Abstract

The Dirichlet boundary value problem for the Stokes operator with L^p data in any dimension on domains with conical singularity (not necessary a Lipschitz graph) is considered. We establish the solvability of the problem for all $p \in (2-\varepsilon, \infty]$ and also its solvability in $C(\overline{D})$ for the data in $C(\partial D)$.

L^p solvability of the Stationary Stokes problem on domains with conical singularity in any dimension *

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1 Introduction

In this paper we study the Stokes system (which is the linearized version of the stationary Navier-Stokes system) on a fixed domain $D \subset \mathbb{R}^n$, for $n \geq 3$. In fact, we establish our result for a both Lamé system ($\nu < 1/2$) and the Stokes system ($\nu = 1/2$).

We want to consider a classical question of the solvability of the L^p Dirichlet problem on the domain D.

Let us recall that *the Dirichlet problem for the system (1.1) is L^p solvable* on the domain D if for all vector fields $f \in L^p(\partial D)$ there is a pair of (u, p) (here $u : D \to \mathbb{R}^n$, $p : D \to \mathbb{R}$) such that

$$\begin{aligned} -\Delta u + \nabla p &= 0, \qquad \text{div } u + (1-2\nu)p = 0 \quad \text{in } D, \\ u\big|_{\partial D} &= f \quad \text{almost everywhere}, \\ u^* &\in L^p(\partial D), \end{aligned} \tag{1.1}$$

*Mathematics Subject Classifications: 35J57, 35J47 and moreover for some $C > 0$ independent of f the estimate

$$\|u^*\|_{L^p(\partial D)} \leq C\|f\|_{L^p(\partial D)} \qquad \text{holds.}$$

Here, the boundary values of u are understood in the nontangential sense, that is we take the limit

$$u\big|_{\partial D}(x) = \lim_{y \to x,\, y \in \Gamma(x)} u(y),$$

over a collection of interior nontangential cones $\Gamma(x)$ of the same aperture and height and vertex at $x \in \partial D$ and u^* is the classical nontangential maximal function defined as

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|, \qquad \text{for all } x \in \partial D.$$

Furthermore, we say that the Dirichlet problem (1.1) is solvable for continuous data, if for all $f \in C(\partial D)$ the vector field u belongs to $C(\overline{D})$ and the estimate

$$\|u\|_{C(\overline{D})} \leq C\|f\|_{C(\partial D)} \qquad \text{holds.}$$

So far, we have not said anything about the domain D, except the requirement on the existence of the interior nontangential cones $\Gamma(x) \subset D$ of the same aperture and height at every boundary point $x \in \partial D$. For symmetry reasons let us also assume the existence of exterior (i.e. in $\mathbb{R}^n \setminus \overline{D}$) nontangential cones, as well as that the domain D is bounded.

A most classical example of a domain D that satisfies these assumptions is a Lipschitz domain which is a domain for which the boundary ∂D can be locally described as a graph of a Lipschitz function.

Another important class of domains satisfying outlined assumptions are so-called generalized polyhedral domains (see [11] and [13]). The precise definition is rather complicated and unnecessary for our purposes what we have in mind are domains that look like polyhedra, however we allow the sides or edges to be curved not just flat. For example, in two dimensions the boundary of such domain will consist of a finite set of vertices joined by C^1 curves meeting at the vertices nontransversally.

At the first sight, one might assume that the class of generalized polyhedral domains is a subset of the class of Lipschitz domains. This is however only true when $n = 2$, when $n > 3$ it is no longer the case. Is it however clear that these classes are related.

The L^p Dirichlet boundary value problem on Lipschitz domains for the Laplacian has a long history starting with pioneering work of Dahlberg [2], Jerison and Kenig [5]. As follows from these results the L^p Dirichlet boundary problem for the Laplacian is solvable for all $p \in (2 - \varepsilon, \infty]$ regardless of dimension. There are two key ingredients to establish this result: the so called Rellich estimates (when $p = 2$) and the maximum principle (when $p = \infty$). Interpolating these two leads to the stated result.

The Stokes (and Lamé) equations are PDE systems not a single (scalar) equation. This implies that the second ingredient of the above approach is not readily available since the maximum principle that holds for PDE equations is not applicable to general PDE systems. In addition, if we go outside the class of Lipschitz domains, the Rellich estimates are also not available.

Despite that in low dimensions $n = 2, 3$ a weak version of the maximum principle does hold [14] which still allows to prove L^p solvability for all $p \in (2 - \varepsilon, \infty]$, provided the domain is Lipschitz. See [3] or [4] for most up-to date approach that works even on Riemannian manifolds. See also [1] for regularity issues related to the Stokes system in Lipschitz domains.

The question whether the same is true if $n > 3$ is open and only partial range of p for which the problem is solvable is known.

In this paper we consider the problem outlined above for domains with a single singular point, a conical vertex. Apart from this point our domain will be smooth. In three dimensions the situation has been considered in a different context before [10] where estimates such as (3.33) in three dimensions are established. We mention also the recent book [13] where strong solutions of the three-dimensional problem (1.1) and stationary Navier-Stokes equation are studied in detail. Our approach is general and works in any dimension n. In principle what we present here is fully extendable to all generalized polyhedral domains. However to avoid technical challenges we only focus on the particular case of an isolated conical singularity.

Our main result shows that in the setting described above (a domain with one conical point) the range of solvability $p \in (2 - \varepsilon, \infty]$ remains true in *any* dimension.

As we stated above the classes of Lipschitz and polyhedral domains are related but neither is a subset of the other. However the result we present is a strong indication that the range $p \in (2 - \varepsilon, \infty]$ should hold even for Lipschitz domains. In fact, the known counterexamples to solvability in L^p are shared by these two classes of domains (see [6] for such examples when $p < 2$).

The paper is organized as follows. In section 2 we establish estimates for eigenvalues of a certain operator pencil for Lamé and Stokes systems in a cone that holds in any dimension. These estimates are in the spirit of work done in [10]. In section 3 we prove estimates for Green's function that are based upon section 2 and finally in section 4 we present our main result Theorem 4.1 and its proof that is based upon the explicit estimates for the Green's function from section 3 and interpolation.

2 The operator pencil for Lamé and Stokes systems in a cone

Let $\Omega \subset \mathbb{S}^{n-1}$. Suppose that $\mathrm{cap}(\mathbb{S}^{n-1} \setminus \Omega) > 0$. Then the first eigenvalue of the Dirichlet problem for the operator $-\Delta_{\mathbb{S}^{n-1}}$ in Ω is positive. We represent this eigenvalue in the form $M(M + n - 2)$, $M > 0$.

