Essential norms and localization moduli of Sobolev embeddings for general domains

J. Lang V. Maz'ya

1 Introduction

Starting with the classical Rellich and Sobolev-Kondrashov theorems (see [18], [9]), it became general knowledge that embedding operators of Sobolev spaces can be either compact or non-compact depending on the smoothness and integrability parameters as well as on properties of the boundary of a domain. In the early 1960s several necessary and sufficient conditions for the boundedness and compactness of Sobolev type embeddings were obtained in [11]–[13], see also [14]. In particular, properties of the embedding operator:

$$E_{p,q}(\Omega) : L_p^1(\Omega) \to L_q(\Omega)$$

were characterized in terms of *relative isoperimetric and isocapacitary inequalities*. Here and in what follows, $L_p^1(\Omega)$ is the space of distributions in the connected open set Ω in $\mathbf{R}^n$, $n > 1$, whose derivatives of the first order belong to $L_p(\Omega)$, $1 \leq p < \infty$.

In the present paper we study non-compact embeddings $E_{p,q}$, where $p \leq q$ (in the opposite case $p > q$, the boundedness of $E_{p,q}$ implies compactness (see [15, Section 8.6]), which makes this case of no interest for us). Various characteristics of non-compact embeddings such as essential norms, limits of the approximation numbers, certain measures of non-compactness, and the constants in the Poincaré type inequalities, were investigated by Amick [2], Edmunds and Evans [4], Evans and Harris [7] (see [5] and [6] for a detailed account of this development), and Yerzakova [19], [20]. Here we define new measures of non-compactness of $E_{p,q}$ and characterize their mutual relations as well as their relations with "local" isoperimetric and isocapacitary constants. In order to describe our results we need to introduce some notation which will be frequently used in the paper.

We supply $L_p^1(\Omega)$ with the norm:

$$\|u\|_{L_p^1(\Omega)} = \|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)},$$

where ∇ stands for the gradient and ω is a non-empty open set with compact closure $\overline{\omega} \subset \Omega$. It is standard that a change of ω leads to an equivalent norm in $L_p^1(\Omega)$. *We often omit Ω in notations of spaces and norms if it causes no ambiguity.*

Among other things, we study the essential norm of the embedding operator $E_{p,q} : L_p^1 \to L_q$, i.e. the number

$$\text{ess}\|E_{p,q}\| = \inf \|E_{p,q} - T\|_{L_p^1 \to L_q},$$

where the infimum is taken over all compact operators $T : L_p^1 \to L_q$.

Another characteristic of $E_{p,q}$ to be dealt with later in this paper is defined by

$$\mathcal{C}(E_{p,q}) = \inf C,$$

where C is a positive constant such that there exist $\rho > 0$ and $K > 0$ subject to the inequality

$$\|u\|_{L_q(\Omega)} \leq C\|u\|_{L_p^1(\Omega)} + K\|u\|_{L_1(\Omega_\rho)} \quad \text{for all } u \in L_p^1\ , \tag{1.1}$$

with

$$\Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \rho\}.$$

Together with the norm $\|E_{p,q}\|$ of the imbedding $E_{p,q}$, its essential norm $\text{ess}\|E_{p,q}\|$ and the number $\mathcal{C}(E_{p,q})$, we shall make use of two numbers $\mathcal{M}_1(E_{p,q})$ and $\mathcal{M}_2(E_{p,q})$. The first of them is defined by

$$\mathcal{M}_1(E_{p,q}) = \lim_{s \to 0} \sup \left\{ \frac{\|u\|_{L_q}}{\|u\|_{L_p^1}} : u \in L_p^1\ ,\ u = 0 \text{ on } \Omega_s \right\}.$$

The definition of $\mathcal{M}_2$ is as follows

$$\mathcal{M}_2(E_{p,q}) = \lim_{\varrho \to 0} \sup_{x \in \partial\Omega} \sup \left\{ \frac{\|u\|_{L_q}}{\|u\|_{L_p^1}} : u \in L_p^1\ ,\ \text{supp}\ u \subset B(x, \varrho) \right\},$$

where $B(x, \varrho)$ is the open ball with radius ϱ centered at x.

These two characteristics of $E_{p,q}$ differ in the ways of localization of the functions involved and it seems appropriate to call $\mathcal{M}_1$ and $\mathcal{M}_2$ *the localization moduli* of the embedding $E_{p,q}$.

In section 2 we show that

$$\text{ess}\|E_{p,q}\| = \mathcal{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) \tag{1.2}$$

provided $1 \leq p \leq q < pn/(n-p)$ if $n > p$ and $1 \leq q < \infty$ for $p \geq n$. We also prove that the three quantities in (1.2) are equal to $\mathcal{M}_2(E_{p,q})$ under the additional assumption $p < q$. The last fact fails if $p = q$ as shown in an example of a domain Ω for which $\mathcal{M}_2(E_{p,q}) = 0$ and $\|E_{p,p}\| = \text{ess}\|E_{p,p}\| = \mathcal{M}_1(E_{p,p}) = \mathcal{C}(E_{p,p}) = \infty$ (see Sect. 3).

In section 4, we assume that Ω is a bounded C^1 domain and that $q = pn/(n-p)$, and $n > p \geq 1$. In this case we find an explicit value for $\mathcal{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) = \mathcal{M}_2(E_{p,q})$.

The results obtained in Sect. 2 are readily extended in Sect. 5 to the embedding of $L_p^1(\Omega)$ to the space $L_q(\Omega, \mu)$, where μ is a Radon measure.

Next, we turn to domains with a power cusp on the boundary and find explicit formulae for the measures of non-compactness under consideration and apply these results to the Neumann problem for a particular Schrödinger operator (Sections 6 and 7).

In the final section we show relations between our measures of non-compactness and local isocapacitary and isoperimetric constants. In particular, we obtain

$$\begin{aligned}\operatorname{ess}\|E_{1,q}\| = \mathcal{C}(E_{1,q}) = \mathcal{M}_1(E_{1,q}) = \mathcal{M}_2(E_{1,q})\\ = \lim_{s\to 0}\sup_{g\subset\Omega\setminus\overline{\Omega_s}}\frac{(\mathrm{mes}_n(g))^{1/q}}{\mathcal{H}_{n-1}(\Omega\cap\partial g)},\end{aligned}$$

where $1<q<n/(n-1)$, g is a relatively closed subset of Ω such that $\Omega\cap\partial g$ is a smooth surface, and $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional area. This together with results from Sect. 6 yields explicit values of the local isoperimetric constants for power cusps.

2 Localization moduli and their properties

Let us discuss relations between the moduli. First of all, obviously,

$$\mathcal{M}_1(E_{p,q}) \geq \mathcal{M}_2(E_{p,q}). \tag{2.1}$$

Lemma 2.1 *Let Ω be a domain in $\mathbf{R}^n$ with $\mathrm{mes}_n(\Omega)<\infty$. Suppose that $1\leq p<\infty$ and $1\leq q<\infty$. Then for any $u\in L_p^1(\Omega)$ the following estimate holds:*

$$\|u\|_{L_q(\Omega)} \leq (\mathcal{M}_1(E_{p,q})+\varepsilon)\|u\|_{L_p^1(\Omega)} + C(\varepsilon)\|u\|_{L_{\max\{q,p\}}(\omega_\varepsilon)}, \tag{2.2}$$

where ε is an arbitrary positive number, $C(\varepsilon)$ is a positive function of ε, and ω_ε is an open set with smooth boundary and compact closure $\overline{\omega_\varepsilon}\subset\Omega$.

