Quasilinear elliptic equations on noncompact Riemannian manifolds

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Abstract

The existence of solutions to a class of quasilinear elliptic problems on noncompact Riemannian manifolds is investigated. Boudary value problems, with homogeneous Neumann conditions, in possibly irregular Euclidean domains are included as a special instance. A nontrivial solution is shown to exist under a an unconventional growth condition on the right-hand side, which depends on the geometry of the underlying manifold. The identification of the critical growth is a crucial step in our analysis, and entails the use of the isocapacitary function of the manifold. A condition involving its isoperimetric function is also provided.

1 Introduction

The present paper is concerned with the existence of solutions to semilinear elliptic equations on an n-dimensional Riemannian manifold M, whose weak formulation reads

$$\int_M |\nabla u|^{p-2}\nabla u \cdot \nabla v \, d\mathcal{H}^n = \int_M f(u)\, v \, d\mathcal{H}^n \tag{1.1}$$

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for every test function v in the Sobolev space $W^{1,p}(M)$. Here, $p \in (1, \infty)$, ∇ stands for the gradient operator on M, $|\nabla u|$ denotes its length, determined by the scalar product " $\cdot$ " associated with the Riemannian metric on M, and $\mathcal{H}^n$ is the volume measure on M induced by the metric. The function $f : \mathbb{R} \to \mathbb{R}$ is continuous, and satisfies suitable growth conditions for the right-hand side of (1.1) to be well defined for every test function v.
Throughout, we assume that M is connected, orientable and

$$\mathcal{H}^n(M) < \infty \tag{1.2}$$

Although compact manifolds are included as a special case, the main emphasis will be on the noncompact case. Its treatment calls for new inequalities of Sobolev type, whose form is patterned on the geometry of M.

Equation (1.1) encompasses problems of diverse nature, depeding on analytic-geometric properties of M. For instance, if the space of smooth compactly supported functions on M is dense in $W^{1,p}(M)$ – this certainly holds when M is a complete Riemannian manifold – then (1.1) corresponds to the weak form of the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) \qquad \text{on } M. \tag{1.3}$$

In the case when M is an open subset Ω of a Riemannian manifold, and in particular of the Euclidean space $\mathbb{R}^n$, equation (1.1) amounts to the definition of weak solution to the Neumann boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.4}$$

where $\mathbf{n}$ denotes the unit normal vector to $\partial\Omega$.

A necessary condition for the existence of a solution to problem (1.1) is that $f(t_0) = 0$ for some $t_0 \in \mathbb{R}$. This is easily seen on choosing a constant test function v in (1.1). Thus, the function $u = t_0$ is trivially a solution to (1.1). The aim of the present paper is to exhibit minimal conditions on f guaranteeing also the existence of a nontrivial, namely non-constant, solution.

Elliptic equations involving nonlinearities of this kind have been extensively investigated in the literature. In particular, the case when M is a bounded open set $\Omega \subset \mathbb{R}^n$, and homogenenous Dirichlet boundary conditions are prescribed, is very well understood. Define $F : \mathbb{R} \to \mathbb{R}$ as

$$F(t) = \int_0^t f(r)\, dr \quad \text{for } t \in \mathbb{R}. \tag{1.5}$$

Then methods from critical point theory – a mountain pass theorem by Ambrosetti-Rabinowitz [AR], for example – ensure that there exists a nontrivial solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.6}$$

provided that f fulfils the standard assumptions:

$$\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0, \tag{1.7}$$

$$F(t) > 0 \quad \text{for } t \neq 0, \tag{1.8}$$

$$\liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > p, \tag{1.9}$$

coupled, when $p \leq n$, with a growth condition near infinity dictated by the Sobolev embedding theorem:

$$\lim_{t\to\pm\infty} \frac{f(t)}{|t|^{q-1}} = 0, \tag{1.10}$$

for some $q < p^*$, or some $q < \infty$, according to whether $p < n$ or $p = n$. Here, $p^* = \frac{np}{n-p}$, the usual Sobolev conjugate of p.
Assumption (1.10) is essentially sharp, inasmuch as the Pohozaev identity prevents a nontrivial solution to the Dirichlet problem to exist in any starshaped domain Ω, if $f(t) = |t|^{q-2}t$ for some $q > p^*$.

The situation is similar when Neumann problems of the form (1.4), in smooth Euclidean domains Ω, are in question – see e.g. the papers [AlOm, BP, GP, MMT, MoTa], where improvements and variants also appear as far as conditions (1.7)–(1.9) are concerned. The same conditions, and the same growth condition (1.10), come into play for equation (1.3) on a compact manifold M. This follows, via a suitable version of the mountain pass theorem, from the fact that the critical exponent in the appropriate Sobolev embedding is still p^* in both cases.

By contrast, the picture is drastically different when dealing with Euclidean or Riemannian domains with non-smooth boundary, or with entire non-compact Riemannian manifolds. Indeed, the Sobolev embedding fails in these frameworks, at least in its standard version. We shall show that, this notwithstanding, the existence of a nontrivial solution to (1.1) can still be established, provided that f has a subcritical growth near infinity depending on p and on the geometry of M, under some mild assumption on the latter.

A key step in our approach is an embedding theorem, of Sobolev type, into an Orlicz space built upon a Young function depending on M. The behavior of such a Young function, and hence the admissible growth of f, can be conveniently described in terms of the so-called isocapacitary function $\nu_{M,p}$ of M. This is the largest function of the measure of any subset G of M, not exceeding $\frac{\mathcal{H}^n(M)}{2}$, that bounds from below its condenser capacity $C_{M,p}(G)$ – see Section 2 for a definition. Thus,

$$\nu_{M,p}(\mathcal{H}^n(G)) \leq C_{M,p}(G) \tag{1.11}$$

for every measurable set $G \subset M$ with $\mathcal{H}^n(G) \leq \frac{\mathcal{H}^n(M)}{2}$. Inequality (1.11) is referred to as the isocapacitary inequality relative to M. The isocapacitary function was introduced in [Ma2] to characterize Sobolev type inequalities for functions whose gradient is integrable to the power p in an open set in $\mathbb{R}^n$. Its use in a priori estimates for Neumann problems for elliptic equations on irregular Euclidean domains has been pointed out in [CM1, ACMM]. Isocapacitary inequalities have also been exploited in the analysis of elliptic problems on Riemannian manifolds [Gr1, Gr2, CM2, CM3].

The decay of $\nu_{M,p}(s)$ as $s \to 0^+$ is the sole piece of information on $\nu_{M,p}$ that is relevant for our present applications. It depends on p and on the regularity of the geometry of M. Loosely speaking, a complicated geometry of M results in a faster decay of $\nu_{M,p}$. If M is a compact manifold, then

$$\nu_{M,p}(s) \approx \begin{cases} s^{\frac{n-p}{n}} & \text{if } 1 \leq p < n, \\ \left(\log \frac{1}{s}\right)^{1-n} & \text{if } p = n, \\ 1 & \text{if } p > n, \end{cases} \tag{1.12}$$

as $s \to 0^+$, where the notation " $\approx$ " means that the two sides of equation (1.12) are bounded by each other, up to multiplicative constants. This is the slowest, and hence best possible, decay of

$\nu_{M,p}$. Equation (1.12) also holds if M is a bounded open set with regular boundary in $\mathbb{R}^n$. On the other hand, $\nu_{M,p}(s)$ is in general just dominated by (a constant times) the right-hand side of (1.12), but can decay essentially faster.

In fact, only a lower bound for $\nu_{M,p}$ is needed in view of our purposes. In other words, the same conclusions about problem (1.1) hold for manifolds whose isocapacitary functions admit the same bound from below. Thus, instead of just focusing on single manifolds, our results will be formulated for classes of manifolds defined as

$$\mathcal{C}_p(\nu) = \{M : \nu_{M,p}(s) \geq \nu(s) \text{ for } s \text{ near } 0\}\,. \tag{1.13}$$

Here, $\nu : (0,\infty) \to [0,\infty)$ is a prescribed quasi-concave function, in the sense that it is increasing and

$$\frac{\nu(s)}{s} \quad \text{is non-increasing.} \tag{1.14}$$

In particular, ν is continuous. The notation $\nu(0^+) = \lim_{s\to 0^+} \nu(s)$ will be adopted in what follows. Note that, if $M \in \mathcal{C}_p(\nu)$ for some function ν as above, then $\inf_{s\in\left(0,\frac{\mathcal{H}^n(M)}{2}\right)} \frac{\nu_{M,p}(s)}{s} > 0$.
Let us also incidentally observe that the optimal choice of ν for a single manifold M is the function given by

$$s \inf_{r\in(0,s)} \frac{\nu_{M,p}(r)}{r} \quad \text{for } s > 0. \tag{1.15}$$

Indeed, this is the largest quasi-concave function that does not exceed $\nu_{M,p}$, as shown by an easy variant of an argument from [Mu, Lemma 3.2].

The main result of this paper can be stated as follows.

Theorem 1.1 *Let ν be a quasi-concave function. Assume that $M \in \mathcal{C}_p(\nu)$, and that f fulfils* (1.7)–(1.9). *Assume, in addition, that either*

$$\nu(0^+) > 0, \tag{1.16}$$

or

$$\nu(0^+) = 0 \quad \text{and} \quad \lim_{t\to\infty} \nu^{-1}(t^{-p})\, t\, f(kt) = 0 \quad \text{for every } k \in \mathbb{R}, \tag{1.17}$$

where ν^{-1} stands for the inverse of ν. Then, there exists a nontrivial (i.e. non-constant) solution to problem (1.1).

Observe that condition (1.16) is a counterpart, in the present general setting, of the assumption $p > n$ for manifolds M whose isocapacitary function obeys (1.12). The growth condition on f prescribed by the limit in (1.17) can be regarded as a balance between the exponent p and the (ir)regularity of the geometry of M, and replaces the customary growth imposed by the limit in (1.10).

Criteria for the existence of a nontrivial solution to (1.1) can also be given in terms of the isoperimetric function λ_M of M. The function λ_M has a transparent geometric character, and it is usually easier to investigate than $\nu_{M,p}$. The price to pay is that the resulting condition is somewhat stronger than that involving $\nu_{M,p}$, and may not be applicable to certain manifolds with complicated geometric configurations, to which, instead, Theorem 1.1 applies – see Examples 4.4 and 4.5, Section

4, in this connection.
In analogy with $\nu_{M,p}$, the isoperimetric function λ_M is defined as the largest function of the measure of any subset G of M, not exceeding $\frac{\mathcal{H}^n(M)}{2}$, that bounds from below its perimeter $P(G)$ in M. This is the content of the relative isoperimetric inequality on M, which takes the form

$$\lambda_M(\mathcal{H}^n(G)) \leq P(G) \tag{1.18}$$

for every measurable set $G \subset M$ with $\mathcal{H}^n(G) \leq \frac{\mathcal{H}^n(M)}{2}$. The function λ_M was introduced in [Ma1] to characterize Sobolev embeddings for functions whose gradient is merely integrable with power 1 in open sets in $\mathbb{R}^n$.