Consider the cone $\mathcal{K} = \{x \in \mathbb{R}^n; 0 < |x| < \infty, x/|x| \in \Omega\}$. Or goal is to understand the solutions of the system (2.2)-(2.3)

$$-\Delta U + \nabla P = 0, \qquad \mathrm{div}\ U + (1 - 2\nu)P = 0 \quad \text{in } \mathcal{K} \tag{2.2}$$

with the boundary condition

$$U\big|_{\partial K\setminus\{0\}} = 0, \tag{2.3}$$

in the form $U(x) = r^{\lambda_0}u(\omega)$, $P(x) = r^{\lambda_0-1}p(\omega)$. This requires study of spectral properties of a certain operator pencil $\mathcal{L}(\lambda)$ defined below (2.9).

Here ν is the so-called the Poisson ratio. If $\nu < 1/2$ the equation (2.2) can be written in more classical (elasticity) form

$$\Delta U + (1-2\nu)^{-1}\nabla\nabla\cdot U = 0 \quad \text{in } \mathcal{K}. \tag{2.4}$$

The case $\nu = 1/2$ corresponds to the Stokes system

$$-\Delta U + \nabla P = 0, \qquad \operatorname{div} U = 0 \quad \text{in } \mathcal{K} \tag{2.5}$$

with the boundary condition

$$U\big|_{\partial K\setminus\{0\}} = 0. \tag{2.6}$$

Now we define the pencil. We write (2.2) in the polar form for $U = r^{\lambda}(u_u, u_\omega)$. After multiplying by $r^{2-\lambda}$ we obtain:

$$\begin{aligned} -\Delta_{\mathbb{S}^{n-1}}u_r &- (\lambda+1)(\lambda+n-1)u_r - \frac{\lambda-1}{1-2\nu}[(\lambda+n-1)u_r \\ &+ \nabla_\omega\cdot u_\omega] + 2[(\lambda+n-1)u_r + \nabla_\omega\cdot u_\omega] = 0, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} Lu_\omega &- (\lambda+1)(\lambda+n-1)u_\omega - \frac{1}{1-2\nu}[(\lambda+n-1)\nabla_\omega u_r \\ &+ \nabla_\omega(\nabla_\omega\cdot u_\omega)] + 2[(\lambda+n-1)u_\omega - \nabla_\omega u_r] = 0, \end{aligned} \tag{2.8}$$

where L is the second order differential operator acting on vector fields on $\mathbb{S}^{n-1}$ that arises from the Navier-Stokes equation on the sphere ($L = -\Delta_{\mathbb{S}^{n-1}} - \nabla_\omega(\nabla_\omega\cdot\,) - 2$ and $\Delta_{\mathbb{S}^{n-1}}$ is the Hodge Laplacian on $\mathbb{S}^{n-1}$) (see also section 3.2.3 of [9] for the corresponding calculation in three dimensions).

Hence we arrive at the matrix differential operator $\mathcal{L}(\lambda)\begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = 0$, where:

$$\mathcal{L}(\lambda)\begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = \begin{pmatrix} -\Delta_{\mathbb{S}^{n-1}}u_r - \frac{2-2\nu}{1-2\nu}(\lambda-1)(\lambda+n-1)u_r + \frac{3-4\nu-\lambda}{1-2\nu}\nabla_\omega\cdot u_\omega \\ L_\nu u_\omega - (\lambda-1)(\lambda+n-1)u_\omega - \frac{n+1-4\nu+\lambda}{1-2\nu}\nabla_\omega u_r \end{pmatrix}. \tag{2.9}$$

Here $L_\nu = L - (1-2\nu)^{-1}\nabla_\omega(\nabla_\omega\cdot\,)$. This defines the operator pencil. The pencil operator is self-adjoint if and only if $\operatorname{Re}\lambda = -\frac{n-2}{2}$.

From now on we consider arbitrary $\nu \le 1/2$. In particular, this includes the case of the Stokes system we are mostly interested in.

For arbitrary real t consider

$$\phi(t) = (t+1)(t+n-1)(2t+n-2) - (3-4\nu-t)(M-t)(M+t+n-2). \tag{2.10}$$

Let $t(M)$ be the smallest solution of the equation

$$\phi(t) = (n-1)(2t+n-2)$$

in the interval $-(n-2)/2 < t < M$. We claim that $t(M) > 0$.

Indeed, for t in the interval $[-(n-2)/2, 0]$ we have

$$\phi(t) < (t+1)(t+n-1)(2t+n-2) < (n-1)(2t+n-2),$$

hence there is no solution in this interval. On the other hand $\phi(M) = (M+1)(M+n-1)(2M+n-2) > (n-1)(2M+n-2)$, hence the solution always exists and is positive.

The following result extends a similar three-dimensional result in [10] (see also Section 5.5.4 in [9]).

Theorem 2.1 *The strip determined by the inequality*

$$\left| Re\ \lambda + \frac{n-2}{2} \right| < \min\{1, t(M)\} + \frac{n-2}{2} \tag{2.11}$$

does not contain any eigenvalues of the pencil $\mathcal{L}(\lambda)$ in $\mathcal{K}$, provided $\nu \le 1/2$. What this means is that the boundary value problem (2.2)-(2.3) has no solution of the form

$$U(x) = r^{\lambda_0} u(\omega), \qquad P(x) = r^{\lambda_0 - 1} p(\omega), \tag{2.12}$$

for λ_0 in this strip.

Proof. We will only consider the case when Re $\lambda > -(n-2)/2$. This is enough, since by Theorem 3.2.1 of [9] λ_0 is an eigenvalue of the pencil if and only if $-(n-2) - \overline{\lambda_0}$ is an eigenvalue. Also there is no eigenvalue on the line Re $\lambda = -(n-2)/2$.

Let us therefore consider a pair (U, P) of the form (2.12) that solves the boundary value problem (2.2)-(2.3) in $\mathcal{K}$. For arbitrary $\varepsilon > 0$ we consider the domain

$$\mathcal{K}_\varepsilon = \{x \in \mathcal{K} : \varepsilon < |x| < 1/\varepsilon\}.$$

Then (2.5) implies

$$-\int_{\mathcal{K}_\varepsilon} \Delta U \cdot \overline{U} dx + \int_{\mathcal{K}_\varepsilon} \nabla P \cdot \overline{U} dx = 0.$$

Integrating by parts and using the equation for the divergence of U we obtain that

$$\int_{\mathcal{K}_\varepsilon} |\nabla U|^2 dx + \varepsilon^{n-1} \int_{r=\varepsilon} \overline{U} \cdot \partial_r U d\omega - \varepsilon^{-n+1} \int_{r=1/\varepsilon} \overline{U} \cdot \partial_r U d\omega \tag{2.13}$$

$$+(1-2\nu) \int_{\mathcal{K}_\varepsilon} |P|^2 dx - \varepsilon^{n-1} \int_{r=\varepsilon} P\overline{U}_r d\omega + \varepsilon^{-n+1} \int_{r=1/\varepsilon} P\overline{U}_r d\omega = 0.$$

