Bojarski–Meyers Estimate for Solutions to Zaremba Problem

Yurij A. Alkhutov[1†], Gregory A. Chechkin[2,3,4,5*†] and Vladimir G. Maz'ya[6,7*†]

[1]A.G. and N.G. Stoletov Vladimir State University, Stroitelej st., 11, Vladimir, Russia.
[2*]M.V. Lomonosov Moscow State University, Leninskie gory, 1, Moscow, Russia.
[3]Institute of Mathematics with Computing Center - Subdivision of the Ufa Federal Research Center of Russian Academy of Science, Ufa, Russia.
[4]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.
[5]ORCID 0000-0002-7654-5803.
[6]Linköpings Universitet, Linköping, Sverige.
[7]RUDN University, Miklukho-Maklaya St, 6 , Moscow, Russia.

*Corresponding author(s). E-mail(s): chechkin@mech.math.msu.su; vladimir.mazya@liu.se;
Contributing authors: yurij-alkhutov@yandex.ru;
[†]These authors contributed equally to this work.

Abstract

The variational solution to the Zaremba problem for divergent linear second order elliptic equation with measurable coefficients is considered. The problem is set in a local Lipschitz graph domain. An estimate in $L_{2+\delta}$, $\delta > 0$, for the gradient of a solution is obtained. An example of the problem with the Dirichlet data supported by a fractal set of zero $(n-1)$-dimensional measure and non-zero p-capacity, $p > 1$, is constructed.

Keywords: Bojarski–Meyers estimates, Zaremba problem, p-capacity, Lipschitz graph domain

MSC Classification: 35A23 , 35D30 , 35J15 , 35J25

1 Introduction

The Zaremba boundary value problem for elliptic equations is of interest both from purely mathematical and applied points of view. It appears in elasticity theory of bodies with partially fasten boundaries, in modelling of conductivity of corroded electrical contacts, in biological problems connected with permeability of perforated membranes, in percolation problems etc. One of applications is the Zaremba problem with rapidly alternating boundary conditions (see, for instance, [26], [27] and the references in this monograph).

For the Laplace equation, the Zaremba problem in a three-dimensional bounded domain with a smooth boundary and inhomogeneous Dirichlet and Neumann conditions was first considered in [1], where the classical solvability of the problem was established by methods of potential theory. Study of properties of solutions to the Zaremba problem for second-order elliptic equations with variable regular coefficients goes back to [2]. In particular, it is shown in [2] that at the junction of the Dirichlet and Neumann data, the smoothness of the solutions is lost. For divergent second-order uniformly elliptic equations with measurable coefficients, integral and pointwise estimates for solutions of the Zaremba problem under fairly general assumptions about the domain boundary are treated in [3] (see also [4] and [5]).

The question of higher integrability of the gradient of solutions to elliptic equations is classical (see, for instance, [6], [7], [8], [9], [10], [11] et al, where this phenomenon was treated).

The L_p-integrability, $p > 2$, of the gradient of a solution to the Zaremba problem, was studied in [12]. The authors of [12] deal with the Poisson equation in a square with frequent change of the Dirichlet and Neumann boundary conditions. Such estimates are useful in the homogenization theory; these estimates improve the rate of convergence of solutions to the given problem with small parameter to solutions of the homogenized (limit) problem (see [13] for a similar problem in a domain perforated along the boundary).

Some results of the present paper were announced in [14].

The present paper is devoted to the Zaremba problem for an elliptic equation in a bounded Lipschitz graph domain $D \in \mathbb{R}^n$, with the operator

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u). \tag{1}$$

Here the matrix $a(x) = \{a_{ij}(x)\}$ has measurable components and satisfies $a_{ij} = a_{ji}$ and

$$\alpha^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \alpha|\xi|^2 \text{ for almost all } x \in D \text{ and for all } \xi \in \mathbb{R}^n \tag{2}$$

with some $\alpha > 1$. Before formulating the Zaremba problem, we define the Sobolev space of functions $W_2^1(D, F)$, where $F \subset \partial D$ is a closed set, as a completion of functions infinitely differentiable in the closure of D and equal to zero in a neighborhood of F, by norm

$$\| u \|_{W_2^1(D,F)} = \Big(\int_D v^2\, dx + \int_D |\nabla v|^2\, dx \Big)^{1/2}.$$

A priori, the functions $v \in W_2^1(D,F)$ are assumed to satisfy the inequality

$$\int_D v^2\, dx \leq C \int_D |\nabla v|^2\, dx. \tag{3}$$

We consider a variational statement of the classical Zaremba problem

$$\mathcal{L}u = l \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \tag{4}$$

where $G = \partial D \setminus F$, and $\frac{\partial u}{\partial \nu} = \sum\limits_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \nu_i$ is the outward conormal derivative of the function u, and l is a linear functional on the space $W_2^1(D,F)$.

A variational solution to problem (4) is a function $u \in W_2^1(D,F)$ subject the equality

$$\int_D a\nabla u \cdot \nabla \varphi\, dx = -l(\varphi), \tag{5}$$

where $\varphi \in W_2^1(D,F)$ is an arbitrary function.

