Chapter 1
Uniform asymptotics of Green's kernels for mixed and Neumann problems in domains with small holes and inclusions

V. Maz'ya and A. Movchan

To the memory of S.L. Sobolev

Abstract :
Uniform asymptotic approximations of Green's kernels for the harmonic mixed and Neumann boundary value problems in domains with singularly perturbed boundaries are obtained. We consider domains with small holes (in particular cracks) or inclusions. Formal asymptotic algorithms are supplied with rigorous estimates of the remainder terms.
Keywords: Uniform asymptotic approximations, Green's functions, singularly perturbed domains, mixed boundary value problems, Neumann's problem.

Acknowledgement: This work was supported by the UK Engineering and Physical Sciences Research Council via the research grant EP/F005563/1. We would like to thank Michael Nieves (University of Liverpool) for providing a numerical example mentioned in Introduction.

1.1 Introduction

There is a wide range of applications in physics and structural mechanics involving perforated domains and bodies with defects of different types. Direct numerical treatment of such problems is sometimes inefficient, especially for the situations when the right-hand sides in the equations and/or boundary conditions have singularities. Asymptotic approximations are important for problems of this kind and sometimes can be directly incorporated into computational algorithms if desirable.

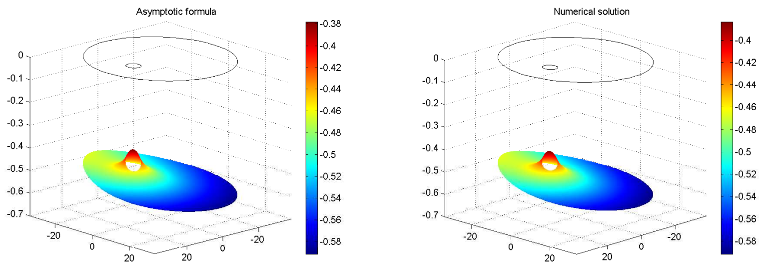
Asymptotic formulae for Green's kernels of several classical boundary value problems under small variations of a domain were obtained in the pioneering paper [2] by Hadamard. These asymptotic approximations are related to the case of a *regularly perturbed domain*, when the boundary $\partial\Omega_\varepsilon$ of the perturbed domain approximates the limit boundary $\partial\Omega$ in such a way that the angle between the outward normals at nearby points of $\partial\Omega$ and $\partial\Omega_\varepsilon$ is small.

Asymptotic approximations in [2] are not uniform with respect to the independent variables. Results on uniform asymptotic approximations of Green's kernels in various *singularly perturbed domains* are formulated in [5]. Detailed derivation and analysis of uniform asymptotic formulae for Green's functions of the *Dirichlet problem* for the operator $-\Delta$ in n-dimensional domains with small holes are given in [6]. In particular, the asymptotic approximation, obtained in [6], for Green's function of the Dirichlet problem in a two-dimensional domain Ω_ε with an inclusion $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$ has the form

$$
\begin{aligned}
G_\varepsilon(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + g(\boldsymbol{\xi},\boldsymbol{\eta}) + g(\boldsymbol{\xi},\infty) + g(\infty,\boldsymbol{\eta}) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|}{r_F} \\
-\frac{2\pi}{\log(\varepsilon r_F {R_\Omega}^{-1})}\left(G(\mathbf{x},0) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\xi}|}{r_F} - g(\boldsymbol{\xi},\infty)\right) \\
\times\left(G(0,\mathbf{y}) + \frac{1}{2\pi}\log\frac{|\boldsymbol{\eta}|}{r_F} - g(\infty,\boldsymbol{\eta})\right) + O(\varepsilon),
\end{aligned}
\tag{1.1}
$$

where $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x},\ \boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$, G and g are Green's functions of 'model' interior and exterior Dirichlet problems in 'limit' domains Ω and $\mathbf{R}^2 \setminus F$, independent of ε; R_Ω and r_F are the inner (with respect to $\mathbf{O}$) and outer conformal radii of Ω and F, respectively (see [8], Appendix G).

Approximations of this type are readily applicable to numerical simulations. For example, in Fig. 1 we show the regular part of Green's function G_ε in a two-dimensional domain with a small circular inclusion. The results on two diagrams are practically indistinguishable, while in Fig. 1a the data are obtained via the uniform asymptotic approximation, whereas Fig. 1b presents the result of independent finite element computations produced in COMSOL.



(a) (b)

Fig. 1.1 (a) Regular part of Green's function, computed via the asymptotic formula (1.1). (b) A finite element computation (in COMSOL) for the regular part of Green's function.

The aim of the present article is to derive and justify asymptotic approximations of Green's kernels for singularly perturbed domains whose boundary, or some part of it, supports the *Neumann boundary condition.* Although the corresponding asymptotic formulae to be obtained and (1.1) are of similar nature, the former have some new features and require individual treatments. We also derive simpler asymptotic formulae, which become efficient when certain constraints are imposed on the independent variables.

Sections 1.2 and 1.3 deal with the Dirichlet-Neumann problems in two-dimensional domains with small holes, inclusions or cracks. Section 1.4 gives the uniform approximation of Green's function for the Neumann problem in the domain of the same type. Finally, in Section 1.5 we formulate similar asymptotic approximations of Green's kernels in three-dimensional domains with small holes or small inclusions.

1.2 Mixed boundary value problem in a planar domain with a small hole or a crack

Let Ω be a bounded domain in $\mathbb{R}^2$, which contains the origin $\mathbf{O}$, and let F be a compact set in $\mathbb{R}^2$, $\mathbf{O} \in F$. We suppose that the boundary $\partial\Omega$ is smooth. This constraint is not essential and can be considerably weakened. We assume, without loss of generality, that diam $F = 1/2$, and that $\text{dist}(\mathbf{O}, \partial\Omega) = 1$. We also introduce the set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$, with ε being a small positive parameter. The boundary ∂F is required to be piecewise smooth, with the angle openings from the side of $\mathbb{R}^2 \setminus F$ belonging to $(0, 2\pi]$. In the case of a crack, ∂F and ∂F_ε are treated as two-sided. We assume that $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ is connected, and in the sequel we refer to it as a domain with a small hole (or possibly a small crack).

Let $G_\varepsilon^{(N)}$ denote Green's function of the operator $-\Delta$, with the Neumann data on ∂F_ε and the Dirichlet data on $\partial\Omega$. In other words, $G_\varepsilon^{(N)}$ is a solution of the problem

$$\Delta_x G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon, \tag{1.2}$$

$$G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon, \tag{1.3}$$

$$\frac{\partial G_\varepsilon^{(N)}}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \ \mathbf{y} \in \Omega_\varepsilon. \tag{1.4}$$

Here and elsewhere *the Neumann condition is understood in the variational sense.*

In this section, we construct an asymptotic approximation of $G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y})$, uniform with respect to $\mathbf{x}$ and $\mathbf{y}$ in Ω_ε.

1.2.1 Special solutions of model problems

While constructing the asymptotic approximation of $G_\varepsilon^{(N)}$, we use the variational solutions $G(\mathbf{x},\mathbf{y}), \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}), \zeta(\varepsilon^{-1}\mathbf{x})$ and $\mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$ of certain model problems in the limit domains Ω and $\mathbb{R}^2 \setminus F$. It is standard that all solutions, introduced in this subsection, exist and are unique. We describe these solutions.

1. Let G be *Green's function for the Dirichlet problem in Ω*:

$$G(\mathbf{x},\mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x}-\mathbf{y}|^{-1} - H(\mathbf{x},\mathbf{y}), \tag{1.5}$$

where H is the regular part of G, i.e. a unique solution of the Dirichlet problem

$$\Delta_x H(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega, \tag{1.6}$$

$$H(\mathbf{x},\mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x}-\mathbf{y}|^{-1}, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega. \tag{1.7}$$

2. We introduce the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. The notation ζ is used for a unique special solution of the Dirichlet problem:

$$\Delta\zeta(\boldsymbol{\xi}) \ = 0 \ \text{ in } \ \mathbb{R}^2 \setminus F, \tag{1.8}$$

$$\zeta(\boldsymbol{\xi}) \ = 0 \ \text{ for } \ \boldsymbol{\xi} \in \partial F, \tag{1.9}$$

$$\zeta(\boldsymbol{\xi}) \ = (2\pi)^{-1} \log |\boldsymbol{\xi}| + \zeta_\infty + O(|\boldsymbol{\xi}|^{-1}), \quad \text{as } |\boldsymbol{\xi}| \to \infty, \tag{1.10}$$

where ζ_∞ is constant.

Also, it can be shown that ζ is the limit of Green's function $\mathcal{G}$ of the exterior Dirichlet problem in $\mathbb{R}^2 \setminus F$

$$\zeta(\boldsymbol{\eta}) = \lim_{|\boldsymbol{\xi}|\to\infty} \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}), \tag{1.11}$$

where

$$\Delta_\xi \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F, \tag{1.12}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi} \in \partial F, \ \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F, \tag{1.13}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) \ \text{ is bounded as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F. \tag{1.14}$$

Representation (1.11) follows from Green's formula applied to ζ and $\mathcal{G}$. Here and elsewhere $B_R = \{\mathbf{X} \in \mathbb{R}^2 : \ |\mathbf{X}| < R\}$. We derive

$$\begin{aligned}\zeta(\boldsymbol{\eta}) &= -\lim_{R\to\infty} \int_{B_R\setminus F} \zeta(\boldsymbol{\xi}) \Delta_\xi \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) d\boldsymbol{\xi} \\ &= \lim_{R\to\infty} \int_{|\boldsymbol{\xi}|=R} \Big(\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) \frac{\partial\zeta(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - \zeta(\boldsymbol{\xi}) \frac{\partial\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})}{\partial|\boldsymbol{\xi}|} \Big) dS_\xi\end{aligned}$$

$$= (2\pi)^{-1} \lim_{R\to\infty} \int_{|\boldsymbol{\xi}|=R} \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta})|\boldsymbol{\xi}|^{-1} dS_\xi = \mathcal{G}(\infty,\boldsymbol{\eta}), \qquad (1.15)$$

which yields (1.11).

3. Let $\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta})$ be *the Neumann function* in $\mathbb{R}^2 \setminus F$ defined by

$$\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta}) = (2\pi)^{-1} \log |\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - h_N(\boldsymbol{\xi},\boldsymbol{\eta}), \qquad (1.16)$$

where h_N is the regular part of $\mathcal{N}$ subject to

$$\Delta_\xi h_N(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F, \qquad (1.17)$$

$$\frac{\partial h_N}{\partial n_\xi}(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{1}{2\pi}\frac{\partial}{\partial n_\xi}(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}), \quad \boldsymbol{\xi} \in \partial F,\ \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F, \qquad (1.18)$$

$$h_N(\boldsymbol{\xi},\boldsymbol{\eta}) \to 0, \quad \text{as } |\boldsymbol{\xi}| \to \infty,\ \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F. \qquad (1.19)$$

We note that the Neumann function $\mathcal{N}$ used here, is symmetric. This follows from Green's formula applied to $U(\mathbf{X}) := \mathcal{N}(\mathbf{X},\boldsymbol{\xi})$ and $V(\mathbf{X}) := \mathcal{N}(\mathbf{x},\boldsymbol{\eta})$, where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are arbitrary fixed points in $\mathbb{R}^2 \setminus F$. We have

$$U(\boldsymbol{\eta}) - V(\boldsymbol{\xi}) = \lim_{R\to\infty} \int_{B_R \setminus F} \Big\{ V(\mathbf{X})\Delta_x U(\mathbf{X}) - U(\mathbf{X})\Delta_X V(\mathbf{X}) \Big\} d\mathbf{X}$$

$$= \lim_{R\to\infty} \int_{|\mathbf{X}|=R} \{ V(\mathbf{X}) \frac{\partial}{\partial |\mathbf{X}|} U(\mathbf{X}) - U(\mathbf{X}) \frac{\partial}{\partial |\mathbf{X}|} V(\mathbf{X}) \} dS_X$$

$$= - \lim_{R\to\infty} (4\pi^2 R)^{-1} \int_{|\mathbf{X}|=R} \Big\{ (\log|\mathbf{X}-\boldsymbol{\eta}|^{-1} + O(R^{-1})) \Big(\frac{\mathbf{X}\cdot(\mathbf{X}-\boldsymbol{\xi})}{|\mathbf{X}-\boldsymbol{\xi}|^2} + O(R^{-2}) \Big)$$

$$-(\log|\mathbf{X}-\boldsymbol{\xi}|^{-1} + O(R^{-1})) \Big(\frac{\mathbf{X}\cdot(\mathbf{X}-\boldsymbol{\eta})}{|\mathbf{X}-\boldsymbol{\eta}|^2} + O(R^{-2}) \Big) \Big\} dS_x = 0.$$

Thus,

$$0 = U(\boldsymbol{\eta}) - V(\boldsymbol{\xi}) = \mathcal{N}(\boldsymbol{\eta},\boldsymbol{\xi}) - \mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta}).$$

4. The *vector of dipole fields* $\boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = (\mathcal{D}_1(\boldsymbol{\xi}), \mathcal{D}_2(\boldsymbol{\xi}))^T$ is a solution of the exterior Neumann problem

$$\Delta \boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = 0 \quad \text{in } \mathbb{R}^2 \setminus F, \qquad (1.20)$$

$$\frac{\partial \mathcal{D}_j}{\partial n}(\boldsymbol{\xi}) = n_j \quad \text{for } \boldsymbol{\xi} \in \partial F,\ j=1,2, \qquad (1.21)$$

$$\mathcal{D}_j(\boldsymbol{\xi}) \to 0 \quad \text{as } |\boldsymbol{\xi}| \to \infty,\ j=1,2, \qquad (1.22)$$

were n_1, n_2 are components of the unit normal on ∂F.