Here U_r denotes the radial part of the vector U and $d\omega$ is the surface measure on S^{n-1}. Now using the assumption (2.12) we obtain

$$\int_{\varepsilon}^{1/\varepsilon} r^{2Re\,\lambda_0+n-3}dr \int_\Omega \left(|\nabla_\omega u|^2 + |\lambda_0|^2|u|^2 + (1-2\nu)|p|^2\right) d\omega \tag{2.14}$$
$$+\left(\varepsilon^{2Re\,\lambda_0+n-2} - \varepsilon^{-(2Re\,\lambda_0+n-2)}\right)\left(\lambda_0\int_\Omega |u|^2 d\omega - \int_\Omega p\overline{u}_r d\omega\right) = 0.$$

We take the real part of equation (2.14) and then integrate in r. This gives us

$$\int_\Omega |\nabla_\omega u|^2 d\omega \quad + \quad \left((Re\,\lambda_0)^2 + (Im\,\lambda_0)^2 - (2Re\,\lambda_0 + n - 2)Re\,\lambda_0\right)\int_\Omega |u|^2 d\omega$$
$$+(1-2\nu)\int_\Omega |p|^2 dx \quad + \quad (2Re\,\lambda_0 + n-2)Re\int_\Omega p\overline{u}_r d\omega = 0,$$

which can be simplified to

$$\int_\Omega |\nabla_\omega u|^2 d\omega \quad + \quad \left((Im\,\lambda_0)^2 - Re\,\lambda_0(Re\,\lambda_0+n-2)\right)\int_\Omega |u|^2 d\omega$$
$$+(1-2\nu)\int_\Omega |p|^2 dx \quad + \quad (2Re\,\lambda_0+n-2)Re\int_\Omega p\overline{u}_r d\omega = 0. \tag{2.15}$$

Now we use the original equations to get an expression for the pressure p. It follows that

$$r\partial_r P = x\cdot\nabla P = x\cdot\Delta U = \Delta(x\cdot U) - 2\,\mathrm{div}\,U = \Delta(rU_r) + 2(1-2\nu)P.$$

Again using (2.12) we obtain that

$$\begin{aligned} r\partial_r P &= r^{\lambda_0-1}(\lambda_0-1)p(\omega), \\ \Delta(rU_r) &= \Delta(r^{\lambda_0+1}u_r(\omega)) = r^{\lambda_0-1}\left(\Delta_{\mathbb{S}^{n-1}}u_r(\omega) + (\lambda_0+1)(\lambda_0+n-1)u_r(\omega)\right). \end{aligned}$$

Hence $p(\omega) = -(3-4\nu-\lambda_0)^{-1}(\Delta_{\mathbb{S}^{n-1}}u_r(\omega) + (\lambda_0+1)(\lambda_0+n-1)u_r(\omega))$. Consequently,

$$Re\int_\Omega p\overline{u}_r\, d\omega = Re\,(3-4\nu-\lambda_0)^{-1}\int_\Omega |\nabla_\omega u_r|^2 d\omega - Re\frac{(\lambda_0+1)(\lambda_0+n-1)}{3-4\nu-\lambda_0}\int_\Omega |u_r|^2 d\omega.$$

From this identity and (2.15) after multiplying by $|3-4\nu-\lambda_0|^2$ we obtain:

$$\begin{aligned} 0 &= |3-4\nu-\lambda_0|^2\left[\int_\Omega |\nabla_\omega u|^2 d\omega + \left((Im\,\lambda_0)^2 - Re\,\lambda_0(Re\,\lambda_0+n-2)\right)\int_\Omega |u|^2 d\omega\right] \\ &+ |3-4\nu-\lambda_0|^2(1-2\nu)\int_\Omega |p|^2 dx \\ &+ (2Re\,\lambda_0+n-2)(3-4\nu-Re\,\lambda_0)\int_\Omega |\nabla_\omega u_r|^2 d\omega \\ &- (2Re\,\lambda_0+n-2)Re\left[(\lambda_0+1)(\lambda_0+n-1)(3-4\nu-\overline{\lambda}_0)\right]\int_\Omega |u_r|^2 d\omega. \end{aligned} \tag{2.16}$$

Now we use the fact that $M(M+n-2)$ is the first eigenvalue of the Dirichlet problem for Laplacian on Ω. Hence the inequality

$$M(M+n-2)\int_\Omega |u|^2 d\omega \le \int_\Omega |\nabla_\omega u|^2 d\omega$$

holds. Also

$$\begin{aligned} & Re\left[(\lambda_0+1)(\lambda_0+n-1)(3-4\nu-\overline{\lambda}_0)\right] \\ = \; & (Re\,\lambda_0+1)(Re\,\lambda_0+n-1)(3-4\nu-Re\,\lambda_0)-|Im\,\lambda_0|^2(Re\,\lambda_0+n+3-4\nu) \\ \le \; & (Re\,\lambda_0+1)(Re\,\lambda_0+n-1)(3-4\nu-Re\,\lambda_0). \end{aligned}$$

Using these two inequalities and the fact that $2Re\,\lambda_0+n-2\ge 0$ we get that

$$\begin{aligned} & |3-4\nu-\lambda_0|^2\left[M(M+n-2)-Re\,\lambda_0(Re\,\lambda_0+n-2)\right]\int_\Omega |u|^2 d\omega \\ + \; & |3-4\nu-\lambda_0|^2(1-2\nu)\int_\Omega |p|^2 dx \\ + \; & (2Re\,\lambda_0+n-2)(3-4\nu-Re\,\lambda_0)\int_\Omega |\nabla_\omega u_r|^2 d\omega \qquad (2.17) \\ \le \; & (2Re\,\lambda_0+n-2)(Re\,\lambda_0+1)(Re\,\lambda_0+n-1)(3-4\nu-Re\,\lambda_0)\int_\Omega |u_r|^2 d\omega. \end{aligned}$$

We obtain further simplification by using the inequality $|3-4\nu-\lambda_0|^2 \ge (3-4\nu-Re\,\lambda_0)^2$. This gives us

$$\begin{aligned} & (3-4\nu-Re\,\lambda_0)(M-Re\,\lambda_0)(M+Re\,\lambda_0+n-2)\int_\Omega |u|^2 d\omega \\ + \; & (3-4\nu-Re\,\lambda_0)(1-2\nu)\int_\Omega |p|^2 dx \\ + \; & (2Re\,\lambda_0+n-2)\int_\Omega |\nabla_\omega u_r|^2 d\omega \qquad (2.18) \\ \le \; & (2Re\,\lambda_0+n-2)(Re\,\lambda_0+1)(Re\,\lambda_0+n-1)\int_\Omega |u_r|^2 d\omega. \end{aligned}$$

We write $|u|^2=|u_r|^2+|u_\omega|^2$. This gives

$$\begin{aligned} & (3-4\nu-Re\,\lambda_0)(M-Re\,\lambda_0)(M+Re\,\lambda_0+n-2)\int_\Omega |u_\omega|^2 d\omega \\ + \; & (3-4\nu-Re\,\lambda_0)(1-2\nu)\int_\Omega |p|^2 dx \\ + \; & (2Re\,\lambda_0+n-2)\int_\Omega |\nabla_\omega u_r|^2 d\omega \le \phi(Re\,\lambda_0)\int_\Omega |u_r|^2 d\omega, \qquad (2.19) \end{aligned}$$

where ϕ is defined by (2.10).