By (3), the space $W_2^1(D,F)$ can be endowed with a norm containing only the gradient. Then each element from the Sobolev space can be put into a one-to-one isometric correspondence with its gradient, that is, an element from $(L_2(D))^n$. Using the Hahn-Banach theorem, as, for example, in section 1.1.15 from the monograph [15], on the form of a functional in Sobolev spaces, one can show that the functional l can be written as

$$l(\varphi) = -\sum_{i=1}^{n} \int_D f_i \varphi_{x_i}\, dx, \tag{6}$$

where $f_i \in L_2(D)$. Hence, (5) can be rewritten in the form

$$\int_D a\nabla u \cdot \nabla \varphi\, dx = \int_D f \cdot \nabla \varphi\, dx. \tag{7}$$

The Riesz representation Theorem combined with inequality (3) for functions $v \in W_2^1(D,F)$ implies the unique solvability of problem (7).

Let us discuss the inequality (3). For this aim we need the notion of p-capacity. Given a compact set $K \subset \mathbb{R}^n$, the capacity $C_p(K)$, with $1 \leq p < n$, is defined by

$$C_p(K) = \inf \left\{ \int\limits_{\mathbb{R}^n} |\nabla\varphi|^p \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geq 1 \text{ on } K \right\}. \tag{8}$$

Denoting by $\mathcal{Q}_d$ an open cube with edge length d and faces parallel to the coordinate axes, assuming that the Lipschitz domain D has diameter d and $D \subset \mathcal{Q}_d$. Next, we need the notion of the capacity $C_p(K, \mathcal{Q}_{2d})$ of the compact set $K \subset \overline{\mathcal{Q}}_d$ with respect to the cube $\mathcal{Q}_{2d}$, which is defined by

$$C_p(K, \mathcal{Q}_{2d}) = \inf \left\{ \int\limits_{\mathcal{Q}_{2d}} |\nabla\varphi|^p \, dx : \ \varphi \in C_0^\infty(\mathcal{Q}_{2d}), \ \varphi \geq 1 \text{ on } K \right\}. \tag{9}$$

Due to Maz'ya (see [15]) for the functions $v \in W_2^1(D, F)$ the inequality (3) holds if and only if $C_2(F) > 0$ for $n > 2$ and $C_2(F, \mathcal{Q}_{2d}) > 0$ for $n = 2$.

2 Gradient estimate

Let $x_0 \in \mathbb{R}^n$ and let $B_r^{x_0}$ stand for an open n-dimensional ball of radius r, centered at x_0. We choose $p = 2n/(n+2)$ for $n > 2$ and $p = 3/2$ for $n = 2$. Let us assume the following condition to be valid: for an arbitrary point $x_0 \in F$ the inequality

$$C_p(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-p} \tag{10}$$

holds for $r \leq r_0$, where r_0 is some positive constant. Here the positive constant c_0 is independent of x_0 and r.

We assume that for every point $x_0 \in \partial D$ there exists an open cube Q centered at x_0 whose faces are parallel to coordinate axes, the edge length does not depend on x_0, and in some Cartesian coordinate system with origin x_0 the set $Q \cap \partial D$ is the graph of a Lipschitz function $x_n = g(x_1, \ldots, x_{n-1})$ with a Lipschitz constant independent of x_0. The edge length of such cubes is assumed to be $2R_0$, and the Lipschitz constant of the corresponding functions g is denoted by L. Without loss of generality we suppose that the set $Q \cap D$ is located above the graph of the function g.

Let us now formulate the main assertion, assuming that the constant r_0 from condition (10) does not exceed the constant R_0 involved in the definition of the Lipschitz domain.

Theorem 1 *If $f \in \Big(L_{2+\delta_0}(D)\Big)^n$, where $\delta_0 > 0$, then there are positive constants $\delta(n, \delta_0) < \delta_0$ and C such that for solution of problem (4) the following estimate holds:*

$$\int\limits_D |\nabla u|^{2+\delta} dx \leq C \int\limits_D |f|^{2+\delta} \, dx, \tag{11}$$

where C depends only on δ_0, space dimension n, ellipticity constant α from (2), c_0 from (10), and also constants L and R_0 involved in the definition of the Lipschitz property of the domain D.

Remark 1 Denote by $mes_{n-1}(E)$ the $(n-1)$-dimensional Lebesque measure of the set $E \subset \partial D$. Instead of condition (10) one can use

$$mes_{n-1}(F \cap \overline{B}_r^{x_0}) \ge c_0 r^{n-1}. \tag{12}$$

We show that condition (29) implies (10). Let us use the S.L. Sobolev–V.P. Il'in inequality (see [16])

$$\|u\|_{L_q(\mathbb{R}^{n-1})} \le K(n,p)\|\nabla u\|_{L_p(\mathbb{R}^n)}, \qquad q = \frac{p(n-1)}{n-p},$$

where u is an arbitrary function from $C_0^\infty(\mathbb{R}^n)$. Setting $u = 1$ on a compact set S and minimizing the right hand side, we arrive at

$$(mes_{n-1}(S))^{\frac{n-p}{n-1}} \le K(n,p)\, C_p(S). \tag{13}$$

Hence, (10) holds.

