x^\sigma u(x)|. \tag{2.2}$$

In the domain G^-, we define similar spaces $N_\beta^{l,\alpha}(G^-)$ and $C_\beta^{l,\alpha}(G^-)$. Suppose that the ball $B(R,0)$ of radius R contains $\overline{G^+}$. Let χ denote a function from the space $C^\infty(\mathbb{R}^3)$, equal to unity on $B(R,0)$ and to zero on $\mathbb{R}^3 \backslash B(R+1,0)$. The function u in G^- belongs to $N_\beta^{l,\alpha}(G^-)$ (respectively, $C_\beta^{l,\alpha}(G^-)$) if and only if the norm (2.1) (respectively, (2.2)) for $u\chi$ and the norm

$$\sup_{x\in G^-} |x|^{l+1+\alpha}[V]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x\in G^-} |x||v(x)|$$

for $v = (1-\chi)u$ are finite.

The spaces of traces on Γ for functions from $C_\beta^{l,\alpha}(G^+)$ and $N_\beta^{l,\alpha}(G^+)$ will be denoted by $C_\beta^{l,\alpha}(\Gamma)$ and $N_\beta^{l,\alpha}(\Gamma)$.

Finally, let $N_\beta^{l,\alpha}(G)$ and $C_\beta^{l,\alpha}(G)$ refer to the spaces of functions u in $G = G^+ \cup G^-$ whose restrictions $u^\pm$ on $G^\pm$ belong to $N_\beta^{l,\alpha}(G^\pm)$ and $C_\beta^{l,\alpha}(G^\pm)$ respectively and

$$\|u\|_{N_\beta^{l,\alpha}(G)} = \sum_\pm \|u^\pm\|_{N_\beta^{l,\alpha}(G^\pm)}, \quad \|u\|_{C_\beta^{l,\alpha}(G)} = \sum_\pm \|u^\pm\|_{C_\beta^{l,\alpha}(G^\pm)}.$$

2.2 Boundary value problems

We consider the interior first boundary value problem for the three-dimensional Lamé equations of the linear isotropic elasticity theory

$$\Delta^* u = 0 \quad \text{in } G^+, \qquad u = f \quad \text{on } \Gamma. \tag{2.3}$$

Here, $u(x) = (u_1(x), u_2(x), u_3(x))^T$ and $f(x) = (f_1(x), f_2(x), f_3(x))^T$ are displacement vectors in G^+ and on Γ, the symbol T is used for the transposition of a matrix,

$$\Delta^* = \mu\Delta + (\lambda+\mu)\nabla \,\mathrm{div},$$

and λ, μ are the Lamé constants, $\mu > 0$, $\lambda + \mu > 0$.

We introduce an auxiliary problem which is the exterior second boundary value problem for the Lamé equations

$$\Delta^* v = 0 \quad \text{in } G^-, \qquad \mathcal{J} v = g \quad \text{on } \Gamma\backslash 0, \tag{2.4}$$

where v and g are vector-valued functions with three components, $\mathcal{J} = \mathcal{J}(\partial_x, n_x)$ is the matrix differential operator with elements

$$\mathcal{J}_{ij}(\partial_x, n_x) = \mu\delta_{ij}\frac{\partial}{\partial n_x} + \lambda n_i(x)\frac{\partial}{\partial x_j} + \mu n_j(x)\frac{\partial}{\partial x_i},$$

called the *traction operator*. Here, $n_x = (n_1(x), n_2(x), n_3(x))$ is the normal vector to the surface Γ directed outward with respect to G^+, δ_{ij} is the Kronecker symbol, and $\partial/\partial n_x$ is the normal derivative.

In order to formulate solvability theorems, it is necessary to consider certain boundary value problems in domains Ω^+ and $\Omega^- = S^2\backslash\overline{\Omega^+}$ on the unit sphere S^2 depending on the complex parameter γ (cf. [5, 6]). These problems can be obtained by substitution of the vector-valued functions $u(x) = |x|^\gamma\varphi(x/|x|)$ and $v(x) = |x|^\gamma\psi(x/|x|)$ into Equations (2.3) and (2.4) with $f = 0$ and $g = 0$. In what follows, these problems will be denoted by $p(\gamma)$ and $q(\gamma)$ respectively.

Theorem 2.1 (cf. [4])**.** *Let* δ^+ *be the largest number such that the strip* $|\operatorname{Re}\gamma + 1/2| < 1/2 + \delta^+$ *contains no eigenvalues of the problem* $p(\gamma)$*. If* $l \geqslant 1$, $|\beta - \alpha - 1/2| < 1/2 + \delta^+$, *then there exists a unique solution* $u \in N^{l,\alpha}_{\beta+l}(G^+)$ *of the problem* (2.3) *for all vector-valued functions* $f \in N^{l,\alpha}_{\beta+l}(\Gamma)$*. This solution can be expressed in the form*

$$u(x) = \int\limits_\Gamma \mathcal{P}^+(x,\xi) f(\xi) ds_\xi. \tag{2.5}$$

The operator $P^+ : N^{l,\alpha}_{\beta+l}(\Gamma) \ni f \to u \in N^{l,\alpha}_{\beta+l}(G^+)$ *defined by* (2.5) *is bounded. The following estimates hold for the derivatives of the kernel* $\mathcal{P}^+(x,\xi)$:

$$|\partial^\sigma_x \partial^\tau_\xi \mathcal{P}^+(x,\xi)| \leqslant \begin{cases} c|x|^{\sigma^+ - |\sigma| - \varepsilon} |\xi|^{-2-\delta^+ - |\tau| + \varepsilon}, & |x| < d|\xi|, \\ c|x-\xi|^{-|\tau|-|\sigma|-2}, & d|\xi| < |x| < d^{-1}|\xi|, \\ c|x|^{-\delta^+ - |\sigma| - 1 + \varepsilon} |\xi|^{\delta^+ - |r| - 1 - \varepsilon}, & |x| > d^{-1}|\xi|. \end{cases}$$

Here, d *is a fixed number in* $(0,1)$, τ *and* σ *are multiindices,* ε *is any sufficiently small positive number, and* $|\mathcal{A}|$ *means max* $|\mathcal{A}_{ij}|$, *where* $\mathcal{A}_{ij}$ *are entries of the matrix* $\mathcal{A}$, $1 \leqslant i,j \leqslant 3$.