Using the equation div $U+(1-2\nu)P=0$ we get that $\nabla_\omega\cdot u_\omega+(\lambda_0+n-1)u_r+(1-2\nu)p=0$. Integrating this over Ω we conclude that

$$\int_\Omega\left(u_r+\frac{1-2\nu}{\lambda_0+n-1}p\right)d\omega=0. \qquad (2.20)$$

Consider therefore the following minimization problem for the functional

$$\int_\Omega |\nabla_\omega v|^2\, d\omega + \frac{(3-4\nu - Re\,\lambda_0)(1-2\nu)}{2Re\,\lambda_0 + n - 2}\int_\Omega |q|^2 d\omega \tag{2.21}$$

for pairs of functions $(v,q) \in W_2^{1,0}(\Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \left(v + \frac{1-2\nu}{\lambda_0 + n - 1} q\right) d\omega = 0 \quad \text{and} \quad \|v\|_{L^2(\Omega)} = 1. \tag{2.22}$$

Let us denote by $\Theta(\Omega, \lambda)$ that minimum of this functional and let (v_0, q_0) be a pair of functions realizing this minimum. We choose q_1 to be arbitrary $L^2(\Omega)$ function orthogonal to 1. Then the pair $(v_0, q_0 + \alpha q_1)$ for any $\alpha \in \mathbb{R}$ satisfies (2.22). Inserting this pair into (2.21) we obtain

$$Re \int_\Omega q_0 \cdot \overline{q_1}\, d\omega = 0.$$

Consequently q_0 is a constant and we can restrict ourselves to constant q in the above formulated variational problem. From this

$$\Theta(\Omega,\lambda) \quad = \quad \inf\left\{ \int_\Omega |\nabla_\omega v|^2\, d\omega + \frac{(3-4\nu-Re\,\lambda_0)|\lambda_0+n-1|^2}{(1-2\nu)(2Re\,\lambda_0+n-2)|\Omega|}\left|\int_\Omega v\, d\omega\right|^2 \right\}, \tag{2.23}$$

where the infimum is taken over all $W_2^{1,0}(\Omega)$ functions v with $L^2(\Omega)$ norm 1.

When $\nu = 1/2$ this minimization problem takes slightly different form and simplifies to

$$\Theta(\Omega) = \inf\left\{ \int_\Omega |\nabla_\omega v|^2\, d\omega : \int_\Omega v\, d\omega = 0,\ v\big|_{\partial\Omega} = 0 \text{ and } \|v\|_{L^2(\Omega)} = 1 \right\}. \tag{2.24}$$

Note that is this case the minimization problem is independent of λ.

Considering Re $\lambda_0 \in \left(-\frac{n-2}{2}, 3-4\nu\right]$ we see the number

$$C = \frac{(3-4\nu-Re\,\lambda_0)|\lambda_0+n-1|^2}{(1-2\nu)(2Re\,\lambda_0+n-2)|\Omega|} > 0.$$

Writing v for the problem (2.23) in the form $v = c_0 + v_1$, where c_0 is a constant function and v_1 has average 0 over Ω we see that

$$\int_\Omega |\nabla_\omega v|^2\, d\omega + C\left|\int_\Omega v\, d\omega\right|^2 = \int_\Omega |\nabla_\omega v_1|^2\, d\omega + C|\Omega|^2 c_0^2.$$

Using (2.24) the integral $\int_\Omega |\nabla_\omega v_1|^2\, d\omega$ can be further estimated from below by $\Theta(\Omega)\|v_1\|^2_{L^2(\Omega)} = \Theta(\Omega)(1 - \|c_0\|^2_{L^2(\Omega)})$.

Hence

$$\Theta(\Omega,\lambda) = \inf\{\Theta(\Omega)(1-\|c_0\|^2_{L^2(\Omega)}) + C|\Omega|^2 c_0^2;\ c_0 \in \mathbb{C} \text{ and } \|c_0\|^2_{L^2(\Omega)} \le 1\}.$$

It follows that the infimum will be attained either when $c_0 = 0$ or $\|c_0\|_{L^2(\Omega)} = 1$. Hence

$$\Theta(\Omega,\lambda) = \min\{\Theta(\Omega), C|\Omega|\},$$

or

$$\Theta(\Omega,\lambda)=\min\left\{\Theta(\Omega),\frac{(3-4\nu-Re\,\lambda_0)|\lambda_0+n-1|^2}{(1-2\nu)(2Re\,\lambda_0+n-2)}\right\}. \tag{2.25}$$

Looking at (2.24) we can see that the function $\Omega\mapsto\Theta(\Omega)$ does not increase as Ω increases. It follows that $\Theta(\Omega)\geq\Theta(\mathbb{S}^{n-1})$. This minimum for the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$ is known explicitly and equals to $n-1$. This and (2.25) implies that for Re $\lambda\leq 1$ we have that $\Theta(\Omega,\lambda)\geq n-1$.

Using this for $v=u_r$ in (2.19) we see that

$$\begin{aligned}&(3-4\nu-Re\,\lambda_0)(M-Re\,\lambda_0)(M+Re\,\lambda_0+n-2)\int_\Omega|u_\omega|^2d\omega\\ \leq\;&[\phi(Re\,\lambda_0)-(n-1)(2Re\,\lambda_0+n-2)]\int_\Omega|u_r|^2d\omega.\end{aligned} \tag{2.26}$$

It follows that for all $-(n-2)/2<\mathrm{Re}\,\lambda_0<t(M)$ the term on the righthand side of the inequality (2.26) is not positive, but the term on the lefthand side is nonnegative. Hence both terms have to vanish, i.e., $u_r=u_\omega=0$ or u is constant. Given that u vanishes on $\partial\Omega$ we get that $U(x)=r^{\lambda_0}u(\omega)=0$ everywhere. This establishes the claim.

Theorem 2.2 *Consider any (energy) solution of the system (1.1) in $\mathcal{K}\cap B(0,1)$ such that*

$$u\big|_{\partial\mathcal{K}\cap B(0,1)}=0,\qquad u\big|_{\partial B(0,1)\cap\mathcal{K}}\in C(\Omega).$$

Then

$$u\in C^\alpha(\overline{\mathcal{K}\cap B(0,1/2)}),\quad\text{and}\quad|\nabla u(x)|\leq C|x|^{\alpha-1}\|u\|_{C(\overline{\partial B(0,1)\cap\mathcal{K}})}, \tag{2.27}$$

for all $|x|\leq 1/2$ and some $\alpha>0$, $C>0$ independent of u.