As in the case of $\nu_{M,p}$, only the behavior of λ_M at 0 plays a role in our results, and, in particular, a lower bound for it. We shall thus consider classes $\mathcal{I}(\lambda)$ of manifolds defined as

$$\mathcal{I}(\lambda) = \{M : \lambda_M(s) \geq \lambda(s) \text{ near } 0\} \tag{1.19}$$

for some function $\lambda : (0, \infty) \to [0, \infty)$. The quasi-concavity of ν translates into the assumption that λ be an increasing function satisfying:

$$\liminf_{s\to 0^+} \frac{\lambda(s)^{p'}}{s} \int_s^{\frac{\mathcal{H}^n(M)}{2}} \frac{dr}{\lambda(r)^{p'}} > p-1\,. \tag{1.20}$$

Given such a function λ, define

$$\Lambda(s) = \left(\int_s^{\frac{\mathcal{H}^n(M)}{2}} \frac{dr}{\lambda(r)^{p'}}\right)^{1-p} \quad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(M)}{2}\big). \tag{1.21}$$

Corollary 1.2 *Let λ be an increasing function satisfying* (1.20)*. Assume that $M \in \mathcal{I}(\lambda)$, and that f fulfils* (1.7)–(1.9)*. Assume, in addition, that either*

$$\Lambda(0^+) > 0, \tag{1.22}$$

or

$$\Lambda(0^+) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \Lambda^{-1}(t^{-p})\, t\, f(kt) = 0 \quad \textit{for every } k \in \mathbb{R}, \tag{1.23}$$

where Λ^{-1} stands for the inverse of Λ. Then, there exists a nontrivial solution to problem (1.1)*.*

Corollary 1.2 can be derived from Theorem 1.1, owing to the inequality

$$\nu_{M,p}(s) \geq \left(\int_s^{\frac{\mathcal{H}^n(M)}{2}} \frac{dr}{\lambda_M(r)^{p'}}\right)^{1-p} \quad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(M)}{2}\big), \tag{1.24}$$

which follows via an analogous argument as in [Ma3, Proposition 6.3.5/1], and to the fact that assumption (1.20) implies the quasi-concavity of Λ near zero.

2 Geometry-depending embedding theorems

After recalling a few basic definitions and properties from the theory of Young functions and Orlicz spaces, we establish here a crucial Sobolev type embeddings, whose Orlicz target is modelled on the geometry of the underlying manifold M.

A function $A : [0, \infty) \to [0, \infty]$ is called a Young function if it has the form

$$A(t) = \int_0^t a(\tau)d\tau \qquad \text{for } t \geq 0, \tag{2.1}$$

for some non-decreasing, left-continuous function $a : [0, \infty) \to [0, \infty]$ which is neither identically equal to 0 nor to ∞. Clearly, any convex (non trivial) function from $[0, \infty)$ into $[0, \infty]$, which is left-continuous and vanishes at 0, is a Young function.
One has that

$$\frac{A(t)}{t} \leq a(t) \leq \frac{A(2t)}{t} \qquad \text{for } t > 0. \tag{2.2}$$

Moreover,

$$kA(t) \leq A(kt) \quad \text{for } k \geq 1 \text{ and } t \geq 0. \tag{2.3}$$

The Young conjugate $\widetilde{A}$ of A is defined by

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geq 0\} \qquad \text{for} \qquad t \geq 0\,.$$

Note the representation formula

$$\widetilde{A}(t) = \int_0^t a^{-1}(\tau)d\tau \qquad \text{for } t \geq 0, \tag{2.4}$$

where a^{-1} denotes the (generalized) left-continuous inverse of the function a appearing in (2.1). One can show that

$$s \leq A^{-1}(s)\widetilde{A}^{-1}(s) \leq 2s \qquad \text{for } s \geq 0, \tag{2.5}$$

where A^{-1} and $\widetilde{A}^{-1}$ stand for the generalized right-continuous inverses of A and $\widetilde{A}$, respectively. Hence, by (2.2),

$$a(t) \leq 2\widetilde{A}^{-1}(A(2t)) \qquad \text{for } t \geq 0. \tag{2.6}$$

A Young function A is said to dominate another Young function B globally if there exists a positive constant c such that

$$B(t) \leq A(ct) \tag{2.7}$$

for $t \geq 0$. The function A is said to dominate B near infinity if there exists $t_0 \geq 0$ such that (2.7) holds for $t \geq t_0$. If A and B dominate each other globally [near infinity], then they are called equivalent globally [near infinity]. This terminology will also be adopted for merely nonnegative functions, which are not necessarily Young functions.
A Young function B is said to increase essentially more slowly than another Young function A near infinity, if

$$\lim_{t\to\infty} \frac{B(\lambda t)}{A(t)} = 0 \;\; \text{for every } \lambda > 0\,. \tag{2.8}$$

Condition (2.8) is equivalent to

$$\lim_{t\to\infty} \frac{A^{-1}(t)}{B^{-1}(t)} = 0\,. \tag{2.9}$$

It follows from (2.9) and (2.5) that B increases essentially more slowly than A near infinity if and only if $\widetilde{A}$ increases essentially more slowly than $\widetilde{B}$ near infinity.

The Orlicz space $L^A(M)$, associated with a Young function A, is the Banach function space of those measurable functions $u : M \to \mathbb{R}$ for which the Luxemburg norm

$$\|u\|_{L^A(M)} = \inf\left\{\lambda > 0 : \int_M A\left(\frac{|u|}{\lambda}\right) d\mathcal{H}^n \leq 1\right\} \tag{2.10}$$

is finite. In particular, $L^A(M) = L^p(M)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(M) = L^\infty(M)$ if $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = \infty$ for $t > 1$.
The Luxemburg norm is equivalent, up to absolute multiplicative constants, to the Orlicz norm given by

$$\|u\|_{\mathcal{L}^A(M)} = \sup\left\{\int_M |uv|\, d\mathcal{H}^n : \int_M \widetilde{A}(|v|)\, d\mathcal{H}^n \leq 1\right\}. \tag{2.11}$$

In fact, one has that

$$\|u\|_{L^A(M)} \leq \|u\|_{\mathcal{L}^A(M)} \leq 2\|u\|_{L^A(M)} \tag{2.12}$$

for every measurable function u on M.
The Hölder type inequality

$$\int_M |uv|\, d\mathcal{H}^n \leq 2\|u\|_{L^A(M)}\|v\|_{L^{\widetilde{A}}(M)} \tag{2.13}$$

holds for every $u \in L^A(M)$ and $v \in L^{\widetilde{A}}(M)$.
If A dominates B globally, then

$$\|u\|_{L^B(M)} \leq c\|u\|_{L^A(M)} \tag{2.14}$$

for every $u \in L^A(M)$, where c is the same constant as in (2.7). If A dominates B near infinity, then inequality (2.14) continues to hold for some constant $c = c(A, B, \mathcal{H}^n(M))$.
We refer the reader to the monographs [BS, RR1, RR2] for more details and proofs on these topics.

Let $p \in [1, \infty]$. We denote by $V^{1,p}(M)$ the Sobolev type space defined as

$$V^{1,p}(M) = \{u : \ u \text{ is weakly differentiable on } M \text{ and } |\nabla u| \in L^p(M)\ \}. \tag{2.15}$$

Given any open set $\omega \subset M$, whose closure $\overline{\omega}$ in M is compact, the functional

$$\|u\|_{V^{1,p}(M)} = \|\nabla u\|_{L^p(M)} + \|u\|_{L^p(\omega)} \tag{2.16}$$

defines a norm in $V^{1,p}(M)$. Different choices of ω result in equivalent norms in $V^{1,p}(M)$.
The standard Sobolev space $W^{1,p}(M)$ is then given by

$$W^{1,p}(M) = V^{1,p}(M) \cap L^p(M), \tag{2.17}$$

and is equipped with the norm

$$\|u\|_{W^{1,p}(M)} = \|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)} \tag{2.18}$$

As usual, $W_0^{1,p}(M)$ stands for the closure in $W^{1,p}(M)$ of the set of smooth compactly supported functions on M. Also, we call $W_\perp^{1,p}(M)$ the Banach subspace of $W^{1,p}(M)$ defined as

$$W_\perp^{1,p}(M) = \{u \in W^{1,p}(M) : u_M = 0\}, \tag{2.19}$$

where

$$u_M = \frac{1}{\mathcal{H}^n(M)} \int_M u \, d\mathcal{H}^n \,, \tag{2.20}$$

the mean value of u over M. Clearly,

$$W^{1,p}(M) = \mathbb{R} \oplus W_\perp^{1,p}(M). \tag{2.21}$$

We denote by $P(G)$ the perimeter in M of a measurable set $G \subset M$, in the sense of geometric measure theory. Recall that, if the boundary ∂G of G in M is smooth, then

$$P(G) = \mathcal{H}^{n-1}(\partial G) \,,$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on M induced by its Riemannian metric. In the special case when M is an open subset Ω of $\mathbb{R}^n$, and $\partial_{\mathbb{R}^n} G \cap \Omega$ is smooth, one has that $P(G) = \mathcal{H}^{n-1}(\partial_{\mathbb{R}^n} G \cap \Omega)$, where $\partial_{\mathbb{R}^n} G$ stands for the boundary of G in the whole of $\mathbb{R}^n$.
The isoperimetric function $\lambda_M : \big[0, \frac{\mathcal{H}^n(M)}{2}\big] \to [0, \infty]$ of M is defined as

$$\lambda_M(s) = \inf\Big\{P(G) : \, G \subset M, \, s \le \mathcal{H}^n(G) \le \tfrac{\mathcal{H}^n(M)}{2}\Big\} \qquad \text{for } s \in \big[0, \tfrac{\mathcal{H}^n(M)}{2}\big]. \tag{2.22}$$

The isoperimetric inequality (1.18) is a straightforward consequence of definition (2.22).
Given $p \in [1, \infty)$, the standard p-capacity of a set $G \subset M$ can be defined as

$$C_p(G) = \inf\left\{\int_M |\nabla u|^p \, d\mathcal{H}^n : u \in W_0^{1,p}(M), u \ge 1 \text{ in some neighbourhood of } G\right\}. \tag{2.23}$$

A property concerning the pointwise behavior of functions is said to hold C_p-quasi everywhere in M, briefly C_p-q.e., if it is fulfilled outside a set of p-capacity zero.
Each function $u \in W^{1,p}(M)$ has a representative – the precise representative – with the property that, for every $\varepsilon > 0$, there exists a set $A \subset M$, with $C_p(A) < \varepsilon$, such that $u_{|M\setminus A}$ is continuous in $M \setminus A$. Throughout, we assume that every function $u \in W^{1,p}(M)$ agrees with its precise representative. It is well-known that, under this assumption, one can just require that $u \ge 1$ in G on the right-hand side of (2.23) [MZ, Corollary 2.25]. Accordingly, the condenser capacity $C_{M,p}(G)$ of a set $G \subset M$ is defined as

$$C_{M,p}(G) = \inf\left\{\int_M |\nabla u|^p \, d\mathcal{H}^n : u \in W^{1,p}(M), u \ge 1 \; C_p\text{-q.e. in } G, \text{ and } \mathcal{H}^n(\{u > 0\}) \le \tfrac{\mathcal{H}^n(M)}{2}\right\}. \tag{2.24}$$

The p-isocapacitary function $\nu_{M,p} : \big[0, \frac{\mathcal{H}^n(M)}{2}\big] \to [0, \infty]$ of M is defined by

$$\nu_{M,p}(s) = \inf\Big\{C_{M,p}(G) : \, G \subset M, \, s \le \mathcal{H}^n(G) \le \tfrac{\mathcal{H}^n(M)}{2}\Big\} \qquad \text{for } s \in \big[0, \tfrac{\mathcal{H}^n(M)}{2}\big]. \tag{2.25}$$

One has that

$$\nu_{M,1} = \lambda_M \,, \tag{2.26}$$

as shown by an analogous argument as in [Ma3, Lemma 2.2.5].