The example, that condition (29) does not generally follow from (10) is given at the beginning of section 3.

Proof The proof of the Theorem is based on the inner and near-boundary estimates of the gradient of solutions to problem (4). First, the higher integrability of the gradient of the solution to problem (4) is obtained for a solution in a neighborhood of the boundary of the domain D.

Setting $Q_{R_0} = \{x : \ |x_i| < R_0, \ i = 1, \dots, n\}$ and using the definition of a Lipschitz domain, for an arbitrary boundary point $x_0 \in \partial D$ consider a local Cartesian coordinate system with the origin at x_0 such that the part of the boundary ∂D falling into the Q_{R_0} cube is given in this coordinate system by the equation $x_n = g(x')$, where $x' = (x_1, \dots, x_{n-1})$, and g is a Lipschitz function with the Lipschitz exponent L. It is assumed that the region $D_{R_0} = Q_{R_0} \cap D$ is located on the set of those points where $x_n > g(x')$. Next, we introduce a new coordinate system in Q_{R_0} by performing a non-degenerate transformation of variables

$$y' = x', \ y_n = x_n - g(x') \tag{14}$$

It is clear that the part of the boundary $Q_{R_0} \cap \partial D$ is transformed into a piece of the hyperplane

$$P_{R_0} = \{y : \ |y_i| < R_0, \ i = 1, \dots, n-1, \ y_n = 0\}$$

For what follows, we note that the domain $\widetilde{Q}_{R_0}$ contains the cube

$$K_{R_0} = \{y : \ |y_i| < (1 + \sqrt{n-1}L)^{-1} R_0, \ i = 1, \dots, n\}. \tag{15}$$

Indeed, if $y \in \widetilde{Q}_{R_0}$ and $|y_i| < \delta R_0$ for some $\delta \in (0,1)$ and $i = 1, \dots, n-1$, then

$$y_n \in (-R_0 - g(y'), R_0 - g(y')),$$

and since the function g is Lipschitz and $g(0) = 0$, then $|g(y')| \le L|y'| < \sqrt{n-1}L\delta R_0$. Hence,

$$(-R_0(1 - \sqrt{n-1}L\delta), R_0(1 - \sqrt{n-1}L\delta)) \subset (-R_0 - g(y'), R_0 - g(y')).$$

Choosing here $\delta = \frac{1}{1+\sqrt{n-1}L}$, we arrive at $K_{R_0} \subset \widetilde{Q}_{R_0}$.

In the semicube $K_{R_0}^+ = K_{R_0} \cap \{y : \ y_n > 0\}$ contained in the image of the domain $D \cap Q_{R_0}$, problem (4) takes the form

$$\widetilde{\mathcal{L}}u = \tilde{l} \quad \text{in} \quad K_{R_0}^+, \quad u = 0 \text{ on } \widetilde{F}_{R_0}, \quad \frac{\partial u}{\partial \widetilde{\nu}} = 0 \text{ on } \widetilde{G}_{R_0}. \tag{16}$$

We keep the original notation for its solution. Here

$$\widetilde{\mathcal{L}}u := \operatorname{div}(b(y)\nabla u) \tag{17}$$

is uniformly elliptic operator with symmetric matrix $b(y) = \{b_{ij}(y)\}$, satisfying the condition

$$\beta^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \text{ for almost all } y \in K^+_{R_0} \text{ and for all } \xi \in \mathbb{R}^n, \tag{18}$$

where the constant β depends only on α, which appear in (2), and the Lipschitz constant L of g. The vector function f in (6), becomes $\tilde{f}$ defined by the formulae

$$\tilde{f}(y) = (\tilde{f}_1(y), \dots, \tilde{f}_n(y)), \text{ where } \tilde{f}_i(y) = f_i(y', y_n + g(y')) \text{ as } i = 1, \dots, n-1,$$
$$\tilde{f}_n(y) = \sum_{i=1}^{n-1} \frac{\partial g(y')}{\partial y_i} f_i(y', y_n + g(y')) + f_n(y', y_n + g(y')). \tag{19}$$

The sets $\widetilde{F}_{R_0}$ and $\widetilde{G}_{R_0}$ from (16) are such that $\widetilde{F}_{R_0} = \widetilde{F} \cap P_{R_0} \cap K_{R_0}$ and $\widetilde{G}_{R_0} = \widetilde{G} \cap P_{R_0} \cap K_{R_0}$, where $\widetilde{F}$, $\widetilde{G}$ are the images of the sets $F \cap Q_{R_0}$ and $G \cap Q_{R_0}$ respectively, and $\frac{\partial u}{\partial \widetilde{\nu}}$ means the outer conormal derivative of the function u associated with operator (17) .