Proof. Let η denote a smooth function on $\mathbf{R}^+$, such that $0\leq\eta\leq 1$ and $\eta(t)=0$ for $t\in(0,1)$ and $\eta(t)=1$ for $t\geq 2$. By $d(x)$ we denote the distance from $x\in\Omega$ to $\partial\Omega$. Let us introduce the cut-off function $H_s(x)=\eta(d(x)/s)$. We write

$$\|u\|_{L_q} \leq \|H_s u\|_{L_q} + \|(1-H_s)u\|_{L_q}$$

and note that the second term does not exceed

$$\sup\left\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L_p^1(\Omega)}} : v\in L_p^1(\Omega),\ \mathrm{supp}\, v\subset\Omega\setminus\overline{\Omega}_{2s}\right\}\|(1-H_s)u\|_{L_p^1}.$$

Hence, for sufficiently small $s=s(\varepsilon)$

$$\begin{aligned}\|u\|_{L_q} \leq\ & (\mathcal{M}_1(E_{p,q})+\varepsilon)(\|u\|_{L_p^1}+\|u\nabla H_s\|_{L_p})\\ & +\|H_s u\|_{L_q}.\end{aligned}$$

Since the supports of H_s and its derivatives are in Ω_s the result follows. ■

Corollary 2.2 *Let Ω be a domain in $\mathbf{R}^n$ with $\mathrm{mes}_n(\Omega) < \infty$. Suppose that $1 \le p < \infty$ and $1 \le q < \infty$. Then*

$$\mathcal{M}_1(E_{p,q}(\Omega)) < \infty$$

if and only if

$$\|E_{p,q}\|_{L^1_p(\Omega)\to L_q(\Omega)} < \infty.$$

Proof. By Lemma 2.1 $\mathcal{M}_1(E_{p,q}(\Omega)) < \infty$ implies $\|E_{p,q}\|_{L^1_p(\Omega)\to L_q(\Omega)} < \infty$. The converse is obvious. ∎

Lemma 2.3 *Let Ω be a domain in $\mathbf{R}^n$ with $\mathrm{mes}_n(\Omega) < \infty$.*

(i) *If $1 \le q < \infty$ and $1 \le p < \infty$, then*

$$\mathcal{M}_1(E_{p,q}) \le \mathrm{ess}\|E_{p,q}\|.$$

(ii) *Let $1 \le q < pn/(n-p)$ if $1 \le p < n$ and $1 \le q < \infty$ if $n \le p < \infty$. Then*

$$\mathrm{ess}\|E_{p,q}\| \le \mathcal{M}_1(E_{p,q}).$$

Proof. (i) By F we denote an operator of finite rank acting from $L^1_p(\Omega)$ into $L_q(\Omega)$ and given by

$$Fu = \sum_{1\le j\le N} c_j(u)\varphi_j,$$

where $\varphi_j \in L_q(\Omega)$ and c_j are continuous functionals on $L^1_p(\Omega)$. Let $\varepsilon > 0$. We choose the operator F to satisfy

$$\varepsilon + \mathrm{ess}\|E_{p,q}\| \ge \|E_{p,q} - F\|. \tag{2.3}$$

Without loss of generality, we can assume that the functions φ_j have compact supports in Ω. Hence, there exists a positive $s(\varepsilon)$ such that $Fu = 0$ on $\Omega\setminus\overline{\Omega_{s(\varepsilon)}}$ for all $u \in L^1_p(\Omega)$. Let $s \in (0, s(\varepsilon))$. By (2.3), for all $u \in L^1_p(\Omega)$ vanishing on Ω_s,

$$\varepsilon + \mathrm{ess}\|E_{p,q}\| \ge \frac{\|u\|_{L_q(\Omega\setminus\overline{\Omega_{s(\varepsilon)}})}}{\|u\|_{L^1_p(\Omega)}} = \frac{\|u\|_{L_q(\Omega)}}{\|u\|_{L^1_p(\Omega)}}.$$

The required lower estimate for $\mathrm{ess}\|E_{p,q}\|$ follows from the definition of $\mathcal{M}_1(E_{p,q})$.

(ii) Let ε be an arbitrary positive number and let $s > 0$ be so small that

$$\sup\left\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L^1_p(\Omega)}} : v \in L^1_p(\Omega),\ v = 0 \text{ on } \Omega_s\right\} \le \mathcal{M}_1(E_{p,q}) + \varepsilon.$$

We introduce a domain ω with smooth boundary and compact closure $\overline{\omega}$, $\overline{\omega} \subset \Omega$, such that $\mathrm{mes}_n(\Omega\setminus\omega) = \delta\,\mathrm{mes}_n(\Omega\setminus\overline{\Omega}_s)$ with any $\delta \in (0,1)$. By χ_ω we denote the characteristic function of ω. By the compactness of the embedding $L^1_p(\Omega) \to L_q(\omega)$ we have

$$\mathrm{ess}\|E_{p,q}\| \le \sup_{u\in L^1_p} \frac{\|u-\chi_\omega u\|_{L_q(\Omega)}}{\|u\|_{L^1_p(\Omega)}} = \sup_{u\in L^1_p} \frac{\|u\|_{L_q(\Omega\setminus\omega)}}{\|u\|_{L^1_p(\Omega)}}. \tag{2.4}$$

It is obvious that for any positive T

$$|u| \le \min\{T,|u|\} + (|u|-T)_+,$$

where f_+ means the nonnegative part of f. This implies

$$\|u\|_{L_q(\Omega\setminus\omega)} \le \|\min\{T,|u|\}\|_{L_q(\Omega\setminus\omega)} + \|(|u|-T)_+\|_{L_q(\Omega\setminus\omega)}. \tag{2.5}$$

We use the notation $\mathcal{M}_t = \{x : |u(x)| > t\}$ and choose T as

$$T = \inf\{t>0 : \mathrm{mes}_n\mathcal{M}_t < \mathrm{mes}_n(\Omega\setminus\overline{\Omega_s})\}.$$

Then for $q<\infty$

$$\begin{aligned} \|\min\{T,|u|\}\|_{L_q(\Omega\setminus\omega)} &\le T\mathrm{mes}_n(\Omega\setminus\omega)^{1/q} \\ &= (\delta\mathrm{mes}_n(\Omega\setminus\overline{\Omega_s}))^{1/q}T \le \delta^{1/q}\|u\|_{L_q(\Omega)}. \end{aligned}$$

Hence,

$$\sup_{u\in L^1_p(\Omega)} \frac{\|\min\{T,|u|\}\|_{L_q(\Omega\setminus\omega)}}{\|u\|_{L^1_p(\Omega)}} \le \delta^{1/q}\|E_{p,q}\|.$$

Combining this inequality with (2.4) and (2.5), we arrive at

$$\mathrm{ess}\|E_{p,q}\| \le \sup_{u\in L^1_p(\Omega)} \frac{\|(|u|-T)_+\|_{L_q(\Omega\setminus\omega)}}{\|u\|_{L^1_p(\Omega)}} + \delta^{1/q}\|E_{p,q}\|.$$

Let σ be an arbitrary positive number and let H_σ be the function introduced in the proof of Lemma 2.1. Then

$$\|(|u|-T)_+\|_{L_q(\Omega)} \le \|(|u|-T)_+(1-H_\sigma)\|_{L_q(\Omega)} + \|(|u|-T)_+H_\sigma\|_{L_q(\Omega)}. \tag{2.6}$$

In order to estimate the first term in the right-hand side, we take σ so small that

$$\sup\left\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L^1_p(\Omega)}} : v\in L^1_p(\Omega); v=0 \text{ on } \Omega_{2\sigma}\right\} \le \mathcal{M}_1(E_{p,q})+\varepsilon.$$

We normalize u by $\|u\|_{L^1_p(\Omega)} = 1$. Then

$$\begin{aligned} \|(|u|-T)_+(1-H_\sigma)\|_{L_q(\Omega)} &\le (\mathcal{M}_1(E_{p,q})+\varepsilon) \\ &\quad\times\Big(1+\|(|u|-T)_+\nabla H_\sigma\|_{L_p(\Omega)}\Big). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we see that

$$\begin{aligned}\|(u-T)_+\|_{L_q(\Omega)} \quad \leq \quad & (\mathcal{M}_1(E_{p,q})+\varepsilon) \\ & +C(\Omega)(\|u\|_{L_q(\mathcal{M}_T\cap\Omega_\sigma)}+\|u\|_{L_p(\mathcal{M}_T\cap\Omega_\sigma)})\end{aligned} \tag{2.8}$$

with a constant $C(\Omega)$ depending only on Ω, σ, p and q but not on s. Using the compactness of the restriction operator $L^1_p(\Omega) \to L_{\max\{p,q\}}(\overline{\Omega}_\sigma)$ and the equality $\mathrm{mes}_n \mathcal{M}_T \leq \mathrm{mes}_n(\Omega \setminus \overline{\Omega}_s)$, we conclude that the two norms on the right-hand side of (2.8) tend to zero as $s \to 0$. Therefore

$$\limsup_{s\to 0} \|(|u|-T)_+\|_{L_q(\Omega)} \leq \mathcal{M}_1(E_{p,q})+\varepsilon$$

and hence

$$\mathrm{ess}\|E_{p,q}\| \leq \mathcal{M}_1(E_{p,q})+\varepsilon+\delta^{1/q}\|E_{p,q}\|.$$

The proof is completed by using the arbitrariness of ε and δ. ■

Theorem 2.4 *Let Ω be a domain in $\mathbf{R}^n$ with $\mathrm{mes}_n(\Omega) < \infty$. Suppose that $1 \leq p \leq q < pn/(n-p)$ if $1 \leq p < n$ and $1 \leq q < \infty$ if $n \leq p < \infty$. Then*

$$\mathrm{ess}\|E_{p,q}\| = \mathcal{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}).$$

Proof. In the case $q < pn/(n-p)$, $n > p$ and q arbitrary when $p \geq n$ the trace map from $L^1_p(\Omega)$ into $L_{\max\{p,q\}}(\omega_\delta)$ is compact. Then from [10, Theorem 16.4] it follows that

$$\|u\|_{L_{\max(p,q)}(\omega_\delta)} \leq \varepsilon\|u\|_{L^1_p(\Omega)} + C(\varepsilon)\|u\|_{L_1(\omega_\delta)},$$

where $\varepsilon > 0$ is an arbitrary small number. This together with (2.2) allows us to obtain

$$\mathcal{C}(E_{p,q}) \leq \mathcal{M}_1(E_{p,q}).$$

Let $\varepsilon > 0$ be an arbitrary positive number. Then

$$\|u\|_{L_q(\Omega)} \leq (\mathcal{C}(E_{p,q})+\varepsilon)\|u\|_{L^1_p(\Omega)} + c(\varepsilon)\|u\|_{L_1(\Omega_{s(\varepsilon)})}$$

and hence

$$\mathcal{M}_1(E_{p,q}) \leq \mathcal{C}(E_{p,q})+\varepsilon.$$

■

We shall see that, in dealing with $\mathcal{M}_2(E_{p,q})$, we must distinguish between the cases $p < q$ and $p = q$.