Remark 2.1 (cf. [7])**.** Let $\omega(\omega+1)$ be the first eigenvalue of the Dirichlet problem for the Beltrami operator on Ω^+, $\omega > 0$. The

$$\delta^+ > \min\Big\{1, \frac{(3-4\nu)\omega}{\omega + 6 - 4\nu}\Big\},$$

where ν is the Poisson coefficient, i.e., $\nu = \lambda/2(\lambda+\mu)$.

Lemma 2.1 (cf. [1])**.** *If* $0 < \alpha - \beta < 1$, $\alpha - \beta < \delta^+$, $l = 1, 2, \ldots$, *then the function* u *defined by* (2.5) *is a unique solution to the problem* (2.3) *from the space* $C^{l,\alpha}_{\beta+l}(G^+)$ *for all* $f \in C^{l,\alpha}_{\beta+l}(G^+)$*. Moreover, the following estimate holds*:

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G^+)} \leqslant C\|f\|_{C^{l,\alpha}_{\beta+l}(\Gamma)}.$$

The following assertion is essentially contained in [8].

Theorem 2.2. *Let* ν^- *be the largest number such that the strip* $|\operatorname{Re}\gamma + 1/2| < 1/2 + \nu^-$ *contains no eigenvalues of the problem* $q(\gamma)$ *with exception of* $\gamma = 0$ *and* $\gamma = -1$*. If* $|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\}$, $l = 1, 2, \ldots$, *then there exists a unique solution* $v \in N^{l,\alpha}_{\beta+l}(G^-)$ *of the problem* (2.4) *for all vector-valued functions* $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$*. This solution can be represented in the form*

$$v(x) = \int\limits_\Gamma \mathcal{Q}^-(x,\xi) g(\xi) ds(\xi). \tag{2.6}$$

The operator $Q^- : N^{l-1,\alpha}_{\beta+l}(\Gamma) \ni f \to u \in N^{l,\alpha}_{\beta+l}(G^-)$ *defined by* (2.6) *is bounded. The following estimates for the kernel* $\mathcal{Q}^-(x,\xi)$ *are valid*:

$$|\partial^\sigma_x \partial^\tau_\xi \mathcal{Q}^-(x,\xi)| \leqslant |\mathcal{Q}^-(x,\xi)| \leqslant \begin{cases} c|x|^{-|\sigma|} |\xi|^{-1-|\tau|} (\,|x|/|\xi|\,)^{\nu-\varepsilon}, & |x| < d|\xi|, \\ c|x-\xi|^{-1-|\sigma|-|\tau|}, & d|\xi| < |x| < d^{-1}|\xi|, \\ c|x|^{-1-|\sigma|} |\xi|^{-|\tau|} (\,|\xi|/|x|\,)^{\nu-\varepsilon}, & |x| > d^{-1}|\xi| \end{cases}$$

if the points x *and* ξ *lie in a neighborhood of the vertex* 0 *and* $\nu = \min\{0, \nu^-\}$.

We note that $-1/2 < \nu^- < 1$. (The left inequality follows from the solvability of the problem (2.4) in the energy space, and the right inequality follows from the fact that the rigid displacement vector satisfies the homogeneous problem (2.4).)

Lemma 2.2 (cf. [3])**.** *Let Γ be defined near the vertex of a cone by the equation $x_3 = h(x_1, x_2)$, where h is a positive homogeneous function of order* 1*, smooth on $\mathbb{R}^2\setminus\{0\}$. Then there are only two eigenvalues $\gamma_0 = 0$ and $\gamma_1 = -1$ of the problem $q(\gamma)$ in the strip $-1 \leqslant \operatorname{Re}\gamma \leqslant 0$. These eigenvalues have multiplicity* 3 *and the corresponding Jordan chains for $q(\gamma)$ consist only of eigenfunctions. The vector-valued functions $w = \text{const}$ form the eigenspace associated with the eigenvalue γ_0.*

Lemma 2.2 enables one to give a more precise description of the behavior of $\mathcal{Q}^-(x,\xi)$.

Theorem 2.3. *Let the surface Γ satisfy the assumptions of Lemma* 2.2*. Then*

$$\begin{aligned}
&\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(0,\xi) + \mathcal{R}^-(x,\xi), \quad |x| < d|\xi|,\\
&\mathcal{Q}^-(x,\xi) = \mathcal{Q}^-(x,0) + (\mathcal{R}^-(\xi,x))^T, \quad |x| > d^{-1}|\xi|,\\
&\mathcal{Q}^-(x,0) = (\mathcal{Q}^-(0,x))^T = |x|^{-1}\mathcal{A}^-(x/|x|\,)\mathcal{B}^- + \mathcal{C}^-(x)
\end{aligned}$$

if the points x and y lie in a neighborhood of 0*. Here, $\mathcal{A}^-(x/|x|\,)$ is the matrix whose columns are eigenfunctions corresponding to the eigenvalue $\gamma = -1$, $\mathcal{B}^-$ is a constant matrix, and the matrices $\mathcal{R}^{-1}(x,\xi)$, $\mathcal{C}^{-1}(x)$ satisfy the inequalities*

$$\begin{aligned}
&|\partial_x^\sigma \partial_\xi^\tau \mathcal{R}^-(x,\xi)| \leqslant c|x|^{-|\sigma|}|\xi|^{-1-|\tau|}(\,|x|/|\xi|\,)^{\nu^- -\varepsilon},\\
&|\partial_x^\sigma \mathcal{C}^-(x)| \leqslant c|x|^{\nu^- -|\sigma|-\varepsilon}.
\end{aligned}$$

Lemma 2.3. *There exists a matrix $\mathcal{A}^+(x/|x|\,)$ with smooth components in $\overline{G^+}$ such that*

$$\begin{aligned}
&\Delta^*(\,|x|^{-1}\mathcal{A}^*(x/|x|\,) = 0 \quad \text{in } G^+,\\
&\mathcal{A}^+(x/|x|\,) = \mathcal{A}^-(x/|x|\,) \quad \text{on } \partial\Omega^+.
\end{aligned}$$

This lemma follows from the fact that $\gamma = -1$ is not an eigenvalue of the pencil $p(\gamma)$ (cf. [6]).