Similarly, if u is an (energy) solution of the system (1.1) in $\mathcal{K}\setminus\overline{B(0,1)}$ such that

$$u\big|_{\partial\mathcal{K}\setminus B(0,1)}=0,\qquad u\big|_{\partial B(0,1)\cap\mathcal{K}}\in C(\Omega).$$

Then

$$|u(x)|\leq C|x|^{2-n-\alpha}\|u\|_{C(\overline{\partial B(0,1)\cap\mathcal{K}})},\quad|\nabla u(x)|\leq C|x|^{1-n-\alpha}\|u\|_{C(\overline{\partial B(0,1)\cap\mathcal{K}})}, \tag{2.28}$$

for all $|x|\geq 2$ and some $\alpha>0$, $C>0$ independent of u.

Proof. This is a consequence of Theorem 2.1. Indeed, we first look for solutions of (1.1) in the separated form $U(x)=r^\lambda u(\omega)$, $P(x)=r^{\lambda-1}p(\omega)$. Choose, any $\alpha\in(0,\min\{1,t(M)\})$. Then by Theorem 2.1, Re $\lambda>\alpha$ and any (energy) solution of the system (1.1) on $\mathcal{K}\cap B(0,1)$ has the following asymptotic representation by the Kondrat'ev's theorem [7]:

$$u(r,\omega)=r^{\lambda_0}\sum_{i=1}^m\sum_{j=0}^{k_i}c_{ij}\frac{1}{j!}(\log r)^j u_j^i(\omega)+O(r^{\mathrm{Re}\,\lambda_0-\varepsilon}),\qquad\text{for all }\varepsilon>0. \tag{2.29}$$

Here $u_0^i(\omega)$ are the eigenfunctions of the pencil $\mathcal{L}(\lambda_0)$ corresponding to an eigenvalue λ_0 with the smallest real part (m is the multiplicity of this eigenvalue) and $u_1^i(\omega), u_2^i(\omega), \dots, u_{k_i}^i(\omega)$ are the generalized eigenfunctions (a Jordan chain). Given our assumption that $\partial\Omega$ is smooth, these are C^∞ vector-valued functions.

Similarly, in the second case working on $\mathcal{K} \setminus \overline{B(0,1)}$ we have

$$u(r,\omega) = r^{2-n-\lambda_0} \sum_{i=1}^{m} \sum_{j=0}^{k_i} c_{ij} \frac{1}{j!} (\log r)^j u_j^i(\omega) + O(r^{2-n-\text{Re }\lambda_0+\varepsilon}), \tag{2.30}$$

for all $\varepsilon > 0$. Hence (2.27) holds as can be seen from (2.29) and (2.28) holds by (2.30).

3 Estimates for Green's function

We shall consider estimates for the Green's function for the Lamé and Stokes systems on a cone $\mathcal{K}$ in a spirit of the three dimensional result of Maz'ya and Plamenevskii [10] and Theorem 11.4.7 of [13].

Let $D \subset \mathbb{R}^n$ be a bounded domain that is smooth everywhere except at a single point (without loss of generality we can assume this point is 0). In a small neighborhood of this point we will assume the domain looks like a cone $\mathcal{K}$ defined above. That is for some $\delta > 0$

$$D \cap B(0,\delta) = \mathcal{K} \cap B(0,\delta), \tag{3.31}$$

where $\mathcal{K} = \{x \in \mathbb{R}^n; 0 < |x| < \infty, x/|x| \in \Omega\}$. Recall that $\Omega \subset \mathbb{S}^{n-1}$. We will assume that Ω is open and nonempty.

Let us denote the vector δ_j with components $(\delta_{1j}, \delta_{2j}, \dots, \delta_{nj})$ where δ_{ij} denotes the Kronecker symbol. Consider the fundamental solution (g^j, p^j) of our system in D, that is the solution of the problem

$$\begin{aligned} -\Delta_x g^j(x,\xi) + \nabla_x p^j(x,\xi) &= \delta(x-\xi)\delta_j, \\ \text{div}_x g^j(x,\xi) + (1-2\nu)p^j(x,\xi) &= 0, \qquad x,\xi \in D. \end{aligned} \tag{3.32}$$

We claim that the following holds:

$$\begin{aligned} |\nabla_\xi g^j(x,\xi)| &\le c|x-\xi|^{-(n-1)}, && \text{if } |x| < |\xi| < 2|x|, \\ |\nabla_\xi g^j(x,\xi)| &\le c|x|^{\alpha}|\xi|^{-(\alpha+n-1)}, && \text{if } 2|x| < |\xi|, \\ |\nabla_\xi g^j(x,\xi)| &\le c|\xi|^{\alpha-1}|x|^{-(\alpha+n-2)}, && \text{if } 2|\xi| < |x|. \end{aligned} \tag{3.33}$$

Here $\alpha > 0$ is a small constant as in Theorem 2.2.

We claim it suffices to establish estimates (3.33) for the fundamental solution in the unbounded cone $\mathcal{K}$ with zero Dirichlet boundary conditions at $\partial\mathcal{K}$ and infinity. Here the fundamental solution in the unbounded cone $\mathcal{K}$ is a solution of (3.33) in $\mathcal{K}$ that decays at infinity, i.e., $g^j(x,\xi) \to 0$ as $|x| \to \infty$ sufficiently fast so that $g^j(.,\xi)$ is an L^2 function outside the pole at $x = \xi$.

We will explain the step of going from $\mathcal{K}$ to D in detail below. Let us now work on $\mathcal{K}$. The existence and uniqueness of Green's function in an infinite cone for general elliptic boundary value problems was established in Theorem 7.2 of [8], in particular the first estimate (3.33) follows directly from this Theorem.

