BEHAVIOUR OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR THE BIHARMONIC OPERATOR AT A BOUNDARY POINT

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1°. Introduction. According to the classical result by Wiener [1], [2] the regularity of a boundary point 0 for the Laplace equation in a domain $\Omega \subset R^n$, $n > 2$ is equivalent to the divergence of the series

$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}(C_{2^{-k}} \setminus \Omega)$$

where $C_\rho = \{x \in R^n : \rho/2 \le |x| \le \rho\}$ and cap is the harmonic capacity. Wiener's theorem was extended (sometimes only with respect to sufficiency) to different classes of linear and quasilinear second order partial differential equations ([3] - [11] and others). However, results of this type for higher order equations seem to be unknown.

In the present paper we study the behaviour near a boundary point of solutions to the Dirichlet problem with zero boundary data for the equation $\Delta^2 u = f$, $f \in C_0^\infty(\Omega)$, $\Omega \subset R^n$. The proof covers only dimensions $n = 4,5,6,7$ (the case $n < 4$ is not interesting). We show in particular that the condition

$$\sum_{k=1}^{\infty} 2^{k(n-4)} \operatorname{cap}_2(C_{2^{-k}} \setminus \Omega) = \infty \ , \quad n = 5,6,7,$$

where cap_2 is the so called biharmonic capacity, guarantees the continuity of the solution at the point 0. This result follows from an estimate of the modulus of continuity. Such estimates, formulated in terms of the rate of divergence of Wiener's series were known only for second order equations ([12], [7], [9], [13]).

In the last section we obtain some pointwise estimates for the Green function $G(x,y)$ of the Dirichlet problem for Δ^2 valid without any restrictions on the boundary $\partial\Omega$. In particular it is proved that $|G(x,y)| \le c|x-y|^{4-n}$ where $n = 5,6,7$ and c is a positive constant depending only on n.

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2°. Preliminaries and definitions. Let Ω denote an open subset of Euclidean space R^n with a compact closure $\bar{\Omega}$ and a boundary $\partial\Omega$. Let 0 be a point of $\partial\Omega$ and $B_\rho = \{x : |x| < \rho\}$, $C_\rho = B_\rho \setminus B_{\rho/2}$. We denote by $c, c_1, \dots$ positive constants depending only on n and write $\nabla_l = \{\partial^l / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}\}$, $\nabla_1 = \nabla$. We consider only real functions.

Let $\overset{\circ}{W}{}_2^2(\Omega)$ be the closure of the space $C_0^\infty(\Omega)$ in the norm $\|\nabla_2 u\|_{L_2(\Omega)}$.

We introduce the biharmonic capacity of a compact e with respect to an open domain G, $G \supset e$:

$$\mathrm{cap}_2(e;G) = \inf \left\{ \int_G |\nabla_2 u|^2 \, dx : u \in C_0^\infty(G), \right.$$
$$\left. u = 1 \text{ in a neighbourhood of } e \right\}.$$

We write $\mathrm{cap}_2(e)$ instead of $\mathrm{cap}_2(e;R^n)$.

Let Γ denote the fundamental solution for the biharmonic operator, i.e.

$$\Gamma(x) = \frac{|x|^{4-n}}{2(n-4)(n-2)\omega_n} \quad \text{if } n > 4, \tag{1}$$

$$\Gamma(x) = (4\omega_4)^{-1} \log \frac{d}{|x|} \quad \text{if } n = 4,$$

where $\omega_n = \mathrm{mes}_{n-1} \partial B_1$ and d is a constant.

<u>3°. "Weighted" positivity of Δ^2.</u>

<u>Lemma 1.</u> Let $u \in \overset{\circ}{W}{}_2^2(\Omega) \cap C^\infty(\Omega)$ and $4 \le n \le 7$. Then for every point $p \in \Omega$ (and in the case $n = 4$ for any d satisfying $d \ge \mathrm{diam}\,(\mathrm{supp}\, u)$) we have

$$u(p)^2 + c \int_\Omega \left[(\nabla_2 u(x))^2 + \frac{(\nabla u(x))^2}{|p-x|^2} \right] \Gamma(x-p) \, dx \le \tag{2}$$
$$\le 2 \int_\Omega \Delta u(x) . \Delta(u(x)\, \Gamma(x-p)) \, dx.$$