1.2.2 The dipole matrix $\boldsymbol{\mathcal{P}}$

The dipole fields $\mathcal{D}_j, j = 1, 2$, defined in (1.20)–(1.22), allow for the asymptotic representation (see, for example, [8])

$$\mathcal{D}_j(\boldsymbol{\xi}) = \frac{1}{2\pi} \sum_{k=1}^{2} \frac{\mathcal{P}_{jk}\xi_k}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-2}), \tag{1.23}$$

where $|\boldsymbol{\xi}| > 2$, and $\boldsymbol{\mathcal{P}} = (\mathcal{P}_{jk})_{j,k=1}^2$ is the *dipole matrix.*

The symmetry of $\boldsymbol{\mathcal{P}}$ can be verified as follows. Let B_R be a disk of sufficiently large radius R, centered at the origin. We apply Green's formula to $\xi_j - \mathcal{D}_j(\boldsymbol{\xi})$ and $\mathcal{D}_k(\boldsymbol{\xi})$ in $B_R \setminus F$, and deduce

$$\int_{\partial B_R} \Big\{(\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\frac{\partial D_k(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi})\frac{\partial}{\partial|\boldsymbol{\xi}|}(\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\Big\}dS$$
$$= -\int_{\partial F} (\xi_j - \mathcal{D}_j(\boldsymbol{\xi}))\frac{\partial \mathcal{D}_k(\boldsymbol{\xi})}{\partial n}dS, \tag{1.24}$$

where $\partial/\partial n$ is the normal derivative in the direction of the interior normal with respect to F. In the limit, as $R \to \infty$, the integral in the left-hand side of (1.24) tends to $-\mathcal{P}_{kj}$, whereas the integral in the right-hand side becomes

$$-\int_{\partial F} \xi_j \frac{\partial \xi_k}{\partial n} dS + \int_{\partial F} \mathcal{D}_j(\boldsymbol{\xi})\frac{\partial \mathcal{D}_k(\boldsymbol{\xi})}{\partial n}dS$$
$$= \delta_{jk}\text{meas}(F) + \int_{\mathbb{R}^2\setminus F} \nabla\mathcal{D}_j(\boldsymbol{\xi}) \cdot \nabla\mathcal{D}_k(\boldsymbol{\xi})\ d\boldsymbol{\xi},$$

where meas(F) stands for the two-dimensional Lebesgue measure of the set F. Thus, the representation for components of the dipole matrix takes the form

$$P_{kj} = -\delta_{jk}\text{meas}(F) - \int_{\mathbb{R}^2\setminus F} \nabla\mathcal{D}_j(\boldsymbol{\xi}) \cdot \nabla\mathcal{D}_k(\boldsymbol{\xi})\ d\boldsymbol{\xi}, \tag{1.25}$$

which implies that the *dipole matrix* $\boldsymbol{\mathcal{P}}$ *for the hole* F *is symmetric and negative definite.*

1.2.3 Pointwise estimate of a solution to the exterior Neumann problem

In this subsection, we make use of the function spaces $L_2^1(\mathbb{R}^2 \setminus F)$, $W_p^1(\mathbb{R}^2 \setminus F)$ and $W_p^{-1/p}(\partial F)$. The first of them is the space of distributions whose gradients belong to $L_2(\mathbb{R}^2 \setminus F)$. The second one is the usual Sobolev's space consisting of functions in $L_p(\mathbb{R}^2 \setminus F)$ with distributional first derivatives in $L_p(\mathbb{R}^2 \setminus F)$. Finally, $W_p^{-1/p}(\partial F)$ stands for the dual of the space of traces on ∂F of functions in $W^1_{p'}(\mathbb{R}^2 \setminus F)$, $p + p' = pp'$.

The following pointwise estimate will be used repeatedly in the sequel.

Lemma 1. *Let $U \in L^1_2(\mathbb{R}^2 \setminus F)$ be a solution of the exterior Neumann problem*

$$\Delta U(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, \tag{1.26}$$

$$\frac{\partial U}{\partial n}(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \partial F, \tag{1.27}$$

$$U(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty, \tag{1.28}$$

where $\partial/\partial n$ is the normal derivative on ∂F, outward with respect to $\mathbb{R}^2 \setminus F$, and $\varphi \in L_\infty(\partial F)$,

$$\int_{\partial F} \varphi(\boldsymbol{\xi}) ds_\xi = 0. \tag{1.29}$$

We also assume that

$$\int_{\partial F} U(\boldsymbol{\xi}) \frac{\partial \zeta}{\partial n}(\boldsymbol{\xi}) ds_\xi = 0, \tag{1.30}$$

where ζ is the same as in (1.11). *Then*

$$\sup_{\boldsymbol{\xi} \in \mathbb{R}^2 \setminus F} \{(|\boldsymbol{\xi}| + 1)|U(\boldsymbol{\xi})|\} \leq C \|\varphi\|_{L_\infty(\partial F)}, \tag{1.31}$$

where C is a constant depending on ∂F.

Proof. Let B_r denote the disk of radius r centered at $\mathbf{O}$ and let $W^1_2(B_r \setminus F)$ be the space of restrictions of functions in $W^1_2(\mathbb{R}^2 \setminus F)$ to $B_r \setminus F$. By the W^1_p local coercivity result [7], $U \in W^1_p(B_2 \setminus F)$ for any $p \in (1, 4)$, and

$$\|U\|_{W^1_p(B_2 \setminus F)} \leq C\Big(\|\varphi\|_{W_p^{-1/p}(\partial F)} + \|U\|_{L_2(B_3 \setminus F)}\Big). \tag{1.32}$$

The first term in the right-hand side of (1.32) satisfies

$$\|\varphi\|_{W_p^{-1/p}(\partial F)} \leq C \|\varphi\|_{L_\infty(\partial F)}. \tag{1.33}$$

It follows from (1.26) and (1.27) that

$$\|\nabla U\|^2_{L_2(\mathbb{R}^2 \setminus F)} = \int_{\partial F} U(\boldsymbol{\xi}) \varphi(\boldsymbol{\xi}) dS \leq \|U\|_{L_2(\partial F)} \|\varphi\|_{L_2(\partial F)}. \tag{1.34}$$

Note that by Sobolev's trace theorem

$$\|U\|_{L_q(\partial F)} \leq C \|U\|_{W^1_2(B_2 \setminus F)} \tag{1.35}$$

for any $q < \infty$ (see, for instance, Theorem 1.4.5 in [4]). It follows from our assumptions on F that

$$\left|\frac{\partial\zeta(\boldsymbol{\xi})}{\partial n}\right| \le C(\delta(\boldsymbol{\xi}))^{-1/2}, \tag{1.36}$$

where $\delta(\boldsymbol{\xi})$ is the distance from $\boldsymbol{\xi} \in \partial F$ to the nearest angle vertex on ∂F. Hence

$$\Big|\int_{\partial F} U(\boldsymbol{\xi})\frac{\partial\zeta(\boldsymbol{\xi})}{\partial n}dS\Big| \le C\|U\|_{L_q(\partial F)} \tag{1.37}$$

for any $q > 2$. This inequality, together with (1.35), shows that the left-hand side in (1.37) is a semi-norm, continuous in $W_2^1(B_2 \setminus F)$. Besides,

$$\int_{\partial F}\frac{\partial\zeta}{\partial n}(\boldsymbol{\xi})dS = \lim_{R\to\infty}(2\pi)^{-1}\int_{|\boldsymbol{\xi}|=R}\frac{\partial}{\partial|\boldsymbol{\xi}|}\log|\boldsymbol{\xi}|\;dS = 1.$$

Now, Sobolev's equivalent normalizations theorem (see Section 1.1.15 in [4]) implies that the norm in $W_2^1(B_2 \setminus F)$ is equivalent to the norm

$$\|\nabla U\|_{L_2(B_2\setminus F)} + \Big|\int_{\partial F} U(\boldsymbol{\xi})\frac{\partial\zeta}{\partial n}(\boldsymbol{\xi})dS\Big|.$$

Combining this fact with (1.35) and using (1.30), we arrive at

$$\|U\|_{L_2(\partial F)} \le C\|\nabla U\|_{L_2(\mathbb{R}^2\setminus F)}. \tag{1.38}$$

Then, (1.34) and (1.38) yield

$$\|\nabla U\|_{L_2(\mathbb{R}^2\setminus F)} + \|U\|_{L_2(\partial F)} \le C\|\varphi\|_{L_2(\partial F)}. \tag{1.39}$$

By (1.35), the norm in $W_2^1(B_3 \setminus F)$ is equivalent to the norm

$$\|\nabla U\|_{L_2(B_3\setminus F)} + \|U\|_{L_2(\partial F)}.$$

Hence

$$\|U\|_{L_2(B_3\setminus F)} \le C\Big(\|\nabla U\|_{L_2(\mathbb{R}^2\setminus F)} + \|U\|_{L_2(\partial F)}\Big), \tag{1.40}$$

which, together with (1.39), gives

$$\|U\|_{L_2(B_3\setminus F)} \le C\|\varphi\|_{L_2(\partial F)}. \tag{1.41}$$

Substituting estimates (1.33) and (1.41) into (1.32), we arrive at

$$\|U\|_{W_p^1(B_2\setminus F)} \le C\|\varphi\|_{L_\infty(\partial F)}. \tag{1.42}$$

Recalling that $W_p^1(B_2 \setminus F)$ is embedded into $C(\overline{B_2 \setminus F})$ for $p > 2$, by another Sobolev's theorem (see Theorem 1.4.5 in [4]), we obtain

$$\sup_{B_2\setminus F}|U| \le C\|\varphi\|_{L_\infty(\partial F)}. \tag{1.43}$$

Since $U(\boldsymbol{\xi}) \to 0$ as $|\boldsymbol{\xi}| \to \infty$ (see (1.29) and (1.30)), we have the Poisson's formula

$$U(\boldsymbol{\xi}) = \frac{1}{\pi} Re \int_0^{2\pi} \frac{U(1,\theta')}{\rho e^{i(\theta-\theta')} - 1} d\theta', \quad \boldsymbol{\xi} = \rho e^{i\theta}, \tag{1.44}$$

which, together with (1.43), implies for $|\boldsymbol{\xi}| > 1$ that

$$\big(1 + |\boldsymbol{\xi}|\big)|U(\boldsymbol{\xi})| \le C \max_{\boldsymbol{\xi} \in \partial B_1} |U(\boldsymbol{\xi})| \le C \|\varphi\|_{L_\infty(\partial\omega)}. \tag{1.45}$$

Applying (1.43) once more, we complete the proof. □

1.2.4 Asymptotic properties of the regular part of the Neumann function in $\mathbb{R}^2 \setminus F$

Lemma 1 proved in the previous section enables one to describe the asymptotic behaviour of the function h_N defined in (1.17)–(1.19).

Lemma 2. *The solution* $h_N(\boldsymbol{\xi}, \boldsymbol{\eta})$ *of problem* (1.17)–(1.19) *satisfies the estimate*

$$\left| h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{\boldsymbol{\mathcal{D}}(\boldsymbol{\eta}) \cdot \boldsymbol{\xi}}{2\pi |\boldsymbol{\xi}|^2} \right| \le \text{Const } (1 + |\boldsymbol{\eta}|)^{-1} |\boldsymbol{\xi}|^{-2} \tag{1.46}$$

as $|\boldsymbol{\xi}| > 2$ *and* $\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F$.