Theorem 2.5 *Let Ω be a domain in $\mathbf{R}^n$ with $\mathrm{mes}_n(\Omega) < \infty$ and $1 \leq p < q < \infty$. Suppose that $q < pn/(n-p)$ for $p < n$ and $1 \leq q < \infty$ for $p \geq n$. Then*

$$\mathrm{ess}\|E_{p,q}\| = \mathcal{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) = \mathcal{M}_2(E_{p,q}).$$

Proof. By (2.1) and Theorem 2.4 it is sufficient to assume that $\mathcal{M}_2(E_{p,q}) < \infty$ and to prove the inequality

$$\mathcal{M}_2(E_{p,q}) \geq \mathcal{C}(E_{p,q}).$$

Fix $\rho > 0$ and let $\{\mathcal{B}_i\}_{i\geq 1}$ be an open covering of $\overline{\Omega \setminus \Omega_{\rho/2}}$ by balls of radius ρ centered at $\partial\Omega$. Also let the multiplicity of the covering be finite and depend only on n. Obviously the collection $\{\mathcal{B}_i\}_{i\geq 1}$ supplemented by the set $\overline{\Omega}_{\rho/2}$ forms an open covering of Ω which has a finite multiplicity as well. We introduce a family of non-negative functions $\{\eta_i\}_{i\geq 0}$, such that $\eta_0 \in C_0^\infty(\Omega_{\rho/2})$ and $\eta_i \in C_0^\infty(\mathcal{B}_i)$ for $i \geq 1$, and

$$\sum_{i\geq 0} \eta_i(x)^p = 1 \text{ on } \Omega. \tag{2.9}$$

The estimates to be obtained in what follows will be first proved for an arbitrary function $u \in L_p^1(\Omega) \cap L_\infty(\Omega)$. Since $L_p^1(\Omega) \cap L_\infty(\Omega)$ is dense in $L_p^1(\Omega)$ (see [14, Theorem 1.1.5/1]) these estimates remain valid.

Clearly,

$$\begin{aligned} \|u\|^p_{L_q(\Omega)} &= \|\sum_{i\geq 0} |\eta_i u|^p\|_{L_{q/p}(\Omega)} \\ &\leq \sum_{i\geq 0} \||\eta_i u|^p\|_{L_{q/p}(\Omega)} = \sum_{i\geq 0} \|\eta_i u\|^p_{L_q(\Omega)}. \end{aligned} \tag{2.10}$$

Given $\varepsilon > 0$ and sufficiently small ρ, we have

$$\sup_{i\geq 1} \sup \left\{ \frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L_p^1(\Omega)}} : v \in L_p^1(\Omega), v = 0 \text{ on } \Omega \setminus \mathcal{B}_i \right\} \leq \mathcal{M}_2(E_{p,q}) + \varepsilon.$$

Therefore, the right-hand side in (2.10) does not exceed

$$(\mathcal{M}_2(E_{p,q}) + \varepsilon)^p \sum_{i\geq 1} \|\eta_i u\|^p_{L_p^1(\Omega)} + \|\eta_0 u\|^p_{L_q(\Omega)}.$$

Using the elementary inequality

$$(a+b)^p \leq (1+\varepsilon)a^p + c(\varepsilon)b^p \tag{2.11}$$

for positive a and b, we see that

$$\begin{aligned} \|u\|^p_{L_q(\Omega)} \leq\ & (\mathcal{M}_2(E_{p,q}) + \varepsilon)^p \Big\{ (1+\varepsilon) \sum_{i\geq 1} \|\eta_i \nabla u\|^p_{L_p(\Omega)} \\ & + C(\varepsilon, \rho)\|u\|^p_{L_p(\Omega)} \Big\} + \|\eta_0 u\|^p_{L_q(\Omega)}. \end{aligned} \tag{2.12}$$

We note further that by (2.9) the sum over $i \geq 1$ in (2.12) does not exceed $\|\nabla u\|^p_{L_p(\Omega)}$. Since $q^{-1} > p^{-1} - n^{-1}$ and the support of η_0 is separated from $\partial\Omega$, then by the compactness of the trace mapping $L^1_p(\Omega)$ into $L_q(\Omega_{\rho/2})$ we have the estimate

$$\|\eta_0 u\|_{L_q(\Omega)} \le \delta \|\eta_0 u\|_{L^1_p(\Omega)} + c(\delta,\rho)\|\eta_0 u\|_{L_1(\Omega)}, \tag{2.13}$$

where δ is an arbitrary small number.

Let τ be a positive number independent of ρ. Since $q > p$, we conclude that

$$\|u\|_{L_p(\Omega)} \le (\mathrm{mes}_n(\Omega \setminus \overline{\Omega}_\tau))^{p^{-1}-q^{-1}} \|u\|_{L_q(\Omega\setminus\overline{\Omega}_\tau)} + \|u\|_{L_p(\Omega_\tau)} \tag{2.14}$$

and

$$\|u\|_{L_p(\Omega_\tau)} \le \varepsilon \|u\|_{L^1_p(\Omega)} + c(\varepsilon,\tau)\|u\|_{L_1(\Omega_\tau)}. \tag{2.15}$$

Choosing τ and δ sufficiently small and using (2.12)–(2.15) we arrive at the inequality

$$\|u\|_{L_q(\Omega)} \le (\mathcal{M}_2(E_{p,q}) + c\varepsilon)\|u\|_{L^1_p(\Omega)} + c(\varepsilon)\|u\|_{L_1(\Omega_\tau)},$$

with a constant c independent of ε and u. This completes the proof. ■

3 Counterexample for the case $p = q$

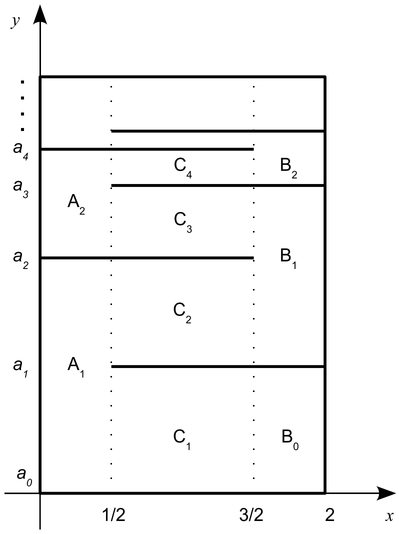
In the next example, we show that the condition $p < q$ in Theorem 2.5 cannot be removed. To be more precise, for any $p \in (1,\infty)$ we construct a planar domain for which $\|E_{p,p}\| = \mathrm{ess}\|E_{p,p}\| = \mathcal{M}_1(E_{p,p}) = \mathcal{C}(E_{p,p}) = \infty$ and $\mathcal{M}_2(E_{p,p}) = 0$.

Example: Let $\Omega \subset \mathbf{R}^2$ be the union of rectangles

$$\begin{aligned} A_i &= (0,1/2] \times (a_{2i-2}, a_{2i}), \quad i > 0, \\ B_0 &= [3/2,2) \times (0, a_1), \\ B_i &= [3/2,2) \times (a_{2i-1}, a_{2i+1}), \quad i > 0, \\ C_i &= [1/2,3/2] \times (a_{i-1}, a_i), \quad i > 0, \end{aligned}$$

where

$$a_0 = 0 \quad \text{and} \quad a_i = \sum_{n=1}^{i} n^{-p} \text{ for } i > 0.$$



Clearly, $\mathrm{mes}_2(\Omega) < \infty$. For each integer $j > 0$ we define the continuous function $f_j(x)$ as a function which is zero on

$$(\cup_{n \le j} A_n) \cup (\cup_{n \le j} B_n) \cup (\cup_{n < 2j} C_n),$$

$f_j(x) = i$ and $f_j(x) = i + 1$ on A_{j+i} and B_{j+i} respectively, and linear on C_{2j+i} with $|\nabla f_j| = 1$ for $i \ge 0$.