Proof. Let (r, ω) be the spherical coordinates with the center p and let G denote the image of Ω under the mapping $x \to (t, \omega)$ where $t = -\log r$. Since

$$r^2 \Delta u = r^{2-n} (r\,\partial/\partial r) \left[r^{n-2} (r\,\partial/\partial r) u \right] + \delta_\omega u$$

where δ_ω is the Beltrami operator on the unit sphere S^{n-1} we get for the function $v(t,\omega) = u(x)$

$$r^2 \Delta u = v_{tt} - (n-2) v_t + \delta_\omega v = Lv.$$

Consider first the case $n > 4$. By a simple computation

$$c(n) \int_\Omega \Delta u(x) . \Delta(u(x)\, \Gamma(x-p)) \, dx = \int_G e^{(4-n)t} Lv . L(v e^{(n-4)t}) \, dt \, d\omega = \tag{3}$$

$$= \int_G (v_{tt}-(n-2)v_t+\delta_\omega v)(v_{tt}+(n-6)v_t-2(n-4)v+\delta_\omega v)\, dt d\omega$$

where $c(n) = 2(n-2)(n-4)\omega_n$. We remark that

$$(4) \qquad 2\int_G v_t v \, dt\, d\omega = \int_{S^{n-1}} v(\infty,\omega)^2\, d\omega = \omega_n u(p)^2.$$

The following identities are also obvious:

$$(5) \qquad \int_G v_t\, \delta_\omega v\, dt\, d\omega = 0, \qquad \int_G v_t v_{tt}\, dt\, d\omega = 0.$$

Thus the last integral in (3) becomes

$$(6) \qquad \int_G \left[v_{tt}^2-(n-2)(n-6)v_t^2-2(n-4)v_{tt}v+2v_{tt}\,\delta_\omega v+(\delta_\omega v)^2- \right.$$
$$\left. - 2(n-4)v\,\delta v\right] dt\, d\omega + \frac{c(n)}{2} u(p)^2.$$

After integrating by parts we rewrite (6) as

$$(7) \qquad \int_G \left\{v_{tt}^2+(\delta_\omega v)^2+2v_t(-\delta_\omega v_t)+2(n-4)v(-\delta_\omega v) + \right.$$
$$\left. + \left[5-(n-5)^2\right]v_t^2\right\} dt\, d\omega + \frac{c(n)}{2} u(p)^2.$$

Using the former variables (r,ω) we obtain

$$\int_\Omega \left[u_{rr}^2+ \frac{2}{r^2}(\nabla_\omega u_r)^2+2\,\frac{n-4}{r^4}(\nabla_\omega u)^2+ \frac{(7-n)(n-3)}{r^2}u_r^2\right] \frac{dx}{r^{n-4}} + \frac{c(n)}{2} u(p)^2.$$

This completes the proof of (2) for $n = 5,6$. In the case $n = 7$ one can use the inequality

$$\int_\Omega u_{rr}^2 \frac{dx}{r^{n-4}} \geq \int_\Omega u_r^2 \frac{dx}{r^{n-2}}$$

which is a corollary of the one-dimensional inequality

$$\int_0^\infty w(r)^2 r\, dr \leq \int_0^\infty w'(r)^2 r^{(3)}\, dr.$$

Now let $n = 4$. We have

$$\int_\Omega 4\omega_4 \quad \Delta u(x).\Delta(u(x)\,\Gamma(x-p))\, dx = \int_\Omega \Delta u(x)\Delta(u(x)\log\frac{d}{|x-p|})\, dx =$$
$$= \int_G Lv.L((\ell+t)v)\, dt\, d\omega$$

where $\ell = \log d$. The last integral is equal to

$$\int_G (\ell+t)(Lv)^2\,dt\,d\omega + 2\int_G (v_t-v)Lv\,dt\,d\omega. \tag{8}$$

Applying (4) and (5) we rewrite (8) in the form

$$\int_G (\ell+t)(Lv)^2\,dt\,d\omega + 2\int_G [(\nabla_\omega v)^2-v_t^2]\,dt\,d\omega + 2\omega_4 u(p)^2. \tag{9}$$