Proof. The leading-order approximation of the harmonic function $h_N(\boldsymbol{\xi}, \boldsymbol{\eta})$, as $|\boldsymbol{\xi}| \to \infty$, is sought in the form

$$(2\pi)^{-1} |\boldsymbol{\xi}|^{-2} (C_1 \xi_1 + C_2 \xi_2).$$

Applying Green's formula in $B_R \setminus F$ to $h_N(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j$, and taking the limit, as $R \to \infty$, we obtain

$$\lim_{R \to \infty} \int_{|\mathbf{x}| = R} \Big\{ h_N(\boldsymbol{\xi}) \frac{\partial (\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j)}{\partial |\boldsymbol{\xi}|} + (\xi_j - \mathcal{D}_j(\boldsymbol{\xi})) \frac{\partial h_N(\boldsymbol{\xi})}{\partial |\boldsymbol{\xi}|} \Big\} dS_\xi$$

$$= \int_{\partial F} (\mathcal{D}_j(\boldsymbol{\xi}) - \xi_j) \frac{\partial h_N(\boldsymbol{\xi})}{\partial n} dS_\xi, \tag{1.47}$$

where $\partial/\partial n$ is the normal derivative in the direction of the inward normal with respect to F. As $R \to \infty$, the the left-hand side of (1.47) becomes

$$\frac{1}{2\pi} \lim_{R \to +\infty} \int_{|\mathbf{x}| = R} \Big\{ -2 \frac{(C_1 \xi_1 + C_2 \xi_2) \xi_j}{R^3} \Big\} dS_\xi$$

$$= -\frac{1}{\pi} \lim_{R \to +\infty} \int_0^{2\pi} (C_1 \cos\theta + C_2 \sin\theta) R^{-1} \xi_j d\theta = -C_j. \tag{1.48}$$

Taking into account the definition of the dipole fields $\mathcal{D}_j$ (see (1.20)–(1.22)) and the definition of the regular part h_N of Neumann's function (see (1.17)–(1.19)) in $\mathbb{R}^2 \setminus F$, we can reduce the integral $\mathcal{I}$ in the right-hand side of (1.47) to the form

$$\mathcal{I} = \frac{1}{2\pi}\Big\{ \int_{\partial F} \Big(\mathcal{D}_j(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi}\Big(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big) - \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} \frac{\partial}{\partial n_\xi}\mathcal{D}_j(\boldsymbol{\xi})\Big) dS_\xi + \int_{\partial F}\Big(n_j \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} - \xi_j \frac{\partial}{\partial n_\xi}\Big(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big)\Big) dS_\xi\Big\}. \tag{1.49}$$

The second integral in (1.49) equals zero. Applying Green's formula to the first integral in (1.49) we obtain

$$\frac{1}{2\pi}\int_{\partial F}\Big(\mathcal{D}_j(\boldsymbol{\xi}) \frac{\partial}{\partial n_\xi}\Big(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1}\Big) - \log|\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1} \frac{\partial}{\partial n_\xi}\mathcal{D}_j(\boldsymbol{\xi})\Big) dS_\xi = -\mathcal{D}_j(\boldsymbol{\eta}). \tag{1.50}$$

Hence, it follows from (1.48)–(1.50) that

$$C_j = \mathcal{D}_j(\boldsymbol{\eta}),\ \ j = 1, 2. \tag{1.51}$$

We note that the function

$$h_N(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\eta})\cdot\nabla_\xi\big(\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1}\big) \tag{1.52}$$

is harmonic in $\mathbb{R}^2 \setminus F$, both in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, and it vanishes at infinity. Using (1.21) and (1.18), we obtain

$$\begin{aligned} \frac{\partial}{\partial n_\eta}\Big(h_N(\boldsymbol{\xi},\boldsymbol{\eta}) \ &+\ \boldsymbol{\mathcal{D}}(\boldsymbol{\eta})\cdot\nabla_\xi\big(\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1}\big)\Big) \\ &= \frac{\partial}{\partial n_\eta} h_N(\boldsymbol{\xi},\boldsymbol{\eta}) + \mathbf{n}\cdot\nabla_\xi\big(\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1}\big) \\ &= -\mathbf{n}\cdot\nabla_\xi\big\{\frac{1}{2\pi}\log(|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}|^{-1})\big\} \\ &= -\frac{1}{2\pi|\boldsymbol{\xi}|^2}\mathbf{n}\cdot\Big\{\boldsymbol{\eta} - \frac{2\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^{-1})\Big\} \end{aligned} \tag{1.53}$$

as $\boldsymbol{\eta} \in \partial F$ and $|\boldsymbol{\xi}| > 2$. We also note that

$$\int_{\partial F}\frac{\partial}{\partial n_\eta}\Big(h_N(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\eta})\cdot\nabla_\xi\big(\frac{1}{2\pi}\log|\boldsymbol{\xi}|^{-1}\big)\Big) dS_\eta = 0.$$

Consider the problem (1.26)–(1.28) in the formulation of Lemma 1, where the variable $\boldsymbol{\xi}$ is replaced by $\boldsymbol{\eta}$, the differentiation is taken with respect to components of $\boldsymbol{\eta}$, and the function U is changed for (1.52), with fixed $\boldsymbol{\xi}$. In this case, the right-hand side φ in (1.27) is replaced by

$$\frac{\partial}{\partial n_\eta} h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) + \mathbf{n} \cdot \nabla_\xi (\frac{1}{2\pi} \log |\boldsymbol{\xi}|^{-1}).$$

Then using (1.53) and applying Lemma 1, we obtain (1.46). □

Using the notion of the dipole matrix, from (1.23) and Lemma 2 we derive the following asymptotic representation of h_N.

Corollary 1. *Let* $|\boldsymbol{\xi}| > 2$, *and* $|\boldsymbol{\eta}| > 2$. *Then*

$$h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{4\pi^2} \sum_{j,k=1}^{2} \frac{\mathcal{P}_{jk} \xi_j \eta_k}{|\boldsymbol{\xi}|^2 |\boldsymbol{\eta}|^2} + O\Big(\frac{|\boldsymbol{\xi}| + |\boldsymbol{\eta}|}{|\boldsymbol{\xi}|^2 |\boldsymbol{\eta}|^2} \Big). \tag{1.54}$$

1.2.5 Maximum modulus estimate for solutions to the mixed problem in Ω_ε, with the Neumann data on ∂F_ε

In the sequel, when estimating the remainder term in the asymptotic representation of $G_\varepsilon(\mathbf{x}, \mathbf{y})$, we use the following assertion.

Lemma 3. *Let u be a function in $C(\overline{\Omega}_\varepsilon)$ such that ∇u is square integrable in a neighbourhood of ∂F_ε. Also, let u be a solution of the mixed boundary value problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.55}$$

$$u(\mathbf{x}) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{1.56}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{1.57}$$

where $\varphi \in C(\partial\Omega)$, $\psi_\varepsilon \in L_\infty(\partial F_\varepsilon)$, *and*

$$\int_{\partial F_\varepsilon} \psi_\varepsilon(\mathbf{x}) ds = 0. \tag{1.58}$$

Then there exists a positive constant C, independent of ε and such that

$$\|u\|_{C(\overline{\Omega}_\varepsilon)} \le \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{1.59}$$

Proof. (a) We introduce the inverse operator

$$\mathfrak{N} : \psi \to v \tag{1.60}$$

for the boundary value problem

$$\begin{aligned} \Delta v(\boldsymbol{\xi}) &= 0,\ \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, & (1.61)\\ \frac{\partial v}{\partial n}(\boldsymbol{\xi}) &= \psi(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in \partial F, & (1.62)\\ v(\boldsymbol{\xi}) &\to 0,\ \text{as } |\boldsymbol{\xi}| \to \infty, & (1.63) \end{aligned}$$

where $\psi \in L_\infty(\partial F)$, and

$$\int_{\partial F} \psi(\boldsymbol{\xi}) ds_\xi = 0. \tag{1.64}$$

In the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$, the operator $\mathfrak{N}_\varepsilon$ is defined by

$$(\mathfrak{N}_\varepsilon \psi_\varepsilon)(\mathbf{x}) = (\mathfrak{N}\psi)(\boldsymbol{\xi}), \tag{1.65}$$

where $\psi_\varepsilon(\mathbf{x}) = \varepsilon^{-1}\psi(\varepsilon^{-1}\mathbf{x})$.

(b) We look for the solution u of (1.55)–(1.58) in the form

$$u = V(\mathbf{x}) + W(\mathbf{x}), \tag{1.66}$$

where $V = \mathfrak{N}_\varepsilon \psi_\varepsilon$, and the function W satisfies the problem

$$\Delta W(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.67}$$

$$\frac{\partial W}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \tag{1.68}$$

$$W(\mathbf{x}) = \varphi(\mathbf{x}) - V(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{1.69}$$

By Lemma 1, we have

$$\max_{\overline{\Omega}_\varepsilon} |V| = \max_{\overline{\Omega}_\varepsilon} |\mathfrak{N}_\varepsilon \psi_\varepsilon| \le \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{1.70}$$

Hence, as follows from (1.69) and (1.70)

$$\max_{\partial\Omega} |W| \le \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}, \tag{1.71}$$

and by the weak maximum principle for variational solutions (see, for example, [1], pages 215-216) of (1.67)–(1.69) we obtain

$$\max_{\overline{\Omega}_\varepsilon} |W| \le \|\varphi\|_{C(\partial\Omega)} + \varepsilon C \|\psi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{1.72}$$

The result follows from (1.70), (1.72) combined with (1.66). □

1.2.6 Approximation of Green's function $G_\varepsilon^{(N)}$

The required approximation of $G_\varepsilon^{(N)}$ is given in the next Theorem.

Theorem 1. *Green's function $G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y})$ for the boundary value problem* (1.2)–(1.4)*, with the Neumann data on ∂F_ε and the Dirichlet data on $\partial\Omega$, has the asymptotic representation*

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$

$$+\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0) + r_\varepsilon(\mathbf{x},\mathbf{y}), \tag{1.73}$$

where

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \le \text{Const}\ \varepsilon^2 \tag{1.74}$$

uniformly with respect to $\mathbf{x},\mathbf{y}\in\Omega_\varepsilon$. Here, G, $\mathcal{N}$, $\boldsymbol{\mathcal{D}}$ and H are the same as in Section 1.2.1*.*

Proof. We begin with the formal argument leading to (1.73). First, we note that

$$N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + (2\pi)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) = -h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}),$$

and then represent $G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y})$ in the form

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + \rho_\varepsilon(\mathbf{x},\mathbf{y}). \tag{1.75}$$

By the direct substitution of (1.75) into (1.2)–(1.4) and using Lemma 2, we deduce that $\rho_\varepsilon(\mathbf{x},\mathbf{y})$ satisfies the boundary value problem

$$\begin{aligned} \Delta_x\rho_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \\ \rho_\varepsilon(\mathbf{x},\mathbf{y}) &= h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ &= \frac{\varepsilon}{2\pi}\boldsymbol{\mathcal{D}}\Big(\frac{\mathbf{y}}{\varepsilon}\Big)\cdot\frac{\mathbf{x}}{|\mathbf{x}|^2} + O(\varepsilon^2), \quad \text{for } \mathbf{x}\in\partial\Omega, \mathbf{y}\in\Omega_\varepsilon, \end{aligned} \tag{1.76}$$

and

$$\begin{aligned} \frac{\partial\rho_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) &= \frac{\partial}{\partial n_x}H(\mathbf{x},\mathbf{y}) \\ &= \mathbf{n}\cdot\nabla_x H(0,\mathbf{y}) + O(\varepsilon), \quad \text{for } \mathbf{x}\in\partial F_\varepsilon, \mathbf{y}\in\Omega_\varepsilon. \end{aligned} \tag{1.77}$$

Hence, by (1.6), (1.7) and (1.20)–(1.22), the leading-order approximation of ρ_ε is

$$\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0),$$

which, together with (1.75), leads to (1.73).

Now, we prove the remainder estimate (1.74). The direct substitution of (1.73) into (1.2)–(1.4) yields the boundary value problem for r_ε:

$$\begin{aligned}
\Delta_x r_\varepsilon(\mathbf{x},\mathbf{y}) &= 0, \quad \text{for } \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, && (1.78)\\
r_\varepsilon(\mathbf{x},\mathbf{y}) &= h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\
&\quad -\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y}) - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0), && (1.79)\\
&\quad \text{for } \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon,\\
\frac{\partial r_\varepsilon(\mathbf{x},\mathbf{y})}{\partial n_x} &= \mathbf{n}\cdot\nabla_x H(\mathbf{x},\mathbf{y}) - \varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})\Big)\\
&\quad -\varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)\Big), && (1.80)\\
&\quad \text{for } \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon.
\end{aligned}$$

We note that every term in the right-hand side of (1.80) has zero average on ∂F_ε, and hence

$$\int_{\partial F_\varepsilon} \frac{\partial r_\varepsilon(\mathbf{x},\mathbf{y})}{\partial n_x} dS_x = 0. \tag{1.81}$$

It follows from Lemma 2 that

$$|h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)| \le \text{Const }\varepsilon^2, \tag{1.82}$$

uniformly with respect to $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$. Since $|\boldsymbol{\mathcal{D}}(\boldsymbol{\xi})| \le \text{Const } |\boldsymbol{\xi}|^{-1}$, as $|\boldsymbol{\xi}|\to\infty$, and $\nabla_x H(0,\mathbf{y})$ is smooth on Ω_ε, we deduce

$$|\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})| \le \text{Const }\varepsilon^2 \tag{1.83}$$

uniformly with respect to $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$. By (1.82) and (1.83), the modulus of the right-hand side in (1.79) is bounded by Const ε^2, uniformly in $\mathbf{x}\in\partial\Omega$ and $\mathbf{y}\in\Omega_\varepsilon$.

It also follows from the definition of the dipole fields $\mathcal{D}_j(\boldsymbol{\xi}), j=1,2$, and the smoothness of the function $H(\mathbf{x},\mathbf{y})$ for all $\mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon$ that

$$\Big|\mathbf{n}\cdot\nabla_x H(\mathbf{x},\mathbf{y}) - \varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x H(0,\mathbf{y})\Big)\Big| \le \text{Const }\varepsilon, \tag{1.84}$$

and

$$\Big|\varepsilon\frac{\partial}{\partial n_x}\Big(\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y H(\mathbf{x},0)\Big)\Big| \le \text{Const }\varepsilon, \tag{1.85}$$

uniformly with respect to $\mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon$. These estimates imply that the modulus of the right-hand side in (1.80) is bounded by Const ε, uniformly in $\mathbf{x}\in\partial F_\varepsilon$ and $\mathbf{y}\in\Omega_\varepsilon$.