Lemma 2.4 (cf. [1])**.** *Let $0 < \alpha - \beta < \nu^-$, and let l be a positive integer. If $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then the function* (2.6) *is a unique solution of the problem* (2.4) *in the space $C^{l,\alpha}_{\beta+l}(G^-)$. Moreover, the following estimate holds*:

$$\|v\|_{C^{l,\alpha}_{\beta+l}(G^-)} \leqslant \|f\|_{N^{l-1,\alpha}_{\beta+l}(\Gamma)}.$$

2.3 Representations of the inverse operators for the integral equations

Let $W\psi$ denote the double layer potential with density ψ defined by

$$(W\psi)(x) = -\frac{1}{4\pi}\int\limits_\Gamma (\mathcal{J}(\partial_\xi, n_\xi)\Phi(x,\xi))^T \psi(\xi)dS_\xi, \quad x \in G,$$

and let $W_0\psi$ be the value of $W\psi$ on Γ.

Lemma 2.5. *Let $0 < \beta - \alpha < 2$, and let l be a positive integer. If $\psi \in N^{l,\alpha}_{\beta+l}(\Gamma)$, then $W\psi \in N^{l,\alpha}_{\beta+l}(G)$ and*

$$(W\psi)^{\pm} = W_0\psi \pm \psi/2, \quad (\mathcal{J}W\psi)^{+} = (\mathcal{J}W\psi)^{-} \tag{2.7}$$

on $\Gamma\backslash 0$, where the symbols $\pm$ mean the traces on $\Gamma\backslash 0$ of functions defined on $G^{\pm}$.

Proof. We can directly show that

$$\sup_{x\in B(1,0)} |x|^{\beta-\alpha}|(W\psi)(x)| \leqslant c\|\psi\|_{N^{l,\alpha}_{\beta}(\Gamma)}.$$

Taking into account that $W\psi$ is a solution of the problem

$$\Delta^* u = 0 \ \text{ in } \ G, \quad u^+ - u^- = \psi \ \text{ on } \ \Gamma,$$
$$(\mathcal{J}u)^+ - (\mathcal{J}u)^- = 0 \ \text{ on } \ \Gamma\backslash 0$$

and applying the local estimates (cf. the proof of Lemma 1.5 in [1]), we arrive at the inclusion $W\psi \in N^{l,\alpha}_{\beta+l}(G)$. The relations (2.7) follow directly from similar relations in the case of smooth surfaces (cf. [5, 9]). □

The proof of the following four assertions is the same as that in [1].

Lemma 2.6. *Let $0 < \beta - \alpha < 1$, and let l be a positive integer. If $\varphi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then $V\varphi \in N^{l,\alpha}_{\beta+l}(G)$ and $(\mathcal{J}V\varphi)^{\pm} = -W_0^*\varphi \pm \varphi/2$, $(V\varphi)^+ = (V\varphi)^-$ on $\Gamma\backslash 0$, where W_0^* is the integral operator on $\Gamma\backslash 0$, formally adjoint of W_0.*

Lemma 2.7. *Let $0 < \beta - \alpha < 1$. If $u \in N^{1,\alpha}_{\beta+1}(G)$ satisfies the system $\Delta^* u = 0$ in G, then $u = V((\mathcal{J}u)^+ - (\mathcal{J}^-)) + W(u^+ - u^-)$.*

Theorem 2.4. *Let $|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\}$, and let l be a positive integer. If $f \in N^{l,\alpha}_{\beta+l}(\Gamma)$, then the equation $(1 + T)\varphi = f$ is uniquely solvable in the space $N^{l,\alpha}_{\beta+l}(\Gamma)$ and*

$$(I + T)^{-1} f = \frac{1}{2}(1 - Q^- \mathcal{J} P^+) f. \tag{2.8}$$

Here, P^+ and Q^- are the operators defined by (2.5) and (2.6).

Theorem 2.5. *Let $|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\}$, and let l be a positive integer. Then the equation $(1 + T^*)\psi = g$ is uniquely solvable in the space $N^{l-1,\alpha}_{\beta+l}(\Gamma)$ for all $g \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$ and*

$$(1 + T^*)^{-1} g = \frac{1}{2}(1 - \mathcal{J} P^+ Q^-) g. \tag{2.9}$$

Lemma 2.8. *Let $|\beta - \alpha - 1/2| < 1/2 + \min\{0, \nu^-\}$, and let l be a positive integer. Then the operators $(1 + T)^{-1}$ and $(1 + T^*)^{-1}$ are continuous in the spaces $N^{l,\alpha}_{\beta+l}(\Gamma)$ and $N^{l-1,\alpha}_{\beta+l}(\Gamma)$ respectively.*

The assertions of the lemma is a consequence of the representations (2.8), (2.9) and Theorems 2.1 and 2.2. The following lemma concerns the operator T defined by

$$(T\psi)(x) = 2(W_0\psi)(x) + (1 - d(x))\psi(x),$$

where $d(x)$ is the identity matrix for $x \in \Gamma\backslash 0$ and $d(0) = 2(W_0 1)(0)$.