We now look at the last estimate in (3.33). Given the homogeneity of g^j on $\mathcal{K}$ we have

$$g^j(x,\xi) = \lambda^{n-2} g^j(\lambda x, \lambda\xi), \text{ for all } \lambda > 0.$$

It follows that

$$g^j(x,\xi) = |x|^{2-n} g^j(x/|x|, \xi/|x|).$$

Fix now $x \in \mathcal{K}$. On domain $|\xi| \le 0.99999|x|$, $\xi \mapsto g^j(x,\xi)$ is just the solution of the adjoint problem to (1.1) - so all we proved for (1.1) also holds for the adjoint equation. In particular (2.27) applies and

$$\begin{aligned} |\nabla_\xi g^j(x,\xi)| &= |x|^{2-n} |\nabla_\xi g^j(x/|x|, \xi/|x|)| \le C|x|^{2-n} \left|\frac{\xi}{x}\right|^{\alpha-1} \frac{1}{|x|} \\ &= C|\xi|^{\alpha-1}|x|^{-(\alpha+n-2)}, \qquad \text{for } |\xi| < 3/4|x|. \end{aligned} \tag{3.34}$$

The second estimate is similar, but we use (2.28). Again we have

$$g^j(x,\xi) = |x|^{2-n} g^j(x/|x|, \xi/|x|),$$

for on domain domain $|\xi| > 1.00001|x|$. As before $\xi \mapsto g^j(x,\xi)$ solves an adjoint problem, so by (2.28) we get for $|\xi| > 4/3|x|$:

$$\begin{aligned} |\nabla_\xi g^j(x,\xi)| &= |x|^{2-n} |\nabla_\xi g^j(x/|x|, \xi/|x|)| \le C|x|^{2-n} \left|\frac{\xi}{|x|}\right|^{1-n-\alpha} \frac{1}{|x|} \\ &= C|x|^{\alpha}|\xi|^{-(\alpha+n-1)}. \end{aligned} \tag{3.35}$$

Having required estimates on $\partial\mathcal{K}$ we show now that the same will be true on the domain D defined at the beginning of this section. The point is that by (3.31) the domains D and $\mathcal{K}$ coincide. Hence, if we denote the Green's function on $\mathcal{K}$ by $\widetilde{g^j}$, then the Greens's function g^j for D can be sought in the form

$$g^j(x,\xi) = \widetilde{g^j}(x,\xi)\varphi(x) + f_j(x,\xi),$$

where $\varphi(x)$ is a smooth cut-off function equal to one on $B(0,\delta/2)$ and vanishing outside $B(0,\delta)$.

Since

$$\begin{aligned} -\Delta_x[g^j(x,\xi)\varphi(x)] + \nabla_x p^j(x,\xi) &= \delta(x-\xi)\delta_j + \rho(x,\xi), \\ \mathrm{div}_x[g^j(x,\xi)\varphi(x)] + (1-2\nu)p^j(x,\xi) &= \tau(x,\xi), \qquad x,\xi \in D, \end{aligned} \tag{3.36}$$

where ρ and τ are are only supported in $D \cap \{\delta/2 < |x| < \delta\}$, we see that f_j must solve

$$\begin{aligned} -\Delta_x f_j(x,\xi) + \nabla_x p^j(x,\xi) &= -\rho(x,\xi), \\ \mathrm{div}_x f_j(x,\xi) + (1-2\nu)p^j(x,\xi) &= -\tau(x,\xi), \qquad x,\xi \in D, \end{aligned} \tag{3.37}$$

and $f_j\big|_{\partial D} = 0$. The main point is that this reduces the problem to dealing with the remainder term f_j. Now however, near the singular vertex Theorem 2.2 applies and away from the vertex the domain ∂D is smooth, hence f_j is smooth there as well, so f_j has the required regularity. See also section 4 of [12] for estimates of this type.

4 The L^p Dirichlet problem

Let D be the domain defined in the previous section. We would like to study the solvability of the classical L^p Dirichlet problem for the Lamé and Stokes systems in the domain D. Let us recall the definition.

Definition 4.1 *Let $1 < p \leq \infty$. We say that the Dirichlet problem for the Lamé system ($\nu < 1/2$) or Stokes system ($\nu = 1/2$) is L^p solvable on the domain D if for all vector fields $f \in L^p(\partial D)$ there is pair (u, p) such that*

$$\begin{aligned} -\Delta u + \nabla p &= 0, \qquad div\ u + (1-2\nu)p = 0 \quad in\ D \\ u\big|_{\partial D} &= f \quad almost\ everywhere, \\ u^* &\in L^p(\partial D), \end{aligned} \tag{4.38}$$

and for some $C > 0$ independent of f

$$\|u^*\|_{L^p(\partial D)} \leq C\|f\|_{L^p(\partial D)}.$$

Here, the boundary values of u are understood in the nontangential sense, that is we take the limit

$$u\big|_{\partial D}(x) = \lim_{y \to x,\, y \in \Gamma(x)} u(y),$$

over a collection of nontangential cones $\Gamma(x)$ of same aperture and vertex at $x \in \partial D$ and u^* is the classical nontangential maximal function defined as

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|, \qquad for\ all\ x \in \partial D.$$

Our main result is

Theorem 4.1 *Let D be the domain defined above and $\nu \leq 1/2$. Then for any $(n-1)/(\alpha+n-2) < p \leq \infty$ (α is same as in (3.33)) the L^p Dirichlet problem for the system (4.38) is uniquely solvable. Moreover, for any such p there exists a constant $C(p) > 0$ such that the solution (u, p) of the problem with boundary data $f \in L^p$ satisfies the estimate*

$$\|u^*\|_{L^p(\partial D)} \leq C(p)\|f\|_{L^p(\partial D)}.$$

Moreover, if $f \in C(\partial D)$, then $u \in C(\overline{\Omega})$ and an estimate

$$\|u\|_{C(\overline{D})} \leq C\|f\|_{C(\partial D)}$$

holds.

Proof: Consider the representation of the solution u by the Green's formula. That is, for $j = 1, 2, \ldots, n$

$$u_j(x) = -\int_{\partial D} \frac{\partial g^j(x,\xi)}{\partial \nu_\xi} f(\xi) d\sigma_\xi, \qquad x \in D, \tag{4.39}$$

where ν_ξ is the outer normal at the boundary point ξ and $d\sigma$ is the $(n-1)$-dimensional area element at ∂D. We will use (3.33) to establish the claim. From this estimate we obtain in the zone $|x|/2 < |\xi| < 2|x|$

$$\left|\frac{\partial g^j(x,\xi)}{\partial \nu_\xi}\right| \le C\frac{R(x)}{|x-\xi|^n},$$

where $R(x) = \mathrm{dist}(x, \partial D)$. From this and other two estimates of (3.33) we obtain:

$$\begin{aligned} |u_j(x)| &\le C\left(|x|^\alpha \int_{E_1} \frac{|f(\xi)|}{|\xi|^{n-1+\alpha}} d\sigma + R(x)\int_{E_2} \frac{|f(\xi)|}{|x-\xi|^n} d\sigma \right. \\ &+ \left. \frac{1}{|x|^{n-2+\alpha}} \int_{E_3} \frac{|f(\xi)|}{|\xi|^{1-\alpha}} d\sigma\right). \end{aligned} \quad (4.40)$$

Let us denote these three integrals by $v^1(x)$, $v^2(x)$, $v^3(x)$, respectively. Here

$$\begin{aligned} E_1 &= \{\xi \in \partial D : 2|x| < |\xi|\}, \\ E_2 &= \{\xi \in \partial D : |x|/2 \le |\xi| \le 2|x|\} \\ E_3 &= \{\xi \in \partial D : 2|\xi| < |x|\}. \end{aligned} \quad (4.41)$$