Using the estimates on ∂F_ε and $\partial\Omega$, just obtained, together with the orthogonality condition (1.81), we deduce that the right-hand sides of problem (1.78)–(1.80) satisfy the conditions of Lemma 3. Applying Lemma 3, we obtain that $\|r_\varepsilon\|_{L_\infty(\Omega_\varepsilon)}$ is dominated by Const ε^2, which completes the proof. $\square$

1.2.7 Simpler asymptotic formulae for Green's function $G_\varepsilon^{(N)}$

Here we formulate two corollaries of Theorem 1. They contain simpler asymptotic formulae, which are efficient for the cases when both $\mathbf{x}$ and $\mathbf{y}$ are distant from F_ε or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε.

Corollary 2. *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula holds*

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \frac{\varepsilon^2}{4\pi^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}}{|\mathbf{y}|^2} \\ &+ \frac{\varepsilon^2}{2\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\nabla_x H(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^2}\boldsymbol{\mathcal{P}}\nabla_y H(\mathbf{x},0)\Big\} \\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2}), \end{aligned} \tag{1.86}$$

where H is the regular part of Green's function G in Ω, and $\boldsymbol{\mathcal{P}}$ is the dipole matrix for F, as defined in (1.23).

Proof. Using (1.54) for the regular part h_N of the Neumann function in $\mathbb{R}^2 \setminus F$, together with the asymptotic representation (1.23) of the dipole fields $\mathcal{D}_j$ in $\mathbb{R}^2 \setminus F$, we obtain

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) &= G(\mathbf{x},\mathbf{y}) - \frac{\varepsilon^2}{4\pi^2}\sum_{j,k=1}^{2}\frac{\mathcal{P}_{jk}x_j y_k}{|\mathbf{x}|^2|\mathbf{y}|^2} + O\Big(\varepsilon^3\frac{|\mathbf{x}|+|\mathbf{y}|}{|\mathbf{x}|^2|\mathbf{y}|^2}\Big) \\ &+ \frac{1}{2\pi}\sum_{j,k=1}^{2}\Big\{\varepsilon^2\mathcal{P}_{jk}\Big(\frac{x_k}{|\mathbf{x}|^2}\frac{\partial H}{\partial x_j}(0,\mathbf{y}) + \frac{y_k}{|\mathbf{y}|^2}\frac{\partial H}{\partial y_j}(\mathbf{x},0)\Big) \\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2})\Big\} + O(\varepsilon^2). \end{aligned} \tag{1.87}$$

Combining the remainder terms and adopting the matrix representation involving the dipole matrix $\boldsymbol{\mathcal{P}}$, we arrive at (1.86). □

The formula (1.86) becomes efficient when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently distant from the small hole F_ε. Compared to (1.73), formula (1.86) does not involve special solutions of model problems in $\mathbb{R}^2 \setminus F$, while the influence of the hole F is seen through the dipole matrix $\boldsymbol{\mathcal{P}}$.

Corollary 3. *The following asymptotic formula for Green's function $G_\varepsilon^{(N)}$ of the boundary value problem* (1.2)–(1.4) *holds:*

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) &= (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - H(0,0) \\ &- (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x H(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y H(\mathbf{x},0) \\ &+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2), \end{aligned} \tag{1.88}$$

for $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (Needless to say, ε^2 in the remainder can be omitted if the interior of F is non-empty and contains the origin.)

Proof. Using the Taylor expansion of $H(\mathbf{x}, \mathbf{y})$ in a neighbourhood of the origin, we obtain

$$\begin{aligned} G_\varepsilon^{(N)}(\mathbf{x}, \mathbf{y}) &= -H(0,0) - \mathbf{x} \cdot \nabla_x H(0, \mathbf{y}) - \mathbf{y} \cdot \nabla_y H(\mathbf{y}, 0) + O(|\mathbf{x}|^2 + |\mathbf{y}|^2) \\ &+ \mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - (2\pi)^{-1} \log \varepsilon \\ &+ \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x H(0, \mathbf{y}) \\ &+ \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y H(\mathbf{x}, 0) + O(\varepsilon^2). \end{aligned} \tag{1.89}$$

By substituting

$$\mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) = (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1} + (2\pi)^{-1} \log \varepsilon - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$$

into (1.89) and rearranging the terms, we arrive at (1.88). □

1.3 Mixed boundary value problem with the Dirichlet condition on ∂F_ε

In the present section, the meaning of the notations Ω, F and F_ε, already used in Section 1.2, will be slightly altered. Hopefully, this will not lead to any confusion. Let Ω be a bounded domain with smooth boundary, and let F stand for an arbitrary compact set in $\mathbb{R}^2$ of positive logarithmic capacity [3]. As in Section 1.2, it is assumed that diam $F = 1/2$, and that $\text{dist}(\mathbf{O}, \partial\Omega) = 1$. We also set $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$.

We consider the mixed boundary value problem in a two-dimensional domain $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$, with the Dirichlet data on ∂F_ε and the Neumann data on $\partial\Omega$.

Green's function $G_\varepsilon^{(D)}$ of this problem is a weak solution of

$$\Delta_x G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon, \tag{1.90}$$

$$G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \ \mathbf{y} \in \Omega_\varepsilon, \tag{1.91}$$

$$\frac{\partial G_\varepsilon^{(D)}}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon. \tag{1.92}$$

Before deriving an asymptotic approximation of $G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y})$, uniform with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$, we outline the properties of solutions of auxiliary model problems in limit domains.

1.3.1 Special solutions of model problems

1. Let $N(\mathbf{x}, \mathbf{y})$ be the Neumann function in Ω, i.e.

$$\Delta N(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega, \tag{1.93}$$

$$\frac{\partial}{\partial n_x}\Big(N(\mathbf{x},\mathbf{y}) + (2\pi)^{-1}\log|\mathbf{x}|\Big) = 0, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega, \tag{1.94}$$

and

$$\int_{\partial\Omega} N(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n_x}\log|\mathbf{x}|ds_x = 0. \tag{1.95}$$

Condition (1.95) implies the symmetry of $N(\mathbf{x},\mathbf{y})$. In fact, let $U(\mathbf{x}) = N(\mathbf{x},\mathbf{z})$ and $V(\mathbf{x}) = N(\mathbf{x},\mathbf{y})$, where $\mathbf{z}$ and $\mathbf{y}$ are fixed points in Ω. Then applying Green's formula to U and V and using (1.93)–(1.95) we deduce

$$U(\mathbf{y}) - V(\mathbf{z}) = \int_{\Omega}\Big(V(\mathbf{x})\Delta_x U(\mathbf{x}) - U(\mathbf{x})\Delta_x V(\mathbf{x})\Big)d\mathbf{x}$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}\Big(U(\mathbf{x})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - V(\mathbf{x})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|)\Big)dS_x$$

$$= \frac{1}{2\pi}\Big\{\int_{\partial\Omega} N(\mathbf{x},\mathbf{z})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|)dS_x - \int_{\partial\Omega} N(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|)\Big)dS_x\Big\} = 0,$$

where $\partial/\partial n_x$ is the normal derivative in the direction of the outward normal on $\partial\Omega$. Hence $N(\mathbf{y},\mathbf{z}) = N(\mathbf{z},\mathbf{y})$.

The regular part of the Neumann function is defined by

$$R(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x},\mathbf{y}). \tag{1.96}$$

Note that

$$R(0,\mathbf{y}) = -(2\pi)^{-2}\int_{\partial\Omega}\log|\mathbf{x}|\frac{\partial}{\partial n}\log|\mathbf{x}|ds_x, \tag{1.97}$$

which is verified by applying Green's formula to $R(\mathbf{x},\mathbf{y})$ and $(2\pi)^{-1}\log|\mathbf{x}|$ as follows:

$$R(0,\mathbf{y}) = \frac{1}{2\pi}\int_{\Omega} R(\mathbf{x},\mathbf{y})\Delta_x(\log|\mathbf{x}|)d\mathbf{x}$$

$$= \frac{1}{2\pi}\int_{\partial\Omega}\Big(R(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - \log|\mathbf{x}|\frac{\partial}{\partial n_x}R(\mathbf{x},\mathbf{y})\Big)ds_x, \tag{1.98}$$

where $\partial/\partial n_x$ is the normal derivative in the outward direction on $\partial\Omega$. Taking into account (1.94), (1.95) and (1.96), we can write (1.98) in the form

$$R(0,\mathbf{y}) = \frac{1}{4\pi^2}\int_{\partial\Omega}\Big(\log|\mathbf{x}-\mathbf{y}|^{-1}\frac{\partial}{\partial n_x}(\log|\mathbf{x}|) - \log|\mathbf{x}|\frac{\partial}{\partial n_x}(\log|\mathbf{x}-\mathbf{y}|^{-1})\Big)ds_x$$

$$+\frac{1}{2\pi}\int_{\partial\Omega}\log|\mathbf{x}|\frac{\partial}{\partial n_x}(N(\mathbf{x},\mathbf{y}))ds_x. \tag{1.99}$$

The first integral in (1.99) is equal to zero, while the second integral in (1.99) is reduced to (1.97) because of the boundary condition (1.94).

As in Section 1.2, the notations $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ will be used for the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$. The corresponding limit domain is $\mathbb{R}^2 \setminus F$.

2. Green's function $\mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta})$ for the Dirichlet problem in $\mathbb{R}^2 \setminus F$ is a unique solution to the problem (1.12)–(1.14). The regular part $h(\boldsymbol{\xi}, \boldsymbol{\eta})$ of Green's function $\mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta})$ is

$$h(\boldsymbol{\xi}, \boldsymbol{\eta}) = (2\pi)^{-1} \log |\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} - \mathcal{G}(\boldsymbol{\xi}, \boldsymbol{\eta}). \tag{1.100}$$

3. Here and in the sequel, $\mathbf{D}(\boldsymbol{\xi})$ denotes a vector function, whose components $D_j,\ j = 1, 2$, satisfy the model problems

$$\Delta D_j(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus F, \tag{1.101}$$

$$D_j(\boldsymbol{\xi}) = \xi_j, \quad \boldsymbol{\xi} \in \partial F, \tag{1.102}$$

$$D_j(\boldsymbol{\xi}) \text{ is bounded as } |\boldsymbol{\xi}| \to \infty. \tag{1.103}$$

We use the notations $D_j^\infty = \lim_{|\boldsymbol{\xi}|\to\infty} D_j(\boldsymbol{\xi})$ and $\mathbf{D}^\infty = (D_1^\infty, D_2^\infty)^T$.

Application of Green's formula to D_j and the function ζ, defined in (1.8)–(1.10), gives

$$D_j^\infty = -\int_{\partial F} \xi_j \frac{\zeta(\boldsymbol{\xi})}{\partial n} dS_\xi. \tag{1.104}$$

Here and in other derivations of this section, $\partial/\partial n$ on ∂F is the normal derivative in the direction of the inward normal with respect to F.

We also find an additional connection between D_j and ζ by analyzing the asymptotic formula (compare with (1.10))

$$\zeta(\boldsymbol{\xi}) = (2\pi)^{-1} \log |\boldsymbol{\xi}| + \zeta_\infty + \frac{1}{2\pi} \sum_{k=1}^{2} \frac{\alpha_k \xi_k}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-2}), \quad |\boldsymbol{\xi}| \to \infty, \tag{1.105}$$

and showing that

$$\alpha_k = -D_k^\infty. \tag{1.106}$$

Let us apply Green's formula to ξ_j and ζ:

$$\begin{aligned} \int_{\partial F} \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial n} dS_\xi &= \int_{\partial F} \Big\{ \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial n} - \zeta(\boldsymbol{\xi}) \frac{\partial \xi_j}{\partial n} \Big\} dS_\xi \\ &= -\lim_{R\to\infty} \int_{|\boldsymbol{\xi}|=R} \Big\{ \xi_j \frac{\partial \zeta(\boldsymbol{\xi})}{\partial |\boldsymbol{\xi}|} - \zeta(\boldsymbol{\xi}) \frac{\partial \xi_j}{\partial |\boldsymbol{\xi}|} \Big\} dS_\xi \\ &= \frac{1}{\pi} \lim_{R\to\infty} \int_{|\boldsymbol{\xi}|=R} \sum_{k=1}^{2} \frac{\alpha_k \xi_k \xi_j}{|\boldsymbol{\xi}|^3} dS_\xi = \alpha_j. \end{aligned} \tag{1.107}$$

Then formulae (1.107) and (1.104) lead to (1.106).

1.3.2 Asymptotic property of the regular part of Green's function in $\mathbb{R}^2 \setminus F$

Asymptotic representation at infinity for the regular part of Green's function in $\mathbb{R}^2 \setminus F$ is given by the following Lemma.

Lemma 4. *The regular part* (1.100) *of* $\mathcal{G}$*satisfies the estimate*

$$\Big| h(\boldsymbol{\xi},\boldsymbol{\eta}) - (2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1} + \zeta(\boldsymbol{\eta}) - \frac{1}{2\pi}\sum_{j=1}^{2}\frac{D_j(\boldsymbol{\eta})\xi_j}{|\boldsymbol{\xi}|^2}\Big| \le \frac{\text{Const}}{|\boldsymbol{\xi}|^2}, \quad (1.108)$$

as $|\boldsymbol{\xi}| > 2$, *and* $\boldsymbol{\eta} \in \mathbb{R}^2 \setminus F$.