We extend the function u satisfying (16), evenly with respect to the hyperplane $\{y : \ y_n = 0\}$. The continued function, for which we again retain the previous notation, will satisfy the following relation:

$$\widetilde{\mathcal{L}}_1 u = l_h \quad \text{in} \quad K_{R_0} \setminus \widetilde{F}_{R_0}, \quad u = 0 \text{ on } \widetilde{F}_{R_0}. \tag{20}$$

Here

$$\widetilde{\mathcal{L}}_1 u := \operatorname{div}(c(y)\nabla u),$$

positive definite matrix $c = \{c_{ij}(y)\}$ is such that the elements $c_{jn}(y) = c_{jn}(y)$ for $j \neq n$ are odd extensions of the elements of the matrices $b_{jn}(y)$ from (17), and all other elements of $c_{ij}(y)$ are even extensions of $b_{ij}(y)$. The components of the vector-function $h = (h_1, \dots, h_n)$ in (20), appearing in the representation of the corresponding functional l_h, are determined by similar relations: its the components $h_i(y)$ for $i = 1, \dots, n-1$ are even extensions of the components $\tilde{f}_i(y)$ from (16), and $h_n(y)$ are odd extension $\tilde{f}_n(y)$. It is clear that the function $u \in W^1_2(K_{R_0})$ from (20) satisfies the integral relation (see (7))

$$\int_{K_{R_0}} c(y)\nabla u \cdot \nabla\varphi \, dy = \int_{K_{R_0}} h \cdot \nabla\varphi \, dy \tag{21}$$

for all functions $\varphi \in W^1_2(K_{R_0}, \widetilde{F}_{R_0})$. Denote by $Q^{y_0}_R$ an open cube centered at the point y_0 with edges of length $2R$ that are parallel to the coordinate axes. It is assumed below that

$$y_0 \in K_{R_0/2} \setminus \partial K_{R_0/2}, \text{ where } R \leq \frac{1}{2} dist(y_0, \partial K_{R_0/2}) \tag{22}$$

and introduce a notation

$$⨍_{Q^{y_0}_R} f \, dx = \frac{1}{|Q^{y_0}_R|} \int_{Q^{y_0}_R} f \, dx,$$

where $|Q^{y_0}_R|$ stands for the n-dimensional Lebesque measure of the cube $Q^{y_0}_R$.

First, consider the case when $Q^{y_0}_{3R/2} \subset K_{R_0} \setminus \widetilde{F}_{R_0}$ and put the test function $\varphi = (u-\lambda)\eta^2$ in (21), where

$$\lambda = \fint_{Q^{y_0}_{3R/2}} u,\ dy,$$

and the cut-off function $\eta \in C_0^\infty(Q^{y_0}_{3R/2})$ satisfies $0 < \eta \le 1$, $\eta = 1$ in $Q^{y_0}_R$ and $|\nabla \eta| \le CR^{-1}$. As a result, by (21), the Cauchy inequality, and the ellipticity condition (18), we arrive at the Caccioppoli-type inequality (see [17])

$$\int_{Q^{y_0}_R} |\nabla u|^2\,dy \le C(n,\alpha,L)\biggl(R^{-2} \int_{Q^{y_0}_{3R/2}} (u-\lambda)^2\,dy + \int_{Q^{y_0}_{3R/2}} |h|^2\,dy \biggr). \tag{23}$$

Further, assuming, that $p = 3/2$ for $n = 2$ and $p = 2n/(n+2)$ for $n > 2$, from the Poincaré–Sobolev inequality

$$\biggl(\fint_{Q^{y_0}_{3R/2}} (u-\lambda)^2\,dx \biggr)^{1/2} \le C(n)R\biggl(\fint_{Q^{y_0}_{3R/2}} |\nabla u|^p\,dx \biggr)^{1/p}$$

together with (23) we derive

$$\biggl(\fint_{Q^{y_0}_R} |\nabla u|^2\,dy \biggr)^{1/2} \le C(n,\alpha,L)\biggl(\biggl(\fint_{Q^{y_0}_{2R}} |\nabla u|^p\,dy \biggr)^{1/p} + \biggl(\fint_{Q^{y_0}_{2R}} |h|^2\,dy \biggr)^{1/2} \biggr). \tag{24}$$

Let us now consider the case when $Q^{y_0}_{3R/2} \cap \widetilde{F}_{R_0} \ne \emptyset$. Choosing a test function $\varphi = u\eta^2$ in the integral identity (21), we arrive at (23) with $\lambda = 0$, i.e.

$$\int_{Q^{y_0}_R} |\nabla u|^2\,dy \le C(n,\alpha,L)\biggl(R^{-2} \int_{Q^{y_0}_{2R}} u^2\,dy + \int_{Q^{y_0}_{2R}} |h|^2\,dy \biggr). \tag{25}$$