For the first integral in (9) we have

$$\int_G (\ell+t)(Lv)^2\,dt\,d\omega = \int_G [v_{tt}^2+4v_t^2+(\delta_\omega v)^2](\ell+t)\,dt\,d\omega +$$
$$+ 2\int_G (v_{tt}\delta_\omega v-2v_t\delta_\omega v-2v_{tt}v_t)(\ell+t)\,dt\,d\omega,$$

and integrating by parts, we get

$$\int_G(\ell+t)(Lv)^2\,dt\,d\omega = \int_G [v_{tt}^2+4v_t^2+(\delta_\omega v)^2+2(\nabla_\omega v_t)^2](\ell+t)\,dt\,d\omega -$$
$$- 2\int_G [(\nabla_\omega v)^2- v_t^2]\,dt\,d\omega .$$

Therefore

$$4\omega_4\int_\Omega \Delta u\cdot\Delta(u\Gamma)\,dx = \int_G [v_{tt}^2+4v_t^2+(\delta_\omega v)^2 +$$
$$+ 2(\nabla_\omega v_t)^2]\,(\ell+t)\,dt\,d\omega + 2\omega_4 u(p)^2.$$

This identity together with the following easily checked one

$$\int_{S^{n-1}} (\delta_\omega v)^2\,d\omega \geq (n-1)\int_{S^{n-1}} (\nabla_\omega v)^2\,d\omega$$

implies

$$2\int_\Omega\Delta u\cdot\Delta(u\Gamma)\,dx \geq c\int_G [(\nabla_2 v)^2 + (\nabla v)^2]\,(\ell+t)\,dt\,d\omega +$$
$$+ u(p)^2 \geq c\int_\Omega [(\nabla_2 u)^2 + \frac{(\nabla u)^2}{|x-p|^2}]\log\frac{d}{|x-p|}\,dx + u(p)^2.$$

The proof is complete.

Lemma 1 fails for $n \geq 8$. Indeed, let the function $u\in C_0^\infty(\Omega\setminus p)$ depend only on $r = |x-p|$. Then (see [7])

$$c(n)\int_\Omega \Delta u(x)\circ\Delta(u(x)\Gamma(x-p))\,dx = \omega_n\int_{-\infty}^{+\infty} v_{tt}^2\,dt - c\int_{-\infty}^{+\infty} v_t^2\,dt$$

where $v(t) = u(e^{-t})$. Therefore rhe estimate (2) is impossible.

4^o. Local estimates. In the next lemma and henceforth we use the notation:

$$M_\rho(u) = \rho^{-n}\int_{\Omega\cap C_{2\rho}} u^2\,dx,$$

$$N_\rho(u) = \int_{\Omega\cap B_{2\rho}} \left[(\nabla_2 u)^2 + \frac{(\nabla u)^2}{|x-p|^2}\right]\Gamma dx$$

where $\Gamma = \Gamma(x-p)$ and we set $d = 3\rho$ for the case $n = 4$ in the definition of Γ.

Lemma 2. Let $\eta\in C_0^\infty(B_{2\rho})$, $\eta = 1$ in a neighbourhood of the ball B_ρ; $u\in \overset{\circ}{W}{}_2^2(\Omega)\cap C_0^\infty(\Omega)$. Then for any point $p\in B_{\rho/2}$

$$\int_\Omega \Delta(\eta^2 u)\Delta(\eta^2 u\Gamma)\,dx \le \int_\Omega \Delta u.\Delta(\eta^4 u\Gamma)\,dx + \tag{10}$$
$$+ c\,M_\rho(u)^{1/2}N_\rho(\eta^2 u)^{1/2} + c\,M_\rho(u).$$

Proof. Since

$$\Delta(\eta^2 u)\Delta(\eta^2 u\Gamma) - \Delta u.\Delta(\eta^4 u\Gamma) =$$
$$= [\Delta,\eta^2]u.\Delta(\eta^2 u\Gamma) - \Delta u\circ[\Delta,\eta^2]\eta^2 u\Gamma =$$
$$= [\Delta,\eta^2]u.[\Delta,\eta^2\Gamma] - \Delta u.\ [[\Delta,\eta^2]\circ\eta^2\Gamma]u$$

(the square brackets denote the commutator of operators), we must estimate the difference of the integrals

$$i_1 = \int_\Omega [\Delta,\eta^2]u.[\Delta,\eta^2\Gamma]u\,dx,\quad i_2 = \int_\Omega \Delta u.[[\Delta,\eta^2],\eta^2\Gamma]u\,dx.$$