Proof. Let

$$\beta(\boldsymbol{\xi},\boldsymbol{\eta}) = h(\boldsymbol{\xi},\boldsymbol{\eta}) - (2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1} + \zeta(\boldsymbol{\eta}) - \frac{1}{2\pi}\sum_{j=1}^{2}\frac{D_j(\boldsymbol{\eta})\xi_j}{|\boldsymbol{\xi}|^2}.$$

We have

$$\Delta_\eta \beta(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \mathbb{R}^2 \setminus F,$$

and

$$\begin{aligned}\beta(\boldsymbol{\xi},\boldsymbol{\eta}) &= -\frac{1}{4\pi}\log\Big(1 - 2\frac{\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{|\boldsymbol{\xi}|^2} + \frac{|\boldsymbol{\eta}|^2}{|\boldsymbol{\xi}|^2}\Big) - \frac{\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2\pi|\boldsymbol{\xi}|^2} \\ &= -\frac{1}{4\pi|\boldsymbol{\xi}|^2}\Big\{|\boldsymbol{\eta}|^2 - 2\frac{(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2}{|\boldsymbol{\xi}|^2} + O(|\boldsymbol{\xi}|^{-1})\Big\} \end{aligned} \quad (1.109)$$

as $\boldsymbol{\eta} \in \partial F$. By (1.8)–(1.10) and Green's formula

$$\beta(\boldsymbol{\xi},\infty) = -\int_{\partial F}\beta(\boldsymbol{\xi},\boldsymbol{\eta})\frac{\partial\zeta(\boldsymbol{\eta})}{\partial n_\eta}dS_\eta,$$

which together with (1.109) and (1.36) implies

$$|\beta(\boldsymbol{\xi},\infty)| \le C\ |\boldsymbol{\xi}|^{-2}.$$

Hence the maximum principle gives (1.108). □

1.3.3 Maximum modulus estimate for solutions to the mixed problem in Ω_ε, with the Dirichlet data on ∂F_ε

Lemma 5. *Let* u *be a function in* $C(\overline{\Omega}_\varepsilon)$ *such that* ∇u *is square integrable in a neighbourhood of* $\partial\Omega$. *Let* u *be a solution of the mixed problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.110}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{1.111}$$

$$u(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{1.112}$$

where $\psi \in C(\partial\Omega),\ \varphi_\varepsilon \in C(\partial F_\varepsilon)$, *and*

$$\int_{\partial\Omega} \psi(\mathbf{x}) ds = 0. \tag{1.113}$$

Then there exists a positive constant C *such that*

$$\|u\|_{C(\Omega_\varepsilon)} \le \|\varphi_\varepsilon\|_{C(\partial F_\varepsilon)} + C\|\psi\|_{C(\partial\Omega)}. \tag{1.114}$$

Proof. (a) First, we introduce the inverse operator

$$\mathfrak{N}_\Omega : \psi \to w \tag{1.115}$$

for the interior Neumann problem in Ω

$$\Delta w(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{1.116}$$

$$\frac{\partial w}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{1.117}$$

with $\psi \in C(\partial\Omega)$ and

$$\int_{\partial\Omega} \psi(\mathbf{x}) dS_x = 0 \ \text{ and } \ \int_{\partial\Omega} w(\mathbf{x}) \frac{\partial}{\partial n}\big(\log|\mathbf{x}|\big) dS_x = 0. \tag{1.118}$$

Applying Green's formula to $w(\mathbf{x})$ and $N(\mathbf{x}, \mathbf{y})$ in Ω we obtain

$$w(\mathbf{y}) = \int_{\partial\Omega} \Big(N(\mathbf{x}, \mathbf{y})\psi(\mathbf{x}) + \frac{1}{2\pi} w(\mathbf{x}) \frac{\partial}{\partial n_x}(\log|\mathbf{x}|)\Big) dS_x.$$

Then the unique solution of (1.116)–(1.118) is given by

$$w(\mathbf{x}) = \int_{\partial\Omega} N(\mathbf{x}, \mathbf{y})\psi(\mathbf{y}) dS_y, \tag{1.119}$$

and

$$\max_{\overline{\Omega}} |w| \le C\|\psi\|_{C(\partial\Omega)}. \tag{1.120}$$

(b) The solution u of (1.110)–(1.112) is sought in the form

$$u(\mathbf{x}) = w(\mathbf{x}) + v(\mathbf{x}), \tag{1.121}$$

where $w = \mathfrak{N}_\Omega \psi$ is defined by (1.119), whereas the second term v satisfies the problem

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.122}$$

$$\frac{\partial v}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{1.123}$$

$$v(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}) - w(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon. \tag{1.124}$$

According to the estimate (1.120) and the maximum principle for variational solutions of (1.122)–(1.124) (see, for example, [1]) we have

$$\max_{\overline{\Omega_\varepsilon}} |v| \le \|\varphi_\varepsilon\|_{C(\partial F_\varepsilon)} + C\|\psi\|_{C(\partial\Omega)}. \tag{1.125}$$

Finally, using the representation (1.121), together with the estimates (1.120) and (1.125), we obtain the result (1.114). This completes the proof. □

1.3.4 Approximation of Green's function $G_\varepsilon^{(D)}$

We give a uniform asymptotic formula for Green's function solving the problem (1.90)-(1.92).

Theorem 2. *Green's function $G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y})$ for problem (1.90)–(1.92) admits the asymptotic representation*

$$G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = \mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + N(\mathbf{x},\mathbf{y}) - (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0)$$
$$+\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + r_\varepsilon(\mathbf{x},\mathbf{y}), \tag{1.126}$$

where $\mathcal{G}, N, R, D$ are defined in (1.12)–(1.14), (1.93)–(1.95), (1.96), (1.101)–(1.103), and

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \le \text{Const }\varepsilon^2,$$

which is uniform with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

Proof. First, we describe the formal argument leading to (1.126). Let $\rho_\varepsilon(\mathbf{x},\mathbf{y}) = G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) - \mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})$. This function satisfies the problem

$$\Delta_x \rho_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \tag{1.127}$$

$$\rho_\varepsilon(\mathbf{x},\mathbf{y}) = 0 \ \text{ when } \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon, \tag{1.128}$$

and

$$\frac{\partial \rho_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = -\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1} - h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})\Big) \tag{1.129}$$
$$= -\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x},\mathbf{y})\Big)$$

$$+\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}|+h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\Big),$$

where $\mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon$. Here $h(\boldsymbol{\xi},\boldsymbol{\eta})$ is the regular part of Green's function $\mathcal{G}$ in $\mathbf{R}^2\setminus F$. Taking into account (1.96), we deduce that

$$\rho_\varepsilon(\mathbf{x},\mathbf{y})=-R(\mathbf{x},\mathbf{y})+R(0,0)+\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y}), \tag{1.130}$$

where $R(\mathbf{x},\mathbf{y})$ is the regular part of the Neumann function $N(\mathbf{x},\mathbf{y})$ in Ω, and $\mathcal{R}_\varepsilon$ is harmonic in Ω_ε and satisfies the boundary conditions

$$\frac{\partial\mathcal{R}_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y})=\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi}\log|\mathbf{x}|+h(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\Big)\ \text{ as }\ \mathbf{x}\in\partial\Omega,\ \mathbf{y}\in\Omega_\varepsilon, \tag{1.131}$$

$$\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y})=\mathbf{x}\cdot\nabla_x R(0,\mathbf{y})+O(\varepsilon^2)\ \text{ as }\ \mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon. \tag{1.132}$$

The asymptotics of $h(\boldsymbol{\xi},\boldsymbol{\eta})$ given by Lemma 4, can be used in evaluation of the right-hand side in (1.131).

The boundary condition (1.132) can be written as

$$\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y})-\varepsilon\mathbf{D}(\boldsymbol{\xi})\cdot\nabla_x R(0,\mathbf{y})=O(\varepsilon^2),$$

for $\mathbf{x}\in\partial F_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon$. In turn, the boundary condition (1.131) is reduced to

$$\frac{\partial}{\partial n_x}\Big\{\mathcal{R}_\varepsilon(\mathbf{x},\mathbf{y})-\varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)\Big\}=O(\varepsilon^2),$$

when $\mathbf{x}\in\partial\Omega,\ \ \mathbf{y}\in\Omega_\varepsilon$. Hence, representation (1.130) of ρ_ε can be updated to the form

$$\begin{aligned}\rho_\varepsilon(\mathbf{x},\mathbf{y})\ =&\ -R(\mathbf{x},\mathbf{y})+R(0,0)\\ &+\ \varepsilon\mathbf{D}(\boldsymbol{\xi})\cdot\nabla_x R(0,\mathbf{y})+\varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)+\mathcal{R}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}),\end{aligned} \tag{1.133}$$

where the principal part of $\mathcal{R}_\varepsilon^{(1)}(\mathbf{x},\mathbf{y})$ compensates for the leading term of the discrepancy $\varepsilon^2\boldsymbol{\xi}\cdot\nabla_x\big(\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)\big)\big|_{\mathbf{x}=0}$ brought by the term $\varepsilon\mathbf{D}(\boldsymbol{\eta})\cdot\nabla_y R(\mathbf{x},0)$ into the boundary condition (1.128) on ∂F_ε. This leads to the required formula (1.126).

For the remainder $r_\varepsilon(\mathbf{x},\mathbf{y})$ in the asymptotic formula (1.126), we verify by the direct substitution that

$$\Delta_x r_\varepsilon(\mathbf{x},\mathbf{y})=0,\quad \mathbf{x},\mathbf{y}\in\Omega_\varepsilon, \tag{1.134}$$

and that the boundary condition (1.91) implies

$$\begin{aligned}r_\varepsilon(\mathbf{x},\mathbf{y})=R(0,\mathbf{y})-R(0,0)+\mathbf{x}\cdot\nabla_x R(0,\mathbf{y})\\ -\varepsilon\mathbf{D}(\mathbf{x}/\varepsilon)\cdot\nabla_x R(0,\mathbf{y})+O(\varepsilon^2)=O(\varepsilon^2)\ \text{as}\ \mathbf{x}\in\partial\omega_\varepsilon,\ \mathbf{y}\in\Omega_\varepsilon,\end{aligned} \tag{1.135}$$

where $\mathbf{D}(\mathbf{x}/\varepsilon) = \varepsilon^{-1}\mathbf{x}$ for $\mathbf{x} \in \omega_\varepsilon$, and formula (1.97) was used to state that $R(0, \mathbf{y})$ is independent of $\mathbf{y}$. In turn, the second boundary condition (1.92), together with formula (1.108), yields

$$\begin{aligned}\frac{\partial r_\varepsilon}{\partial n_x}(\mathbf{x}, \mathbf{y}) &= \frac{\partial}{\partial n_x}\Big(h(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - \frac{1}{2\pi}\log|\mathbf{x}|^{-1}\Big) \\ &\quad -\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\partial}{\partial n_x}\Big(\nabla_y R(\mathbf{x}, 0)\Big) + O(\varepsilon^2) \\ &= -\varepsilon\sum_{j=1}^{2} D_j(\varepsilon^{-1}\mathbf{y})\frac{\partial}{\partial n_x}\Big(\frac{x_j}{2\pi|\mathbf{x}|^2}\Big) \\ &\quad -\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\partial}{\partial n_x}\Big(\nabla_y R(\mathbf{x}, 0)\Big) + O(\varepsilon^2) = O(\varepsilon^2), \qquad (1.136)\end{aligned}$$

as $\mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon$.

It can also be verified that $\int_{\partial\Omega} \frac{\partial}{\partial n_x} r_\varepsilon(\mathbf{x}, \mathbf{y}) dS_x = 0$. Indeed,

$$\begin{aligned}-\int_{\partial\Omega}\frac{\partial}{\partial n_x} r_\varepsilon(\mathbf{x}, \mathbf{y}) dS_x &= \int_{\partial\Omega}\frac{\partial}{\partial n_x}\Big\{\mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + \frac{1}{2\pi}\log\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|} \\ &\quad +\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y})\Big\} dS_x \\ &= \varepsilon\int_{\partial\Omega}\frac{\partial}{\partial n_x}\Big\{\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y\Big((2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1} - N(\mathbf{x}, \mathbf{y})\Big)\Big|_{\mathbf{y}=0}\Big\} dS_x \\ &= \frac{\varepsilon}{2\pi}\int_{\partial\Omega}\frac{\partial}{\partial n_x}\Big\{\mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\mathbf{x}}{|\mathbf{x}|^2}\Big\} dS_x = 0.\end{aligned}$$

Using (1.135), (1.136), together with Lemma 5, we complete the proof. □

1.3.5 Simpler asymptotic representation of Green's function $G_\varepsilon^{(D)}$

Two corollaries, which will be formulated here, follow from Theorem 2. They include simplified asymptotic formulae for the Green's function, which are efficient for the cases when both $\mathbf{x}$ and $\mathbf{y}$ are distant from F_ε or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε.

Corollary 4. *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula* (1.126) *is simplified to the form*

$$\begin{aligned}G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) &= N(\mathbf{x}, \mathbf{y}) - (2\pi)^{-1}\log\varepsilon + \zeta_\infty + R(0, 0) \\ &\quad + (2\pi)^{-1}\log(|\mathbf{x}||\mathbf{y}|) - \frac{\varepsilon}{2\pi}\mathbf{D}^\infty \cdot \Big(\mathbf{x}|\mathbf{x}|^{-2} + \mathbf{y}|\mathbf{y}|^{-2}\Big) \\ &\quad + \varepsilon\mathbf{D}^\infty \cdot \Big(\nabla_x R(0, \mathbf{y}) + \nabla_y R(\mathbf{x}, 0)\Big) \qquad (1.137)\end{aligned}$$

$$+ \; O(\varepsilon^2 |\mathbf{x}|^{-1} |\mathbf{y}|^{-1}),$$

where R is the regular part of Neumann's function N in Ω.

