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ELLIPTIC EQUATIONS IN CONVEX DOMAINS

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To Yury Burago, a friend of my youth,
with love and admiration

ABSTRACT. A short survey of a series of results by the author, partly obtained in collaboration with Yu. Burago.

INTRODUCTION

It is an elementary consequence of the theory of conformal mappings that harmonic functions of two variables with zero either Dirichlet or Neumann data are differentiable at the vertex of any angle less than π. In contrast, reentrant angles produce singularities of the gradient. This is the simplest example of the beneficial influence of a domain's convexity on the regularity of solutions to classical boundary value problems.

In the present article I review several results illustrating the same fenomena obtained with my participation.

I start with a half century old work, joint with Yu. Burago, on the double and single layer harmonic potential theory.

§2 is devoted to L_p-properties of the gradient of solutions to the Neumann problem for the Poisson equation in a convex domain (see [1]). ∇L^p

The next section addresses a class of quasilinear elliptic equations and systems in a convex domain. Here some sharp results obtained in 2014 together with Cianchi are described.

The last section concerns the Dirichlet problem for the biharmonic equation ~~operator~~. It reflects the joint paper [6].

§1. NEUMANN PROBLEM FOR HARMONIC FUNCTIONS INSIDE AND OUTSIDE A CONVEX DOMAIN

I start with description of solvability results for classical boundary value problems for the Laplace equation obtained by Yu. Burago and myself.

Our starting point was a remark made in the famous course of functional analysis by F. Riesz and B. Sz.-Nagy, namely that "in the case of the spatial problem an analog of curves with bounded rotation has not yet been found". These curves form the largest class of contours known by that time for which the classical harmonic potential theory was developed. According to the note by Burago, Maz'ya, Sapozhnikova [2], it turned out that a "proper" generalization of Radon's result to higher dimensions can be achieved in terms of a certain function $\omega(\xi, B)$ replacing in a sense the solid angle at which the set B is seen from the point ξ. Within a few years Yu. Burago and myself developed a comprehencive

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theory of double and single layer potentials for a large class of n-dimensional domains. An exposition of that theory was given in the book [3].

The basic results, i.e., theorems on the solvability of the integral equations of the fundamental boundary value problems, were obtained for domains subject to two conditions

$$\text{(A)} \qquad \sup\left\{ \operatorname{var}\omega(\xi, S\setminus\xi) \,:\, \xi\in S\right\} < \infty,$$

$$\text{(B)} \qquad \lim_{r\to 0}\sup\left\{ \operatorname{var}\omega(\xi, S\cap B_r(\xi)) \,:\, \xi\in S\right\} < \sigma_n/2.$$

Here S is the boundary of the domain Ω with compact closure, var denotes the variation of the charge, $B_r(\xi)$ is the ball with center ξ and radius r, and σ_n is the area of the unit sphere.

We wish to point out that *condition* (A) is a corner-stone of the entire theory, whereas *condition* (B) is solely needed to prove the Fredholm alternative.

An arbitrary convex domain Ω satisfies condition (B). In fact, the closed ball $B_r(p)$ lies in Ω. For every point $q \in \partial\Omega$, we have

1) $\operatorname{var}\omega_{\partial\Omega} = \omega_{\partial\Omega}$ (because the projection from p is bijective);

2) $\omega_{\partial\Omega} \le \frac{1}{2}\omega_n$ $\{$the angle at which $B_r(p)$ is seen from the point $q\}$.

It is trivial to obtain a positive minorant for the last angle depending on ρ and r, which together with 2) guarantees (B).

Therefore, in what follows, I shall always asume Ω to be a convex domain.

I give a short exposition of our results.

Definition 1.1. The harmonic double layer potential with continuous density χ is the function defined for $x \notin S$ by

which

$$(W\chi)(x) = \frac{1}{\sigma_n}\int_S \chi(\xi)\omega(x, d\xi).$$

Theorem 1.1. *For any continuous function χ, the limit values $W_{\pm}\chi$ of the potential $W\chi$ exist from inside and outside of Ω, satisfy*

$$W_{+}\chi = \frac{1}{2}(\chi + T\chi), \tag{1.1}$$

$$W_{-}\chi = \frac{1}{2}(-\chi + T\chi), \tag{1.2}$$

and the doubled direct value of the ~~double layer~~ potential,

$$(T\chi)(x) = \frac{2}{\sigma_n}\int_S \chi(\xi)\omega(x, d\xi),$$

generates a continuous operator in the space of continuous functions $C(S)$.