Theorem 2.1 below is the main result of this section. It provides us with several forms of a Sobolev embedding of $V^{1,p}(M)$ into an Orlicz space $L^A(M)$, under a necessary and sufficient condition on the Young function A depending on $\nu_{M,p}$. In the statement, $\operatorname{med}(u)$ denotes the median of a measurable function $u : M \to \mathbb{R}$, given by

$$\operatorname{med}(u) = \inf\big\{t \in \mathbb{R} : \mathcal{H}^n(\{x \in \Omega : u(x) > t\}) \leq \tfrac{\mathcal{H}^n(M)}{2}\big\}.$$

Furthermore,

$$\operatorname{supp} u = \mathcal{H}^n(\{x \in M : u \neq 0\}).$$

Theorem 2.1 *Let $p \in [1, \infty)$, and let $A : [0, \infty) \to [0, \infty)$ be such that $A(t^{\frac{1}{p}})$ is a Young function (hence, A is a Young function as well). Given $\kappa \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big]$, set*

$$\sigma(\kappa) = \sup_{s \in (0,\kappa)} \frac{1}{\nu_{M,p}(s) A^{-1}(1/s)^p}. \tag{2.27}$$

The following facts are equivalent:
(i) There exists $\kappa \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big]$ such that

$$\sigma(\kappa) < \infty\,. \tag{2.28}$$

(ii) Condition (2.28) holds for every $\kappa \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big]$.
(iii) There exists a constant C such that

$$\|u\|_{L^A(M)} \leq C \|\nabla u\|_{L^p(M)} \tag{2.29}$$

for every $u \in V^{1,p}(M)$ satisfying $\mathcal{H}^n(\operatorname{supp} u) \leq \kappa$.
(iv) There exists a constant C such that

$$\int_M A\left(\frac{|u|}{C\|\nabla u\|_{L^p(M)}}\right) d\mathcal{H}^n \leq 1 \tag{2.30}$$

for every $u \in V^{1,p}(M)$ satisfying $\mathcal{H}^n(\operatorname{supp} u) \leq \kappa$.
(v) There exists a constant C such that

$$\|u - \operatorname{med}(u)\|_{L^A(M)} \leq C \|\nabla u\|_{L^p(M)} \tag{2.31}$$

for every $u \in V^{1,p}(M)$.
(vi) The embedding

$$V^{1,p}(M) \to L^A(M) \tag{2.32}$$

holds.
Moreover, the constant C in inequalities (2.29) – (2.31) can be chosen of the form

$$C = c\sigma(\kappa)^{\frac{1}{p}} \tag{2.33}$$

for some constant $c = c(p)$.
In particular, if $\nu_{M,p}(0^+) > 0$, then (2.28) holds for any Young function A, and hence also when $L^A(M) = L^\infty(M)$. In this case inequality (2.29) reads

$$\|u\|_{L^\infty(M)} \leq C \|\nabla u\|_{L^p(M)}. \tag{2.34}$$

for every $u \in V^{1,p}(M)$ satisfying $\mathcal{H}^n(\operatorname{supp} u) \leq \kappa$, where C is as in (2.33).

Remark 2.2 If $A(t^{\frac{1}{p}})$ is just equivalent to a Young function, say $D(t)$, then a statement analogous to that of Theorem 2.1 holds, with $A(t)$ replaced by $D(t^p)$ in (iii)–(vi). In this case, the constant C in (2.29) – (2.31) also dependis on the relevant equivalence constants. For instance, this generalization applies when the function $\frac{A(t)}{t^p}$ is non-decreasing, since

$$A((t/2)^{\frac{1}{p}}) \leq \int_0^t \frac{A(s^{\frac{1}{p}})}{s}\, ds \leq A(t^{\frac{1}{p}}) \quad \text{for } t > 0,$$

and $\int_0^t \frac{A(s^{\frac{1}{p}})}{s}\, ds$ is a Young function. Embedding (2.32) then yields $V^{1,p}(M) \to L^B(M)$, where

$$B(t) = \int_0^t \frac{A(s)}{s}\, ds \quad \text{for } t \geq 0.$$

The next lemma will be of use in the proof of Theorem 2.1.

Lemma 2.3 *Let $p \in [1, \infty)$. Then there exists a constant $c = c(p)$ such that*

$$\int_M |\nabla u|^p d\mathcal{H}^n \geq c \int_0^\infty C_{M,p}(\{|u| \geq t\})\, d(t^p) \tag{2.35}$$

for every $u \in V^{1,p}(M)$ satisfying $\mathcal{H}^n(\operatorname{supp} u) \leq \frac{\mathcal{H}^n(M)}{2}$.

Proof. The following chain holds owing to the monotonicity of $C_{M,p}$ with respect to set inclusion:

$$\begin{aligned} \int_0^\infty C_{M,p}(\{|u| \geq t\})\, d(t^p) &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} C_{M,p}(\{|u| \geq t\})\, d(t^p) \\ &\leq \sum_{k\in\mathbb{Z}} C_{M,p}(\{|u| \geq 2^k\}) \int_{2^k}^{2^{k+1}} d(t^p) = (2^p - 1) \sum_{k\in\mathbb{Z}} 2^{pk} C_{M,p}(\{|u| \geq 2^k\}). \end{aligned} \tag{2.36}$$

Let $\psi : \mathbb{R} \to [0, 1]$ be the function given by $\psi(t) = \min\{1, \max\{t, 0\}\}$ for $t \in \mathbb{R}$, and define $u_k : M \to \mathbb{R}$ as $u_k = \psi(2^{1-k}|u| - 1)$ for $k \in \mathbb{Z}$. Since ψ is a Lipschitz continuous function, one has that $u_k \in W^{1,p}(M)$. Furthermore, $u_k = 1$ in $\{|u| \geq 2^k\}$, and $u_k = 0$ in $\{|u| \leq 2^{k-1}\}$. In particular, $u_k = 0$ in $M \setminus \operatorname{supp}(u)$. Hence, by definition (2.24),

$$\begin{aligned} \sum_{k\in\mathbb{Z}} 2^{pk} C_{M,p}(\{|u| \geq 2^k\}) &\leq \sum_{k\in\mathbb{Z}} 2^{pk} \int_M |\nabla u_k|^p d\mathcal{H}^n \\ &= \sum_{k\in\mathbb{Z}} 2^{pk} 2^{p(1-k)} \int_{\{2^{k-1} \leq |u| < 2^k\}} |\nabla u|^p d\mathcal{H}^n = 2^p \int_M |\nabla u|^p d\mathcal{H}^n. \end{aligned} \tag{2.37}$$

Coupling inequality (2.36) with (2.37) yields (2.35). □

Proof of Theorem 2.1. *(i) is equivalent to (ii).* Clearly, it suffices to show the (i) implies (ii), the reverse implication being trivial. Assume that there exists $\kappa_0 \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big)$ such that (2.28) holds with $\kappa = \kappa_0$. Hence, in particular, it holds for every $\kappa \in (0, \kappa_0]$. The conclusion will then follow if we prove that there exists a positive constant c such that

$$\nu_{M,p}(s) \geq c \quad \text{for } s \in \big[\kappa_0, \tfrac{\mathcal{H}^n(M)}{2}\big]. \tag{2.38}$$

The connectedness of M ensures that

$$\lambda_M(s) \geq c \quad \text{for } s \in [\kappa, \tfrac{\mathcal{H}^n(M)}{2}], \tag{2.39}$$

for some positive constant c. This follows from an argument analogous to that of [Ma3, Lemma 5.2.4]. Equation (2.38) is a consequence of (2.39) and (1.24).
(i) implies (iii). Assume that (i) is in force. Suppose that $u \in V^{1,p}(M)$ fulfills $\mathcal{H}^n(\operatorname{supp} u) \leq \kappa$. Let B be the Young function defined as $B(t) = A(t^{\frac{1}{p}})$ for $t \geq 0$. Hence, $B(t^p) = A(t)$ for $t \geq 0$, and $B^{-1}(s) = A^{-1}(s)^p$ for $s \geq 0$. Define the distribution function $\mu : [0, \infty) \to [0, \mathcal{H}^n(M)]$ of u as

$$\mu(t) = \mathcal{H}^n(\{x \in M : |u(x)| \geq t\}) \qquad \text{for } t \geq 0.$$

Owing to assumption (2.28) and to inequality (1.11),

$$\frac{1}{\sigma(\kappa)B^{-1}(1/\mu(t))} \leq \nu_{M,p}(\mu(t)) \leq C_{M,p}(\{|u| \geq t\}) \quad \text{for } t > 0. \tag{2.40}$$

Hence, by Lemma 2.3, there exists a constant $c = c(p)$ such that

$$\int_M |\nabla u|^p d\mathcal{H}^n \geq \frac{c}{\sigma(\kappa)} \int_0^\infty \frac{d(t^p)}{B^{-1}(1/\mu(t))}. \tag{2.41}$$