Proof. Estimate (1.108) can be written in the form

$$\begin{aligned} h(\boldsymbol{\xi}, \boldsymbol{\eta}) \;&=\; (2\pi)^{-1} \log(|\boldsymbol{\xi}||\boldsymbol{\eta}|)^{-1} - \zeta_\infty \\ &+\; \frac{\varepsilon}{2\pi} \sum_{j=1}^{2} D_j^\infty \Big(\frac{x_j}{|\mathbf{x}|^2} + \frac{y_j}{|\mathbf{y}|^2} \Big) + O(\varepsilon^2 |\mathbf{x}|^{-1} |\mathbf{y}|^{-1}). \end{aligned} \tag{1.138}$$

Using (1.100), (1.126) and (1.138) we obtain

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) \;&=\; -\frac{1}{2\pi} \log \varepsilon + \frac{1}{2\pi} \log \frac{|\mathbf{x}||\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} + \zeta_\infty \\ &-\; \frac{\varepsilon}{2\pi} \sum_{j=1}^{\infty} D_j^\infty \Big(\frac{x_j}{|\mathbf{x}|^2} + \frac{x_j}{|\mathbf{x}|^2} \Big) + O(\varepsilon^2 |\mathbf{x}|^{-1} |\mathbf{y}|^{-1}) \\ &+\; N(\mathbf{x}, \mathbf{y}) - (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1} + R(0,0) \\ &+\; \varepsilon \mathbf{D}^\infty \cdot \Big(\nabla_y R(\mathbf{x}, 0) + \nabla_x R(0, \mathbf{y}) \Big) \\ &+\; \varepsilon^2 O(|\mathbf{x}|^{-1} + |\mathbf{y}|^{-1}). \end{aligned} \tag{1.139}$$

Rearranging the terms in (1.139) and taking into account that the remainder terms in the above formula are $O(\varepsilon^2 |\mathbf{x}|^{-1} |\mathbf{y}|^{-1})$, we arrive at (1.137). □

Formula (1.137) is efficient when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently distant from F_ε.

The next corollary of Theorem 2 gives the representation of $G_\varepsilon^{(D)}$, which is effective for the case when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε.

Corollary 5. *The following asymptotic formula for Green's function $G_\varepsilon^{(D)}$ of the boundary value problem* (1.90)–(1.92) *holds*

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) \;&=\; \mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - (\mathbf{x} - \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{x})) \cdot \nabla_x R(0, \mathbf{y}) \\ &-\; (\mathbf{y} - \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{x})) \cdot \nabla_y R(\mathbf{x}, 0) \\ &+\; O(|\mathbf{x}|^2 + |\mathbf{y}|^2 + \varepsilon^2), \end{aligned} \tag{1.140}$$

for $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. (The term ε^2 in the remainder can be omitted if the interior of F is nonempty and contains the origin.)

Proof: Using the Taylor expansion of $R(\mathbf{x}, \mathbf{y})$ in a neighbourhood of the origin we reduce the formula (1.126) to the form

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x}, \mathbf{y}) \;&=\; \mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - R(\mathbf{x}, \mathbf{y}) + R(0,0) \\ &+\; \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y}) + O(\varepsilon^2) \\ &=\; \mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ &-\; \mathbf{x} \cdot \nabla_x R(0, \mathbf{y}) - \mathbf{y} \cdot \nabla_y R(\mathbf{x}, 0) + O(|\mathbf{x}|^2 + |\mathbf{y}|^2) \end{aligned} \tag{1.141}$$

$$+ \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + \varepsilon \mathbf{D}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y}) + O(\varepsilon^2).$$

By rearranging the terms in the above formula, we arrive at (1.140). □

1.4 The Neumann function for a planar domain with a small hole or crack

It is noted that in the previous sections, boundary conditions of the Dirichlet type were set at a part of the boundary of Ω_ε. Now, we consider the case when $\partial\Omega_\varepsilon$ is subject to the Neumann boundary conditions. Here, the set F_ε is the same as in Section 1.2.

The *Neumann function* $N_\varepsilon(\mathbf{x}, \mathbf{y})$ for $\Omega_\varepsilon \subset \mathbb{R}^2$ is defined as a solution of the boundary value problem

$$\Delta_x N_\varepsilon(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \Omega_\varepsilon, \tag{1.142}$$

$$\frac{\partial}{\partial n_x}\left(N_\varepsilon(\mathbf{x}, \mathbf{y}) + (2\pi)^{-1} \log|\mathbf{x}|\right) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon, \tag{1.143}$$

$$\frac{\partial N_\varepsilon}{\partial n_x}(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \ \mathbf{y} \in \Omega_\varepsilon. \tag{1.144}$$

In addition, we require the orthogonality condition, which provides the symmetry of $N_\varepsilon(\mathbf{x}, \mathbf{y})$

$$\int_{\partial\Omega} N_\varepsilon(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial n} \log|\mathbf{x}| dS_x = 0. \tag{1.145}$$

The regular part $R_\varepsilon(\mathbf{x}, \mathbf{y})$ of the Neumann function is defined by

$$R_\varepsilon(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log|\mathbf{x} - \mathbf{y}|^{-1} - N_\varepsilon(\mathbf{x}, \mathbf{y}).$$

1.4.1 Special solutions of model problems

As in the previous sections, we consider two limit domains independent of the small parameter ε: the domain Ω (with no hole), and the unbounded domain $\mathbb{R}^2 \setminus F$ that represents scaled exterior of the small hole. As always, the scaled coordinates $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\boldsymbol{\eta} = \varepsilon^{-1}\mathbf{y}$ will be used.

The Neumann function $N(\mathbf{x}, \mathbf{y})$ of Ω is defined by (1.93)–(1.95), and the regular part $R(\mathbf{x}, \mathbf{y})$ of $N(\mathbf{x}, \mathbf{y})$ is the same as in (1.96).

We shall use the vector function $\boldsymbol{\mathcal{D}}$ already defined in Section 1.2.

Another model field to be used is the Neumann function $\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta})$ in $\mathbb{R}^2 \setminus F$, as in (1.16), whose regular part h_N satisfies the problem (1.17)–(1.19).

1.4.2 Maximum modulus estimate for solutions to the Neumann problem in Ω_ε

First, we formulate and prove the auxiliary Lemma required for the forthcoming estimate of the remainder term in the approximation of N_ε.

Lemma 6. *Let u be a function in $C(\overline{\Omega}_\varepsilon)$ such that ∇u is square integrable in a neighbourhood $\partial\Omega_\varepsilon$. Also, let u be a solution of the Neumann boundary value problem*

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.146}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{1.147}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{1.148}$$

where $\psi \in C(\partial\Omega)$, $\varphi_\varepsilon \in L_\infty(\partial F_\varepsilon)$, and

$$\int_{\partial F_\varepsilon} \varphi_\varepsilon(\mathbf{x}) ds = 0 \quad \text{and} \quad \int_{\partial\Omega} \psi(\mathbf{x}) ds = 0. \tag{1.149}$$

We also assume that

$$\Big| \int_{\partial\Omega} u(\mathbf{x}) \frac{\partial}{\partial n}\big(\log|\mathbf{x}|\big) ds \Big| \le \text{const}\ \{\|\psi\|_{C(\partial\Omega)} + \varepsilon\|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}\}. \tag{1.150}$$

Then there exists a positive constant C, independent of ε and such that

$$\|u\|_{C(\Omega_\varepsilon)} \le C\big\{\|\psi\|_{C(\partial\Omega)} + \varepsilon\|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}\big\}. \tag{1.151}$$

Proof. (a) We use the operators $\mathfrak{N}$ and $\mathfrak{N}_\Omega$ of model problems (1.61)–(1.63) and (1.116)–(1.118) introduced in Sections 1.2 and 1.3.

(b) We begin with the case of the homogeneous boundary condition on $\partial\Omega$, i.e.

$$\Delta u_1(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon, \tag{1.152}$$

$$\frac{\partial u_1}{\partial n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{1.153}$$

$$\frac{\partial u_1}{\partial n}(\mathbf{x}) = \varphi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon, \tag{1.154}$$

where the right-hand side φ_ε is such that

$$\int_{\partial F_\varepsilon} \varphi_\varepsilon(\mathbf{x}) ds = 0.$$

The operator $\mathfrak{N}_\varepsilon$ is defined as in (1.65), so that

$$(\mathfrak{N}_\varepsilon \varphi_\varepsilon)(\mathbf{x}) = (\mathfrak{N}\varphi)(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} = \varepsilon^{-1}\mathbf{x}$ and $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-1}\varphi(\varepsilon^{-1}\mathbf{x})$.

The solution u_1 is sought in the form

$$u_1 = \mathfrak{N}_\varepsilon g_\varepsilon - \mathfrak{N}_\Omega\Big(\frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon g_\varepsilon)_{\partial\Omega}\Big), \tag{1.155}$$

where g_ε is an unknown function such that

$$\int_{\partial F} g(\boldsymbol{\xi}) ds_\xi = 0.$$

By Lemma 1, we have

$$|\mathfrak{N}g(\boldsymbol{\xi})| \le C\varepsilon \|g\|_{L_\infty(\partial F)}, \tag{1.156}$$

and

$$\max_{\overline{\Omega}_\varepsilon} |\mathfrak{N}_\varepsilon g_\varepsilon| \le C\varepsilon \|g_\varepsilon\|_{L_\infty(\partial F)}. \tag{1.157}$$

It follows from (1.155) that $\frac{\partial}{\partial n}u_1(\mathbf{x}) = 0$ when $\mathbf{x} \in \partial\Omega$, and on the boundary ∂F_ε we have

$$\varphi_\varepsilon = g_\varepsilon + S_\varepsilon g_\varepsilon, \tag{1.158}$$

where

$$S_\varepsilon g_\varepsilon = -\frac{\partial}{\partial n}\Big(\mathfrak{N}_\Omega\big(\frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon g_\varepsilon)_{\partial\Omega}\big)\Big) \quad \text{on} \quad \partial F_\varepsilon. \tag{1.159}$$

Taking into account Lemma 1 and the definitions of $\mathfrak{N}_\Omega$ and $\mathfrak{N}_\varepsilon$, as in (1.115) and (1.60), (1.65), we deduce that

$$\max_{\partial\Omega} |\nabla(\mathfrak{N}_\varepsilon g_\varepsilon)| \le \text{const } \varepsilon^2 \|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)},$$

and

$$\|S_\varepsilon g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \le \text{const } \ \varepsilon^2 \|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}.$$

Owing to the smallness of the norm of the operator S_ε we can write

$$\|g_\varepsilon\|_{L_\infty(\partial F_\varepsilon)} \le \text{const } \ \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}.$$

Following (1.119), (1.120), (1.155) and (1.157) we deduce (1.150) and

$$\max_{\overline{\Omega}_\varepsilon} |u_1| \le \text{const } \ \varepsilon \|\varphi_\varepsilon\|_{L_\infty(\partial F_\varepsilon)}. \tag{1.160}$$

(c) Next, we consider the problem (1.146)–(1.149) with the homogeneous data on $\partial\omega_\varepsilon$. The corresponding solution u_2 is written in the form

$$u_2 = \mathfrak{N}_\Omega \psi + v, \tag{1.161}$$

where the harmonic function v satisfies zero boundary condition on $\partial\Omega$, whereas the condition (1.154) is replaced by

$$\frac{\partial}{\partial n} v(\mathbf{x}) = -\frac{\partial}{\partial n}\big(\mathfrak{N}_\Omega \psi\big)(\mathbf{x}), \quad \mathbf{x} \in \partial F_\varepsilon,$$

and by part (b)

$$\max_{\overline{\Omega}_\varepsilon} |v| \le \text{const} \quad \|\psi\|_{C(\partial\Omega)}.$$

The function v and hence u_2 satisfy (1.150).

Following (1.119), (1.120) and (1.161) we deduce

$$\max_{\overline{\Omega}_\varepsilon} |u_2| \le \text{const} \quad \|\psi\|_{C(\partial\Omega)}. \tag{1.162}$$

Combining estimates (1.160) and (1.162) we complete the proof. □

1.4.3 Asymptotic approximation of N_ε

Now we state the theorem, which gives a uniform asymptotic formula for the Neumann function N_ε.

Theorem 3. *The Neumann function $N_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\eta})$ of the domain Ω_ε defined in* (1.142)–(1.145) *satisfies*

$$\begin{aligned} N_\varepsilon(\mathbf{x}, \mathbf{y}) = \;& N(\mathbf{x}, \mathbf{y}) - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\ & + \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y}) \\ & + \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x}, 0) + r_\varepsilon(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{1.163}$$

where

$$|r_\varepsilon(\mathbf{x}, \mathbf{y})| \le Const\; \varepsilon^2 \tag{1.164}$$

uniformly with respect to $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$.