Owing to formulas (1.1) and (1.2), continuous solutions of the internal and external Dirichlet problems with prescribed boundary functions φ^{+} and φ^{-} in $C(S)$ may be determined by means of the integral equations

$$\chi + T\chi = 2\varphi_{+}, \tag{1.3}$$

$$-\chi + T\chi = 2\varphi_{-}. \tag{1.4}$$

To pose, in some sense, the Neumann problem for surfaces satisfying condition (A), we need the notion of boundary flow.

Definition 1.2. A function $u \in C^1(\operatorname{int}\Omega)$ is said to possess an inner boundary flow if

(i) for each infinitely differentiable function φ on R^n with compact support and for each sequence of sets $\Omega_m \subset \operatorname{int}\Omega$ with smooth boundaries that converge to Ω, the following limit exists:

$$l_u(\varphi) = \lim_{m\to\infty} \int_{\partial\Omega_m} \varphi(x) \frac{\partial u}{\partial \nu_x} H_{n-1}(dx);$$

(ii) the functional l_u is bounded in the norm of $C(S)$.

Definition 1.3. Let the requirements (i), (ii) of the previous definition be satisfied. The finite charge Σ^+ on S generating the extension of the functional l_u to all of $C(S)$,

$$l_u(\varphi) = \int_S \varphi(x)\Sigma^+(dx),$$

is called the inner boundary flow of the function u.

The outer boundary flow Σ^- of a function $u \in C^1(\mathbb{R}^n \setminus \bar{\Omega})$ is defined similarly.

In our understanding of the internal and external Neumann problems the charges Σ^+ and Σ^- play part of the normal derivatives.

Definition 1.4. Given a finite charge ϱ on $S \subset \mathbb{R}^n$ $(n \geq 3)$, the single layer potential with the charge ϱ is defined by

$$(V\varrho)(x) = \frac{1}{\sigma_n} \int_S r^{2-n} \varrho(d\xi), \quad x \notin S.$$

The function $V\varrho$ turns out to be harmonic in $\mathbb{R}^n \setminus S$.

Theorem 1.2. *Suppose* $\operatorname{mes}_n(S) = 0$, $S = \partial(\mathbb{R}^n \setminus \bar{\Omega})$, *and condition* (A) *is fulfilled. Then the potential* $V\varrho$ *possesses inner and outer boundary flows, which equal*

with

$$-\frac{1}{2}\varrho(B) + \frac{1}{\sigma_n} \int_S \omega(x, B)\varrho(dx),$$
$$\frac{1}{2}\varrho(B) + \frac{1}{\sigma_n} \int_S \omega(x, B)\varrho(dx),$$

B being an arbitrary Borel subset of S.

The condition (A) is not only sufficient but also necessary for the boundary flows of an arbitrary $V\varrho$ to exist.

Thus, looking for the solution of the Neumann problem in the form of a single layer potential leads to the equations

$$-\varrho + T^*\varrho = 2\Sigma^+, \tag{1.5}$$

$$\varrho + T^*\varrho = 2\Sigma^-. \tag{1.6}$$

Here T^* is the operator adjoint of T, acting on the dual space $C^*(S)$ of $C(S)$.

The following theorem is our basic result on the solvability of the above integral equations.

Theorem 1.3. 1) *The integral equation* (1.5) *of the internal Dirichlet problem has a unique solution in* $C(S)$ *for every continuous right-hand side* Σ^-.

2) *The integral equation* (1.6) *of the external Neumann problem is uniquely solvable for every finite charge* Σ^-.

Definition 1.4. A finite charge ϱ is said to belong to the class C_V if the simple layer potential $V\varrho$ generated by ϱ possesses finite and equal limits on S from inside and outside of S.

I conclude with our theorem on the solvability of the Neumann problem in the convex domain Ω and its complement. In its formulation BV stands for the space of functions whose distributional gradients are vector-valued charges.

Theorem 1.4. 1) *For every finite charge* $\Sigma^+ \in C_V$ *with vanishing total mass, there exists, up to an additive constant, exactly one solution of the internal Neumann problem belonging to the class* $C(\bar{\Omega}) \cap BV(\operatorname{int}\Omega)$.