Next,

$$\begin{aligned}
\| |u|^p \|_{\mathcal{L}^B(M)} &= \sup\left\{ \int_M |u|^p |v| \, d\mathcal{H}^n : \int_M \widetilde{B}(|v|) \, d\mathcal{H}^n \leq 1 \right\} \\
&= \sup\left\{ \int_0^\infty \int_{\{|u|^p \geq t\}} |v| d\mathcal{H}^n \, dt : \int_M \widetilde{B}(|v|) \, d\mathcal{H}^n \leq 1 \right\} \\
&= \sup\left\{ \int_0^\infty \int_{\{|u| \geq \tau\}} |v| d\mathcal{H}^n \, d(\tau^p) : \int_M \widetilde{B}(|v|) \, d\mathcal{H}^n \leq 1 \right\} \\
&\leq \int_0^\infty \sup\left\{ \int_{\{|u| \geq \tau\}} |v| d\mathcal{H}^n \; : \int_M \widetilde{B}(|v|) \, d\mathcal{H}^n \leq 1 \right\} d(\tau^p) \\
&= \int_0^\infty \|\chi_{\{|u| \geq \tau\}}\|_{\mathcal{L}^B(M)} d(\tau^p) \leq 2 \int_0^\infty \|\chi_{\{|u| \geq \tau\}}\|_{L^B(M)} d(\tau^p) \\
&= 2 \int_0^\infty \frac{d(\tau^p)}{B^{-1}(1/\mu(\tau))} \leq \frac{2\sigma(\kappa)}{c} \int_M |\nabla u|^p d\mathcal{H}^n,
\end{aligned} \tag{2.42}$$

where we have made use of inequalies (2.12) and (2.41). Since, by (2.12) again,

$$\|u\|_{L^A(M)} = \| |u|^p \|_{L^B(M)}^{\frac{1}{p}} \leq \| |u|^p \|_{\mathcal{L}^B(M)}^{\frac{1}{p}},$$

inequality (2.29) follows.
(iii) implies (i). Let $\kappa \leq \frac{\mathcal{H}^n(M)}{2}$. Assume that G is a measurable set in M such that $\mathcal{H}^n(G) \leq \kappa$, and let $u \in V^{1,p}(M)$ be such that $|u| \geq 1$ q.e. in G and $\mathcal{H}^n(\text{supp} u) \leq \kappa$. By inequality (2.29),

$$C\|\nabla u\|_{L^p(M)} \geq \|u\|_{L^A(M)} \geq \|\chi_G\|_{L^A(M)} = \frac{1}{A^{-1}(1/\mathcal{H}^n(G))}. \tag{2.43}$$

Hence, by definition (2.24),

$$C_{M,p}(G) \geq \frac{1}{C^p A^{-1}(1/\mathcal{H}^n(G))^p}, \tag{2.44}$$

and (2.28) follows, owing to equation (2.25) and the fact that the right-hand side of (2.24) is a non-decreasing function of $\mathcal{H}^n(G)$.
(iii) is equivalent to (iv). This is a consequence of the definition of Luxemburg norm.
(iii) implies (v). This is a consequence of the fact that, for every $u \in V^{1,p}(M)$, the functions $(u-\mathrm{med}(u))_+$ and $(u-\mathrm{med}(u))_-$ satisfy the inequalities $\mathcal{H}^n(\mathrm{supp}(u-\mathrm{med}(u))_+) \leq \frac{\mathcal{H}^n(M)}{2}$ and $\mathcal{H}^n(\mathrm{supp}(u-\mathrm{med}(u))_-) \leq \frac{\mathcal{H}^n(M)}{2}$. Here, the subscripts $+$ and $-$ stand for positive and negative part, respectively.
(v) implies (iii). This follows from the fact that, if $\mathcal{H}^n(\mathrm{supp}(u)) \leq \frac{\mathcal{H}^n(M)}{2}$ then $\mathrm{med}(u) = 0$.
(v) implies (vi). Given any $u \in V^{1,p}(M)$, write $u = u - \mathrm{med}(u) + \mathrm{med}(u)$. By (2.31), $u - \mathrm{med}(u) \in L^A(M)$, and, trivially, $\mathrm{med}(u) \in L^A(M)$. This shows that $V^{1,p}(M) \subset L^A(M)$. Since the identity map from $V^{1,p}(M)$ into $L^A(M)$ is linear and closed, by the closed graph theorem it is also continuous. Hence, (2.32) follows.
(vi) implies (v). Embedding (2.32) tells us that, given any smooth open set $\omega \subset M$, with compact closure in M, there exists a constant C such that

$$\|u-\mathrm{med}(u)\|_{L^A(M)} \leq C\big(\|\nabla u\|_{L^p(M)} + \|u-\mathrm{med}(u)\|_{L^p(\omega)}\big)$$

for every $u \in V^{1,p}(M)$. On the other hand, one classically has that

$$\|u-\mathrm{med}(u)\|_{L^p(\omega)} \leq C'\|\nabla u\|_{L^p(\omega)}$$

for some constant C'. Coupling the last two inequalities yields (2.31). □

We conclude this section with some consequences of Theorem 2.1.

Corollary 2.4 *Let $p \in [1,\infty)$, and let $A:[0,\infty) \to [0,\infty)$ be such that $A(t^{\frac{1}{p}})$ is a Young function. Assume that* (2.28) *holds for some $\kappa \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big]$ (and hence for all $\kappa \in \big(0, \frac{\mathcal{H}^n(M)}{2}\big]$). Then*

$$W^{1,p}(M) = V^{1,p}(M), \tag{2.45}$$

up to equivalent norms. Moreover, there exists a constant $c = c(p)$ such that

$$\|u\|_{L^A(M)} \leq c\,\sigma\big(\tfrac{\mathcal{H}^n(M)}{2}\big)\,\|\nabla u\|_{L^p(M)}, \tag{2.46}$$

and

$$\int_M A\left(\frac{|u|}{c\sigma\big(\frac{\mathcal{H}^n(M)}{2}\big)\|\nabla u\|_{L^p(M)}}\right) d\mathcal{H}^n \leq 1 \tag{2.47}$$

for every $u \in W^{1,p}_\perp(M)$.

Proof. Since we are assuming that $A(t^{\frac{1}{p}})$ is a Young function, the function $A(t^{\frac{1}{p}})/t$ is non-decreasing, and hence the function $A(t)$ dominates t^p near infinity. Consequently, $L^A(M) \to L^p(M)$, and, by embedding (2.32), $V^{1,p}(M) \to L^p(M)$. Thus, $V^{1,p}(M) \to W^{1,p}(M)$. Inasmuch as the reverse embedding holds trivially, equation (2.45) follows.
Inequality (2.46) is a consequence of inequality (2.31), and of the fact that $\|u-u_M\|_{L^A(M)} \leq 2\|u-\mathrm{med}(u)\|_{L^A(M)}$. Inequality (2.47) is equivalent to (2.46), by the definition of Luxemburg norm. □

An application of Theorem 2.1 and Corollary 2.4, with $A(t) = t^p$, yields, in particular, the following result.

Corollary 2.5 *Let $p \in [1, \infty)$. Assume that*

$$\inf_{s \in (0, \frac{\mathcal{H}^n(M)}{2})} \frac{\nu_{M,p}(s)}{s} > 0. \tag{2.48}$$

Then, there exists a constant $C = C(p, M)$ such that

$$\|u - \mathrm{med}(u)\|_{L^p(M)} \leq C\|\nabla u\|_{L^p(M)} \tag{2.49}$$

for every $u \in V^{1,p}(M)$, and

$$\|u\|_{L^p(M)} \leq C\|\nabla u\|_{L^p(M)} \tag{2.50}$$

for every $u \in W^{1,p}_{\perp}(M)$. In particular, the functional

$$\|u\|_{1,p} = \|\nabla u\|_{L^p(M)} + |u_M| \,, \tag{2.51}$$

defines a norm on $W^{1,p}(M)$ equivalent to (2.18).

The compactness of a Sobolev embedding is the subject of the last result of this section.

Corollary 2.6 *Let p and A be as in Theorem 2.1. Assume that* (2.28) *holds. If E is any Young function increasing essentially more slowly than A near infinity, then the embedding*

$$W^{1,p}(M) \to L^E(M) \tag{2.52}$$

is compact.

Proof. By Theorem 2.1,

$$W^{1,p}(M) \to L^A(M). \tag{2.53}$$

A general property of Orlicz-Sobolev embeddings (see e.g. [HL, Theorem 3.4]) yields the compactness of (2.52) from the assumption that E increases essentially more slowly than A near infinity. $\square$

3 Proof of the main result

Let X be a Banach space, and let X^* denote its dual. A functional $I : X \to \mathbb{R}$ is said to satisfy the Palais-Smale condition if

(3.1) any sequence $\{u_k\} \subset X$ such that $\{I(u_k)\}$ is bounded, and $\lim_{k\to\infty} \|I'(u_k)\|_{X^*} = 0$, has a convergent subsequence in X.

A sequence $\{u_k\}$ as in (3.1) will be called a Palais-Smale sequence for the functional I.

The following version of the mountain pass theorem from [R, Theorem 5.3] will be exploited in the proof of Theorem 1.1. In the statement, B_r^X denotes the ball in X, centered at 0 and with radius $r > 0$. Moreover, given $\overline{u} \in X$ and $r > 0$, we set, with abuse of notation,

$$[0, r\overline{u}] = \{u \in X : u = \lambda\overline{u} \text{ for some } \lambda \in [0, r]\}.$$

Under the assumption that $\mathbb{R} \subset X$, we also define

$$Q_r = [-r, r] \oplus [0, r\overline{u}], \tag{3.2}$$

and denote by ∂Q_r the boundary of Q_r as a subset of the (two-dimensional) direct sum of $\mathbb{R}$ with the subspace of X spanned by $\overline{u}$.

Theorem 3.1 *[Mountain Pass Theorem [R]] Let X be a Banach space such that $\mathbb{R} \subset X$, and let Y be a Banach subspace of X such that $X = \mathbb{R} \oplus Y$. Assume that the functional $I : X \to \mathbb{R}$ is of class C^1, satisfies the Palais-Smale condition, and:*

(i) there exist $\rho,\ \alpha > 0$ such that $I_{|\partial B_\rho^X \cap Y} \geq \alpha$,

(ii) there exist $\overline{u} \in \partial B_1^X \cap Y$ and $R > \rho$ such that $I_{|\partial Q_R} \leq 0$.

Then I has a critical point u, satisfying $I(u) = c \geq \alpha$, where

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in Q_R} I(\gamma(u)),$$

and

$$\Gamma = \{\gamma \in C^0(Q_R, X) : \gamma_{|\partial Q_R} = \mathrm{id}_{|\partial Q_R}\}.$$

We shall make use of Theorem 3.1, with $X = W^{1,p}(M)$, endowed with the norm $\|\cdot\|_{1,p}$ given by (2.51), and $I = J$, the energy functional associated with problem (1.1), defined as

$$J(u) = \frac{1}{p}\int_M |\nabla u|^p \, d\mathcal{H}^n - \int_M F(u) d\mathcal{H}^n \tag{3.3}$$

for $u \in W^{1,p}(M)$. Let us point out that condition (2.48) will always be satisfied under our assumptions in what follows, and hence $\|\cdot\|_{1,p}$ is actually a norm equivalent to the usual one in $W^{1,p}(M)$, by Corollary 2.5.

The Sobolev embedding theory developed in the previous section is exploited hereafter to show that certain hypotheses of Theorem 3.1 are actually fulfilled by the functional J under the assumptions of Theorem 1.1.

3.1 Mountain pass geometry

The first main result of this section reads as follows. In the statement, B_ρ^W is an abridged notation for $B_\rho^{W^{1,p}(M)}$.