We begin with the estimate of i_2. Clearly

$$[[\Delta,\eta^2],\eta^2\Gamma]u = 2u\nabla\eta^2\nabla(\eta^2\Gamma) = 4u\eta^2(2\Gamma(\nabla\eta)^2+\eta\nabla\eta\nabla\Gamma).$$

Hence

$$i_2 = \int_\Omega u\Delta(\varphi_2\eta^2 u)\,dx, \tag{11}$$

where $\varphi_2 = 4(2\Gamma(\nabla\eta)^2 + \eta\nabla\eta\cdot\nabla\Gamma)$. In general, we denote further by φ_i the functions from $C_0^\infty(B_{2\rho}\setminus\bar B_\rho)$ satisfying

$$|\nabla_k \varphi_i| \leq c\rho^{i-n-k}, \qquad k = 0,1, \ldots$$

The inequality

$$|i_2| \leq c\, M_\rho(u)^{1/2} N_\rho(\eta^2 u)^{1/2} + c M_\rho(u)$$

is a straightforward consequence of (11). Now we pass to the estimate of i_1. Since

$$[\Delta, \eta^2]u \circ [\Delta, \eta^2 \Gamma]u =$$

$$= (4\eta \nabla\eta \circ \nabla u + u\Delta\eta^2)(2\nabla u \circ \nabla(\eta^2\Gamma) + u\Delta(\eta^2\Gamma)),$$

we have

$$(12) \qquad i_1 = 8\int_\Omega (\nabla u \cdot \nabla\eta)\eta(\nabla(\eta^2\Gamma) \cdot \nabla u)\, dx + \int_\Omega \varphi_0 u^2\, dx,$$

where $\varphi_0 = \Delta\eta^2 \cdot \Delta(\eta^2\Gamma) - \mathrm{div}(\Delta\eta^2 \circ \nabla(\eta^2\Gamma)) -$ $- 2\mathrm{div}(\Delta(\eta^2\Gamma) \cdot \eta\nabla\eta)$. The first term on the right hand side of (12) can be written in the form

$$i_1 = 8\int_\Omega (\nabla u \circ \nabla\eta)(2\Gamma\nabla\eta + \eta\nabla\Gamma) \cdot \nabla(\eta^2 u)\, dx +$$

$$+ 8\int_\Omega u^2\, \mathrm{div}\left\{(\nabla\eta \cdot \nabla(\eta^2\Gamma))\nabla\eta\right\} dx =$$

$$= \int_\Omega u\, \mathrm{div}(\varphi_2 \nabla(\eta^2 u))\, dx + \int_\Omega u^2 \varphi_0\, dx.$$

Hence

$$|i_i| \leq c\, M_\rho(u)^{1/2}\, N_\rho(\eta^2 u)^{1/2} + c\, M_\rho(u),$$

which completes the proof.

Using Lemmas 1 and 2 we get

<u>Corollary 1.</u> Let $4 \leq n \leq 7$, $u \in \mathring{W}_2^2(\Omega)$, $\Delta^2 u = 0$ in $\Omega \cap B_{2\rho}$. Then for all points $p \in B_{\rho/2}$

$$(13) \quad u(p)^2 + \int_{\Omega\cap B_\rho} ((\nabla_2 u)^2 + |x-p|^{-2}(\nabla u)^2)\Gamma(x-p)\, dx \leq c\, M_\rho(u).$$

<u>Corollary 2.</u> Let $4 \leq n \leq 7$ and let the function $u \in \mathring{W}_2^2(\Omega)$ satisfy the equation $\Delta^2 u = 0$ in $\Omega \setminus B_\rho$. Then for all points $p \in \Omega \setminus B_{2\rho}$,

$$(14) \qquad |u(p)| \leq c\left(\frac{\rho}{|p|}\right)^{n-4} M_\rho(u)^{1/2}.$$

Proof. Let G be the image of Ω under the inversion $p \to p|p|^{-2}$. We make use of the Kelvin transform $U(q) = |q|^{4-n} u(q|q|^{-2})$

which maps u into a biharmonic function in $G \cap B_{\rho^{-1}}$. One can easily see that the Kelvin transform preserves the class $\mathring{W}_2^2$. By the inequality (13) for all points $q \in G \cap B_{(2\rho)^{-1}}$

$$U(q)^2 \le c\rho^n \int_{B_{2\rho^{-1}} \setminus B_{\rho^{-1}}} U(y)^2 \, dy$$

or which is the same,

$$|q|^{2(4-n)} u(q|q|^{-2})^2 \le c\rho^n \int_{B_{2\rho^{-1}} \setminus B_{\rho^{-1}}} |y|^{2(4-n)} u(y|y|^{-2})^2 \, dy.$$

Setting here $p = q|q|^{-2}$, $x = y|y|^{-2}$ we obtain the estimate (14).