The graph of each function $f_j(x)$ has the shape of a staircase with slope 1 on C_{2j+i}, and landings on A_{j+i} and B_{j+i} for $i > 0$.

By a simple computation we obtain $\|\nabla f_j\|_{L_p} < \infty$ and $\|f_j\|_{L_p} = \infty$ for each $j > 0$ which implies $\mathcal{M}_1(E_{p,p}) = \infty$. Furthermore, by Theorem 2.5 we have $\mathcal{C}(E_{p,p}) = \mathrm{ess}\|E_{p,p}\| = \infty$.

It remains to show that $\mathcal{M}_2(E_{p,p}) = 0$. Let $x \in \partial\Omega$ and $\rho < 1/4$. By $Q(x, \rho)$ we denote the square $(x - \rho, x + \rho)^2$. By the definition of Ω one obtains that $\Omega \cap Q(x, \rho)$ is a union of open disjoint sets $\{I_i\}$, where I_i is either a rectangle or the union of three rectangles.

Take $f \in L^1_p(\Omega)$ with $\mathrm{supp} f \subset \Omega \cap Q(x, \rho)$. By the Friedrichs inequality we obtain

$$\int_{I_i} |f(x)|^p dx \le (c\rho)^p \int_{I_i} |\nabla f(x)|^p dx.$$

Summing over $\{I_i\}$ we arrive at

$$\left(\int_{\cup I_i} |f(x)|^p dx \right)^{1/p} \le c\rho \left(\int_{\cup I_i} |\nabla f(x)|^p dx \right)^{1/p},$$

and then

$$\sup \left\{ \frac{\|u\|_{L_p}}{\|u\|_{L^1_p}} : u \in L^1_p(\Omega), \mathrm{supp} u \subset Q(x, \rho) \right\} \le c\rho$$

for every $x \in \partial\Omega$. This implies $\mathcal{M}_2(E_{p,p}) = 0$.

4 The critical Sobolev exponent

Here we show that all our measures of non-compactness can be found explicitly if $\partial\Omega$ has a continuous normal and q is the critical Sobolev exponent.

Theorem 4.1 *Let $n > p \geq 1$ and let Ω be a bounded C^1 domain. Then*

$$\mathcal{C}(E_{p,\frac{pn}{n-p}}) = \mathcal{M}_1(E_{p,\frac{pn}{n-p}}) = \mathcal{M}_2(E_{p,\frac{pn}{n-p}}) = 2^{1/n}c(p,n),$$

where

$$c(p,n) = \pi^{-1/2}n^{-1/p}\left(\frac{p-1}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n)\Gamma(1+n/2)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right)^{1/n}$$

is the best constant in the Sobolev inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(\mathbf{R}^n)} \leq c\|\nabla u\|_{L_p(\mathbf{R}^n)}. \tag{4.1}$$

Proof. Let ζ be a radial function in $C_0^\infty(\mathbf{R}^n)$, $\varepsilon > 0$ and $\zeta_\varepsilon(x) = \zeta(x/\varepsilon)$. By $\mathcal{O}$ we denote an arbitrary point at $\partial\Omega$ and we put $\zeta_{\varepsilon,\mathcal{O}}(x) := \zeta_\varepsilon(x - \mathcal{O})$ into the inequality (1.1). We use q^* to denote $pn/(n-p)$. Using the definition of $\mathcal{C}(E_{p,q})$ we obtain

$$\limsup_{\varepsilon\to 0}\frac{\|\zeta_{\varepsilon,\mathcal{O}}\|_{L_{q^*}(\Omega)}}{\|\nabla\zeta_{\varepsilon,\mathcal{O}}\|_{L_p(\Omega)}} \leq \mathcal{C}(E_{p,q^*}).$$

We note the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\|\zeta_{\varepsilon,\mathcal{O}}\|_{L_{q^*}(\Omega)}}{\|\nabla\zeta_{\varepsilon,\mathcal{O}}\|_{L_p(\Omega)}} = \frac{\|\zeta\|_{L_{q^*}(\mathbf{R}^n_+)}}{\|\nabla\zeta\|_{L_p(\mathbf{R}^n_+)}} = \frac{2^{1/n-1/p}\|\zeta\|_{L_{q^*}(\mathbf{R}^n)}}{2^{-1/p}\|\nabla\zeta\|_{L_p(\mathbf{R}^n)}}.$$

In order to obtain the lower estimate for $\mathcal{C}(E_{p,q^*})$ one needs only to recall that the best constant in (4.1) is attained for a radial function when $p > 1$ and a sequence of radial functions for $p = 1$.

Let us turn to the upper estimate for $\mathcal{C}(E_{p,q^*})$. We construct a finite open covering of $\overline{\Omega}$ by balls $\mathcal{B}_i$ of radius ρ with a finite multiplicity depending only on n. Let either $\overline{\mathcal{B}_i} \subset \Omega$ or the center of $\mathcal{B}_i$ be placed on $\partial\Omega$. We introduce a family of non-negative functions $\{\eta_i\}$ such that $\eta_i \in C_0^\infty(\mathcal{B}_i)$ and

$$\sum_i \eta_i(x)^p = 1 \quad \text{on} \quad \Omega.$$

Obviously,

$$\begin{aligned}\|u\|^p_{L_{q^*}(\Omega)} &= \|\sum_i |\eta_i u|^p\|_{L_{q^*/p}(\Omega)} \\ &\leq \sum_i \||\eta_i u|^p\|_{L_{q^*/p}(\Omega)} = \sum_i \|\eta_i u\|^p_{L_{q^*}(\Omega)}.\end{aligned} \tag{4.2}$$

If $\overline{\mathcal{B}}_i \subset \Omega$ then

$$\|\eta_i u\|_{L_{q^*}(\Omega)} \le c(p,n)\|\nabla(\eta_i u)\|_{L_p(\Omega)}.$$

Let ε be an arbitrary positive number. Since ρ is sufficiently small and the domain is in the class C^1 one can easily show that

$$\|\eta_i u\|_{L_{q^*}(\Omega)} \le (2^{1/n}c(p,n)+\varepsilon)\|\nabla(\eta_i u)\|_{L_p(\Omega)}$$

for the ball $\mathcal{B}_i$ centered at a point at $\partial\Omega$. By using (2.11) we see that

$$\begin{aligned}\sum_i \|\nabla(\eta_i u)\|^p_{L_p(\Omega)} &\le \sum_i \Big(\|\eta_i \nabla u\|^p_{L_p(\Omega)} + \|u\nabla\eta_i\|^p_{L_p(\Omega)}\Big)\\ &\le (1+\varepsilon)\sum_i \|\eta_i\nabla u\|^p_{L_p(\Omega)} + C(\varepsilon)\sum_i \|u\nabla\eta_i\|^p_{L_p(\Omega)}. \quad (4.3)\end{aligned}$$

We note that the first sum in (4.3) is equal to $\|\nabla u\|^p_{L_p(\Omega)}$. Now, it follows from (4.2) that

$$\begin{aligned}\|u\|^p_{L_{q^*}(\Omega)} &\le (2^{1/n}c(p,n)+\varepsilon)^p\Big((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)}\\ &\quad + C(\varepsilon)\sum_i \|u\nabla\eta_i\|^p_{L_p(\Omega)}\Big). \quad (4.4)\end{aligned}$$

Furthermore the second sum in the right-hand side of (4.4) can be majorized by $C(\varepsilon,\rho)\|u\|^p_{L_p(\Omega)}$, and then

$$\begin{aligned}\|u\|^p_{L_{q^*}(\Omega)} &\le (2^{1/n}c(p,n)+\varepsilon)^p\Big((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)}\\ &\quad + C(\varepsilon,\rho)\|u\|^p_{L_p(\Omega)}\Big).\end{aligned}$$

Since the embedding $E_{p,p}(\Omega)$ is compact, it follows from [10, Theorem 16.4] that

$$\begin{aligned}\|u\|^p_{L^p_{q^*}(\Omega)} &\le (2^{1/n}c(p,n)+\varepsilon)^p\\ &\quad \times\left((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)} + \Big(\varepsilon\|u\|_{L^1_p(\Omega)} + C(\varepsilon,\varrho,\tau)\|u\|_{L_1(\Omega_\tau)}\Big)^p\right).\end{aligned}$$

Using (2.11) once more, we arrive at (1.1).