Let us estimate the first integral on the right-hand side of (25). Since $Q^{y_0}_{3R/2} \cap \widetilde{F}_{R_0} \ne \emptyset$, it follows that there is a point $z_0 \in Q^{y_0}_{3R/2} \cap \widetilde{F}_{R_0}$ such that $\overline{Q}^{z_0}_{R/2} \subset \overline{Q}^{y_0}_{2R}$. Denote by $z \in F \cap Q_{R_0}$ the inverse image of z_0 under transformation (14). Note that the inverse image of the closed cube $\overline{Q}^{z_0}_{R/2}$ contains the closed ball $\overline{B}^z_{cR}$, where $c = c(L,n) > 0$. By (10), the inequality $C_p(F \cap \overline{B}^z_{cR}) \ge C(L,n,c_0)R^{n-p}$ holds. From here and from the definition of the capacity given in (8), it follows that $C_p(\widetilde{F}_{R_0} \cap \overline{Q}_{2R}) \ge C(L,n,c_0)R^{n-p}$. Therefore, due to Maz'ya (see [15, §14.1.2])

$$\biggl(\fint_{Q^{y_0}_{2R}} u^2\,dy \biggr)^{1/2} \le C(n,p,L,c_0)R\biggl(\fint_{Q^{y_0}_{2R}} |\nabla u|^p\,dy \biggr)^{1/p}. \tag{26}$$

Remark 2 If condition (29) is satisfied, then by the estimate of Proposition 4 in [15, §13.1.1]) we again arrive at estimate (26). Thus, from (25) we obtain the required inequality (24).

In further analysis we use the generalized Gehring Lemma (see [18], [19], and also [20, Chapter VII]).

Proposition 2 (Generalized Gehring Lemma) *Let* (22) *and estimate* (24) *with* $h \in L_{2+\delta_0}(K_{R_0})$ *hold. Then,*

$$\int_{K_{R_0/4}} |u|^{2+\delta}\, dy \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_{K_{R_0/2}} |h|^{2+\delta}\, dy, \tag{27}$$

where $\delta = \delta(n,p,\delta_0)$ *is a positive constant.*

From estimate (26), which is valid for all cubes $Q_R^{y_0}$, Proposition 2, and from (15), we get (27) for $h \in L_{2+\delta_0}(K_{R_0})$, $\delta_0 > 0$. Since the function u is even with respect to the hyperplane $\{y:\ y_n = 0\}$, inequality (27) can be rewritten in the form

$$\int_{K^+_{R_0/4}} |u|^{2+\delta}\, dy \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_{K^+_{R_0/2}} |\tilde{f}|^{2+\delta}\, dy. \tag{28}$$

Make the transformation inverse to (14). We see that the inverse image of the semicube $K^+_{R_0/2}$ is contained in the set D_{R_0}, and the inverse image of the semicube $K^+_{R_0/4}$ contains the set $D_{\theta R_0}$, where $\theta = \theta(n,L) > 0$. By (19) and (28) we have

$$\int_{D_{\theta R_0}} |u|^{2+\delta}\, dx \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_{D_{R_0}} |f|^{2+\delta}\, dx.$$

Passing here to the Cartesian coordinate system with the origin at the point $x_0 \in \partial D$, we obtain

$$\int_{D\cap Q^{x_0}_{\theta R_0}} |u|^{2+\delta}\, dx \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_{D\cap Q^{x_0}_{R_0}} |f|^{2+\delta}\, dx.$$

Since $x_0 \in \partial D$ is an arbitrary boundary point and the boundary of ∂D is compact, it follows that there exists a finite covering of ∂D by the cubes centered in $t_i \in \partial D$, $i = 1,\dots N$ such that the closed set

$$\mathcal{D}_{\theta_1 R_0} = \{x \in D:\ dist(x,\partial D) \le \theta_1 R_0\}, \quad \theta_1 = \theta_1(n,L) > 0,$$

is contained in the union of sets $D \cap Q^{t_i}_{\theta R_0}$, where $t_i \in \partial D$. Therefore, summing up the inequalities

$$\int_{D\cap Q^{t_i}_{\theta R_0}} |u|^{2+\delta}\, dx \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_{D\cap Q^{t_i}_{R_0}} |f|^{2+\delta}\, dx,$$

we get the estimate

$$\int_{\mathcal{D}_{\theta_1 R_0}} |u|^{2+\delta}\, dx \le C(n,\alpha,\delta_0,c_0,L,R_0) \int_D |f|^{2+\delta}\, dx.$$

The inner estimate

$$\int_{D\setminus\mathcal{D}_{\theta_1 R_0}} |u|^{2+\delta}\, dx \le C(n,\alpha,\delta_0,R_0) \int_D |f|^{2+\delta}\, dx$$

is well known and essentially follows from [7]. Finally, combining the last two inequalities, we arrive at (11). The proof is complete. □

Remark 3 Denote by $mes_{n-1}(E)$ the $(n-1)$-dimensional Lebesque measure of the set $E \subset \partial D$. Instead of condition (10) one can use

$$mes_{n-1}(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-1}. \tag{29}$$

We show that condition (29) implies (10). Let us use the S.L. Sobolev–V.P. Il'in inequality (see [16])

$$\|u\|_{L_q(\mathbb{R}^{n-1})} \leq K(n,p)\|\nabla u\|_{L_p(\mathbb{R}^n)}, \qquad q = \frac{p(n-1)}{n-p},$$

where u is an arbitrary function from $C_0^\infty(\mathbb{R}^n)$. Setting $u = 1$ on a compact set S and minimizing the right hand side, we arrive at

$$(mes_{n-1}(S))^{\frac{n-p}{n-1}} \leq K(n,p)\, C_p(S). \tag{30}$$

Hence, (10) holds.