Lemma 2.9. *Let the cone K^+ be explicitly represented in a Cartesian coordinate system. If $0 < \alpha - \beta < \min\{\delta^+, \nu^{-1}\}$, then the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$ are continuous in the spaces $C^{l,\alpha}_{\beta+l}(\Gamma)$ and $C^{l-1,\alpha}_{\beta+l}(\Gamma)$ respectively.*

The assertion of the lemma is a consequence of (2.8), (2.9) and Lemmas 2.1 and 2.4.

2.4 Estimates for the kernels of the integral operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$

Assume that the cone K^+ admits an explicit description in a Cartesian coordinate system.

Theorem 2.6. *Suppose that $0 < \beta - \alpha < 1$, $\kappa = \min\{\delta^+, \nu^-\}$, $s = \mu(\lambda + 2\mu)^{-1}$, where λ and μ are the Lamé constants. Let l be a positive integer. Then*

$$(1+T)^{-1} f = \Big(1 - \frac{1}{1-s^2} T + \frac{1}{1-s^2} T^2 + L\Big) f, \quad f \in N^{l,\alpha}_{\beta+l}(\Gamma),$$

$$(1+T^*)^{-1} g = \Big(1 - \frac{1}{1-s^2} T^* + \frac{1}{1-s^2} (T^*)^2 + M\Big) g, \quad g \in N^{l-1,\alpha}_{\beta+l}(\Gamma).$$

Here, L and M are integral operators on $\Gamma\backslash 0$ with kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ which satisfy

$$|\mathcal{L}(x,y)| \leqslant \begin{cases} c|y|^{-2}, & |x| < |y|/2, \\ c|y|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c|x|^{-1}|y|^{-1}(|y|/|x|)^{\kappa-\varepsilon}, & |x| > 2|y|, \end{cases}$$

$$|\mathcal{M}(x,y)| \leqslant \begin{cases} c|x|^{-1}|y|^{-1}(|x|/|y|)^{\varepsilon}, & |x| < |y|/2, \\ c|x|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c|x|^{-2}(|y|/|x|)^{-\varepsilon}, & |x| > 2|y|, \end{cases}$$

where ε is any sufficiently small positive number.

The proof of this theorem can be obtained from (2.8) and (2.9) by using the estimates for the kernels $\mathcal{P}^+(x,\xi)$ and $\mathcal{Q}^-(x,\xi)$ in a similar manner as in [1, Subsection 2.2]. The only difference is that we have to use the following lemma instead of Lemmas 3.7 and 4.6 in [1].

Let D^+ be a bounded open set in $\mathbb{R}^3$ with smooth boundary Γ, which coincides with the cone K^+ for $1/18 < |x| < 18$. We assume that $0 \notin \overline{D}^+$ and $D^+ \subset G^+$. Let D^+_ρ denote the set $\{x \in \mathbb{R}^3 : \ x/\rho \in D^+\}$, $\rho > 0$, and let Γ_ρ be the boundary of D^+_ρ. We introduce the operator T_ρ in the space $C^{0,\alpha}(\Gamma_\rho)$ of Hölder continuous vector-valued functions defined by the equality $T_\rho\varphi = 2W_0\varphi$, where $W_0\varphi$ is the value of the double layer potential with density φ on Γ_ρ.

Lemma 2.10. *Let $\varphi, \psi \in C^{0,\alpha}(\Gamma_\rho)$, $s = \mu/(\lambda + 2\mu)$. Then*

$$\begin{aligned} (1+T_\rho)^{-1}\varphi &= \Big(1 - \frac{1}{1-s^2} T_\rho + \frac{1}{1-s^2} T_\rho^2 + H_\rho\Big)\varphi, \\ (1+T_\rho^*)^{-1}\psi &= \Big(1 - \frac{1}{1-s^2} T_\rho^* + \frac{1}{1-s^2} (T_\rho^*)^2 + H_\rho^*\Big)\psi, \end{aligned} \qquad (2.10)$$

where H_ρ is an integral operator on Γ_ρ with the kernel $\mathcal{H}_\rho(x,y)$ satisfying

$$|\partial_x^\tau \partial_y^\sigma \mathcal{H}_\rho(x,y)| \leqslant c\rho^{-1}|x-y|^{-1-|\tau|-|\sigma|}.$$

Proof. Let $\rho = 1$. The result of this lemma is essentially proved in [10], where regularizers of the singular equations $(1 + T_\rho)\varphi = f$ and $(1 - T_\rho^*)\psi = g$ were constructed. The relations (2.10) for an arbitrary $\rho > 0$ follow immediately from the estimates for $\partial_x^\alpha \partial_y^\tau H_1(x, y)$ and the equality $H_\rho(x, y) = \rho^{-2} H_1(x/\rho, y/\rho)$. □

3 Invertibility of the Boundary Integral Singular Operators of Elasticity

3.1 Function spaces

Let K^+ be an open cone in $\mathbb{R}^3$ with vertex at the origin, bounded by the surface ∂K^+. Assume that the cone K^+ can be explicitly described in a Cartesian coordinate system. Suppose also that the subset $\Omega^+ = \{x \in K^+ : |x| = 1\}$ of the unit sphere has a smooth boundary.

In what follows, we denote by $\{U_j\}_{1\leqslant j\leqslant N}$ a finite covering of $\partial K^+\backslash 0$ by open sets $U_j \subset \partial K^+\backslash 0$ such that

1) for each U_j there exists a homeomorphism γ_j onto a plane angle V_j and $\gamma_j(tx) = t\gamma_j(x)$ for all $x \in U_j$, $t \in \mathbb{R}^+$,

2) if $U_i \cap U_j = \varnothing$, then the mapping $\gamma_j \circ \gamma_i^{-1}$: $\gamma_i(U_i \cap U_j) \to \gamma_j(U_i \cap U_j)$ is infinitely differentiable.