Hence

$$\mathcal{C}(E_{p,q^*}) \le 2^{1/n}c(p,n),$$

which in combination with the lower estimate for $\mathcal{C}(E_{p,q^*})$, shows that

$$\mathcal{C}(E_{p,q^*}) = 2^{1/n}c(p,n). \quad (4.5)$$

Putting an arbitrary $u \in L^1_p(\Omega)$ vanishing outside $\Omega\setminus\overline{\Omega}_s$ into (1.1) and taking the limit as $s \to 0$, we conclude that

$$\mathcal{C}(E_{p,q^*}) \ge \mathcal{M}_1(E_{p,q^*}). \quad (4.6)$$

Now, let $u \in L^1_p(\Omega)$, $u = 0$ outside the ball $B(x_0, \rho)$ with $x_0 \in \partial\Omega$ and sufficiently small ρ. One can easily construct an extension $\widetilde{u}$ of u onto the whole ball $B(x_0, \rho)$ so that

$$\frac{\|u\|_{L_{q^*}(\Omega)}}{\|u\|_{L^1_p(\Omega)}} \geq 2^{1/n}(1-\varepsilon)\frac{\|\widetilde{u}\|_{L_{q^*}(B(x,\rho))}}{\|\widetilde{u}\|_{L^1_p(B(x,\rho))}}.$$

Choosing u in a such way that its extension $\widetilde{u}$ is an almost minimizing function from the Sobolev inequality (4.1) we arrive at

$$\frac{\|u\|_{L_{q^*}(\Omega)}}{\|u\|_{L^1_p(\Omega)}} \geq 2^{1/n}c(p,n)(1-2\varepsilon)$$

and the definition of $\mathcal{M}_2$ yields

$$\mathcal{M}_2(E_{p,q^*}) \geq 2^{1/n}c(p,n). \tag{4.7}$$

Combining (4.5)–(4.7) and the inequality $\mathcal{M}_2 \leq \mathcal{M}_1$, we complete the proof. ∎

5 Generalization

The previous results hold in a more general situation when there is a compact subset of $\partial\Omega$ which is responsible for the loss of compactness of $E_{p,q}$ and the norm in the target space involves a measure.

Let K be a compact subset of $\partial\Omega$ and let $\partial\Omega \setminus K$ be locally Lipschitz (i.e. each point of $\partial\Omega \setminus K$ has a neighborhood $\mathcal{U} \subset \mathbf{R}^n$ such that there exists a quasi-isometric transformation which maps $\mathcal{U} \cap \Omega$ onto a cube).

Define

$$\Omega^K_s = \{x \in \Omega : \mathrm{dist}(x, K) > s\}.$$

It is obvious that for each $s > 0$,

$$\begin{aligned} L^1_p(\Omega^K_s) \quad &\to \quad L_q(\Omega^K_s) \;\text{ is compact} \\ &\text{if and only if }\; L^1_p(\Omega_s) \;\to\; L_q(\Omega_s) \;\text{ is compact.} \end{aligned} \tag{5.1}$$

Let μ be a measure on Ω. We define the embedding operator

$$E_{p,q}(\mu) : L^1_p(\Omega) \to L_q(\Omega, \mu),$$

where

$$L_q(\Omega, \mu) = \left\{ u : \|u\|_{L_{q,\mu}} = \left(\int_\Omega |u|^q d\mu\right)^{1/q} < \infty \right\}.$$

We say that the measure μ is admissible with respect to K if for every $s > 0$ the embedding $L^1_p(\Omega^K_s) \to L_q(\Omega^K_s, \mu)$ is compact.

Let us note that for $1 < p < q < \infty$ and $p < n$ the admissibility of μ is equivalent to

$$\lim_{\rho\to 0} \sup_{x \in \Omega^K_s} \rho^{q(p-n)/p}\mu(B(x,\rho)) = 0$$

(see [1]), in the case $1 < p = n < q < \infty$ it is equivalent to

$$\lim_{\rho\to 0} \sup_{x\in\Omega_s^K} |\log\rho|^{q(p-1)/p}\mu(B(x,\rho)) = 0.$$

We introduce the modified versions of the localization moduli

$$\mathcal{M}_1(E_{p,q}(\mu), K) = \lim_{s\to 0} \sup\left\{\frac{\|u\|_{L_{q,\mu}}}{\|u\|_{L^1_p}} : u \in L^1_p(\Omega),\ v = 0 \text{ on } \Omega\setminus\overline{\Omega_s^K}\right\},$$

and

$$\mathcal{M}_2(E_{p,q}(\mu), K) = \lim_{\varrho\to 0} \sup_{x\in K} \sup\left\{\frac{\|u\|_{L_{q,\mu}}}{\|u\|_{L^1_p}} : u \in L^1_p(\Omega), \operatorname{supp} u \subset B(x,\varrho)\right\}.$$

In the proofs of Theorem 2.4 and Theorem 2.5, we replace Ω_s, $L_q(\Omega)$, $\mathcal{M}_1(E_{p,q})$ and $\mathcal{M}_2(E_{p,q})$ by Ω_s^K, $L_q(\Omega,\mu)$, $\mathcal{M}_1(E_{p,q}, K, \mu)$ and $\mathcal{M}_2(E_{p,q}, K, \mu)$, respectively. Then we use (5.1) and the definition of the admissible measure μ to obtain the following theorem.

Theorem 5.1 *Let K be a compact subset of $\partial\Omega$ such that $\partial\Omega\setminus K$ is locally Lipschitz, and let μ be an admissible measure with respect to K.*

(i) *Let $1 \le p \le q < pn/(n-p)$ for $n > p$ and let $1 \le q < \infty$ for $p \ge n$. Then*

$$\operatorname{ess}\|E_{p,q}(\mu)\| = \mathcal{C}(E_{p,q}(\mu)) = \mathcal{M}_1(E_{p,q}(\mu), K).$$

(ii) *Let $1 \le p < q < pn/(n-p)$ for $n > p$ and let $1 \le q < \infty$ for $p \ge n$. Then*

$$\operatorname{ess}\|E_{p,q}(\mu)\| = \mathcal{C}(E_{p,q}(\mu)) = \mathcal{M}_1(E_{p,q}(\mu), K) = \mathcal{M}_2(E_{p,q}(\mu), K).$$

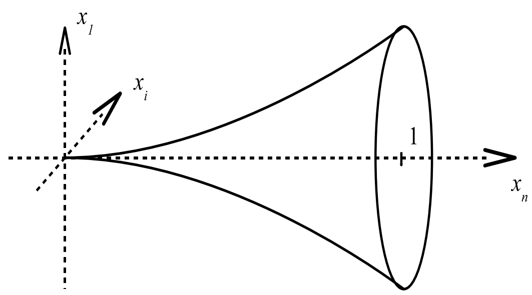
6 The cusp-shaped domains

In this section we find the explicit values of the measures of non-compactness of the embedding $E_{p,q}$ for cusp-shaped domains.

Let ω be a bounded Lipschitz domain in $\mathbf{R}^{n-1}$. Consider the β-cusp

$$\Omega = \{x = (x', x_n) \in \mathbf{R}^n : x' \in x_n^\beta\omega, x_n \in (0,1)\},$$

where $\beta > 1$.



Theorem 6.1 *Let Ω be the β-cusp with $\beta > 1$. Let $p \in [1, \infty)$ and $\gamma \in \mathbf{R}$. We introduce the measure $d\mu = x_n^{-\gamma} dx$ and set*

$$q = \frac{(\beta(n-1)+1-\gamma)p}{\beta(n-1)+1-p}. \tag{6.1}$$

(i) *Let*

$$-\frac{p(\beta-1)(n-1)}{n-p} < \gamma < p \quad \textit{for} \quad 1 < p < n,$$

or

$$\gamma < p < \beta(n-1)+1 \quad \textit{for} \quad n \leq p.$$

Then

$$\mathrm{ess}\|E_{p,q}(\mu)\| = \mathcal{C}(E_{p,q}(\mu)) = \mathcal{M}_1(E_{p,q}(\mu)) = \mathcal{M}_2(E_{p,q}(\mu))$$

$$= (\mathrm{mes}_{n-1}(\omega))^{\frac{p-q}{pq}} \left(\frac{\beta(n-1)+1-p}{p-1}\right)^{\frac{1}{p}-\frac{1}{q}-1} \times$$

$$\times \left(\frac{p}{q(p-1)}\right)^{\frac{1}{q}} \left(\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(p\frac{q-1}{q-p}\right)}\right)^{\frac{q-p}{pq}}.$$

(ii) *Let $1 < p < 1+\beta(n-1)$ and $\gamma = p$. Then*

$$\begin{aligned}\mathrm{ess}\|E_{p,p}(\mu)\| &= \mathcal{C}(E_{p,p}(\mu)) = \mathcal{M}_1(E_{p,p}(\mu)) \\ &= \mathcal{M}_2(E_{p,p}(\mu)) = p(\beta(n-1)+1-p)^{-1}.\end{aligned}$$

(iii) *Let $p = 1$ and $1-\beta < \gamma < 1$. Then*

$$\begin{aligned}\mathrm{ess}\|E_{1,q}(\mu)\| &= \mathcal{C}(E_{1,q}(\mu)) = \mathcal{M}_1(E_{1,q}(\mu)) \\ &= \mathcal{M}_2(E_{1,q}(\mu)) = (\mathrm{mes}_{n-1}(\omega))^{\frac{1-q}{q}} (\beta(n-1)+1-\gamma)^{-1/q},\end{aligned}$$

where q is given by (6.1) *with $p = 1$.*

Proof. (i) By definition of $\mathcal{M}_2$, we have

$$\mathcal{M}_2(E_{p,q}(\mu)) \geq (\mathrm{mes}_n(\omega))^{\frac{p-q}{pq}} \lim_{\gamma \to 0} K_{p,q}(\rho, \beta, \gamma),$$

where

$$K_{p,q}(\rho,\beta,\gamma) := \sup \frac{\left(\int_0^\rho |v(t)|^q t^{\beta(n-1)-\gamma} dt\right)^{1/q}}{\left(\int_0^\rho |v'(t)|^p t^{\beta(n-1)} dt\right)^{1/p}},$$

the supremum being taken over Lipschitz functions on $[0, \rho]$ vanishing at ρ.