3 Example of the set *F*

In this section we give an examples of the set F with zero $(n-1)$-dimensional measure, satisfying condition (29). For simplicity, we restrict ourselves to the planar case, although the same arguments work for the n-dimensional case.

We introduce several auxiliary function spaces. For $p \geq 1$ and $0 < l \leq 1$, we define the Besov space B_p^l as the completion of the set $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\Big(\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} |\varphi(x+y) - 2\varphi(x) + \varphi(x-y)|^p |y|^{-n-pl}\, dx\, dy \Big)^{1/p} + \| \varphi \|_{L_p(\mathbb{R}^n)} .$$

For $1 < p < \infty$ and $l > 0$, we also introduce the Risz potential spaces h_p^l and the Bessel potential spaces H_p^l as the completion of $\varphi \in C_0^\infty(\mathbb{R}^n)$ in the norms

$$\| \varphi \|_{h_p^l} = \| (-\Delta)^{l/2} \varphi \|_{L_p(\mathbb{R}^n)}, \quad \| \varphi \|_{H_p^l} = \| (-\Delta + 1)^{l/2} \varphi \|_{L_p(\mathbb{R}^n)} .$$

Here Δ is the Laplacian and

$$(-\Delta)^{l/2} = F^{-1}|\xi|^l F, \quad (-\Delta+1)^{l/2} = F^{-1}(1+|\xi|^2)^{l/2} F,$$

where $F\varphi(\xi)$ is the inverse Fourier transform

$$F\varphi(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(x)\, dx.$$

It is well known (see Corollary 1 of Theorem 1 [15, Ch.10]), that

$$C_1(n,p) \| \nabla\varphi \|_{L_p(\mathbb{R}^n)} \leq \| (-\Delta)^{1/2} \varphi \|_{L_p(\mathbb{R}^n)} \leq C_2(n,p) \| \nabla\varphi \|_{L_p(\mathbb{R}^n)} \tag{31}$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

For each function space $S_p^l = H_p^l$, or $S_p^l = B_p^l$, or $S_p^l = h_p^l$, we define the capacity of the compact set $K \subset \mathbb{R}^n$ by

$$\mathrm{cap}\,(K, S_p^l) = \inf\{\| u \|_{S_p^l}^p \colon \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geq 1 \text{ on } K\}.$$

We are interested only in the case when $0 < l \leq 1$. The following relations between different capacities are known.

(i) If $diam(K) \leq 1$ and $pl < n$, then (see [21])

$$\mathrm{cap}\,(K, H_p^l) \sim \mathrm{cap}\,(K, h_p^l). \tag{32}$$

(ii) If $1 < p < \infty$, then (see Proposition 4.4.4 in [22])

$$\mathrm{cap}\,(K, H_p^l) \sim \mathrm{cap}\,(K, B_p^l). \tag{33}$$

(iii) If $K \subset \mathbb{R}^n$ and $1 < p < \infty$, then (see [23])

$$\mathrm{cap}\,(K, B_p^l(\mathbb{R}^n)) \sim \mathrm{cap}\,(K, H_p^{l+1/p}(\mathbb{R}^{n+1})). \tag{34}$$

If $K \subset \mathbb{R}^n$, $diam(K) \leq 1$ and $pl < n$, then formulae (32)–(34) imply

$$\mathrm{cap}\,(K, h_p^l(\mathbb{R}^{n+1})) \sim \mathrm{cap}\,(K, H_p^{l-1/p}(\mathbb{R}^n)). \tag{35}$$

Taking in (35) $n = 1$, $l = 1$ and $1 < p < 2$, we get

$$\mathrm{cap}\,(K, h_p^1(\mathbb{R}^2)) \sim \mathrm{cap}\,(K, H_p^{1-1/p}(\mathbb{R}^1)).$$

Note that condition (31) leads to

$$\mathrm{cap}\,(K, h_p^1(\mathbb{R}^2)) \sim C_p(K),$$

where $C_p(K)$ is the defined capacity (8) of the compact K. Hence,

$$C_p(K) \sim \mathrm{cap}\,(K, H_p^{1-1/p}(\mathbb{R}^1)). \tag{36}$$

Let $\{l_j\}$ be a decreasing sequence of positive numbers such that $2l_{j+1} < l_j$ $(j = 1, 2, \dots)$ and let Δ_1 be a closed interval with length $l_1 \leq 1$, located on the Ox_1 axis. Denote by E_1 the subset of Δ_1, which is the union of two closed intervals Δ_2 and Δ_3 with length l_2 and which contains both ends of the interval Δ_1. Thus, we remove from the interval Δ_1 the interval of length $l_1 - 2l_2$ centered in the middle of Δ_1. Next, we repeat the procedure with the intervals Δ_2 and Δ_3 (here the role of l_2 passes to l_3) and thus obtain four closed intervals with length l_3. Let their union be denoted by E_2 and so on. We put $F = \bigcap\limits_{j=1}^{\infty} E_j$ (see Figure 1).