Moreover, we assume that $|x| = |\gamma_j x|$ for all $x \in U_j$, $j = 1, \dots, N$, where $|\cdot|$ on the left-hand side means the norm in $\mathbb{R}^3$ and the same symbol on the right-hand side means the norm in $\mathbb{R}^2$. Let $\{\xi_j\}_{1\leqslant j\leqslant N}$ be a partition of unity on $\partial K^+\backslash 0$ subordinate to the covering $\{U_j\}_{1\leqslant j\leqslant N}$. Suppose that the functions ξ_j are smooth and positive-homogeneous of order 0. We denote by $V^l_{p,\beta}(\partial K^+)$, $1 < p < \infty$, $\beta \in \mathbb{R}$, $l = 0, 1, \dots$, the space of functions with the norm

$$\sum_{1\leqslant j\leqslant N} \Bigg(\sum_{0\leqslant|\sigma|\leqslant l} \int\limits_{\mathbb{R}^2} |x|^{p(\beta-l+|\sigma|)} |\partial_x^\sigma u_j(x)|^p dx \Bigg)^{1/p},$$

where $u_j = \xi_j u \circ \gamma_j^{-1}$ on V_j and $u_j = 0$ outside V_j. Using an equivalent atlas and another partition of unity, we obtain an equivalent norm.

Furthermore, let $N^{l,\alpha}_\delta(\partial K^+)$, $\alpha \in (0, 1)$, $\delta \in \mathbb{R}$, $l = 0, 1, \dots$, be the space of functions with the finite norm

$$\sum_{1\leqslant j\leqslant N} \big(\sup_{x\in\mathbb{R}^2} |x|^\delta [u_j]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x\in\mathbb{R}^2} |x|^{\delta-l-\alpha} |u_j(x)| \big),$$

where

$$[u]^\rho_\Omega = \sup_{x,y\in\Omega} \sum_{|\sigma|=\rho} |x - y|^{[\rho]-p} |\partial_x^\sigma u(x) - \partial_y^\sigma u(y)|,$$

$[\rho]$ is the integer part of ρ, $\sigma = (\sigma_1, \sigma_2)$ is a multiindex of order $|\sigma| = \sigma_1+\sigma_2$, $\partial_x^\sigma = \partial^{|\sigma|}/\partial x_1^{\sigma_1} \partial_{x_2}^{\sigma_2}$.

Let Γ be the boundary of a simply connected domain in $\mathbb{R}^3$ with compact closure. We assume that $0 \in \Gamma$ and $\Gamma\backslash 0$ is a smooth surface. Moreover, let Γ coincide with ∂K^+ in the ball B_ε of radius ε centered at 0.

We introduce the spaces $V^l_{p,\beta}(\Gamma)$ and $N^{l,\alpha}_\delta(\Gamma)$ with the norms

$$\|u\|_{V^l_{p,\beta}(\Gamma)} = \|\eta u\|_{V^l_{p,\beta}(\partial K^+)} + \|(1-\eta)u\|_{W^l_p(\Gamma\backslash B_{\varepsilon/2})},$$

$$\|u\|_{N^{l,\alpha}_\delta(\Gamma)} = \|\eta u\|_{N^{l,\alpha}_\delta(\partial K^+)} + \|(1-\eta)u\|_{C^{l,\alpha}(\Gamma\backslash B_{\varepsilon/2})},$$

where η is a function of class $C^\infty(\mathbb{R}^3)$ such that $\eta = 1$ in $B_{\varepsilon/2}$ and $\eta = 0$ outside B_ε.

3.2 Quasilocal estimates

We set

$$(T\psi)(x) = 2(W_0\psi)(x) = \int\limits_S \mathcal{T}(x,\xi)\psi(\xi)\,ds_\xi, \quad \psi \in L_p(S), \quad p > 1$$

and denote by ζ and χ nonnegative functions of class $C^\infty(\mathbb{R}^3)$ which are equal to 1 in the annulus $\{x : \rho < 2|x| < 4\rho\}$ for some $\rho > 0$ and vanish outside the annulus $\{x : \rho < 4|x| < 16\rho\}$. Moreover, we can assume that the following two properties hold:

(i) $|\partial^{\,\sigma}_{x}\zeta(x)| \leqslant C_\sigma \rho^{|-\sigma|}$ and $|\partial^{\,\sigma}_{x}\chi(x)| \leqslant C_\sigma \rho^{-|\sigma|}$,

(ii) one of the inequalities $|x| < d|\xi|$ or $|\xi| < d|x|$ with some $d \in (0,1)$ holds on the support of the function $(x,\xi) \to \zeta(x)(1-\chi(\xi))$.

The argument used in the proof of Lemmas 2.1–2.4 in [2] leads to the following two assertions.

Lemma 3.1. *Let $1 < \rho < \infty$. If $u \in W^l_p(\Gamma\backslash 0) \cap L_p(\Gamma)$ is a solution of the equation $(1+T)u = \varphi$ (or $(1+T^*)u = \varphi$), then*

$$\|\zeta u\|_{V^l_{p,l}(\Gamma)} \leqslant c\|\chi\varphi\|_{V^l_{p,l}(\Gamma)} + \left(\int\limits_\Gamma \left(\int\limits_\Gamma |\chi(x)|(\,|x| + |\xi|\,)^{-2}|u(\xi)|\,ds_\xi\right)^p ds_x\right)^{1/p}.$$

Here, the symbol $*$ denotes the passage to the formally adjoint operator.