Proposition 3.2 *Let $p \in [1, \infty)$, and let $A : [0, \infty) \to [0, \infty)$ be such that $A(t^{\frac{1}{p}})$ is a Young function. Assume that ν is a quasi-concave function and let $M \in \mathcal{C}_p(\nu)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling properties (1.7) and (1.8). If $\nu(0^+) = 0$, assume, in addition, that*

$$\sup_{s \in (0,\kappa)} \frac{1}{\nu(s) A^{-1}(1/s)^p} < \infty \quad \text{for some } \kappa \in \left(0, \tfrac{\mathcal{H}^n(M)}{2}\right), \tag{3.4}$$

that f fulfills conditions (1.9) and (1.17), and that there exist $\beta > 0$ and $t_1 > 0$ such that

$$F(t) \leq A(\beta|t|) \quad \text{if } |t| \geq t_1. \tag{3.5}$$

Then

$$\lim_{u\in\, W^{1,p}_\perp(M), \|\nabla u\|_{L^p(M)}\to 0} \frac{\int_M F(u)d\mathcal{H}^n}{\|\nabla u\|^p_{L^p(M)}} = 0. \tag{3.6}$$

In particular, there exists $\delta > 0$ such that

$$J_{|\partial B^W_\rho \cap W^{1,p}_\perp(M)} > 0 \quad \text{if } \rho < \delta\,. \tag{3.7}$$

The proof of Proposition 3.2 requires a few preliminary properties that the function f enjoyes owing to the assumptions in force. They are collected in Lemmas 3.3 – 3.5 below.

Lemma 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let F be the function defined by* (1.5)*.*
(i) If conditions (1.7) *and* (1.8) *hold, then for every $\varepsilon > 0$ there exists $t_0 > 0$ such that*

$$F(t) \le \varepsilon |t|^p \quad \text{if} \quad |t| < t_0\,. \tag{3.8}$$

(ii) If conditions (1.8) *and* (1.9) *hold, then there exist $q > p$, $C > 0$ and $t_0 > 0$ such that*

$$F(t) \ge C|t|^q \quad \text{if} \quad |t| \ge t_0\,. \tag{3.9}$$

Lemma 3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (1.8) *and* (1.9)*. Let F be the function defined by* (1.5)*. Then:*
(i) There exist constants $q > p$ and $t_0 > 0$ such that

$$F(r) \le F(s)(r/s)^q \quad \text{if } t_0 \le r \le s, \text{ or } s \le r \le -t_0. \tag{3.10}$$

(ii) *For every $\tau > 0$ there exist constants $q > p$ and $c_0 > 0$ such that*

$$F(t) \le c_0 \eta^q F(t/\eta) \quad \text{if } |t| \ge \tau \text{ and } \eta \in (0,1)\,. \tag{3.11}$$

The proofs of Lemmas 3.3 and 3.4 make use of elementary calculus arguments, and will be omitted for brevity.

Now, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let F be given by (1.5). Define the auxiliary functions $\overline{f} : \mathbb{R} \to [0,\infty)$ as

$$\overline{f}(t) = \max_{s\in[-|t|,|t|]} |f(s)| \quad \text{for } t \in \mathbb{R},$$

and $\overline{F} : [0,\infty) \to [0,\infty)$ as

$$\overline{F}(t) = \int_0^t \overline{f}(s)\, ds \quad \text{for } t \in [0,\infty).$$

Note that $\overline{f}$ is even and non-decreasing in $[0,\infty)$, and hence $\overline{F}$ is a Young function.

Lemma 3.5 *Let $s_0 > 0$, and let $D : (0, s_0) \to (0,\infty)$ be a quasi-concave function such that*

$$\lim_{t\to\infty} tD^{-1}(1/t) = 0\,, \tag{3.12}$$

where D^{-1} stands for the inverse of D. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{t\to\infty} tD^{-1}(1/t)f(kt) = 0 \quad \text{for every } k \in \mathbb{R}\,. \tag{3.13}$$

Then,

$$\lim_{t\to\infty} tD^{-1}(1/t)\overline{f}(kt) = \lim_{t\to\infty} D^{-1}(1/t)F(kt) = \lim_{t\to\infty} D^{-1}(1/t)\overline{F}(|k|t) = 0 \quad \text{for every } k \in \mathbb{R}\,. \tag{3.14}$$

Proof. Let $k \in \mathbb{R}$. By assumption (3.13), for every $\varepsilon > 0$ there exists $M_0 > 0$ such that

$$|f(kt)| \le \frac{\varepsilon}{tD^{-1}(1/t)} \quad \text{if } |t| > M_0 .$$

Since the function $\frac{1}{rD^{-1}(1/r)}$ is non decreasing,

$$\begin{aligned} \overline{f}(kt) &\le \max\left\{ \max_{|r|\le M_0} |f(kr)|, \max_{M_0<|r|\le t} \frac{\varepsilon}{rD^{-1}(1/r)} \right\} \\ &= \max\left\{ \overline{f}(kM_0), \varepsilon\frac{1}{tD^{-1}(1/t)} \right\} \quad \text{if } |t| > M_0 . \end{aligned} \tag{3.15}$$

As a consequence of (3.12), there exists $M_1 > 0$ such that

$$tD^{-1}(1/t) < \frac{\varepsilon}{\overline{f}(kM_0)} \quad \text{if} \quad t \ge M_1 . \tag{3.16}$$

Set $M = \max\{M_0, M_1\}$. From (3.15) and (3.16) we deduce that

$$\overline{f}(kt) \le \frac{\varepsilon}{tD^{-1}(1/t)} \quad \text{if} \quad t > M . \tag{3.17}$$

This establishes the first limit in (3.14). On the other hand, by (3.17),

$$D^{-1}(1/t)|F(kt)| \le D^{-1}(1/t)\overline{f}(kt)|k|t < |k|\varepsilon \quad \text{if } t > M ,$$

and, inasmuch as $\overline{f}$ is a non-decreasing function,

$$D^{-1}(1/t)\overline{F}(|k|t) = D^{-1}(1/t)\int_0^{|k|t} \overline{f}(r)dr \le |k|\varepsilon \quad \text{if } t > M ,$$

whence the second and the third limit in (3.14) follow. □

Proof of Proposition 3.2. Consider first the case when $\nu(0^+) = 0$. Since F and A are continuous and positive for $s \ne 0$, from (3.5) we deduce that for every $t_0 > 0$, there exists $\beta_0 > 0$ such that

$$F(t) \le A\left(\beta_0|t|\right) \quad \text{if } |t| \ge t_0 . \tag{3.18}$$

Fix $\varepsilon > 0$ and choose t_0 in such a way that (3.8) holds. Given $u \in W^{1,p}_\perp(M)$, set $\rho = \int_M |\nabla u|^p \, d\mathcal{H}^n$. Note that, by assumption (3.4), condition (2.48) holds for every $M \in C_p(\nu)$. Thus, by inequality (3.8) and Corollary 2.5, there exists a constant $C = C(p, M)$ such that

$$\frac{\int_{\{|u|\le t_0\}} F(u)d\mathcal{H}^n}{\rho} \le \varepsilon \frac{\int_M |u|^p d\mathcal{H}^n}{\rho} \le \varepsilon C . \tag{3.19}$$

Choose $\tau = t_0$ in (3.11), and ρ in such a way that $\eta = \beta_0 c\sigma(\frac{\mathcal{H}^n(M)}{2})\rho^{\frac{1}{p}} < 1$, where c is the constant appearing in (2.46). Hence, by (3.11), (3.18) and (2.47),

$$\begin{aligned} \frac{\int_{\{|u|>t_0\}} F(u)d\mathcal{H}^n}{\rho} &\le \frac{c_0\eta^q \int_{\{|u|>t_0\}} F\left(\frac{u}{\eta}\right)d\mathcal{H}^n}{\rho} \\ &= c_0\beta_0^q c^q \sigma\left(\frac{\mathcal{H}^n(M)}{2}\right)^q \rho^{\frac{q-p}{p}} \int_{\{|u|>t_0\}} F\left(\frac{u}{\eta}\right) d\mathcal{H}^n(x) \\ &\le c_0\beta_0^q c^q \sigma\left(\frac{\mathcal{H}^n(M)}{2}\right)^q \rho^{\frac{q-p}{p}} \int_{\{|u|>t_0\}} A\left(\frac{|u|}{c\sigma\left(\frac{\mathcal{H}^n(M)}{2}\right)\rho^{\frac{1}{p}}}\right) d\mathcal{H}^n \\ &\le c_0\beta_0^q c^q \sigma\left(\frac{\mathcal{H}^n(M)}{2}\right)^q \rho^{\frac{q-p}{p}} . \end{aligned} \tag{3.20}$$

Equation (3.6) is a consequence of (3.19) and (3.20), owing to the arbitrariness of ε.
Assume next that $\nu(0^+) > 0$, and let u and ρ be as above. Fix $\varepsilon > 0$. Let t_0 be the number appearing in (3.8), and let C be the constant appearing in inequality (2.34). If $\rho < \left(\frac{t_0}{C}\right)^p$, then

$$\frac{\int_M F(u)d\mathcal{H}^n}{\rho} = \frac{\int_{\{|u|\le t_0\}} F(u)d\mathcal{H}^n}{\rho} \le \varepsilon \frac{\int_M |u|^p d\mathcal{H}^n}{\rho} \le \varepsilon C\,. \tag{3.21}$$

Equation (3.6) thus follows also in this case, thanks to the arbitrariness of ε.
Equation (3.7) is a consequence of (3.6). □

Proposition 3.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (1.8) *and* (1.9). *Then there exist $\overline{u} \in \partial B_1^W \cap W_\perp^{1,p}(M)$ and $R > 0$ such that*

$$J_{|\partial Q_R} \le 0\,, \tag{3.22}$$

where Q_R is defined as in (3.2).