2) *For every finite charge* $\Sigma^- \in C_V$ *there exists precisely one solution of the external Neumann problem belonging to the class* $C(\mathbb{R}^n \setminus \operatorname{int}\Omega) \cap BV^{\mathrm{loc}}(\mathbb{R}^n \setminus \bar{\Omega})$ *and tending to zero at infinity.*

Note that ten years later several publications on the solvability of the boundary integral equations (1.3)–(1.6) in the spaces L_p, $1 < p < \infty$, on Lipschitz surfaces appeared (see [7, Section 3]). $V \; L^p$

§2. Neumann problem for the Poisson equation inside and outside of a convex domain

We denote by F a linear functional on the space $L^{1,p'}(\Omega)$, $p + p' = pp'$, $p \in (1, \infty)$. By a distributional solution to the Neumann problem for the Poisson equation $-\Delta u = F$ we mean a function $u \in L^{1,p}(\Omega)$ that is orthogonal to 1 in Ω and satisfies the equality

$$\int_\Omega \nabla u \cdot \nabla\psi \, dx = F(\psi) \quad \forall \psi \in L^{1,p'}(\Omega). \tag{2.1}$$

As is known (cf., for example, [8, Theorem 1.1.15/1]), any linear functional F on $L^{1,p'}(\Omega)$ can be represented as

$$F(\psi) = \int_\Omega \vec{f} \cdot \nabla\psi \, dx, \tag{2.2}$$

where $\vec{f} \in L^p(\Omega)$; moreover,

$$\|F\| = \inf \|\vec{f}\|_{L^p(\Omega)},$$

where the infimum is taken over all vector-valued functions $\vec{f}$ satisfying (2.2) for any $\psi \in L^{1,p'}(\Omega)$. The space of functionals F is denoted by $(L^{1,p'}(\Omega))^*$.

Suppose that $\psi \in L^{1,p}(\Omega)$ and $\operatorname{tr}\psi$ denotes the trace of ψ on $\partial\Omega$. As is known, the set $\{\operatorname{tr}\psi\}$ is the space $B^{1/p,p'}(\partial\Omega)$. We introduce the space of distributions $B^{-1/p,p}(\partial\Omega)$ dual to $B^{1/p,p'}(\partial\Omega)$. It is clear that the mapping

$$L^{1,p'}(\Omega) \ni \psi \longrightarrow \int_{\partial\Omega} h(x) \operatorname{tr}\psi(x) \, ds_x,$$

where $h \in B^{-1/p,p}(\partial\Omega)$ and $h \perp 1$ on $\partial\Omega$, is a linear functional. If $h \in B^{-1/p,p}(\partial\Omega)$, then a particular case of the problem (2.1) is the problem

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = h, \quad u \in L^{1,p}(\Omega),$$

where ν is the unit vector of the outward normal to the boundary $\partial\Omega$.

By the Sobolev embedding theorems (cf., for example, [8, Theorem 1.4.5]), the mapping

$$L^{1,p'}(\Omega) \ni \psi \longrightarrow \int_\Omega f_0(x)\psi(x) \, dx + \int_{\partial\Omega} h(x) \operatorname{tr}\psi(x) \, ds_x$$

is a linear functional if $f_0 \in L^{np/(n+p)}(\Omega)$, $h \in L^{p(n-1)/n}(\partial\Omega)$ for $p' < n$, $f_0 \in L^q(\Omega)$ for any $q > 1$ with $p' = n$ and if

$$\int_\Omega f_0(x) \, dx - \int_{\partial\Omega} h(x) \, ds_x = 0.$$

Using this functional, we can define a weak $L^{1,p}(\Omega)$-solution to the problem

$$-\Delta u = f_0 \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = h. \tag{2.3}$$

Theorem 4.2. *Let Ω be a bounded smooth convex domain in $\mathbb{R}^n$, let $Q \in \partial\Omega$, and let $R \in (0, \operatorname{diam}(\Omega)/5)$. Suppose*

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{3R}(\Omega)), \quad u \in \mathring{W}_2^2(\Omega). \tag{4.14}$$

Then

$$\frac{1}{\rho^n} \fint_{S_\rho(Q)\cap\Omega} |u(x)|^2 d\sigma_x \le \frac{C}{R^n} \fint_{C_{R,4R}(Q)\cap\Omega} |u(x)|^2 dx \quad \text{for every } \rho < R, \tag{4.15}$$

where the constant C depends on the dimension only.

Finally (4.15) leads to Theorem 4.1.

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