Lemma 3.2. *If $u \in C^{l,\alpha}_{\rm loc}(\Gamma\backslash 0) \cap L_p(\Gamma)$, $p > 1$, is a solution of the equation $(1+T)u = \varphi$ (or $(1+T^*)u = \varphi$), then*

$$\|\zeta u\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} \leqslant c\left(\|\chi\varphi\|_{N^{l,\alpha}_{l+\alpha}(\Gamma)} + \int\limits_\Gamma (\,|\xi| + \rho)^{-2}|u(\xi)|\,ds_\xi\right).$$

3.3 Invertibility theorems for the operators $1 + T$ and $1 + T^*$

Let $L_{p,\beta}(\Gamma)$ be the function space equipped with the norm

$$\|u\|_{L_{p,t}(\Gamma)} = \|r^\beta u\|_{L_p(\Gamma)}.$$

Lemma 3.3. *The operators T and T^* are continuous in the space $V^l_{p,\beta+l}(\Gamma)$ for $1 < p < \infty$, $0 < \beta + 2/p < 2$ and in the space $N^{l,\alpha}_{\delta+l+\alpha}$ for $0 < \delta < 2$, $\alpha \in (0,1)$.*

Proof. We show that the operator T is continuous in the space $L^l_{p,\beta}(\Gamma)$ for $0 < \beta + 2/p < 2$. Represent the operator T, defined by the equality

$$(T\varphi)(x) = \int\limits_{\Gamma} \mathcal{T}(x,\xi)\varphi(\xi)\, ds_\xi,$$

as the sum of three integrals T_k over the sets Γ_k, where

$$\Gamma_1 = \{\xi \in \Gamma : \, 2|\xi| < |x|\,\},$$
$$\Gamma_2 = \{\xi \in \Gamma : \, |x| < 2|\xi| < 4|x|\,\},$$
$$\Gamma_3 = \{\xi \in \Gamma : \, |\xi| > 2|x|\,\}.$$

From the definition of $\mathcal{T}(x,\xi)$ it follows that

$$|\mathcal{T}(x,\xi)| \leqslant \begin{cases} c|\xi|^{-2}, & 2|x| < |\xi|, \\ c|x-\xi|^{-2}, & |\xi| < 2|x| < 4|\xi|, \\ c|x|^{-2}, & |x| > 2|\xi|. \end{cases} \tag{3.1}$$

The continuity of T_1 and T_3 in $L^l_{p,\beta}(\Gamma)$ can be obtained as in the proof of Lemma 2.5 in [1]. The continuity of T_2 in $L^l_{p,\beta}(\Gamma)$ is an immediate consequence of the continuity of singular operators in L_p, $p > 1$.

Using the so-called quasilocal estimates (cf. the proof of Lemma 2.6 in [2]), we conclude that T is continuous in $V^l_{p,\beta+l}(\Gamma)$ for $0 < \beta + 2/p < 2$.

Similarly, the continuity of T in the space $N^{l,\alpha}_{\delta+l+\alpha}(\Gamma)$ can be obtained by using quasilocal estimates from the inequality

$$\sup_{x\in\Gamma} |x|^\delta |T\varphi| \leqslant c\|\varphi\|_{N^{0,\alpha}_{\delta+\alpha}(\Gamma)}.$$

The last estimate follows from the continuity of singular operators in the space $C^{0,\alpha}$ of Hölder continuous functions and the estimates (3.1).

Similar arguments lead to the continuity of T^*. □

Lemma 3.4. *Let $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ be the kernels of the operators L and M in Theorem 2.6. If $x \neq y$, then $\mathcal{L}(x,y) = (\mathcal{M}(x,y))^T$, where $\mathcal{M}^Y$ is the adjoint matrix of $\mathcal{M}$.*

Proof. Using the estimates for the kernels $\mathcal{L}(x,y)$ and $\mathcal{M}(x,y)$ (cf. Theorem 2.6), we conclude that the operators $(1+T)^{-1}$ and $(1+T^*)^{-1}$ are continuous in $L_{2,t}(\Gamma)$ and $L_{2,-t}(\Gamma)$ for $-1 < t < 0$ (cf. the proof of Lemma 1.5 in [2]). Hence the required assertion can be obtained by the argument used int he proof of Lemma 3.1 in [2]. □

From Lemma 3.4 and Theorem 2.6 we obtain the following assertion.

Corollary 3.1. *The following estimates hold*:

$$\mathcal{M}(x,y) \leqslant \begin{cases} c\,|x|^{-1}|x|^{-1}(\,|x|/|y|\,)^{\kappa-\varepsilon}, & 2|x| < |y|, \\ c\,|x|^{-1}|x-y|^{-1}, & |y|/2 < |x| < 2|y|, \\ c\,|x|^{-2}, & |x| > 2|y|. \end{cases}$$

These estimates, together with Theorem 2.6 and Lemmas 3.1, 3.2, enable one to establish the continuity of $(1+T)^{-1}$ and $(1+T^*)^{-1}$ in the spaces $V^l_{p,\beta}(\Gamma)$ and $N^{l,\alpha}_{\delta}(\Gamma)$ (cf. the proof of Theorem 2.3 in [2]).

Theorem 3.1. *Let $1 < p < \infty$. Then the following assertions hold.*

(i) *The operator $(1+T)^{-1}$ is continuous in $V^l_{p,\beta+l}(\Gamma)$ for $0 < \beta + 2/p < 1+\kappa$ ($1-\kappa < \beta + 2/p < 2$) as well as in $N^{l,\alpha}_{\delta+l+\alpha+1}(\Gamma)$ for $0 \leqslant \delta < 2$.*

(ii) *The operator $(1+T^*)^{-1}$ is continuous in $V^l_{p,\beta+l}(\Gamma)$ for $1-\kappa < \beta + 2/p < 2$ as well as in $N^{l,\alpha}_{\delta+l+\alpha+1}(\Gamma)$ for $1-\kappa < \delta < 2$.*

3.4 The invertibility theorem for the operator $1+T$ in weighted Hölder spaces with nonhomogeneous norms

Let G^+ be a domain with compact closure $\overline{G}^+$ bounded by Γ, and let $G^- = \mathbb{R}^3 \backslash \overline{G}^+$. We use the notation from Section 2. Consider the transmission problem

$$\begin{aligned} &\Delta^* u = 0 \ \text{ in } \ G, \\ &u^+ - u^- = \varphi \ \text{ on } \ S, \quad (\mathcal{J}u)^+ - (\mathcal{J}u)^- = \varphi \ \text{ on } \ \Gamma\backslash 0. \end{aligned} \tag{3.2}$$