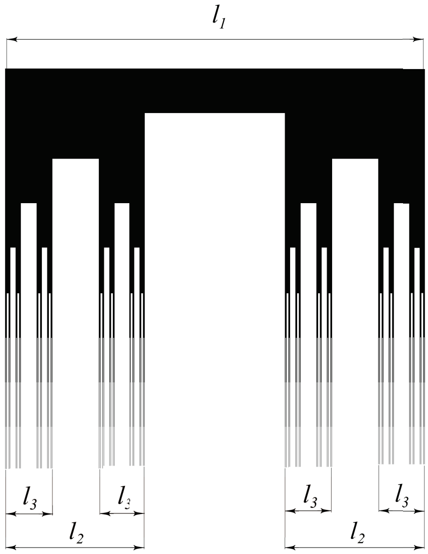

Fig. 1 Construction of a Cantor set.

It follows from the result of [24] (see also [25]) that the statements

$$\mathrm{cap}\,(F, H_p^{1-1/p}(\mathbb{R}^1)) > 0$$

and

$$\sum_{j=1}^{\infty} 2^{\frac{j}{1-p}} l_j^{\frac{2-p}{1-p}} < \infty \tag{37}$$

are equivalent. Thus, under the condition (37), we have

$$C_p(F) > 0. \tag{38}$$

We are interested in the case when $p = 3/2$ and the condition (37) becomes

$$\sum_{j=1}^{\infty} 4^{-j} l_j^{-\frac{1}{4}} < \infty.$$

If we put $l_j = a^{-j+1}$, where $a > 2$, and, hence, $2l_{l+1} < l_j$, then we arrive at the condition

$$\sum_{j=1}^{\infty} \left(\frac{1}{4} a^{1/4}\right)^j a^{-1/4} < \infty.$$

In particular, if $a = 3$, then this series converges and we arrive at the classical Cantor set F.

One-dimensional Lebesgue measure of F is equal to zero. Indeed, on the j-th step we have 2^{j-1} intervals of the length 3^{-j}, i.e. the sum of the lengths of the removed intervals equals to

$$\frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j}=1.$$

Besides, by (38) we have

$$C_{3/2}(F)>0. \tag{39}$$

It remains to show that for an arbitrary point $x_0 \in F$ for $r \le r_0$ the inequality

$$C_{3/2}(F\cap \overline{B}_r^{x_0})\ge c_0 r^{1/2}, \tag{40}$$

where $B_r^{x_0}$ is an open circle of radius r centered at x_0 and the positive constant c_0 is independent of x_0 and r.

Let us recall the definition of the capacity of the set $F_r^{x_0}=F\cap \overline{B}_r^{x_0}$:

$$C_{3/2}(F_r^{x_0})=\inf\left\{\int\limits_{\mathbb{R}^2}|\nabla\varphi|^{3/2}\,dx:\ \varphi\in C_0^{\infty}(\mathbb{R}^2),\ \varphi\ge 1 \text{ on } F_r^{x_0}\right\}. \tag{41}$$

It is clear that the set $F_r^{x_0}$ is the intersection of F with the interval centered at the point x_0 of length r. If $r \le r_0 \le 1/3$, then there is a natural number k_0 such that $3^{-k_0-1}<r\le 3^{-k_0}$. Clearly, $x_0\in F$ belongs to the interval I_{k_0} of length 3^{-k_0-2}. Since $I_{k_0}\cap F\subset F_r^{x_0}$, it follows

$$C_{3/2}(F_r^{x_0})\ge C_{3/2}(I_{k_0}\cap F). \tag{42}$$

Performing now in (41) the homothety transformation

$$y=(x-x_0)/r+x_0,\ \text{where } r=3^{-k_0-2}, \tag{43}$$

using (42), we arrive at the inequality

$$C_{3/2}(F_r^{x_0})\ge 3^{-(k_0+2)/2}C_{3/2}(\widetilde{F}_0)\ge 3^{-1/2}r^{1/2}C_{3/2}(\widetilde{F}_0), \tag{44}$$

where $\widetilde{F}_0$ stands for the image of the set $I_{k_0}\cap F$. It remains to note that under (43) the set $\widetilde{F}_0$ is a shift of the Cantor set F along the axis Ox_1.

By (39) and (44), the required relation (40) holds with constant $c_0=3^{-1/2}$ and $r_0=1/3$.

Acknowledgments. The work of the first author (AYuA) were supported in part by RSF (project 22-21-00292). The third author (MVG) was supported by the RUDN University Strategic Academic Leadership Program.