Proof. Let $\overline{u}$ be any function in $\partial B_1^W \cap W_\perp^{1,p}(M)$ fulfilling

$$\mathcal{H}^n(M_1) > 0 \text{ and } \mathcal{H}^n(M_2) > 0, \tag{3.23}$$

where $M_1 = \{x \in M : \overline{u}(x) \ge 1\} > 0$ and $M_2 = \{x \in M : \overline{u}(x) \le -1\}$. Set $M_+ = \{x \in M : \overline{u}(x) \ge 0\}$ and $M_- = \{x \in M : \overline{u}(x) \le 0\}$. Trivially,

$$\mathcal{H}^n(M_+) \ge \mathcal{H}^n(M_1) > 0 \quad \text{and} \quad \mathcal{H}^n(M_-) \ge \mathcal{H}^n(M_2) > 0\,. \tag{3.24}$$

Fix $t_0 > 0$ in such a waythat condition (3.10) is satisfied, and let

$$R > \max\left\{ \left(\frac{t_0^q}{p\mathcal{H}^n(M_1)F(t_0)}\right)^{\frac{1}{q-p}},\ \left(\frac{t_0^q}{p\mathcal{H}^n(M_2)F(-t_0)}\right)^{\frac{1}{q-p}},\ t_0 \right\}. \tag{3.25}$$

Since $\overline{u} \in \partial B_1^W \cap W_\perp^{1,p}(M)$,

$$\int_M |\nabla(\sigma\overline{u})|^p\, d\mathcal{H}^n = \|\sigma\overline{u}\|_{1,p}^p = \sigma^p \tag{3.26}$$

for $\sigma > 0$. If $\sigma \in [0, R]$, then (1.8) and (3.10) yield

$$J(R+\sigma\overline{u}) = \frac{\sigma^p}{p} - \int_M F(R+\sigma\overline{u})d\mathcal{H}^n \le \frac{R^p}{p} - \int_{M_+} F(R+\sigma\overline{u})d\mathcal{H}^n \tag{3.27}$$

$$\le \frac{R^p}{p} - \frac{F(t_0)}{t_0^q}\int_{M_+}(R+\sigma\overline{u})^q d\mathcal{H}^n \le \frac{R^p}{p} - \frac{F(t_0)R^q\mathcal{H}^n(M_+)}{t_0^q} \tag{3.28}$$

$$\le \frac{R^p}{p} - \frac{R^q}{pR^{q-p}} = 0\,.$$

Similarly,

$$J(-R+\sigma\overline{u}) \le \frac{R^p}{p} - \int_{M_-} F(-R+\sigma\overline{u})d\mathcal{H}^n \le \frac{R^p}{p} - \frac{F(t_0)R^q\mathcal{H}^n(M_-)}{t_0^q} \le 0\,. \tag{3.29}$$

Moreover, if $\sigma \in [0,\, R]$, then

$$J(\sigma + R\overline{u}) = \frac{R^p}{p} - \int_M F(\sigma + R\overline{u})d\mathcal{H}^n \leq \frac{R^p}{p} - \int_{M_1} F(\sigma + R\overline{u})d\mathcal{H}^n \tag{3.30}$$

$$\leq \frac{R^p}{p} - \frac{F(t_0)}{t_0^q}\int_{M_1}(\sigma + R\overline{u})^q d\mathcal{H}^n \leq \frac{R^p}{p} - \frac{F(t_0)R^q\mathcal{H}^n(M_1)}{t_0^q} \leq 0,$$

and, if $\sigma \in [-R,\, 0]$, then

$$J(\sigma + R\overline{u}) \leq \frac{R^p}{p} - \int_{M_2} F(\sigma + R\overline{u})d\mathcal{H}^n \leq \frac{R^p}{p} - \frac{F(t_0)R^q\mathcal{H}^n(M_2)}{t_0^q} \leq 0\,. \tag{3.31}$$

Finally, if $\sigma \in [-R,\, R]$, then

$$J(\sigma) = -\int_M F(\sigma)d\mathcal{H}^n \leq 0\,. \tag{3.32}$$

Inequality (3.22) follows from (3.27) – (3.32). $\square$

3.2 Smoothness of the energy functional

Conditions under which the functional J is of classs C^1 are exhibited in the next result.

Proposition 3.7 *Let $p \in (1, \infty)$, and let A be a Young function such that*

$$W^{1,p}(M) \to L^A(M). \tag{3.33}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that either A is finite-valued and there exists another Young function E increasing essentially more slowly than A near infinity for which (3.34) *holds, or A is infinite for large values of its argument (and hence $L^A(\Omega) = L^\infty(\Omega)$, up to equivalent norms). Then the functional J is of class C^1.*

A couple of preliminary lemmas will be needed in the proof of Proposition 3.7. The first one is establshed in [BC, Lemma 3.5].

Lemma 3.8 *[[BC]] Let E and A be Young functions such that E increases essentially more slowly than A near infinity. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|f(t)| \leq c(1 + \widetilde{E}^{-1}(E(c|t|))) \quad \text{for } t \in \mathbb{R}\,, \tag{3.34}$$

for some constant $c > 0$.
(i) If $u \in L^E(M)$ and $\{u_k\}$ is a bounded sequence in $L^A(M)$ such that $u_k \to u$ in $L^E(M)$, then

$$\lim_{k\to\infty}\int_M f(u_k)(u_k - u)d\mathcal{H}^n = 0\,. \tag{3.35}$$

(ii) If $u \in L^A(M)$ and $\{u_k\}$ is a sequence in $L^A(M)$ such that $u_k \to u$ in $L^A(M)$, then

$$\lim_{k\to\infty}\|f(u_k) - f(u)\|_{L^{\widetilde{A}}(M)} = 0\,.$$

Lemma 3.9 *Assume that p, A, E and f are as in Proposition 3.7. Let $u \in W^{1,p}(M)$. Then*

$$\lim_{k\to\infty} \sup_{v\in W^{1,p}(M)\setminus\{0\}} \frac{\left|\int_M (f(u_k) - f(u))\, v \, d\mathcal{H}^n\right|}{\|v\|_{1,p}} = 0\,,$$

for any sequence $\{u_k\} \subset W^{1,p}(M)$ such that $\lim_{k\to\infty} u_k = u$ in $W^{1,p}(M)$.

Proof. Assume first that A is finite-valued, and (3.34) holds. Owing to (2.13) and (3.33), we have that

$$\begin{aligned} \left|\int_M (f(u_k) - f(u))\, v \, d\mathcal{H}^n\right| &\le 2\|f(u_k) - f(u)\|_{L^{\widetilde{A}}(M)} \|v\|_{L^A(M)} \\ &\le C\|f(u_k) - f(u)\|_{L^{\widetilde{A}}(M)} \|v\|_{W^{1,p}(M)}\,, \end{aligned} \tag{3.36}$$

for some constant C. From (3.33) we also deduce that $\lim_{k\to\infty} u_k = u$ in $L^A(M)$. Hence, the conclusion follows via Lemma 3.8, Part (ii).
If, instead, $L^A(M) = L^\infty(M)$, then $L^{\widetilde{A}}(M) = L^1(M)$ (up to equivalent norms), and the conclusion follows from (3.36) again, thanks to the continuity of f. □

Proof of Proposition 3.7. The functional $\int_M |\nabla u|^p \, d\mathcal{H}^n$ is well known to be of class C^1. It thus suffices to show that the functional $\int_M F(u) d\mathcal{H}^n$ is of class C^1 as well. Let us set

$$L_f(u) = \int_M F(u) d\mathcal{H}^n.$$

Our conclusion will follow if we show that the Gâteaux derivative of L_f, denoted by $(L_f)'_G$, is continuous. Let u, $v \in W^{1,p}(M)$ and let $\varepsilon \in (0,1)$. Then,

$$\frac{1}{\varepsilon}\big(L_f(u+\varepsilon v) - L_f(u)\big) = \int_M \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} d\mathcal{H}^n\,. \tag{3.37}$$

By the continuity of f,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\big[F(u(x) + \varepsilon v(x)) - F(u(x))\big] = f(u(x))v(x) \quad \text{for a.e. } x \in M\,. \tag{3.38}$$

Moreover, for a.e. $x \in M$ there exists $\theta_x \in (0,1)$ such that

$$\frac{1}{\varepsilon}\big[F(u(x) + \varepsilon v(x)) - F(u(x))\big] = f(u(x) + \varepsilon\theta_x v(x))v(x)\,. \tag{3.39}$$

If B is finite-valued, and (3.34) holds, then, by (3.39),

$$\begin{aligned} \frac{1}{\varepsilon}\big|F(u(x) + \varepsilon v(x)) - F(u(x))\big| &\le c\big[1 + \widetilde{E}^{-1}(E(c|u(x) + \varepsilon\theta_x v(x)|))\big]|v(x)| \\ &\le c\big[1 + \widetilde{E}^{-1}(E(c(|u(x)| + |v(x)|)))\big]|v(x)| \end{aligned} \tag{3.40}$$

for a.e. $x \in M$. The right-hand side of (3.40) belongs to $L^1(M)$, since

$$\int_M \widetilde{E}^{-1}(E(c(|u| + |v|)))|v| \, d\mathcal{H}^n \le 2\|\widetilde{E}^{-1}(E(c(|u| + |v|)))\|_{L^{\widetilde{E}}(M)} \|v\|_{L^E(M)} < \infty\,. \tag{3.41}$$

Note that the first inequality (3.41) is a consequence of (2.13), and the second one holds by (3.33) and the assumption that E grows essentially more slowly than A near infinity. Consequently, $(L_f)'_G(u)(v)$ exists for every u, $v \in W^{1,p}(M)$, and, by the dominated convergence theorem,

$$(3.42) \qquad (L_f)'_G(u)(v) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\big[L_f(u+\varepsilon v) - L_f(u)\big] = \int_M f(u)\, v \, d\mathcal{H}^n .$$

If $L^A(M) = L^\infty(M)$, then equation (3.42) is a consequence of the boundedness of u and v, and of the continuity of f.
Having formula (3.42) at disposal, the continuity of $(L_f)'_G$ is a consequence of Lemma 3.9. □

3.3 Palais-Smale condition

The validity of the Palais-Smale condition for the functional J is established in the next proposition.

Proposition 3.10 *Let $p \in (1,\infty)$, and let $A : [0,\infty) \to [0,\infty)$ be such that $A(t^{\frac{1}{p}})$ is a Young function. Let ν be a quasi-concave function, and let $M \in \mathcal{C}_p(\nu)$. Assume that condition* (3.4) *holds. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling conditions* (1.8) *and* (1.9)*. Assume that either A is finite-valued and there exists another Young function E increasing essentially more slowly than A near infinity for which* (3.34) *holds, or A is infinite for large values of its argument. Then the functional J satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\} \subset W^{1,p}(M)$ be a Palais-Smale sequence for J. Then, on taking, if necessary, a subsequence, we may assume that $\lim_{k\to\infty} I(u_k) = c$ for some $c \in \mathbb{R}$. Thus, given any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$(3.43) \qquad c - \varepsilon < J(u_k) < c + \varepsilon \quad \text{for } k > k_0 .$$