<u>5^o Local estimates in terms of capacity.</u>

<u>Lemma 3.</u> Let $4 \le n \le 7$ and let the function $u \in \mathring{W}_2^2$ satisfy the equation $\Delta^2 u = 0$ in $\Omega \cap B_{2\rho}$. Then for all points $p \in B_{\rho/2}$

$$(15) \quad u(p)^2 + \int_{\Omega \cap B_\rho} ((\nabla_2 u)^2 + |x-p|^{-2} (\nabla u)^2) \Gamma(x-p) \, dx \le$$

$$\le \frac{c}{\gamma(\rho)} \int_{\Omega \cap C_{2\rho}} ((\nabla_2 u)^2 + |x-p|^{-2} (\nabla u)^2) \Gamma(x-p) \, dx$$

where $\gamma(\rho) = \rho^{4-n} \operatorname{cap}_2(C_{2\rho} \setminus \Omega)$ for $n > 4$ and $\gamma(\rho) = \operatorname{cap}_2(C_{2\rho} \setminus \Omega; B_{2\rho})$ for $n = 4$; in the case $n = 4$ we set $d = 3\rho$ in the definition of the fundamental solution.

Proof. The results of [14], [15] imply

$$\int_{\Omega \cap C_{2\rho}} u^2 \, dx \le \frac{c\rho^4}{\gamma(\rho)} \int_{\Omega \cap C_{2\rho}} ((\nabla_2 u)^2 + \rho^{-2} (\nabla u)^2) \, dx.$$

Noting that $\rho \ge c|x-p|$, $\Gamma(x-p) \ge c\rho^{4-n}$ for $x \in C_{2\rho}$, $p \in B_{\rho/2}$ and using Corollary 1 we complete the proof.

<u>Lemma 4.</u> Under the conditions of Lemma 3 for $2r < \rho$ it holds

$$(16) \quad \int_{\Omega \cap B_r} [(\nabla_2 u)^2 + |x|^{-2} (\nabla u)^2] \frac{dx}{|x|^{n-4}} \le c M_\rho(u) \exp\left(-c \int_r^\rho \gamma(\tau) \frac{d\tau}{\tau}\right).$$

Proof. By (15), for sufficiently small $\varepsilon > 0$ and $r \le \rho$

$$\int_{\Omega \cap (B_r \setminus B_\varepsilon)} ((\nabla_2 u)^2 + |x-p|^{-2} (\nabla u)^2) \Gamma(x-p) \, dx \le$$

$$\le \frac{c}{\gamma(r)} \int_{\Omega \cap C_{2r}} ((\nabla_2 u)^2 + |x-p|^{-2}(\nabla u)^2)\Gamma(x-p)\,dx.$$

Taking limits with $p \to 0$ and then with $\varepsilon \to +0$ we get

$$\int_{\Omega \cap B_r} ((\nabla_2 u)^2 + |x|^{-2}(\nabla u)^2)|x|^{4-n}\,dx \le$$

$$\le \frac{c_1}{\gamma(r)} \int_{\Omega \cap C_{2r}} ((\nabla u)^2 + |x|^{-2}(\nabla u)^2)|x|^{4-n}\,dx.$$

We denote the left hand side of this inequality by $\psi(r)$ and set $r = 2^{-k}$. Then

(17) $$(1+c_2\gamma(2^{-k}))\psi(2^{-k}) \le \psi(2^{1-k}).$$

Since γ is a bounded function, the estimate (17) is equivalent to

$$\psi(2^{-k}) \le \exp[-c_3\gamma(2^{-k})]\,\psi(2^{1-k}).$$

So for $m > \ell$

(18) $$\psi(2^{-m}) \le \exp[-c_3 \sum_{j=m}^{\ell-1} \gamma(2^{-j})]\ \psi(2^{-\ell}).$$

Let numbers m and ℓ satisfy the inequalities $2^{-m-1} \le r \le 2^{-m}$ and $2^{-\ell} \le \rho \le 2^{1-\ell}$. Then (18) and (13) yield

$$\psi(r) \le c\,\exp\,[-c_3 \sum_{j=m}^{\ell-1} \gamma(2^{-j})] M_\rho(u).$$

Using simple properties of the biharmonic capacity (see for example [15]) we obtain (16) from the last estimate.