Declarations

- Funding: RSF (project 22-21-00292), RSF (project 20-11-20272), RUDN University Strategic Academic Leadership Program
- Conflict of interest/Competing interests: Not applicable
- Ethics approval: Not applicable
- Consent to participate: Not applicable
- Consent for publication: Not applicable
- Availability of data and materials: Not applicable
- Code availability: Not applicable
- Authors' contributions: The authors contributed equally to this work

References

[1] Zaremba, S.: Sur un problème mixte relatif à l'équation de Laplace (French). Bulletin de l'Académie des sciences de Cracovie, Classe des sciences mathématiques et naturelles, serie A, 313–344 (1910)

[2] Fichera, G.: Sul problema misto per le equazioni lineari alle derivate parziali del secondo ordine di tipo ellittico (Italian). Rev. Roumaine Math. Pures Appl. **9**, 3–9 (1964)

[3] Maz'ya, V.G.: Some estimates of solutions of second-order elliptic equations (Russian). Dokl. Akad. Nauk SSSR. **137**(5), 1057–1059 (1961)

[4] Kerimov, T.M., Maz'ya, V.G., Novruzov, A.A.: A criterion for the regularity of the infinitely distant point for the Zaremba problem in a half-cylinder (Russian). Z. Anal. Anwendungen **7**(2), 113–125 (1988)

[5] Maz'ya, V.G.: Boundary Behavior of Solutions to Elliptic Equations in General Domains. EMS Tracts in Mathematics Vol. 30. EMS Publishing House, Zürich (2018)

[6] Bojarski, B.V.: Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients (Russian). Mat. Sb. N.S. **43**(85), 451–503 (1957)

[7] Meyers, N.G.: An L^p-estimate for the gradient of solutions of second order elliptic deivergence equations. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3-e série. **17**(3), 189–206 (1963)

[8] Zhikov, V.V.: On some variational problems. Russian Journal of Mathematical physics. **5**(1), 105–116 (1997)

[9] Breit, D., Cianchi, A., Diening, L., Kuusi, T., Schwarzacher, S.: Pointwise Calderón–Zygmund gradient estimates for the p-Laplace system. J. Math. Pures Appl. **114**, 146–190 (2018)

[10] Acerbi, E., Mingione, G.: Gradient estimates for the $p(x)$-Laplacian system. J. Reine Angew. Math. **584**, 117–148 (2005)

[11] Diening, L., Schwarzsacher, S.: Global gradient estimates for the $p(\cdot)$-Laplacian. Nonlinear Anal. **106**, 70–85 (2014)

[12] Alkhutov, Yu.A., Chechkin, G.A.: Increased integrability of the gradient of the solution to the Zaremba problem for the Poisson equation. Russian Academy of Sciencies. Doklady Mathematics **103**(2), 69–71 (2021) (Translated from Doklady Akademii Nauk **497**(2), 3–6 (2021))

[13] Chechkin, G.A.: The Meyers estimates for domains perforated along the boundary. Mathematics **9**(23), Art number 3015 (2021)

[14] Alkhutov, Yu.A., Chechkin, G.A.: The Meyer's estimate of solutions to Zaremba problem for second–order elliptic equations in divergent form. C R Mécanique **349**(2), 299–304 (2021)

[15] Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations. Springer-Verlag, Berlin (2011)

[16] Il'in, V.P.: On a imbedding theorem for limiting exponent (Russian). Dokl. Akad. Nauk SSSR **96**, 905–908 (1954)

[17] Caccioppoli, R.: Limitazioni integrali per le soluzioni di un'equazione lineare ellitica a derivate parziali (Italian). Giorn. Mat. Battaglini **80**, 186–212 (1950–1951)

[18] Gehring, F.W.: The L_p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)

[19] Giaquinta, M., Modica, G.: Regularity results for some classes of higher order nonlinear elliptic systems. Journ. für die reine und angewandte Math. **311/312**, 145–169 (1979)

[20] Skrypnik, I.V.: Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, Translations of Math. Monographs, V.**139**. AMS, Providence (1994)

[21] Adams, D.R., Meyers, N.G.: Thinness and Wiener criteria for nonlinear potentials. Indiana Univ. Math. J. **22**, 139–158 (1972)

[22] Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory. Springer, Berlin (1996)

[23] Sjödin, T.: Capacities of compact sets in linear subspaces of $\mathbb{R}^n$. Pac. J. Math. **78**, 261–266 (1978)

[24] Maz'ya, V.G., Havin, V.P.: A nonlinear analogue of the Newtonian potential, and metric properties of (p, l)-capacity (Russian). Dokl. Akad. Nauk SSSR **194**, 770–773 (1970)

[25] Maz'ya, V.G., Havin, V.P.: Nonlinear potential theory (Russian). Russ. Math. Surv. **27**, 71–148 (1972) (Translated from Usp. Mat. Nauk **27**, 67–138 (1972))

[26] Chechkin, G.A.: On boundary value problems for a second–order elliptic equation with oscillating boundary conditions (Russian). In: V.N.Vragov (eds) Nonclassical Partial Differential Equations, pp. 95–104. IM SOAN SSSR Press, Novosibirsk (1988)

[27] Chechkin, G.A., Piatnitski, A.L., Shamaev, A.S.: Homogenization. Methods and Applications. Translations of Mathematical Monographs. V.**234**. AMS, Providence (2007)