Furthemore, since $\lim_{k\to\infty} \|J(u_k)\|_{W^{1,p}(M)^*} = 0$, there exists a sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0^+$, such that

$$(3.44) \qquad -\varepsilon_k \|v\|_{1.p} \le \int_M |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla v d\mathcal{H}^n - \int_M f(u_k)\, v\, d\mathcal{H}^n \le \varepsilon_k \|v\|_{1.p}$$

for every $v \in W^{1,p}(M)$. By (1.8) and (1.9), there exist $q > p$ and $t_0 > 0$ such that

$$(3.45) \qquad tf(t) - qF(t) > 0 \quad \text{if } |t| > t_0 .$$

From the second inequality in (3.43) and the first inequality in (3.44), applied with $v = u_k$, we deduce that

$$(3.46) \qquad \tfrac{q-p}{p} \|\nabla u_k\|^p_{L^p(M)} - \int_M (qF(u_k) - f(u_k)u_k) d\mathcal{H}^n \le q(c+\varepsilon) + \varepsilon_k \|u_k\|_{1,p}$$

if $k > k_0$. Owing to equations (3.45) and (3.46), and to the continuity of f, there exists a constant C such that

$$(3.47) \quad \begin{aligned} \tfrac{q-p}{p} \|\nabla u_k\|^p_{L^p(M)} &\le q(c+\varepsilon) + \varepsilon_k \|\nabla u_k\|_{L^p(M)} + \varepsilon_k |(u_k)_M| + \int_{|u_k| \le t_0} (qF(u_k) - f(u_k)u_k) d\mathcal{H}^n \\ &\le C + \varepsilon_k \|\nabla u_k\|_{L^p(M)} + \varepsilon_k |(u_k)_M| \end{aligned}$$

if $k > k_0$. We claim that the sequence $\{u_k\}$ is bounded in $W^{1,p}(M)$. Suppose, by contradiction, that $\{u_k\}$ is unbounded. If $\|\nabla u_k\|_{L^p(M)}$ is unbounded, then we infer from (3.47) that there exists $k_1 \in \mathbb{N}$ such that

$$\|\nabla u_k\|^p_{L^p(M)} < |(u_k)_M| \quad \text{if } k > k_1 . \tag{3.48}$$

On the other hand, by (1.9), there exist a, $b > 0$ such that

$$F(t) \geq a|t|^p - b \quad \text{for } t \in \mathbb{R} . \tag{3.49}$$

Coupling (3.48) with (3.49) tells us that there exist positive constants C_1 and C_2 such that

$$J(u_k) < \tfrac{1}{p}|(u_k)_M| + C_1 - C_2\|u_k\|^p_{L^p(M)} \quad \text{if } k > k_1 .$$

Hence

$$\lim_{k\to\infty} J(u_k) = -\infty . \tag{3.50}$$

This contradicts the first inequality in (3.43). Thus the sequence $\|\nabla u_k\|_{L^p(M)}$ is bounded, and hence the sequence $|(u_k)_M|$ must be unbounded. Consequently, $\|u_k\|_{L^p(M)}$ is unbounded as well. The boundedness of the sequence $\|\nabla u_k\|_{L^p(M)}$ and inequality (3.49) then entail that

$$J(u_k) \leq C_1 - C_2\|u_k\|^p_{L^p(M)} \tag{3.51}$$

for suitable positive constants C and C_2. Inequality (3.51) implies (3.50), and hence a contradiction again.

Altogether, we have shown that the sequence $\{u_k\}$ is bounded in $W^{1,p}(M)$. Consequently, there exists a subsequence, still denoted by $\{u_k\}$, that weakly converges in $W^{1,p}(M)$ to some function u. In particular

$$\lim_{k\to\infty} (u_k)_M = u_M . \tag{3.52}$$

Assumption (3.4) enables us to apply Theorem 2.1, and deduce that $W^{1,p}(M) \to L^A(M)$. Now, if A is finite-valued and (3.34) holds, then by (3.44) with $v = u - u_k$, by Proposition 3.8, Part (*i*), and by the boundedness of $\{u_k\}$ in $W^{1,p}(M)$,

$$\lim_{k\to\infty} \int_M |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u - u_k) d\mathcal{H}^n = 0 . \tag{3.53}$$

Since (3.35) is clearly satisfied also when A jumps to infinity, equation (3.53) holds also in this case. The convexity of the function $|\cdot|$ ensures that

$$\int_M |\nabla u_k|^p \, d\mathcal{H}^n - \int_M |\nabla u|^p \, d\mathcal{H}^n \leq p \int_M |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_k - u) d\mathcal{H}^n \tag{3.54}$$

for $k \in \mathbb{N}$. Hence, by (3.53),

$$\limsup_{k\to\infty} \int_M |\nabla u_k|^p \, d\mathcal{H}^n \leq \int_M |\nabla u|^p \, d\mathcal{H}^n . \tag{3.55}$$

Consequently, the sequence $\{\nabla u_k\}$ is actually strongly convergent to ∇u in $L^p(M)$. Combining this piece of information with (3.52) tells us that $\{u_k\}$ converges to u in $W^{1,p}(M)$. $\square$

3.4 Conclusion

We are now in a position to accomplish the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume first that $\nu(0^+) = 0$. Define the function $B : (0, \infty) \to [0, \infty)$ as

$$B(t) = \frac{1}{\nu^{-1}\left(\frac{1}{t^p}\right)} \quad \text{for } t > 0. \tag{3.56}$$

The function B is continuous and increasing. Moreover, the function

$$t \to \frac{B(t)}{t^p} \quad \text{is non-decreasing.} \tag{3.57}$$

This is a consequence of the fact that the function $\frac{t^p}{\nu^{-1}(t^p)}$ is non-increasing, and the latter property is equivalent to (1.14). Also,

$$\nu(s) = \frac{1}{\left[B^{-1}\left(\frac{1}{s}\right)\right]^p} \quad \text{for } s > 0. \tag{3.58}$$

Owing to (3.58), condition (3.4), and hence (2.28) holds. Thus, as a consequence of Theorem 2.1 and Remark 2.2 (with the role of A and B exchanged),

$$W^{1,p}(M) \to L^A(M), \tag{3.59}$$

where A is the Young function given by

$$A(t) = \int_0^t \frac{B(r)}{r} dr \quad \text{for } t > 0. \tag{3.60}$$

Thanks to property (2.3), and to the fact that, by (3.57), the function $\frac{B(t)}{t}$ is non-decreasing,

$$A(t/2) \leq A(t)/2 \leq B(t) \leq A(t) \quad \text{for } t > 0. \tag{3.61}$$

We claim that, if f satisfies (1.17), then

$$\overline{F} \text{ increases essentially more slowly than } A \text{ near infinity.} \tag{3.62}$$

In order to prove (3.62), let us apply Lemma 3.5 with $D(s) = \nu(s)^{\frac{1}{p}}$. By equation (3.14) of this lemma,

$$\lim_{t\to\infty} \nu^{-1}\left(t^{-p}\right) \overline{F}(kt) = 0 \quad \text{for every } k > 0\,. \tag{3.63}$$

Owing to (3.56), (3.61) and (3.63),

$$\lim_{t\to\infty} \frac{\overline{F}(kt)}{A(t)} = 0 \quad \text{for every } k > 0\,. \tag{3.64}$$

Assertion (3.62) follows from (3.64) and (3.61). The very definition of $\overline{F}$ and inequality (2.6), applied with A replaced by $\overline{F}$, tell us that

$$|f(t)| \leq 2\widetilde{\overline{F}}^{-1}(\overline{F}(2|t|)) \quad \text{for } t \in \mathbb{R}\,. \tag{3.65}$$

Thus, f satisfies the hypotheses of Lemma 3.8, with $E = \overline{F}$ and A defined as in (3.60).
The assumptions of Propositions 3.2, 3.6, 3.7 and 3.10 are thus fulfilled with this choice of A and E.
On the other hand, if $\nu(0^+) > 0$ then these assumptions are trivially satisfied when A is a Young function that equals infinity for large values of its argument.
As a consequence, one can apply Theorem 3.1 and deduce that there exists a critical point u of J, namely a solution u to problem (1.1), such that $J(u) > 0$. Notice that u is actually nontrivial. Indeed, if $u = c$ for some constant c, then $J(u) = -\int_M F(c)d\mathcal{H}^n(x) \leq 0$, and this contradicts the inequality $J(u) > 0$. $\square$

4 Applications and examples

In what follows, we tacitly assume that the nonlinearity f satisfies hypotheses (1.7)–(1.9), and we focus on the growth condition (1.17) in the diverse instances.

4.1 Regular geometry

Assume that M is any compact manifold. Then the isocapacitary function of M fulfils (1.12). From Theorem 1.1 one infers that problem (1.3) admits a nontrivial solution, provided that either

$$1 < p < n \quad \text{and} \quad \lim_{t\to\pm\infty} |t|^{-\frac{np-n+p}{n-p}} f(t) = 0\,, \tag{4.1}$$

or

$$p = n \quad \text{and} \quad \lim_{t\to\pm\infty} e^{-k|t|^{\frac{n}{n-1}}} f(t) = 0 \quad \text{for every } k > 0.. \tag{4.2}$$

If $p > n$, no additional growth condition on f near infinity is needed.
The same conclusions hold for the Neumann problem (1.4) when Ω is a regular bounded open set – a Lipschitz domain, say – in $\mathbb{R}^n$, or on a Riemannin manifold M. Neumann problems in John domains also fall in the present setting. Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a John domain if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq cr \qquad \text{for } r \in [0,l]. \tag{4.3}$$

Let us point out that, even in these customary situations, our result somewhat enhances contributions available in the literature, since condition (4.1) is weaker than (1.10).

4.2 Hölder domains

Consider the Neumann problem (1.4) in a connected bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose boundary is Hölder continuous for some exponent $\alpha \in (0,1)$. Then,

$$\nu_{\Omega,p}(s) \geq \begin{cases} Cs^{1-\frac{\alpha p}{n-1+\alpha}} & \text{if } 1 < p < \frac{n-1}{\alpha} + 1 \\ C\big(\log\frac{1}{s}\big)^{\frac{1-n}{\alpha}} & \text{if } p = \frac{n-1}{\alpha} + 1 \\ C & \text{if } p > \frac{n-1}{\alpha} + 1 \end{cases} \tag{4.4}$$

for s near 0, and for some positive constant C. The first and third lines of equation (4.4) follow via the Sobolev embedding of [La, Theorem] and the equivalence of (2.27) and (2.29). The second one

can be established via a variant in the proof of [La, Theorem].
Owing to inequality (4.4), Theorem 1.1 ensures that a nontrivial solution to the Neumann problem (1.4) exists for any arbitrarily fast growth of f near infinity, if $p > \frac{p-1}{\alpha} + 1$. Otherwise, the existence of such a solution is guaranteed provided that

$$\lim_{t\to\pm\infty} |t|^{-\frac{\alpha p}{n+1+\alpha-\alpha p}} f(t) = 0 \qquad \text{if } 1 < p < \tfrac{n-1}{\alpha} + 1, \tag{4.5}$$

and

$$\lim_{t\to\pm\infty} e^{-k|t|^{\frac{n-1+\alpha}{n-1}}} f(t) = 0 \quad \text{for every } k > 0, \quad \text{if } p = \tfrac{n-1}{\alpha} + 1. \tag{4.6}$$

Notice that the same conclusion can be derived via Corollary 1.2. Indeed, by the equivalence of the Sobolev embedding [La, Theorem] with $p = 1$ and the relative isoperimetric inequality in Ω [Ma3, Corollary 5.2.3],

$$\lambda_\Omega(s) \geq C s^{\frac{n-1}{n-1+\alpha}} \quad \text{for } s \in (0, \mathcal{H}^n(M)/2), \tag{4.7}$$

for some positive constant C (see also [Ci, Theorem 1] for an earlier direct proof of (4.7) when $n = 2$).