<u>6^o. Regularity of a boundary point.</u> We say that a point $0 \in \partial\Omega$ is regular for the biharmonic operator if the solution $u \in \mathring{W}_2^2(\Omega)$ of the equation $\Delta^2 u = f$ with an arbitrary right hand side from $C_0^\infty(\Omega)$ is continuous at 0.

<u>Theorem 1.</u> Let $4 \le n \le 7$ and

(19) $$\int_0 \gamma(\tau)\,\frac{d\tau}{\tau} = \infty$$

where γ is the function introduced in Lemma 3. Then the point 0 is regular for Δ^2. Moreover if $u \in \mathring{W}_2^2(\Omega)$ and $\Delta^2 u = 0$ in $\Omega \cap B_{2\rho}$ for some $\rho > 0$ then there exists a constant c such that

$$(20) \qquad \lim_{r \to 0} \exp\left(c \int_r^{\rho} \gamma(\tau) \frac{d\tau}{\tau}\right) \sup_{|p|<r} |u(p)| = 0.$$

Proof. According to (15) we have for all $p \in B_{r/2}$ with $r \leq \rho$

$$(21) \qquad u(p)^2 \leq \frac{c}{\gamma(r)} \int_{\Omega \cap C_{2r}} ((\nabla_2 u)^2 + |x|^{-2}(\nabla u)^2)|x|^{4-n}\, dx.$$

Let $S(r) = \sup\{u(p)^2 : p \in B_{r/2}\}$. From (21) it follows that

$$\int_0^{r/2} S(\tau)\gamma(\tau) \frac{d\tau}{\tau} \leq c \int_0^r \frac{d\tau}{\tau} \int_{\Omega \cap C_{2\tau}} ((\nabla_2 u)^2 + |x|^{-2}(\nabla u)^2)|x|^{4-n} dx =$$

$$= c \int_0^{r/2} \frac{d\tau}{\tau} \int_{\tau}^{2\tau} R^3\, dR \int_{S^{n-1}} ((\nabla_2 u)^2 + R^{-2}(\nabla u)^2)\, d\omega$$

which by the change of integration order becomes

$$\int_0^{r/2} S(\tau)\gamma(\tau) \frac{d\tau}{\tau} \leq c \int_{\Omega \cap B_r} ((\nabla_2 u)^2 + |x|^{-2}(\nabla u)^2)|x|^{4-n}\, dx.$$

Using this estimate and Lemma 4 we obtain

$$(22) \qquad \int_0^{r/2} S(\tau)\gamma(\tau) \frac{d\tau}{\tau} \leq c\, M_\rho(u) \exp\left(-c \int_r^{\rho} \gamma(\tau) \frac{d\tau}{\tau}\right).$$

Let

$$\xi(\tau) = \int_{\tau}^{\rho} \gamma(t) \frac{dt}{t}.$$

The inequality (22) assumes the form

$$\int_{\xi(r/2)}^{\infty} S(\tau(\xi))\, d\xi \leq c\, M_\rho(u) \exp(-c\,\xi(r)).$$

Since the function $\xi \to S(\tau(\xi))$ decreases and $\xi(r) \geq \xi(r/2) - c$, $c > 0$ we conclude

$$\xi(r/2) S(\xi^{-1}(2\xi(r/2))) \leq \int_{\xi(r/2)}^{2\xi(r)} S(\tau(\xi)) d\xi \leq c\, M_\rho(u) \exp(-c\,\xi(r/2)),$$

where ξ^{-1} is the inverse function to $\xi(\tau)$. We set $R = \xi^{-1}(2\xi(r/2))$. Then

$$\xi(R) \exp\left(\frac{c}{2}\xi(R)\right) S(R) \leq 2c\, M_\rho(u)$$

for all $R \leq \xi^{-1}(2\xi(\rho/4))$. Therefore

$$\lim_{R \to 0} \exp(\frac{c}{4}\xi(R))S(R) = 0.$$

The result follows.