Proof. We begin with a formal argument leading to the approximation (1.163). Consider the first three terms in the right-hand side of (1.163) and let

$$r_\varepsilon^{(1)}(\mathbf{x}, \mathbf{y}) = N_\varepsilon(\mathbf{x}, \mathbf{y}) - N(\mathbf{x}, \mathbf{y}) + h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) - \varepsilon \boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) \cdot \nabla_x R(0, \mathbf{y}). \tag{1.165}$$

The function $r_\varepsilon^{(1)}$ is harmonic in Ω_ε, and the direct substitution into the boundary conditions (1.143) and (1.144) gives

$$\begin{aligned} \frac{\partial r_\varepsilon^{(1)}}{\partial n_x}(\mathbf{x}, \mathbf{y}) \;=\; & -\frac{\partial}{\partial n_x}\Big(\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{y}|^{-1}\Big) + \frac{\partial}{\partial n_x}\big(h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y})\big) \\ & + \mathbf{n} \cdot \nabla_x R(0, \mathbf{y}) - \varepsilon \frac{\partial}{\partial n_x} \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x R(0, \mathbf{y}) + O(\varepsilon) \end{aligned}$$

$$= O(\varepsilon), \quad \text{for } \mathbf{x} \in \partial F_\varepsilon,\ \mathbf{y} \in \Omega_\varepsilon, \tag{1.166}$$

and

$$\begin{aligned}\frac{\partial r_\varepsilon^{(1)}}{\partial n_x}(\mathbf{x},\mathbf{y}) \;=\; & \frac{\partial}{\partial n_x}\big(h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})\big) + O(\varepsilon^2)\\ &= \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \frac{\partial}{\partial n_x}\nabla_y R(\mathbf{x},0) + O(\varepsilon^2),\\ &\text{for } \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega_\varepsilon. \end{aligned} \tag{1.167}$$

Thus, $r_\varepsilon^{(1)}$ can be approximated as

$$r_\varepsilon^{(1)}(\mathbf{x},\mathbf{y}) = \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y R(\mathbf{x},0) + O(\varepsilon^2),$$

and together with the representation (1.165), this leads to the required formula (1.163).

Finally, the direct substitution of (1.163) into (1.142)–(1.144) yields that the remainder term $r_\varepsilon(\mathbf{x},\mathbf{y})$ satisfies the problem (1.146)–(1.149), with

$$\max_{\mathbf{x}\in\partial\Omega} |\psi(\mathbf{x},\mathbf{y})| \le \text{Const}\ \varepsilon^2$$

and

$$\max_{\mathbf{x}\in\partial F_\varepsilon} |\varphi_\varepsilon(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y})| \le \text{Const}\ \varepsilon$$

for all $\mathbf{y} \in \Omega_\varepsilon$. Then the estimate (1.164) follows from Lemma 6. □

1.4.4 Simpler asymptotic representation of Neumann's function N_ε

Two corollaries, formulated in this section, follow from Theorem 3. They include asymptotic formulae for the Neumann's function, which are efficient when either both $\mathbf{x}$ and $\mathbf{y}$ are distant from F_ε or both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε.

Corollary 6. *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then*

$$\begin{aligned} N_\varepsilon(\mathbf{x},\mathbf{y}) \;=\; & N(\mathbf{x},\mathbf{y}) - \frac{\varepsilon^2}{4\pi^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}^T}{|\mathbf{y}|^2}\\ &+ \frac{\varepsilon^2}{2\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^2}\boldsymbol{\mathcal{P}}\nabla_x R(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^2}\boldsymbol{\mathcal{P}}\nabla_y R(\mathbf{x},0)\Big\}\\ &+ \varepsilon^2 O(|\mathbf{x}|^{-2} + |\mathbf{y}|^{-2}), \end{aligned} \tag{1.168}$$

where R is the regular part of Neumann's function N in Ω, and $\boldsymbol{\mathcal{P}}$ is the dipole matrix for F, as defined in (1.23).

Proof. The proof is similar to that of Corollary 2, and it uses formula (1.54) for the regular part h_N of the Neumann function in $\mathbb{R}^2 \setminus F$, together with the asymptotic representation (1.23) of the dipole fields $\mathcal{D}_j$ in $\mathbb{R}^2 \setminus F$. □

Next, we state a proposition similar to Corollaries 3 and 5 formulated earlier for Green's functions $G_\varepsilon^{(D)}$ and $G_\varepsilon^{(N)}$.

Corollary 7. *Neumann's function N_ε, defined by* (1.142)–(1.145)*, satisfies the asymptotic formula*

$$
\begin{aligned}
N_\varepsilon(\mathbf{x}, \mathbf{y}) = &\ (2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|^{-1} - R(0,0) - h_N(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) \\
&- \Big(\mathbf{x} - \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\Big) \cdot \nabla_x R(0, \mathbf{y}) - \Big(\mathbf{y} - \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\Big) \cdot \nabla_y R(\mathbf{x}, 0) \\
&+ O(|\mathbf{x}|^2 + |\mathbf{y}|^2 + \varepsilon^2),
\end{aligned}
\tag{1.169}
$$

for $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. (As in Corollaries 2 *and* 4*, ε^2 in the remainder can be omitted if the interior of F is nonempty and contains the origin.)*

Proof. The proof is similar to that of Corollary 3, and it employs the linear approximation of the regular part R of Neumann's function in a neighbourhood of the origin. □

Although, the formulation of Corollary 7 is valid for all $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$, the asymptotic formula (1.169) becomes effective when both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε.

1.5 Asymptotic approximations of Green's kernels for mixed and Neumann's problems in three dimensions

This section includes asymptotic formulae for Green's kernels $G_\varepsilon^{(D)}, G_\varepsilon^{(N)}$ and N_ε in $\Omega_\varepsilon \subset \mathbb{R}^3$. The special solutions of model problems differ from the corresponding solutions used for the two-dimensional case. The uniform asymptotic formulae of Green's kernels are accompanied by simpler representations, which are efficient when certain constraints are imposed on the independent variables. The proofs, which do not require new ideas compared with the two-dimensional case, are omitted.

1.5.1 Special solutions of model problems in limit domains

Here, we describe the functions $G, \mathcal{G}, N, \mathcal{N}$, defined in the limit domains and used for the approximation of Green's kernels.

1. The notation G is used for Green's function of the Dirichlet problem in $\Omega \subset \mathbb{R}^3$:

$$G(\mathbf{x},\mathbf{y}) = (4\pi|\mathbf{x}-\mathbf{y}|)^{-1} - H(\mathbf{x},\mathbf{y}). \tag{1.170}$$

Here H is the regular part of G, and it is a unique solution of the Dirichlet problem

$$\Delta_x H(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega, \tag{1.171}$$

$$H(\mathbf{x},\mathbf{y}) = (4\pi|\mathbf{x}-\mathbf{y}|)^{-1}, \quad \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega. \tag{1.172}$$

2. Green's function $\mathcal{G}$ for the Dirichlet problem in $\mathbb{R}^3 \setminus F$ is defined as a unique solution of the problem

$$\Delta_\xi \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) + \delta(\boldsymbol{\xi}-\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{1.173}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\xi} \in \partial F,\ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{1.174}$$

$$\mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}) \to 0 \ \text{ as } |\boldsymbol{\xi}| \to \infty \text{ and } \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F. \tag{1.175}$$

Here F is a compact set of positive harmonic capacity.

The regular part h of Green's function $\mathcal{G}$ is

$$h(\boldsymbol{\xi},\boldsymbol{\eta}) = (4\pi|\boldsymbol{\xi}-\boldsymbol{\eta}|)^{-1} - \mathcal{G}(\boldsymbol{\xi},\boldsymbol{\eta}). \tag{1.176}$$

3. The components of the vector field $\mathbf{D}(\boldsymbol{\xi}) = (D_1(\boldsymbol{\xi}), D_2(\boldsymbol{\xi}), D_3(\boldsymbol{\xi}))$ (compare with (1.101)–(1.103)), for $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus F$, satisfy the problem

$$\Delta D_j(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^3 \setminus F, \tag{1.177}$$

$$D_j(\boldsymbol{\xi}) = \xi_j, \quad \boldsymbol{\xi} \in \partial F, \tag{1.178}$$

$$D_j(\boldsymbol{\xi}) \to 0, \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty. \tag{1.179}$$

We shall use the matrix $\boldsymbol{\mathcal{T}} = (\mathcal{T}_{jk})_{j,k=1}^3$ of coefficients in the asymptotic representation of D_j at infinity

$$D_j(\boldsymbol{\xi}) = \frac{1}{4\pi}\sum_{k=1}^{3} \frac{\mathcal{T}_{jk}\xi_k}{|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}). \tag{1.180}$$

The symmetry of $\boldsymbol{\mathcal{T}}$ is verified by applying Green's formula in $B_R \setminus F$ to $\xi_j - D_j(\boldsymbol{\xi})$ and $D_k(\boldsymbol{\xi})$ and taking the limit $R \to \infty$. We have

$$\int_{\partial B_R} \Big\{(\xi_j - D_j(\boldsymbol{\xi}))\frac{\partial D_k(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi})\Big(\frac{\xi_j}{|\boldsymbol{\xi}|} - \frac{\partial D_j(\boldsymbol{\xi})}{\partial|\boldsymbol{\xi}|}\Big)\Big\} dS$$
$$+ \int_{\partial F} D_k(\boldsymbol{\xi})\Big(\frac{\partial D_j(\boldsymbol{\xi})}{\partial n} - n_j\Big) dS = 0, \tag{1.181}$$

where $\partial/\partial n$ is the normal derivative in the direction of the interior normal with respect to F. As $R \to \infty$, the first integral $\mathcal{I}(\partial B_R)$ in the left-hand side of (1.181) gives

$$\lim_{R\to\infty} \mathcal{I}(\partial B_R) = \lim_{R\to\infty} \int_{\partial B_R} \Big\{ \xi_j \frac{\partial D_k(\boldsymbol{\xi})}{\partial |\boldsymbol{\xi}|} - D_k(\boldsymbol{\xi}) \frac{\xi_j}{|\boldsymbol{\xi}|} \Big\} dS$$

$$= -\frac{3}{4\pi} \int_{\partial B_1} \sum_{q=1} \mathcal{T}_{kq} \xi_q \xi_j dS = -\mathcal{T}_{kj}. \tag{1.182}$$

The second integral $\mathcal{I}(\partial F)$ in the left-hand side of (1.181) becomes

$$\mathcal{I}(\partial F) = -\int_{\partial F} \xi_k n_j dS + \int_{\partial F} D_k(\boldsymbol{\xi}) \frac{\partial D_j(\boldsymbol{\xi})}{\partial n} dS$$

$$= \delta_{jk} \ \mathrm{meas}_3(F) + \int_{\mathbb{R}^3 \setminus F} \nabla D_k(\boldsymbol{\xi}) \cdot \nabla D_j(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{1.183}$$

where $\mathrm{meas}_3(F)$ is the three-dimensional Lebesgue measure of F. Using (1.182) and (1.183) we deduce

$$\mathcal{T}_{kj} = \delta_{jk} \ \mathrm{meas}_3(F) + \int_{\mathbb{R}^3 \setminus F} \nabla D_k(\boldsymbol{\xi}) \cdot \nabla D_j(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{1.184}$$

which implies that $\boldsymbol{\mathcal{T}}$ is *symmetric and positive definite.*

4. The Neumann function $N(\mathbf{x}, \mathbf{y})$ in $\Omega \subset \mathbb{R}^3$ and its regular part are defined as follows

$$\Delta N(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \ \ \mathbf{x}, \mathbf{y} \in \Omega \subset \mathbf{R}^3, \tag{1.185}$$

$$\frac{\partial}{\partial n_x} \Big(N(\mathbf{x}, \mathbf{y}) - (4\pi)^{-1} |\mathbf{x}|^{-1} \Big) = 0, \ \ \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega, \tag{1.186}$$

and

$$\int_{\partial\Omega} N(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial n_x} |\mathbf{x}|^{-1} ds_x = 0, \tag{1.187}$$

where the last condition (1.95) implies the symmetry of $N(\mathbf{x}, \mathbf{y})$. The regular part of the Neumann function in three dimensions is defined by

$$R(\mathbf{x}, \mathbf{y}) = (4\pi)^{-1} |\mathbf{x} - \mathbf{y}|^{-1} - N(\mathbf{x}, \mathbf{y}). \tag{1.188}$$

5. In this section, the notation $\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta})$ will be used for the Neumann function in $\mathbb{R}^3 \setminus F$, where F is a compact closure of a domain with a smooth boundary, and $\mathcal{N}$ is defined by

$$\mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\eta}) = (4\pi)^{-1} |\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} - h_N(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{1.189}$$

where h_N is the regular part of $\mathcal{N}$ subject to

$$\Delta_\xi h_N(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0, \ \ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{1.190}$$

$$\frac{\partial h_N}{\partial n_\xi}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{4\pi} \frac{\partial}{\partial n_\xi} (|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}), \ \ \boldsymbol{\xi} \in \partial F, \ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F, \tag{1.191}$$

$$h_N(\boldsymbol{\xi},\boldsymbol{\eta}) \to 0, \quad \text{as } |\boldsymbol{\xi}| \to \infty, \ \boldsymbol{\eta} \in \mathbb{R}^3 \setminus F. \tag{1.192}$$

The smoothness assumption on ∂F here and in the sequel is introduced for the simplicity of proofs and can be considerably weakened. In particular, the case of a piece-wise smooth planar crack can be included.

We note that the Neumann function $\mathcal{N}$ just defined is symmetric, i.e. $\mathcal{N}(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathcal{N}(\boldsymbol{\eta},\boldsymbol{\xi})$.

