Strong singularities of solutions to nonlinear elliptic equations

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1 Introduction

I am happy to contribute to the Festschrift in honour of Springer's Editorial Director Dr. Catriona Byrne.

My short paper is dedicated to an area in the Theory of Nonlinear Elliptic Partial Differential Equations which is not sufficiently explored at present and is promising for future interesting discoveries.

Before turning to the mathematical stuff I wish to share some memories related to my collaboration with Dr. Byrne and perhaps the best way to do that is to reproduce my letter written on July 28, 2006.

> Dear Catriona,
> I congratulate you with your quarter of the century with Springer Verlag. During this period of time I always felt your friendly and qualified support. I remember quite well your assistance in publishing "Sobolev Spaces", 1985, my first Springer book. Another example is your editing of my English in the Lecture Notes volume of 1997 which was far beyond your formal duties. More recently you gave me an invaluable help with publication of the volume "Differential Equations with Operator Coefficients", 1999. I am fortunate that our collaboration continues even now.
> I also congratulate Springer Verlag with having such an excellent member of the stuff, devoted, charming, highly efficient and full of energy.
> I wish you, dear Catriona, good health, happiness and many years of fruitful work for the benefit of the world mathematical community.
> Yours truly,
> Vladimir Maz'ya

The intensity of my collaboration with Catriona and with Springer Verlag in general can be illustrated by the following list of books.

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1. V. Maz'ya, Sobolev Spaces, 1985.

2. V. Kozlov, V. Maz'ya, Theory of Higher-Order Sturm-Liouville Equations, Lecture Notes, No. 1659, 1997.

3. V. Kozlov, V. Maz'ya, Differential Equations with Operator Coefficients, 1999.

4. G. Kresin, V. Maz'ya, Sharp Real-Part Theorems, Lecture Notes, No. 1903, 2007.

5. V. Maz'ya, T. O. Shaposhnikova, Theory of Sobolev Multipliers with Applications to Differential and Integral Operators, 2009.

6. V. Maz'ya, Sobolev Spaces with Applications to Elliptic Partial Differential Equations, 2011.

7. V. Maz'ya, A. Movchan, M. Nieves, Green's Kernels and Meso-Scale Approximations in Perforated Domains, Lecture Notes, No. 2077, 2013.

2 Open problem

Construction of asymptotic formulae for solutions to linear elliptic boundary value problems in strips, cylinders or domains with angular and conic boundary points has been developed in numerous publications (see, for instance [1] – [5] and the bibliography there). Less attention was paid to the asymptotics of solutions to nonlinear boundary value problems. In more detail properties of solutions to the p-Laplace equation were investigated (see [6] – [14]). In the case of weak singularities, when the problem can be linearized at infinity or near a singular point, boundary value problems for semilinear and more general quasilinear equations were considered in [15] – [17]. As for solutions with strong singularities, the situation is quite different. Since the principal terms of the asymptotics depend on the nonlinear operator as a whole, the direct linearization is ompossible. This case was dealt with in [18] – [20]. A formal asymptotic representation of solutions to the Dirichlet problem for the Riccati equation near an angular point was given in [21]. Description of asymptotic behavior of all solutions to the Neumann problem for the two-dimensional Riccati equation near an angular point was obtained in [22].

Now I turn to an open problem concerning the Riccati equation in a strip. Consider the quasi-linear equation

$$u_{xx} + u_{yy} + au_x^2 + bu_y^2 = 0, \quad a, b = const > 0 \tag{1}$$

in the half-strip $\{(x, y) : x > 1, 0 < y < l\}$ with the boundary conditions

$$u(x, 0) = u(x, l) = 0. \tag{2}$$

Apriori assuming that the solution is bounded, one can show by a standard linearization argument that the solution is asymptotically equivalent to

$$C \exp\Big(-\frac{k\pi x}{l}\Big) \sin \frac{k\pi y}{l}, \qquad k = 1, 2, \ldots,$$

as $x \to +\infty$, where $C = const$.

The hypothesis I propose to justify is as follows. Suppose that the function u, solving the problem (1)–(2), is not bounded at infinity. Show that the uniform asymptotic formula holds

$$u(x, y) = \frac{1}{b} \log\Big[\exp\Big(\sqrt{\frac{b}{a}}\frac{\pi x}{l}\Big) \sin \frac{\pi y}{l} + \frac{\cos\Big(\sqrt{\frac{b-a}{a}}\big(\frac{\pi y}{l} - \frac{\pi}{2}\big)\Big)}{\cos\Big(\sqrt{\frac{b-a}{a}}\frac{\pi}{2}\Big)}\Big] + O\big(\exp(-\delta x)\big), \tag{3}$$

where $\delta = const > 0$ and

$$(b-a)a^{-1} \neq m^2, \qquad m = 1, 2, \ldots . \tag{4}$$

Obviously, here the main term of the asymptotics depends both on linear and non-linear parts of the elliptic operator.

Note that (3) is global in the sense that it does not contain boundary layer terms, unlike the asymptotics obtained in [21]. (Obviously, the angle in [21] should be replaced by the strip using a conformal mapping.)

There are other interesting questions related to solutions of (1)–(2). For example, what happens with the asymptotics without the restriction (4)? Also, how will the asymptotics look like if a and b in (1) are of different signs or if they depend on the point (x, y)? The answers to these questions are far from being obvious and, as far as I know, there exist no general methods for their treatment at present.

In conclusion I would like to attract reader's attention to other unsolved problems in analysis and partial differential equations collected in [23].

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