An immediate consequence of Theorem 1 is

Corollary 3. If $4 \leq n \leq 7$ and

$$\lim_{r \to 0} \frac{1}{\log \frac{1}{r}} \int_r^{\rho} \gamma(\tau) \frac{d\tau}{\tau} > 0$$

then the solution $u \in \mathring{W}_2^2(\Omega)$ of the equation $\Delta^2 u = f$ with $f \in C_0^\infty(\Omega)$ satisfies the inequality $|u(x)| \leq c|x|^\alpha$, $\alpha > 0$ in a neighbourhood of 0.

<u>7°. Examples of regular points for Δ^2.</u> The proof of the following assertions can be performed in the same way as the proofs of analogous facts for (p,l)-capacity in [9], p. 53-55.

If $n = 4$ and the point 0 belongs to a continuum which is a part of $R^n \setminus \Omega$ then $\gamma(\tau) \geq \text{const} > 0$ and consequently the condition of Corollary 3 holds.

Let the exterior of Ω in a neighbourhood of the point 0 contain the domain $\{x: 0 < x_n < 1,\ x_1^2 + \dots + x_{n-1}^2 < f(x_n)^2\}$, where $f(t)$ is an increasing positive continuous function on $(0,1)$ such that $f(0) = f'(0) = 0$. Then $\gamma(\tau) \geq c|\log f(\tau)|^{-1}$ for $n = 5$ and $\gamma(\tau) \geq c[\tau^{-1}f(\tau)]^{n-5}$ for $n > 5$.

Hence the point 0 is regular for Δ^2, if

$$\int_0 |\log f(\tau)|^{-1}\tau^{-1}\, d\tau = \infty \qquad \text{for } n = 5,$$

$$\int_0 [\tau^{-1}f(\tau)]^{n-5}\tau^{-1}\, d\tau = \infty \qquad \text{for } n = 6,7.$$

<u>8°. Estimates for the Green function.</u> Let $G(x,y)$ be the Green function of the Dirichlet problem for the biharmonic operator.

<u>Theorem 2.</u> Let $5 \leq n \leq 7$ and $d_y = \text{dist}(y, \partial\Omega)$. Then

$$(23) \qquad |G(x,y) - \Gamma(x-y)| \leq c\, d_y^{4-n} \quad \text{if } |x-y| \leq d_y,$$

$$|G(x,y)| \leq c|x-y|^{4-n} \quad \text{if } |x-y| > d_y,$$

and consequently $|G(x,y)| \leq c|x-y|^{4-n}$ for all $x \in \Omega$, $y \in \Omega$.

Proof. Let $B(y) = \{x: |x-y| < d_y\}$ and $aB(y) = \{x: |x-y| < ad_y\}$.

We denote by η a function from $C_0^\infty[0,1)$ equal to unity on the segment $[0,1/2)$ and set

$$H(x,y) = G(x,y) - \eta\left(\frac{|x-y|}{d_y}\right)\Gamma(x-y).$$

Obviously the function $x \to H(x,y)$ belongs to the class $\overset{\circ}{W}{}_2^2(\Omega)\cap C^\infty(\Omega)$, the support of the function $x \to \Delta_x^2 H(x,y)$ lies in $B(y)\setminus \frac{1}{2}B(y)$ and $|\Delta_x^2 H(x,y)| \le d_y^{-n}$. Applying Lemma 1 to the function $x \to H(x,y)$ we get

$$H(p,y)^2 \le 2 \int_{B(y)\cap\Omega} \Delta_x^2 H(x,y)\cdot H(x,y)\,\Gamma(x-p)\,dx.$$

Therefore

$$\sup_{p\in 2B(y)\cap\Omega} H(p,y)^2 \le \tag{24}$$

$$\le \sup_{x\in B(y)\cap\Omega} |H(x,y)| \sup_{p\in 2B(y)\cap\Omega} \int_{B(y)\cap\Omega} |\Delta_x^2 H(x,y)|\,\Gamma(x-p)\,dx,$$

and hence

$$\sup_{p\in 2B(y)\cap\Omega} |H(p,y)| \le c d_y^{-n} \sup_{p\in 2B(y)\cap\Omega} \int_{B(y)\cap\Omega} \Gamma(x-p)\,dx \le c\, d_y^{4-n}. \tag{25}$$

Since $\Delta_p^2 H(p,y) = 0$ for $p \bar{\in} B(y)$ we obtain from (25) and Corollary 2 (in which 0 must be substituted by p) for $p \bar{\in} 2B(y)$

$$|H(p,y)| \le c\left(\frac{d_y}{|p-y|}\right)^{n-4} \sup_{x\in 2B(y)\cap\Omega} |H(x,y)| \le c|p-y|^{4-n}.$$

The result follows.