6. The definition of the dipole vector field $\boldsymbol{\mathcal{D}}(\boldsymbol{\xi}) = (\mathcal{D}_1(\boldsymbol{\xi}), \mathcal{D}_2(\boldsymbol{\xi}), \mathcal{D}_3(\boldsymbol{\xi}))$ is similar to (1.20)–(1.22), with $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus F$. The components of the three-dimensional dipole matrix $\boldsymbol{\mathcal{P}} = (\mathcal{P}_{jk})_{j,k=1}^3$ appear in the asymptotic representation of $\mathcal{D}_j(\boldsymbol{\xi})$ at infinity

$$\mathcal{D}_j(\boldsymbol{\xi}) = \frac{1}{4\pi} \sum_{k=1}^{3} \frac{\mathcal{P}_{jk}\xi_k}{|\boldsymbol{\xi}|^3} + O(|\boldsymbol{\xi}|^{-3}). \tag{1.193}$$

Similar to Section 1.2.2, it can be proved the *the dipole matrix $\boldsymbol{\mathcal{P}}$ for the hole F is symmetric and negative definite.*

1.5.2 Approximations of Green's kernels

The following assertions hold for uniform asymptotic approximations in three-dimensional domains with small holes (or cracks) or inclusions.

Theorem 4. *Green's function $G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y})$ for the mixed problem with the Neumann data on ∂F_ε and the Dirichlet data on $\partial\Omega$, has the asymptotic representation*

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) + \varepsilon^{-1}\mathcal{N}(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) - (4\pi)^{-1}|\mathbf{x}-\mathbf{y}|^{-1}$$
$$+\varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_x H(0,\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_y H(\mathbf{x},0) + r_\varepsilon(\mathbf{x},\mathbf{y}), \tag{1.194}$$

where $\boldsymbol{\mathcal{D}}$ is the three-dimensional dipole vector function in $\mathbb{R}^3 \setminus F$, and $\mathcal{N}$ is the Neumann function in $\mathbb{R}^3 \setminus F$, vanishing at infinity. Here

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \le \text{Const } \varepsilon^2 \tag{1.195}$$

uniformly with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

The proof follows the same algorithm as in Theorem 1.

Now we give the analogues of Corollaries 2 and 3 formulated earlier in Section 1.2.7.

Corollary 8. *Let* $\min\{|\mathbf{x}|,|\mathbf{y}|\} > 2\varepsilon$. *Then the asymptotic formula* (1.194) *is simplified to the form*

$$G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y})$$

$$
\begin{aligned}
&+ \frac{\varepsilon^3}{4\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\nabla_x H(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3}\boldsymbol{\mathcal{P}}\nabla_y H(\mathbf{x},0)\Big\} \\
&- \frac{\varepsilon^3}{(4\pi)^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{P}}\frac{\mathbf{y}}{|\mathbf{y}|^3} \\
&+ O(\varepsilon^2 + \varepsilon^4(|\mathbf{x}| + |\mathbf{y}|)|\mathbf{x}|^{-3}|\mathbf{y}|^{-3}), \qquad (1.196)
\end{aligned}
$$

where H is the regular part of Green's function G in Ω, and $\boldsymbol{\mathcal{P}}$ is the dipole matrix for F, as defined in (1.193).

The next assertion is similar to Corollary 3 of Section 1.2.7.

Corollary 9. *The following asymptotic formula for Green's function $G_\varepsilon^{(N)}$ holds*

$$
\begin{aligned}
G_\varepsilon^{(N)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}\mathcal{N}\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) - H(0,0) \\
&- (\mathbf{x} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x H(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y H(\mathbf{x},0) \\
&+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2), \qquad (1.197)
\end{aligned}
$$

for $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (As in Corollary 3, ε^2 in the remainder can be omitted if the interior of F is nonempty and contains the origin.)

In turn, for the case when the Neumann and Dirichlet boundary conditions are set on $\partial\Omega$ and ∂F_ε, respectively, the modified version of formula (1.126) is given by

Theorem 5. *The Green's function $G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y})$ for the mixed problem with the Dirichlet data on ∂F_ε and the Neumann data on $\partial\Omega$, admits the asymptotic representation*

$$
G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) = \varepsilon^{-1}\mathcal{G}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + N(\mathbf{x},\mathbf{y}) - (4\pi)^{-1}|\mathbf{x}-\mathbf{y}|^{-1} + R(0,0)
$$

$$
+\varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) + r_\varepsilon(\mathbf{x},\mathbf{y}), \qquad (1.198)
$$

where

$$
|r_\varepsilon(\mathbf{x},\mathbf{y})| \le \text{Const } \varepsilon^2,
$$

which is uniform with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$.

The proof is similar to that of Theorem 2. We note that unlike the two-dimensional case, in three dimensions no orthogonality condition is required to ensure the decay of the solution of the exterior Dirichlet problem in $\mathbb{R}^3 \setminus F$.

The analogues of Corollaries 4 and 5 are formulated as follows.

Corollary 10. *Let $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. Then the asymptotic formula* (1.198) *is simplified to the form*

$$
\begin{aligned}
G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) &= N(\mathbf{x},\mathbf{y}) + R(0,0) \\
&+ \frac{\varepsilon^3}{4\pi}\Big\{\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{T}}\nabla_x R(0,\mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3}\boldsymbol{\mathcal{T}}\nabla_y R(\mathbf{x},0)\Big\} \\
&- \frac{\varepsilon^3}{(4\pi)^2}\frac{\mathbf{x}^T}{|\mathbf{x}|^3}\boldsymbol{\mathcal{T}}\frac{\mathbf{y}}{|\mathbf{y}|^3}
\end{aligned}
$$

$$+ \, O(\varepsilon^2 + \varepsilon^4(|\mathbf{x}| + |\mathbf{y}|)|\mathbf{x}|^{-3}|\mathbf{y}|^{-3}), \tag{1.199}$$

where R is the regular part of Neumann's function N in Ω, and $\boldsymbol{\mathcal{T}}$ is the matrix of coefficients in (1.180).

The next assertion is similar to Corollary 5 of Section 1.3.5.

Corollary 11. *The following asymptotic formula for Green's function $G_\varepsilon^{(D)}$ holds*

$$\begin{aligned} G_\varepsilon^{(D)}(\mathbf{x},\mathbf{y}) &= \varepsilon^{-1}\mathcal{G}\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) \\ &- (\mathbf{x} - \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{x}))\cdot\nabla_x R(0,\mathbf{y}) - (\mathbf{y} - \varepsilon\mathbf{D}(\varepsilon^{-1}\mathbf{y}))\cdot\nabla_y R(\mathbf{x},0) \\ &+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2), \end{aligned} \tag{1.200}$$

for $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. (The term ε^2 in the remainder can be omitted if the interior of F is nonempty and contains the origin.)

Finally, we consider the *Neumann function* $N_\varepsilon(\mathbf{x},\mathbf{y})$ for $\Omega_\varepsilon \subset \mathbb{R}^3$. Here, $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$, and F_ε is the small hole with a smooth boundary. We define N_ε as a solution of the following boundary value problem

$$\Delta_x N_\varepsilon(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{x},\mathbf{y} \in \Omega_\varepsilon, \tag{1.201}$$

$$\frac{\partial}{\partial n_x}\left(N_\varepsilon(\mathbf{x},\mathbf{y}) - (4\pi)^{-1}|\mathbf{x}|^{-1}\right) = 0, \quad \mathbf{x} \in \partial\Omega, \ \mathbf{y} \in \Omega_\varepsilon, \tag{1.202}$$

$$\frac{\partial N_\varepsilon}{\partial n_x}(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \partial F_\varepsilon, \ \mathbf{y} \in \Omega_\varepsilon. \tag{1.203}$$

In addition, we require the orthogonality condition, which provides the symmetry of $N_\varepsilon(\mathbf{x},\mathbf{y})$

$$\int_{\partial\Omega} N_\varepsilon(\mathbf{x},\mathbf{y})\frac{\partial}{\partial n}|\mathbf{x}|^{-1} dS_x = 0. \tag{1.204}$$

The asymptotic approximation of N_ε is given by

Theorem 6. *The Neumann function $N_\varepsilon(\boldsymbol{\xi},\boldsymbol{\eta})$ for the domain Ω_ε, defined in* (1.201)–(1.204) *satisfies the asymptotic formula*

$$\begin{aligned} N_\varepsilon(\mathbf{x},\mathbf{y}) = \ & N(\mathbf{x},\mathbf{y}) - \varepsilon^{-1}h_N(\varepsilon^{-1}\mathbf{x},\varepsilon^{-1}\mathbf{y}) + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{x})\cdot\nabla_x R(0,\mathbf{y}) \\ & + \varepsilon\boldsymbol{\mathcal{D}}(\varepsilon^{-1}\mathbf{y})\cdot\nabla_y R(\mathbf{x},0) + r_\varepsilon(\mathbf{x},\mathbf{y}), \end{aligned} \tag{1.205}$$

where

$$|r_\varepsilon(\mathbf{x},\mathbf{y})| \le Const\ \varepsilon^2 \tag{1.206}$$

uniformly with respect to $\mathbf{x},\mathbf{y} \in \Omega_\varepsilon$. Here $\boldsymbol{\mathcal{D}}$ is the three-dimensional dipole vector function in $\mathbb{R}^3 \setminus F$, and h_N is the regular part of the Neumann function $\mathcal{N}$ in $\mathbb{R}^3 \setminus F$, vanishing at infinity. The Neumann function N in Ω and its regular part R are the same as in (1.185)–(1.188).

The proof follows the same algorithm as in Theorem 3.

At last, we formulate the analogues of Corollaries 6 and 7 for the Neumann problem in Ω_ε.

Corollary 12. *Let* $\min\{|\mathbf{x}|, |\mathbf{y}|\} > 2\varepsilon$. *Then* $N_\varepsilon(\mathbf{x}, \mathbf{y})$ *is approximated in the form*

$$\begin{aligned} N_\varepsilon(\mathbf{x}, \mathbf{y}) &= N(\mathbf{x}, \mathbf{y}) - \frac{\varepsilon^3}{(4\pi)^2} \frac{\mathbf{x}^T}{|\mathbf{x}|^3} \boldsymbol{\mathcal{P}} \frac{\mathbf{y}^T}{|\mathbf{y}|^3} \\ &+ \frac{\varepsilon^3}{4\pi} \Big\{ \frac{\mathbf{x}^T}{|\mathbf{x}|^3} \boldsymbol{\mathcal{P}} \nabla_x R(0, \mathbf{y}) + \frac{\mathbf{y}^T}{|\mathbf{y}|^3} \boldsymbol{\mathcal{P}} \nabla_y R(\mathbf{x}, 0) \Big\} \qquad (1.207) \\ &+ O(\varepsilon^2 + \varepsilon^4 (|\mathbf{x}| + |\mathbf{y}|) |\mathbf{x}|^{-3} |\mathbf{y}|^{-3}), \end{aligned}$$

where R *is the regular part of Neumann's function in* Ω, *and* $\boldsymbol{\mathcal{P}}$ *is the dipole matrix for* F, *as defined in* (1.193).

When both $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close to F_ε the asymptotic approximation of N_ε is given in the next assertion.

Corollary 13. *Neumann's function* N_ε *satisfies the asymptotic formula*

$$\begin{aligned} N_\varepsilon(\mathbf{x}, \mathbf{y}) &= \varepsilon^{-1} \mathcal{N} \varepsilon^{-1} \mathbf{x}, \varepsilon^{-1} \mathbf{y}) - R(0, 0) \\ &- (\mathbf{x} - \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1} \mathbf{x})) \cdot \nabla_x R(0, \mathbf{y}) - (\mathbf{y} - \varepsilon \boldsymbol{\mathcal{D}}(\varepsilon^{-1} \mathbf{y})) \cdot \nabla_y R(\mathbf{x}, 0) \\ &+ O(\varepsilon^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2), \qquad (1.208) \end{aligned}$$

for $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$. *The term* ε^2 *in the remainder can be omitted if the interior of* F *is nonempty and contains the origin.*

References

1. D.Gilbarg, N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, 1983, Springer, Berlin-Heidelberg-New York.
2. J.Hadamard, Sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Mémoire couronné en 1907 par l'Académie des Sciences* **33**, No 4, 515-629.
3. N.S.Landkof, Foundations of Modern Potential Theory, 1972, Spinger, Berlin-Heidelberg-New York.
4. V.Maz'ya, Sobolev Spaces, 1985, Spinger, Berlin-Heidelberg-New York.
5. V.Maz'ya, A.Movchan, Uniform asymptotic formulae for Green's kernels in regularly and singularly perturbed domains. *C.R.Acad.Sci. Paris, Ser. Mathematique*, 343(2006), 185-190.
6. V.Maz'ya, A.Movchan, Uniform asymptotic formulae for Green's functions in singularly perturbed domains. *J. Comp. Appl. Math.*, 228 (2007), 194-206.
7. V.Maz'ya, B.Plamenevskii, Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. *In: Elliptic boundary value problems by V.Maz'ya et al. AMS Translations, Series* 2, v. 123, 1-56.
8. G.Pólya, G.Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton, 1951.

Affiliations of the authors:

V. Maz'ya:
Department of Mathematical Sciences, University of Liverpool,
Liverpool L69 3BX, U.K., and
Department of Mathematics,
Ohio State University, 231 W 18th Avenue, Columbus,
OH 43210, USA, and
Department of Mathematics,
Linköping University, SE-581 83 Linköping, Sweden

A. Movchan:
Department of Mathematical Sciences, University of Liverpool,
Liverpool L69 3BX, U.K.