Making the substitution $t = \tau/\lambda$, with $\lambda > 0$, we derive

$$K_{p,q}(\rho,\beta,\gamma) = \sup \frac{\left(\int_0^{\lambda\rho} |v(\tau)|^q \tau^{\beta(n-1)-\gamma} d\tau\right)^{1/q}}{\left(\int_0^{\lambda\rho} |v'(\tau)|^p \tau^{\beta(n-1)} d\tau\right)^{1/p}},$$

where the supremum is taken over all Lipschitz functions on $[0, \lambda\rho]$ vanishing at $\lambda\rho$. This implies that $K_{p,q}(\rho,\beta,\gamma)$ is constant in ρ and then

$$K_{p,q}(\rho,\beta,\gamma) = \sup \frac{\left(\int_0^{\infty} |v(t)|^q t^{\beta(n-1)-\gamma} dt\right)^{1/q}}{\left(\int_0^{\infty} |v'(t)|^p t^{\beta(n-1)} dt\right)^{1/p}},$$

where the supremum is extended over all Lipschitz functions, with compact support on $[0,\infty)$. By substitution $t = \tau^c$, where $c = (1-p)(\beta(n-1)+1-p)^{-1}$, we obtain

$$K_{p,q}(\rho,\beta,\gamma) = \left(\frac{\beta(n-1)+1-p}{p-1}\right)^{\frac{q-p}{pq}-1} \sup \frac{\left(\int_0^{\infty} |v(\tau)|^q \tau^{(\beta(n-1)-\gamma+1)c-1} d\tau\right)^{\frac{1}{q}}}{\left(\int_0^{\infty} |v'(\tau)|^p d\tau\right)^{\frac{1}{p}}},$$

where the supremum is taken over all Lipschitz functions with compact support on $[0,\infty)$. Since

$$(\beta(n-1)-\gamma+1)c - 1 = -1 - \frac{q(p-1)}{p},$$

it follows by Bliss' inequality [3], that

$$K_{p,q}(\rho,\beta,\gamma) = \left(\frac{\beta(n-1)+1-p}{p-1}\right)^{\frac{q-p}{pq}-1} \left(\frac{p}{q(p-1)}\right)^{\frac{1}{q}} \left(\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(p\frac{q-1}{q-p}\right)}\right)^{\frac{q-p}{pq}},$$

which gives the required lower estimate for the essential norm of $E_{p,q}(\mu)$.

We now derive an upper bound for the essential norm of $E_{p,q}(\mu)$. Let us introduce the mean value of u over the cross section $\{(x', x_n) : x' \in x_n^\beta \omega\}$ by

$$\overline{u}(x_n) := \frac{1}{\mathrm{mes}_{n-1}\omega} \int_{\mathbf{R}^{n-1}} \chi_\omega(z) u(x_n^\beta z, x_n) dz. \tag{6.2}$$

Let $u \in L_p^1(\Omega)$. By the triangle inequality

$$\|u\|_{L_q(\Omega,\mu)} \le \|\overline{u}\|_{L_q(\Omega,\mu)} + \|u - \overline{u}\|_{L_q(\Omega,\mu)}. \tag{6.3}$$

We estimate the first term in the right-hand side:

$$\begin{aligned} \|\overline{u}\|_{L_q(\Omega,\mu)} &= \left(\int_0^1 \int_{x_n^\beta \omega} |\overline{u}(x_n)|^q dx' x_n^{-\gamma} dx_n\right)^{1/q} \qquad (6.4) \\ &= (\mathrm{mes}_{n-1}\omega)^{1/q} \left(\int_0^1 |\overline{u}(x_n)|^q x_n^{\beta(n-1)-\gamma} dx_n\right)^{1/q} \\ &\le K_{p,q}(1,\beta,\gamma)(\mathrm{mes}_{n-1}\omega)^{1/q} \left(\int_0^1 |(\overline{u})'(x_n)|^p x_n^{\beta(n-1)} dx_n\right)^{1/p}. \end{aligned}$$

Since

$$
\begin{aligned}
(\overline{u})'(x_n) &= \frac{\partial}{\partial x_n}\left(\frac{1}{\mathrm{mes}_{n-1}\omega}\int_{\mathbf{R}^{n-1}}\chi_\omega(z)u(zx_n^\beta,x_n)dz\right)\\
&= \frac{1}{\mathrm{mes}_{n-1}\omega}\int_{\mathbf{R}^{n-1}}\chi_\omega(z)\left(\partial_n u+\beta x_n^{\beta-1}(z,\nabla_{x'})u\right)(zx_n^\beta,x_n)dz
\end{aligned}
$$

it follows that

$$
|(\overline{u})'(x_n)|\le|(\overline{\partial_n u})(x_n)|+(\mathrm{diam}\ \omega)\beta x_n^{\beta-1}|(\overline{\nabla_{x'}u})(x_n)|.
$$

Using the last inequality we obtain

$$
\begin{aligned}
&\left(\int_0^\rho|(\overline{u})'(x)|^p x^{\beta(n-1)}dx\right)^{1/p} \qquad (6.5)\\
&\quad\le\left(\int_0^\rho|(\overline{\partial_n u})(x)|^p x^{\beta(n-1)}dx\right)^{1/p}\\
&\qquad+(\mathrm{diam}\ \omega)\beta\rho^{\beta-1}\left(\int_0^\rho|(\overline{\nabla_{x'}u})(x)|^p x^{\beta(n-1)}dx\right)^{1/p}.
\end{aligned}
$$

Since

$$
\left(\int_0^\rho|\overline{v}(x_n)|^p x_n^{\beta(n-1)}dx_n\right)^{1/p}\le(\mathrm{mes}_{n-1}\omega)^{-1/p}\left(\int_0^\rho\int_{x_n^\beta\omega}|v|^p dx'dx_n\right)^{1/p}, \qquad (6.6)
$$

we have for sufficiently small ρ

$$
\left(\int_0^\rho|(\overline{u})'(x_n)|^p x_n^{\beta(n-1)}dx_n\right)^{1/p}\le(\mathrm{mes}_{n-1}\omega)^{-1/p}\|\nabla u\|_{L_p(\Omega_\rho^0)}, \qquad (6.7)
$$

where $\Omega_\rho^0=\{x\in\Omega: x_n<\rho\}$. Then for every $\varepsilon>0$ there exists a sufficiently small $\rho>0$ such that

$$
\|\overline{u}\|_{L_q(\Omega_\rho^0,\mu)}\le(K_{p,q}(1,\beta,\gamma)(\mathrm{mes}_{n-1}\omega)^{\frac{p-q}{pq}}+\varepsilon)\|\nabla u\|_{L_p(\Omega_\rho^0)}. \qquad (6.8)
$$

Let us estimate the second term in the right-hand side of (6.3). Consider a sequence $\{z_k\}_{k\ge0}$ given by

$$
z_0\le1,\quad z_{k+1}+z_{k+1}^\beta=z_k,\quad k\ge0.
$$

One can easily verify that $z_k\searrow0$, $z_{k+1}z_k^{-1}\to1$. Moreover $z_{k+1}^\beta z_k^{-\beta}\to1$.

Choosing z_0 to be sufficiently small, we obtain $z_{k+1}/2<z_k<2z_{k+1}, k\ge1$. Set

$$
C_k=\{x=(x',x_n)\in\mathbf{R}^n: x_n\in(z_{k+1},z_k),\ x'\in x_n^\beta\omega\},\quad k\ge1.
$$

It follows from the construction of C_k that

$$
\|u-\overline{u}\|_{L_q(C_k,\mu)}\le2^{\gamma/q}z_k^{-\gamma/q}\|u-\overline{u}\|_{L_q(C_k)}. \qquad (6.9)
$$

We obtain by Sobolev's theorem

$$\|u-\overline{u}\|_{L_q(C_k)} \le c z_k^{\beta n(\frac{1}{q}-\frac{1}{p})} \left(\|u-\overline{u}\|_{L_p(C_k)} + z_k^{\beta} \|\nabla(u-\overline{u})\|_{L_p(C_k)} \right), \quad (6.10)$$

where c depends on ω, n, p, q and is independent of k.