<u>Theorem 3.</u> Let $n = 4$, $d_y = \mathrm{dist}(y,\partial\Omega)$, let Ω be a domain with a diameter $\mathcal{D}$ and

$$\Gamma(x-y) = (4\omega_4)^{-1}\log\frac{\mathcal{D}}{|x-y|}.$$

Then

$$|G(x,y) - \Gamma(x-y)| \le c_1\log\frac{\mathcal{D}}{d_y} + c_2 \qquad \text{if } |x-y| \le d_y,$$

$$|G(x,y)| \le c_3\log\frac{\mathcal{D}}{d_y} + c_4 \qquad \text{if } |x-y| > d_y.$$

Proof. Proceeding in the same way as in the proof of Theorem 2 we come to (24). Hence

$$\sup_{p \in 2B(y)\cap\Omega} |H(p,y)| \leq c d_y^{-4} \sup_{p \in 2B(y)\cap\Omega} \int_{B(y)\cap\Omega} \Gamma(x-p)\, dx \leq$$

$$\leq c_1 \log \frac{\mathcal{D}}{d_y} + c_2$$

which together with Corollary 2 gives for $p \bar{\in} 2B(y)$

$$|H(p,y)| \leq c \sup_{x \in 2B(y)\cap\Omega} |H(p,y)| \leq c(c_1 \log \frac{\mathcal{D}}{d_y} + c_2).$$

Since $G(p,y) = H(p,y)$ for $p \bar{\in} 2B(y)$ the result follows.

References

[1] N. Wiener, Certain notions in potential theory, J. Math. and Phys. 3 (1924), 24-51.

[2] N. Wiener, The Dirichlet problem, J. Math. and Phys., 3 (1924), 127-146.

[3] O.A. Oleynik, On Dirichlet problem for elliptic equations, Matem. Sb., 24, N 1 (1949), 3-14 (Russian).

[4] R.M. Hervé, Recherches axiomatiques sur la theorie des fonctions surharmoniques et du potentiel, Ann. Inst. Fourier, 12 (1962), 415-517.

[5] N.V. Krylov, On the first boundary value problem for elliptic equations, Diff. Uravneniya 3, N 2 (1967), 315-325 (Russian).

[6] W. Littman, G. Stampacchia, H.F. Weinberger, Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa, 17 (1963), 43-77.

[7] E.M. Landis, The second order elliptic and parabolic equations, "Nauka", Moscow, 1971 (Russian).

[8] A.A. Novruzov, On regularity of boundary points for elliptic equations with continuous coefficients, Vestn. Moskovskogo Universiteta, N 6 (1971), 18-25 (Russian).

[9] V.G. Maz'ja, On continuity at a boundary point of the solution of quasilinear elliptic equations. Vestnik Leningradskogo Universiteta, 25 (1970), 42-55 (Russian).

[10] R. Gariepy, W. Ziemer, A gradient estimate at the boundary for solutions of quasilinear elliptic equations, Bull. Amer. Math. Soc., 82 (1976), 629-631.

[11] A.A. Novruzov, On some questions of qualitative theory of linear and quasilinear elliptic equations of second order. Doklady AN SSSR, 204, N 5 (1972), 1053-1056. (Russian).

[12] V.G. Maz'ja, On the behavior near the boundary of the Dirichlet problem for second order elliptic equations in divergence form. Matem. Zametki, 2, N 2 (1967), 209-220 (Russian).

[13] A.A. Novruzov, On the modulus of continuity of the Dirichlet problem at a regular boundary point. Matem. Zametki, 12, N 1 (1972), 62-72 (Russian).

[14] V.G. Maz'ja, On the Dirichlet problem for arbitrary order elliptic equations in unbounded domains. Doklady AN SSSR, 150, N 6 (1963), 1221-1224 (Russian).

[15] V.G. Maz'ja, On (p, ℓ)-capacity, embedding theorems and the spectrum of a selfadjoint elliptic operator. Izv. AN SSSR, ser. matem., 37 (1973), 356-385 (Russian).

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