Lemma 3.5. *Let $0 < \alpha - \beta < 1$, and let l be a positive integer. If $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$ and $\psi \in N^{l-1,\alpha}_{\beta+l}(\Gamma)$, then there exists one and only one solution $u \in C^{l,\alpha}_{\beta+l}(G)$ of the problem* (3.2) *and the following estimate holds*:

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G)} \leqslant c\left(\|\varphi\|_{C^{l,\alpha}_{\beta+l}(\Gamma)} + \|\varphi\|_{N^{l-1,\alpha}_{\beta+l}(\Gamma)}\right).$$

Proof. The problem (3.2) with $\varphi = 0$ and $\psi = 0$ is equivalent tot he equation $\Delta^* u = 0$ in $\mathbb{R}^3\backslash 0$. Consider the operator pencil s_γ on the unit sphere S^2 defined by

$$(s_\gamma v)(\theta) = r^{-\gamma+2} \Delta^* r^\gamma v(\theta),$$

where v is a vector-valued function on S^2. It is known that the eigenvalues of s_γ are integers. Moreover, the multiplicity of the eigenvalue $\gamma = 0$ is equal to 3 and the Jordan chains corresponding to $\gamma = 0$ consist only of the eigenfunctions $v = \text{const}$. Taking into account the fact that the problem (3.2) is solvable in weighted Hölder spaces with homogeneous norms and making use of the asymptotic representation of solution of the problem (3.2) near the conic vertex 0 (cf. [5, 8]), we arrive at the assertion of the lemma. □

Theorem 3.2. *Let $\alpha \in (0,1)$, and let l be a positive integer. The operators T and $(1+T)^{-1}$ are continuous in the space $C^{l,\alpha}_{l+\beta}(\Gamma)$ for $0 < \alpha - \beta < 1$ and $0 < \alpha - \beta < \kappa$ respectively.*

Proof. Let $u \in C^{l,\alpha}_{\beta+l}(G)$ be a solution of the problem (3.2) for $\psi = 0$ and $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$. By Lemma 3.5,

$$\|u\|_{C^{l,\alpha}_{\beta+l}(G)} \leqslant c\,\|\varphi\|_{C^{l,\alpha}_{\beta+l}(\Gamma)}. \tag{3.3}$$

From Lemma 2.7 it follows that u can be represented in the form $u = W(u^+ - u^-) = W\varphi$, where $W\varphi$ is the double layer potential. From this and (3.3), together with the relation $T\varphi =$

$2(W\varphi)^+ - \varphi$ (cf. Lemma 2.5) we conclude that the operator T is continuous in $C^{l,\alpha}_{\beta+l}(\Gamma)$ for $0 < \alpha - \beta < 1$.

The continuity of $(1+T)^{-1}$ is an immediate consequence of the representation for $(1+T)^{-1}$ in terms of inverse operators of boundary value problems (Theorem 2.4) and the continuity of these operators (Lemmas 2.1 and 2.4). □

3.5 Pointwise estimates for the derivatives of the kernels $\mathcal{L}(x, y)$ and $\mathcal{M}(x, y)$ of the operators L and M

Theorems 3.1, 3.2 and Corollary 3.1 lead to the following theorem (cf. the proof of Theorem 3.1 in [2]).

Theorem 3.3. *Let points x and y lie in the same coordinate neighborhood. Then*

$$|\partial^{\,\sigma}_{x'}\partial^{\,\tau}_{y'}\mathcal{L}(x,y)| \leqslant \begin{cases} c\,|x|^{-|\sigma|}|y|^{-2-|\tau|}, & 2|x| < |y|, \\ c\,|x|^{-1}|x-y|^{-1-|\tau|-|\sigma|}, & |y| < 2|x| < 4|y|, \\ c\,|x|^{-1-|\sigma|-\kappa+\varepsilon}|y|^{-1-|\tau|+\kappa-\varepsilon}, & |x| > 2|y|, \end{cases} \tag{3.4}$$

where ε is any small positive number, x' and y' are the local coordinates of x and y.

3.6 Continuity of $(1+T)^{-1}$ in $C^{0,\alpha}$

Theorem 3.4. *The operators T and $(1+T)^{-1}$ are continuous in $C^{\,0,\alpha}(\Gamma)$ for $0 < \alpha < 1$ and for $0 < \alpha < \kappa$ respectively.*

Proof. Let $\varphi \in C^{\,0,\alpha}(\Gamma)$. The argument used in the proof of Lemma 3.3 shows that

$$\sup_{x\in\Gamma} |(T(\varphi - \varphi(0)))(x)| \leqslant c\,\|\varphi\|_{C^{0,\alpha}(\Gamma)}.$$

We used the relation $\varphi(y) - \varphi(0) = O(\,|y|^\alpha)$ and the fact that the singular operator is bounded in the space of Hölder continuous functions.

According to the identity $T(\varphi(0)) = \varphi(0)$, we have

$$\sup_{x\in\Gamma} |\,(T\varphi)(x)| \leqslant c\,\|\varphi\|_{C^{0,\alpha}(\Gamma)}.$$

It remains to establish the inequality

$$[T\varphi]^\alpha_\Gamma \leqslant c\,[\varphi]^\alpha_\Gamma. \tag{3.5}$$

Let $x, z \in \Gamma$, and let $\rho = 2|x - z|$. Taking into account the equality

$$\int_\Gamma \mathcal{T}(x,\xi)\,ds_\xi = 1$$

on Γ, where 1 is the identity 3×3 matrix, we write

$$(T\varphi)(x) - (T\varphi)(z) = \sum_{1\leqslant k\leqslant 4} I_k,$$

where

$$I_1 = \int\limits_{\Gamma\cap B(r,x)} \mathcal{T}(x,\xi)(\varphi(\xi)-\varphi(x))\,ds_\xi,$$

$$I_2 = -\int\limits_{\Gamma\cap B(r,x)} \mathcal{T}(x,\xi)(\varphi(\xi)-\varphi(z))\,ds_\xi,$$

$$I_3 = \int\limits_{\Gamma\cap B(r,x)} \mathcal{T}(z,\xi)(\varphi(\xi)-\varphi(z))\,ds_\xi,$$

$$I_4 = \int\limits_{\Gamma\setminus B(r,x)} (\mathcal{T}(x,\xi)-\mathcal{T}(z,\xi))(\varphi(\xi)-\varphi(x))\,ds_\xi.$$