We deal with these three terms separately. We introduce

$$v^{i,*}(x) = \sup_{y\in\Gamma(x)} |v^i(y)|, \qquad \text{for all } x \in \partial D \text{ and } i = 1, 2, 3.$$

It follows that

$$u^*(x) \le C(v^{1,*}(x) + v^{2,*}(x) + v^{3,*}(x)) \qquad \text{for all } x \in \partial D.$$

Lemma 4.2 *There exists $C > 0$ such that*

$$\begin{aligned} \|v^{1,*}\|_{L^\infty(\partial D)} &= \|v^1\|_{L^\infty(D)} \le C\|f\|_{L^\infty(\partial D)} \\ \|v^{1,*}\|_{L^{1,w}(\partial D)} &\le C\|f\|_{L^1(\partial D)}. \end{aligned} \quad (4.42)$$

Here $L^{p,w}$ is the weak-L^p space equipped with the norm

$$\|f\|_{L^{p,w}} = \left(\sup_{\lambda>0} \lambda^p \sigma(\{\xi : |f(\xi)| > \lambda\})\right)^{1/p}.$$

Proof of the lemma: The definition of v^1 implies that

$$\begin{aligned} v^1(x) &\le |x|^\alpha \int_{E_1} \frac{|f(\xi)|}{|\xi|^{n-1+\alpha}} d\sigma \\ &\le |x|^\alpha \|f\|_\infty \int_{|\xi|>2|x|} |\xi|^{-(n-1+\alpha)} d\sigma \le C\|f\|_\infty, \end{aligned} \quad (4.43)$$

since the integral is bounded by $C|x|^{-\alpha}$. From this the first claim follows. On the other hand, when $f \in L^1$ we use a trivial estimate $|\xi|^{-(n-1+\alpha)} \le C|x|^{-(n-1+\alpha)}$ which gives us

$$v^1(x) = |x|^\alpha \int_{E_1} \frac{|f(\xi)|}{|\xi|^{n-1+\alpha}} d\sigma \le C|x|^{-(n-1)} \int_{E_1} |f(\xi)|\sigma \le C\|f\|_1 |x|^{-(n-1)}. \quad (4.44)$$

From this we can estimate $v^{1,*}(\xi)$ for $\xi \in \partial D$. We realize that for any $x \in \Gamma(\xi)$ we have $|x| \geq |\xi|$, hence

$$v^{1,*}(\xi) \leq C\|f\|_1 |\xi|^{-(n-1)}.$$

It follows that

$$\begin{aligned} \sigma(\{\xi : v^{1,*}(\xi) > \lambda\}) &\leq \sigma(\{\xi : C\|f\|_{L^1}|\xi|^{-(n-1)} > \lambda\}) \\ &= \sigma\left(\left\{\xi : |\xi|^{n-1} < \frac{C\|f\|_{L^1}}{\lambda}\right\}\right) \leq C\frac{\|f\|_{L^1}}{\lambda}, \end{aligned} \tag{4.45}$$

since the surface measure of a ball $\{\xi : |\xi| \leq R\}$ is proportional to R^{n-1}. From this the fact that $v^{1,*}$ belongs to the weak-L^1 follows.

Lemma 4.3 *There exist $C(p) > 0$ such that for all $p > \frac{n-1}{n-2+\alpha}$*

$$\begin{aligned} \|v^{3,*}\|_{L^\infty(\partial D)} &= \|v^3\|_{L^\infty(D)} \leq C\|f\|_{L^\infty(\partial D)} \\ \|v^{3,*}\|_{L^{p,w}(\partial D)} &\leq C\|f\|_{L^p(\partial D)}. \end{aligned} \tag{4.46}$$

Proof of the lemma: The definition of v^3 implies that for $q = p/(p-1)$

$$\begin{aligned} v^3(x) &= |x|^{-(n-2+\alpha)} \int_{E_3} \frac{|f(\xi)|}{|\xi|^{1-\alpha}} d\sigma \\ &\leq C|x|^{-(n-2+\alpha)}\|f\|_{L^p(E_3)} \left(\int_{E_3} \frac{1}{|\xi|^{q(1-\alpha)}} d\sigma\right)^{1/q}. \end{aligned} \tag{4.47}$$

In polar coordinates

$$\int_{E_3} \frac{1}{|\xi|^{q(1-\alpha)}} d\sigma \approx \int_0^{|x|/2} r^{n-2-q(1-\alpha)} dr < \infty \tag{4.48}$$

if and only if $n - 2 - q(1-\alpha) > -1$ or $p > \frac{n-1}{n-2+\alpha}$. Assuming that (4.47) gives us that

$$v^3(x) \leq C|x|^{-(n-2+\alpha)}\|f\|_{L^p(E_3)} \left(|x|^{n-1-q(1-\alpha)}\right)^{1/q} \leq C\|f\|_{L^p(\partial D)}|x|^{-\frac{n-1}{p}}. \tag{4.49}$$

When $p = \infty$ the first part of (4.46) follows. For $\frac{n-1}{n-2+\alpha} < p < \infty$ we observe that as before

$$v^{3,*}(\xi) = \sup_{x \in \Gamma(\xi)} v^3(x) \leq C\|f\|_{L^p(\partial D)}|\xi|^{-\frac{n-1}{p}}.$$

Hence

$$\begin{aligned} \sigma(\{\xi : v^{3,*}(\xi) > \lambda\}) &\leq \sigma(\{\xi : C\|f\|_{L^p}|\xi|^{-(n-1)/p} > \lambda\}) \\ &= \sigma\left(\left\{\xi : |\xi|^{n-1} < \frac{C\|f\|_{L^p}^p}{\lambda^p}\right\}\right) \leq C\frac{\|f\|_{L^p}^p}{\lambda^p}. \end{aligned} \tag{4.50}$$

This gives the second estimate of (4.46).

Lemma 4.4 *There exist $C>0$ such that*

$$\|v^{2,*}\|_{L^\infty(\partial D)} \;=\; \|v^2\|_{L^\infty(D)} \le C\|f\|_{L^\infty(\partial D)}. \tag{4.51}$$

Proof of the lemma: For any $x\in D$:

$$v^2(x) = R(x)\int_{E_2}\frac{|f(\xi)|}{|x-\xi|^n}d\sigma \le \|f\|_{L^\infty(D)}R(x)\int_{E_2}|x-\xi|^{-n}d\sigma. \tag{4.52}$$

We need to consider how the set E_2 looks. Clearly for every $x\in E_2$, $|x-\xi|\ge R(x)$. Since $E_2=\{\xi\in\partial D: |\xi|\in[|x|/2, 2|x|]\}$ we can parameterize E_2 and think about it as a cylinder $B\times[0,A]$, where B is an $n-2$-dimensional set of diameter at most $2|x|$. In this parametrization for $\xi=(b,s)\in B\times[0,A]$ we have $|x-\xi|\approx R(x)+s$. It follows that

$$\begin{aligned} R(x)\int_{E_2}|x-\xi|^{-n}d\sigma &\approx R(x)\int_0^A\frac{R(x)^{n-2}}{(R(x)+s)^n}ds \\ \le\;\; R(x)^{n-1}&\int_{R(x)}^\infty s^{-n}ds \le C. \end{aligned} \tag{4.53}$$

Hence that

$$v^2(x)\le C\|f\|_{L^\infty(D)},$$

with constant $C>0$ independent of the point $x\in D$.