And by the Poincaré inequality we have

$$\int_{x_n^\beta \omega} |u(x', x_n) - \overline{u}(x', x_n)|^p dx' \le c x_n^{\beta p} \int_{x_n^\beta \omega} |\nabla_{x'} u(x', x_n)|^p dx$$

for almost all $x_n \in (0,1)$.

Hence it follows from (6.10) and the previous inequality that

$$\|u-\overline{u}\|_{L_q(C_k)} \le c z_k^{\beta(1-\frac{n}{p}+\frac{n}{q})} \left(\|\nabla u\|_{L_p(C_k)} + \|\nabla \overline{u}\|_{L_p(C_k)} \right). \quad (6.11)$$

We deduce from (6.7)

$$\|\nabla \overline{u}\|_{L_p(C_k)} = \|\partial_n \overline{u}\|_{L_p(C_k)} \le \|\nabla u\|_{L_p(C_k)}. \quad (6.12)$$

Combining (6.9), (6.11) and (6.12) implies

$$\|u-\overline{u}\|_{L_q(C_k,\mu)} \le c z_k^{-\frac{\gamma}{q}+\beta(1-\frac{n}{p}+\frac{n}{q})} \|\nabla u\|_{L_p(C_k)}. \quad (6.13)$$

Using (6.13) and the inequality

$$\left(\sum_k a_k^q \right)^{1/q} \le \left(\sum_k a_k^p \right)^{1/p}, \quad a_k \ge 0, \ q \ge p,$$

we conclude

$$\left(\sum_{k=l}^{\infty} \|u-\overline{u}\|^q_{L_q(C_k,\mu)} \right)^{1/q} \le c \left(\sum_{k=l}^{\infty} z_k^{[-\frac{\gamma}{q}+\beta(1-\frac{n}{p}+\frac{n}{q})]p} \|\nabla u\|^p_{L_p(C_k)} \right)^{1/p}.$$

Since $-\frac{\gamma}{q}+\beta(1-\frac{n}{p}+\frac{n}{q}) > 0$, it follows that for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\|u-\overline{u}\|_{L_q(\Omega_\rho^0,\mu)} \le \varepsilon \|\nabla u\|_{L_p(\Omega_\rho^0)}. \quad (6.14)$$

Combining (6.3), (6.8) and (6.14) gives the upper estimate for the essential norm of $E_{p,q}(\mu)$.

(ii) Let us recall the Hardy inequality (see [8, Theorem 330])

$$\int_0^\infty \Big(\int_x^\infty \quad f(t) \quad dt \Big)^p x^{\beta(n-1)-p} dx$$
$$\le \quad \Big(\frac{p}{\beta(n-1)+1-p} \Big)^p \int_0^\infty f^p(x) x^{\beta(n-1)} dx, \quad (6.15)$$

where $[p/(\beta(n-1)+1-p)]^p$ is the best constant. Then replacing Bliss' inequality in (i) by Hardy's inequality (6.15) with appropriate changes in the proof of (i), we obtain (ii).

(iii) As in (ii) we replace Bliss' inequality by Hardy's inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^{\beta(n-1)-\gamma}dt\right)^{1/q} \leq c\int_0^\infty f(x)x^{\beta(n-1)}dx$$

with the best constant

$$c = (\beta(n-1)+1-\gamma)^{-1/q}$$

and repeat the proof of (i). ■

7 Finiteness of the negative spectrum of a Schrödinger operator on β-cusp domains

Let $\Omega \subset \mathbf{R}^n$ be the β-cusp domain with $\beta > 1/(n-1)$. In this section we study the Neumann problem for the Schrödinger operator

$$\begin{aligned} -\Delta_N u - \frac{\alpha}{x_n^2}u &= f, \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega \setminus \{0\}, \end{aligned} \tag{7.1}$$

where $\alpha = \text{const} > 0$ and ν is a normal. The corresponding quadratic form is given by

$$A_\alpha(u,u) = \int_\Omega |\nabla u|^2 dx - \alpha\int_\Omega \frac{|u|^2}{x_n^2}dx. \tag{7.2}$$

As in section 6 we denote $d\mu = x_n^{-2}dx$. Since the space $L_2^1(\Omega)$ is continuously embedded into $L_2(\Omega,\mu)$ when $\beta > 1/(n-1)$ (see [14, Section 5.1.2]), the form A_α is well defined on $L_2^1(\Omega)$.

We start by showing the semi-boundedness of A_α which guarantees the existence of the Friedrichs extension of (7.1).

Theorem 7.1 *A_α is semi-bounded if and only if $\alpha \leq [(\beta(n-1)-1)/2]^2$.*

Proof. Let $\overline{u}$ be the mean value of u over the cross section (see (6.2)), then by the triangle inequality

$$\|u\|_{L_2(\Omega,\mu)} \leq \|\overline{u}\|_{L_2(\Omega,\mu)} + \|u-\overline{u}\|_{L_2(\Omega,\mu)}. \tag{7.3}$$

Combining (6.4), (6.5) and (6.6) (with $p = q = 2$ and $\gamma = -2$) we obtain

$$\begin{aligned} \|\overline{u}\|_{L_2(\Omega_\rho^0,\mu)} \leq\ & \text{ess}\|E_{2,2}(\mu)\|\big(\|\partial_n u\|_{L_2(\Omega_\rho^0)} \\ & +(\text{diam}_{n-1}\omega)\beta\rho^{n-1}\|\nabla_{x'}u\|_{L_2(\Omega_\rho^0)}\big), \end{aligned} \tag{7.4}$$

where $\Omega_\rho^0 = \{x = (x', x_n) \in \mathbf{R}^n : x \in \Omega \text{ and } x_n < \rho\}$.

Let us estimate the second term in the right-hand side of (7.3):

$$\|u-\overline{u}\|^2_{L_2(\Omega^0_\rho,\mu)} = \int_0^\rho \int_{x_n^\beta \omega} |u(x',x_n)-\overline{u}(x_n)|^2 dx' \frac{dx_n}{x_n^2}.$$

Using the Poincaré inequality on the cross line we see that

$$\begin{aligned}\|u-\overline{u}\|^2_{L_2(\Omega^0_\rho,\mu)} &\le (\mathrm{mes}_{n-1}(\omega))^2 \int_0^\rho \int_{x_n^\beta \omega} |\nabla_{x'} u(x',x_n)|^2 x_n^{2\beta-2} dx' dx_n \\ &\le \rho^{2\beta-2} (\mathrm{mes}_{n-1}(\omega))^2 \|\nabla_{x'} u\|^2_{L_2(\Omega^0_\rho)}.\end{aligned}$$

Therefore by (7.3) and (7.4) we obtain

$$\|u\|_{L_2(\Omega^0_\rho,\mu)} \le \mathrm{ess}\|E_{2,2}(\mu)\| \|\nabla u\|_{L_2(\Omega^0_\rho)}$$

for sufficiently small $\rho > 0$.

Then

$$\begin{aligned}\|u\|^2_{L_2(\Omega,\mu)} &\le \|u\|^2_{L_2(\Omega^0_\rho,\mu)} + \|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho,\mu)} \\ &\le (\mathrm{ess}\|E_{2,2}(\mu)\|)^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho)} \\ &\le (2/(\beta(n-1)-1))^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega)},\end{aligned} \tag{7.5}$$

which gives the semi-boundedness for $\alpha \le [(\beta(n-1)-1)/2]^2$.

Let $\alpha > [(\beta(n-1)-1)/2]^2$. We set $d = \alpha - [(\beta(n-1)-1)/2]^2 > 0$. It follows from Theorem 6.1 that

$$\lim_{\varrho\to 0} \sup \left\{ \frac{\int \frac{|u|^2}{x_n^2} dx}{\int |\nabla u|^2 dx} : u \in L^1_p(\Omega), \mathrm{supp}\ u \subset \Omega^0_\rho \right\} = [2/(\beta(n-1)-1)]^2.$$

Then there exists a sequence of functions $\{u_i\}_{i=1}^\infty$ such that $\mathrm{supp}(u_i) \subset \Omega^0_{1/i}$ and

$$\left(\alpha - \frac{d}{2}\right) \int_\Omega \frac{|u_i|^2}{x_n^2} dx > \int_\Omega |\nabla u_i|^2 dx.$$

Hence,

$$\begin{aligned}\int_\Omega |\nabla u_i|^2 dx &- \alpha \int_\Omega \frac{|u_i|^2}{x_n^2} dx \\ &< -\frac{d}{2} \int_\Omega \frac{|u_i|^2}{x_n^2} dx \\ &\le -\frac{i^2 d}{2} \int_\Omega |u_i|^2 dx.\end{aligned}$$

Therefore A_α is not semi-bounded when $\alpha > [(\beta(n-1)-1)/2]^2$. ■

The next theorem gives a condition for finiteness of the negative spectrum of A_α.