Since the integral

$$\int\limits_{\Gamma\cap B(r,x)} \mathcal{T}(z,\xi)\,d_\xi$$

is bounded, the estimate (3.5) follows from the inequalities

$$|I_1|+|I_2| \leqslant c\,[\varphi]^\alpha_\Gamma \int\limits_{\Gamma\cap B(r,x)} |x-\xi|^{-2+\alpha} ds_\xi \leqslant cr^\alpha[\varphi]^\alpha_\Gamma,$$

$$|I_4| \leqslant cr[\varphi]^\alpha_\Gamma \int\limits_{\Gamma\setminus B(r,x)} |x-\xi|^{-3+\alpha} ds_\xi \leqslant cr^\alpha[\varphi]^\alpha_\Gamma.$$

The continuity of the operator $(1+T)^{-1}$ can be obtained in a similar way by using (3.4) (cf. the proof of Theorem 3.2 in [2]). □

Remark 3.1. If the cone K^+ cannot be prescribed in a Cartesian coordinate system, then the estimates for the kernel $\mathcal{L}(x,y)$ in Theorem 3.3 hold if κ is replaced by $\kappa' = \min\{0,\kappa\}$. The term $(\,|x|/|y|)^{\kappa'-\varepsilon}$ should be added to the right-hand side of (3.4) for $2|x| < |y|$. Theorem 3.1 takes the following form.

Theorem 3.5. 1. *Suppose that* $1 < p < \infty$, $-\kappa' < \beta + 2/p < 1 + \kappa$. *Then the operators* $(1+T)^{-1}$ *and* $(1+T^*)^{-1}$ *are continuous in the spaces* $L_{p,\beta}(\Gamma)$ *and* $L_{p,\beta+1}(\Gamma)$ *respectively.*

2. *Suppose that* $1 < p < \infty$, $-\kappa' < \beta+2/p < 1+\kappa'$, $\alpha \in (0,1)$, $-\kappa' < \delta < 1+\kappa'$, $l = 0,1,\ldots$. *Then the operators* $(1+T)^{-1}$ *and* $(1+T^*)^{-1}$ *are continuous in the spaces* $V^l_{p,\beta+l}(\Gamma)$, $N^{l,\alpha}_{\delta+l+\alpha}(\Gamma)$ *and* $V^l_{p,\beta+l+1}(\Gamma)$, $N^{l,\alpha}_{\delta+l+\alpha+1}(\Gamma)$ *respectively.*

References

1. N. V. Grachev and V. G. Maz'ya, "Estimates for kernels of inverse operators of integral equations of elasticity theory on surfaces with conic points" [in Russian], *Probl. Mat. Anal.* **66**, 39-65 (2012); English transl.: *J. Math. Sci., New York* **186**, No. 2, 179-208 (2012).

2. N. V. Grachev and V. G. Maz'ya, "Invertibility of boundary integral operators of elasticity on surfaces with conic points" [in Russian], *Probl. Mat. Anal.* **66**, 67-88 (2012); English transl.: *J. Math. Sci., New York* **186**, No. 2, 209-233 (2012).

3. V. Kozlov and V. Maz'ya "Spectral properties of operator pencils generated by elliptic boundary value problems in a cone" [in Russian], *Funkt. Anal. Prilozh.* **22**, No. 2, 38–46 (1988); English transl.: *Funct. Anal. Appl.* **22**, No. 2, 114–121 (1988).

4. V. Maz'ya, "Boundary integral equations," In: *Encycl. Math. Sci.* **27**, 127–223 (1991).

5. V. Kondrat'ev, "Boundary value problems for elliptic equations in domains with conical or angular points" [in Russian], *Tr. Mosk. Mat. O-va* **16**, 209–292 (1967); English transl.: *Trans. Moscow Math. Soc.* **16**, 227–313 (1967).

6. V. Maz'ya and B. Plamenevskii, "The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries," *Z. Anal. Anwen.* I. **2**, No. 4, 335–359 (1983); II. **2**, No. 6, 523–551 (1983).

7. V. Maz'ya and B. Plamenevskii, "Properties of solutions of three-dimensional problems of the theory of elasticity and hydrodynamics in domains with isolated singular points" [in Russian], *Din. Sploshnoj Sredy* **50**, 99–120 (1981); English transl.: *Transl. Am. Math. Soc.* **123**, 109–123 (1984).

8. V. Maz'ya and B. Plamenevskii, "On the asymptotics of the fundamental solutions of elliptic boundary value problems in regions with conical points" [in Russian], *Probl. Mat. Anal.* **1**, 100–145 (1979); Engl. transl.: *Sel. Math. Sov.* **4**, 363–397 (1985).

9. V. Maz'ya and B. Plamenevskii, "L_p-estimates of the solutions of elliptic boundary value problems in domains with edges" [in Russian], *Tr. Moskov. Mat. O-va* **37**, 49–93 (1978); English transl.: *Trans. Mosc. Math. Soc.* **1**, 49–97 (1980).

10. V. Maz'ya and V. D. Sapozhnikova, "A remark on the regularization of the singular system of isotropic elasticity" [in Russian] *Vestnik Leningr. Univ.* No. 7, 165–167 (1964): Correction: No. 19, 160 (1977).

11. V. D. Kupradze, *Potential Methods in Elasticity Theory* [in Russian], Fizmatgiz, Moscow (1963).

12. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity* [in Russian], Nauka, Moscow (1976).

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