To handle $p<\infty$ we need to introduce further splitting. Recall that $v^{2,*}(x)$ for a boundary point $x\in\partial D$ is defined as a supremum of v^2 over a nontangential cone $\Gamma(x)$ with vertex at x. The points $y\in\Gamma(x)$ are of two kinds. The first kind are points for which $|y-x|\le|x|$ (these are near the vertex x). The second kind are points $|y-x|>|x|$, for these we can make a simple observation that $R(y)\approx|y-x|\approx|y|$. To distinguish these two we introduce

$$\begin{aligned} w(x) &= \sup\{v^2(y): y\in\Gamma(x) \text{ and } |y-x|\le|x|\}, \\ z(x) &= \sup\{v^2(y): y\in\Gamma(x) \text{ and } |y-x|>|x|\}. \end{aligned} \tag{4.54}$$

It follows that pointwise $v^{2,*}(x)\le w(x)+z(x)$ for $x\in\partial D$ and hence

$$\|v^{2,*}\|_{L^p(\partial D)}\le\|w\|_{L^p(\partial D)}+\|z\|_{L^p(\partial D)},\qquad \text{for any } 1\le p\le\infty.$$

Let us denote by $B_{a,b}$ the part of the boundary of ∂D such that

$$B_{a,b}=\{\xi\in\partial D; a\le|\xi|\le b\}\qquad\text{for } 0<a<b. \tag{4.55}$$

Due to our assumption that near 0 our domain looks like a cone, it follows that for $a,b,\lambda b<\delta$ the sets $B_{a,b}$ and $B_{\lambda a,\lambda b}$ ($\lambda>0$) are simple rescales of each other, that is

$$B_{\lambda a,\lambda b}=\lambda B_{a,b}, \tag{4.56}$$

where the multiplication of a set by a scalar is understood in the usual sense. We claim that for any $\lambda>0$ and $1<p\le\infty$ we have an estimate

$$\|w\|_{L^p(B_{\lambda,2\lambda})}\le C_p\|f\|_{L^p(B_{\lambda/2,8\lambda})}. \tag{4.57}$$

It is enough to establish this for a single value of $\lambda > 0$, since then due to the rescaling argument the statement must be true for all $\lambda > 0$ small as follows from (4.56). The (4.57) holds, since the Dirichlet problem for the Lamé (or Stokes) system is solvable for all $1 < p \leq \infty$, provided the domain is C^1. As the sets $B_{\lambda,2\lambda}$ and $B_{\lambda/2,8\lambda}$ are outside the singularity (vertex at 0), our domain can be modified near the vertex outside these sets, so that the whole domain is C^1. Then the solvability for all $p > 1$ is used to get (4.57).

Setting $\lambda = 2^{-n}\delta$ and summing over n we get:

$$\begin{aligned} \|w\|^p_{L^p(\partial D\cap B(0,\delta/8))} &= \sum_{n=8}^{\infty} \|w\|^p_{L^p(B_{2^{-n-1}\delta,2^{-n}\delta})} \\ &\leq C_p^p \sum_{n=8}^{\infty} \|f\|^p_{L^p(B_{2^{-n-2}\delta,2^{-n+3}\delta})} \leq 4C_p^p\|f\|^p_{L^p(\partial D\cap B(0,\delta))}. \end{aligned} \tag{4.58}$$

This is the necessary estimate for w.

Looking at z, let us pick a point $y \in \Gamma(x)$ for $x \in \partial D$ such that $|y - x| > |x|$. We need to estimate $v^2(y)$:

$$v^2(y) = R(y)\int_{E_2} \frac{|f(\xi)|}{|y-\xi|^n} d\sigma.$$

Clearly, due to the fact that $R(y) \approx |y|$ and since for $\xi \in E_2$: $|\xi| \approx |y|$ we see that for any $\xi \in E_2$ we have $|y - \xi| \approx |y|$. Hence

$$v^2(y) \leq C|y|^{1-n}\int_{E_2} |f(\xi)|d\sigma \leq C|y|^{1-n}\|f\|_{L^1(\partial D)}.$$

It follows that for $x \in \partial D$:

$$z(x) = \sup\{v^2(y) : y \in \Gamma(x) \text{ and } |y - x| > |x|\} \leq C|x|^{1-n}\|f\|_{L^1(\partial D)}. \tag{4.59}$$

By the same argument as in (4.45) it follows that z belongs to a weak $L^{1,w}(\partial D)$. Hence we can claim that

Lemma 4.5 *For any $1 < p \leq \infty$ there exists $C_p > 0$ such that*

$$\|v^{2,*}\|_{L^p(\partial D)} \leq C_p\|f\|_{L^p(\partial D)}. \tag{4.60}$$

Proof. Consider first a mapping $f \mapsto z_f$, where for given f, we define $z = z_f$ by (4.59). This is not a linear mapping, but it is sublinear, that is

$$z_{f+g} \leq z_f + z_g.$$

By Lemma 4.4 this mapping is bounded on L^∞ and also as we have just show maps L^1 to weak $L^{1,w}$. By the Marcinkiewicz interpolation theorem (which only requires sublinearity, not linearity) this mapping is therefore bounded on any L^p, $p > 1$, and we have the estimate:

$$\|z_f\|_{L^p} \leq C_p\|f\|_{L^p}.$$

As we already know this for w we see that

$$\|v^{2,*}\|_{L^p} \leq \|w\|_{L^p} + \|z\|_{L^p} \leq C_p\|f\|_{L^p},$$

for all $p > 1$.

Proof of Theorem 4.1. We use the Marcinkiewicz interpolation theorem in the same spirit as we did above. By Lemmas 4.2 and 4.3 it follows that for all $p > 1$:

$$\|v^{1,*}\|_{L^p} \leq C_p\|f\|_{L^p},$$

and for all $p > \frac{n-1}{n-2+\alpha}$:

$$\|v^{3,*}\|_{L^p} \leq C_p\|f\|_{L^p}.$$

Combining the estimates for $v^{1,*}$, $v^{2,*}$ and $v^{3,*}$ yields the desired claim, since

$$\|u^*\|_{L^p} \leq C(\|v^{1,*}\|_{L^p} + \|v^{2,*}\|_{L^p} + \|v^{3,*}\|_{L^p}).$$

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