4.3 Cusp shaped domains

We deal here with Neumann problem (1.4) in the cusp-shaped set

$$\Omega = \{x \in \mathbb{R}^n : |x'| < \vartheta(x_n), 0 < x_n < L\}$$

(Figure 1). Here, $x = (x', x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $L > 0$ and $\vartheta : [0, L] \to [0, \infty)$ is a differentiable convex function such that $\vartheta(0) = 0$. Let $\Theta : [0, L] \to [0, \infty)$ be the function given by

$$\Theta(\rho) = n\omega_n \int_0^\rho \vartheta(r)^{n-1} dr \qquad \text{for } \rho \in [0, L],$$

where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$. By [Ma3, 4.3.5/1],

$$\nu_{\Omega,p}(s) \approx \left(\int_{\Theta^{-1}(s)}^{\Theta^{-1}(\mathcal{H}^n(\Omega))} \vartheta(r)^{\frac{1-n}{p-1}} dr\right)^{1-p} \quad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\big). \tag{4.8}$$

Consider, for instance, the case when

$$\vartheta(r) \approx r^\delta (\log \tfrac{1}{r})^\beta \quad \text{near } 0,$$

where either $\delta > 1$ and $\beta \in \mathbb{R}$, or $\delta = 1$ and $\beta \leq 0$. Equation (4.8) tells us that

$$\nu_{\Omega,p}(s) \approx \begin{cases} s^{1-\frac{p}{\delta(n-1)+1}} \big(\log \frac{1}{s}\big)^{\frac{p\beta(n-1)}{\delta(n-1)+1}} & \text{if } 1 < p < \delta(n-1)+1, \\ \big(\log \frac{1}{s}\big)^{(\beta-\delta)(n-1)} & \text{if } p = \delta(n-1)+1,\ \beta < \delta, \\ \big(\log\log \frac{1}{s}\big)^{-\delta(n-1)} & \text{if } p = \delta(n-1)+1,\ \beta = \delta, \\ 1 & \text{if } p = \delta(n-1)+1,\ \beta > \delta, \text{ or } p > \delta(n-1)+1, \end{cases} \tag{4.9}$$

for s near 0. Hence, by Theorem 1.1, a notrivial solution to the Neumann problem (1.4) exists, provided that

$$\lim_{t\to\pm\infty} |t|^{1-\frac{p[\delta(n-1)+1]}{\delta(n-1)+1-p}} (\log|t|)^{-\frac{p\beta(n-1)}{\delta(n-1)+1-p}} f(t) = 0 \quad \text{if } 1 < p < \delta(n-1)+1, \tag{4.10}$$

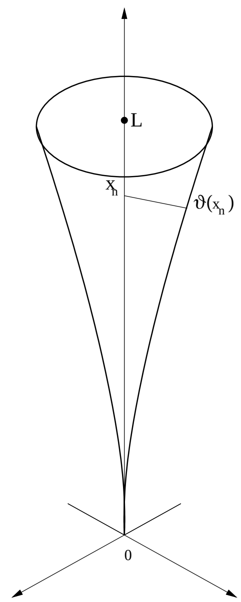

Figure 1: A cusp shaped domain

$$\lim_{t\to\pm\infty} e^{-k|t|^{\frac{p}{(\delta-\beta)(n-1)}}} f(t) = 0 \quad \text{for every } k > 0, \quad \text{if } p = \delta(n-1)+1,\ \beta < \delta, \tag{4.11}$$

and

$$\lim_{t\to\pm\infty} e^{-e^{k|t|^{\frac{p}{\delta(n-1)}}}} f(t) = 0 \quad \text{for every } k > 0, \quad \text{if } p = \delta(n-1)+1,\ \beta = \delta. \tag{4.12}$$

In the remaining cases, namely if either $p = \delta(n-1)+1$ and $\beta > \delta$, or $p > \delta(n-1)+1$, no additional growth condition near infinity is needed on f.
The same conclusions can also be reached via Corollary 1.2, since, by [Ma3, Example 3.3.3/1],

$$\lambda_\Omega(s) \approx \vartheta(\Theta^{-1}(s))^{n-1} \qquad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\big), \tag{4.13}$$

and hence

$$\nu_{\Omega,p}(s) \approx \bigg(\int_s^{\frac{\mathcal{H}^n(\Omega)}{2}} \frac{dr}{\lambda_\Omega(r)^{p'}}\bigg)^{1-p} \quad \text{for } s \in \big(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\big). \tag{4.14}$$

4.4 γ-John domains

The class of γ-John domains extends the class of the usual John domains in that r is allowed to be replaced with r^γ, for some $\gamma \geq 1$, on the right-hand side of inequality (4.4). Namely, a bounded open set Ω in $\mathbb{R}^n$ is a γ-John domain if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq c r^\gamma \qquad \text{for } r \in [0,l].$$

This replacement enlarges the class in such a way that, if $\gamma > 1$, the criterion of Theorem 1.1 can be applied to the Neumann problem (1.4) for certain irregular γ-John domains Ω for which, by contrast, Corollary 1.2 fails.
Indeed, assume that $1 < p < n$ and that Ω is a γ-John domain with

$$1 \le \gamma < \frac{p}{n-1} + 1. \tag{4.15}$$

Then the Poincaré inequality from [KM, Theorem 2.3] and the equivalence of (2.27) and (2.29) ensure that

$$\nu_{\Omega,p}(s) \approx s^{\frac{\gamma(n-1)+1-p}{n}} \quad \text{for } s \text{ near } 0. \tag{4.16}$$

From Theorem 1.1 we then obtain a non trivial solution to the Neumann problem (1.4), provided that

$$\lim_{t\to\pm\infty} |t|^{1-\frac{np}{(\gamma(n-1)+1-p}} f(t) = 0\,. \tag{4.17}$$

On the other hand, the isoperimetric inequality established in the proof of [HK, Corollary 5] tells us that, if

$$1 \le \gamma < \frac{n}{n-1}, \tag{4.18}$$

then

$$\lambda_\Omega(s) \approx s^{\frac{\gamma(n-1)}{n}} \quad \text{for } s \text{ near } 0. \tag{4.19}$$

An application of Corollary 1.2 yields the existence of a nontrivial solution to the Neumann problem (1.4) under the assumption that

$$\lim_{t\to\pm\infty} |t|^{1-\frac{np}{p(n-1)\gamma-np+n}} f(t) = 0\,. \tag{4.20}$$

Conditions (4.18) and (4.20) are more restrictive than (4.15) and (4.17), respectively. Thus, the use of the isoperimetric function actually leads to weaker conclusions than that of the isocapacitary function in this class of very irregular domains.

4.5 A family of manifolds with clustering submanifolds

We conclude by dealing with problem (1.1), for $p \in (1,2]$, on a class of noncompact surfaces M in $\mathbb{R}^3$, that are reminiscent of a planar domain appearing in an example in [CH]. They provide us with further examples where the existence of nontrivial solutions to (1.1) can be shown via Theorem 1.1, but not by Corollary 1.2.
The main feature of the surfaces in question is that they contain a sequence of mushroom-shaped submanifolds $\{N^k\}$ clustering at some point (Figure 2). The submanifolds $\{N^k\}$ are not obtained just by dilation of each other. The diameter of the head and the length of the neck of N^k decay to 0 as 2^{-k} when $k \to \infty$, whereas the width of the neck of N^k decays to 0 as $\sigma(2^{-k})$, where σ is a function such that

$$\lim_{s\to 0} \frac{\sigma(s)}{s} = 0.$$

The space of compactly supported functions in such manifolds M is dense in $W^{1,p}(M)$. This is the content of [CM2, Proposition 2.8] for $p = 2$; a close inspection of the proof of that proposition

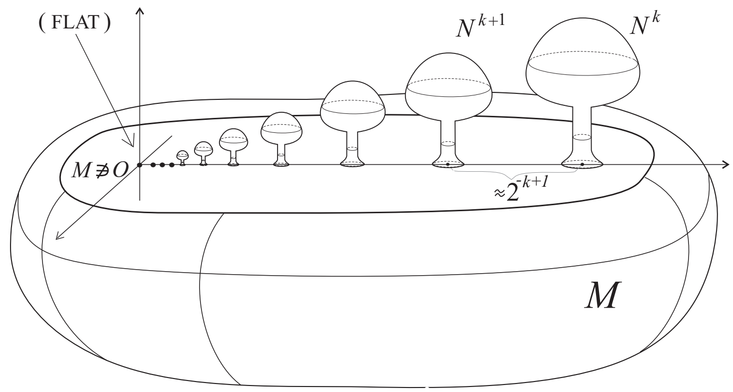

Figure 2: A manifold with a family of clustering submanifolds

reveals that the same conclusion also holds for any $p \in [1, 2)$. Consequently, problem (1.1) is just the weak formulation of the equation (1.3).

The isoperimetric and isocapacitary functions of M depend on the behavior of σ at 0. Loosely speaking, a faster decay to 0 of the function $\sigma(s)$ as $s \to 0$ results in a faster decay to 0 of $\lambda_M(s)$ and $\nu_{M,p}(s)$, and hence in a manifold M with a more irregular geometry.

The asymptotic behavior of the isoperimetric and isocapacitary functions of M can be described as follows [CM3, Propositions 7.1 and 7.2]. Assume that $\sigma : [0, \infty) \to [0, \infty)$ is an increasing function such that

$$\sigma(2s) \le c\sigma(s) \quad \text{for } s \ge 0, \tag{4.21}$$

for some constant $c > 0$, and

$$\frac{s^{\beta+1}}{\sigma(s)} \quad \text{is non-increasing} \tag{4.22}$$

for some $\beta > 0$. If

$$\frac{s^2}{\sigma(s)} \quad \text{is non-decreasing,} \tag{4.23}$$

then

$$\lambda_M(s) \approx \sigma(s^{1/2}) \qquad \text{near } 0. \tag{4.24}$$

Under the (weaker) assumption that

$$\frac{s^{p+1}}{\sigma(s)} \quad \text{is non-decreasing,} \tag{4.25}$$

one has that

$$\nu_{M,p}(s) \approx \sigma(s^{1/2})s^{-\frac{p-1}{2}} \qquad \text{near } 0. \tag{4.26}$$

Since the right-hand side of (4.26) is quasi concave, Theorem 1.1 can be exploited to discuss the existence of a nontrivial solution to equation (1.3), depending on the balance, prescribed by (1.17), between the growth of f near infinity, and the decay of the right-hand side of (4.26) at zero. For instance, if

$$\sigma(s) = s^{\alpha} \quad \text{for } s \geq 0,$$

with $1 < \alpha \leq p+1$, then $\nu_{\Omega,p}(s) \approx s^{\frac{\alpha+1-p}{2}}$ near 0. By Theorem 1.1, equation (1.3) has a nontrivial solution provided that

$$\lim_{t \to \pm\infty} |t|^{1-\frac{2p}{\alpha+1-p}} f(t) = 0\,.$$

By contrast, due to assumption (4.23), we are entitled to apply Corollary 1.2 only if $1 < \alpha \leq 2$. Moreover, it requires the stronger growth condition

$$\lim_{t \to \pm\infty} |t|^{1-\frac{2p}{\alpha p+2-2p}} f(t) = 0\,.$$

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