Theorem 7.2 *If $\alpha < [(\beta(n-1)-1)/2]^2$, then the negative spectrum of A_α is finite.*

Proof. Let $\alpha < [(\beta(n-1)-1)/2]^2$ and M be a linear infinite-dimensional manifold in $L_2^1(\Omega)$. Take

$$\varepsilon < \frac{1-\alpha[2/(\beta(n-1)-1)]^2}{(2+2\alpha)}. \tag{7.6}$$

It follows from (7.5) that

$$\|u\|^2_{L_2(\Omega,\mu)} \le (2/(\beta(n-1)-1))^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho)},$$

for sufficiently small $\rho > 0$, where $\Omega^0_\rho = \{x = (x', x_n) \in \mathbf{R}^n : x \in \Omega \text{ and } x_n < \rho\}$.

Since the restriction $L_2^1(\Omega) \to L_2(\Omega \setminus \Omega^0_\rho)$ is compact, there exists a finite rank operator $F : L_2^1(\Omega) \to L_2(\Omega\setminus\Omega^0_\rho)$, for which

$$\|u - Fu\|_{L_2(\Omega\setminus\Omega^0_\rho)} \le \left(\frac{\varepsilon}{c(\rho)}\right)^{1/2} \|u\|_{L_2^1(\Omega)}.$$

Note that

$$\begin{aligned}\|u\|^2_{L_2(\Omega,\mu)} \le\ & (2/(\beta(n-1)-1))^2\|\nabla u\|^2_{L_2(\Omega^0_\rho)} \\ & + c(\rho)\left(\left(\frac{\varepsilon}{c(\rho)}\right)^{1/2}\|u\|_{L_2^1(\Omega)} + \|Fu\|_{L_2(\Omega\setminus\Omega^0_\rho)}\right)^2.\end{aligned}$$

Let $M^\perp \subset M$ be define by

$$M^\perp := \{u : Fu = 0 \text{ and } u \in M\}.$$

Then $M^\perp$ is a linear infinite-dimensional manifold in $L_2^1(\Omega)$ and for every $u \in M^\perp$ we have

$$\begin{aligned}\|u\|^2_{L_2(\Omega,\mu)} &\le (2/(\beta(n-1)-1))^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + \varepsilon\|u\|^2_{L_2^1(\Omega)} \\ &\le \left[(2/(\beta(n-1)-1))^2 + 2\varepsilon\right] \|\nabla u\|^2_{L_2(\Omega)} + 2\varepsilon\|u\|^2_{L_2(\Omega)}.\end{aligned} \tag{7.7}$$

Combining (7.2), (7.6) and (7.7) we obtain for each $u \in M^\perp \subset M$

$$\begin{aligned}A_\alpha(u,u) &= \int_\Omega |\nabla u|^2 dx - \alpha\int_\Omega \frac{|u|^2}{x_n^2}dx \\ &\ge [(2/(\beta(n-1)-1))^2 + 2\varepsilon]^{-1}\int_\Omega \frac{|u|^2}{x_n^2}dx - \alpha\int_\Omega\frac{|u|^2}{x_n^2}dx \\ &\quad - 2\varepsilon[(2/(\beta(n-1)-1))^2+2\varepsilon]^{-1}\int_\Omega |u|^2 dx > 0.\end{aligned}$$

Therefore, there does not exist a linear manifold of infinite dimension on which $A_\alpha(u,u) < 0$. This together with [14, Lemma 2.5.4/2] completes the proof. ∎

8 Relations of measures of non-compactness with local isocapacitary and isoperimetric constants

Let E and F denote arbitrary relatively closed disjoint subsets of Ω. We introduce the p-capacitance of the conductor $\Omega \setminus (E \cup F)$ as

$$\operatorname{cap}_{p,\Omega}(E,F) = \inf\{\|\nabla u\|_{L_p}^p : u \geq 1 \text{ on } E \text{ and } u = 0 \text{ on } F,$$
$$u \text{ is locally Lipsichitz in } \Omega\},$$

and we define the local isocapacitary constants

$$S(p,q,\mu,\Omega) = \lim_{s\to 0} \sup \left\{ \frac{(\mu(E))^{1/q}}{(\operatorname{cap}_{p,\Omega}(E,\overline{\Omega_s}))^{1/p}} : E \subset \Omega \setminus \overline{\Omega_s} \right\},$$

and

$$\widetilde{S}(p,q,\mu,\Omega) = \lim_{\rho\to 0} \sup_{x\in\partial\Omega} \sup \left\{ \frac{(\mu(E))^{1/q}}{(\operatorname{cap}_{p,\Omega}(E,\Omega \setminus B(x,\rho)))^{1/p}} : E \subset \Omega \cap B(x,\rho) \right\}.$$

Theorem 8.1 *Let $1 \leq p \leq q < pn/(n-p)$ if $n > p$ and $1 \leq q < \infty$ if $p \geq n$. Then*

$$S(p,q,\mu,\Omega) \leq \operatorname{ess}\|E_{p,q}(\mu)\| \leq K(p,q) S(p,q,\mu,\Omega). \tag{8.1}$$

When additionally $p < q$ then

$$\widetilde{S}(p,q,\mu,\Omega) \leq \operatorname{ess}\|E_{p,q}(\mu)\| \leq K(p,q) \widetilde{S}(p,q,\mu,\Omega), \tag{8.2}$$

where

$$K(p,q) = \begin{cases} \left(\dfrac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})} \right)^{(q-p)/pq}, & \text{when } 1 < p < q; \\ (p-1)^{(1-p)/p}, & \text{when } 1 < p = q; \\ 1, & \text{when } 1 = p \leq q. \end{cases}$$

Proof. The left-hand side inequality in (8.1) follows immediately from the definition of $\mathcal{M}_1$.

The right-hand side inequality in (8.1) is a consequence of the capacitary inequality

$$\left(\int_0^\infty \left(\operatorname{cap}_{p,\Omega}(\{x : |u(x)| \geq t\}, F)\right)^{q/p} d(t^q) \right)^{1/q} \leq K(p,q) \left(\int_\Omega |\nabla u|^p dx \right)^{1/p},$$

where $u = 0$ on F which is a relatively compact subset of Ω (see [17, Proposition 1/5]). The inequality (8.2) follows by the same arguments when $\mathcal{M}_1$ is replaced by $\mathcal{M}_2$. ■

In the case $p = 1$ the p-capacity can be replaced by the $(n-1)$-dimensional area $\mathcal{H}_{n-1}$. By g we denote relatively closed subsets of Ω such that $\Omega \cap \partial g$ are smooth surfaces. We introduce the local isoperimetric constants

$$T(q, \mu, \Omega) = \lim_{s \to 0} \sup \left\{ \frac{(\mu(g))^{1/q}}{\mathcal{H}_{n-1}(\Omega \cap \partial g)} : g \subset \Omega \setminus \overline{\Omega_s} \right\},$$

and

$$\widetilde{T}(q, \mu, \Omega) = \lim_{s \to 0} \sup_{x \in \partial\Omega} \sup \left\{ \frac{(\mu(g))^{1/q}}{\mathcal{H}_{n-1}(\Omega \cap \partial g)} : g \subset \Omega \cap B(x, \rho) \right\}.$$

Theorem 8.2 *If $1 < q < n/(n-1)$ then*

$$T(q, \mu, \Omega) = \widetilde{T}(q, \mu, \Omega) = \mathrm{ess}\|E_{1,q}(\mu)\|,$$

and if $q = 1$ then

$$T(q, \mu, \Omega) = \mathrm{ess}\|E_{1,q}(\mu)\|.$$

Proof. It follows from [16, Theorem 3.5] that $\mathcal{M}_1(E_{1,q}) = T(q, \Omega)$ which together with Theorem 2.4 completes the proof. ■

In view of Theorem 2.4 and Theorem 2.5 the role of the essential norm of $E_{p,q}(\mu)$ can be played by $\mathcal{M}_1(E_{p,q}(\mu))$ and $\mathcal{C}(E_{p,q}(\mu))$ and additionally by $\mathcal{M}_2(E_{p,q}(\mu))$ when $p < q$.

The following corollary, which is an immediate consequence of Theorem 6.1 (iii) and the previous theorem, gives the explicit values of the local isoperimetric constants for power cusps.

Corollary 8.3 *Let $\Omega \subset \mathbf{R}^n$ be the β-cusp domain with $\beta > 1$ and $\gamma \in (1-\beta, 1)$. We introduce the measure $d\mu = x_n^{-\gamma} dx$ and set*

$$q = \frac{\beta(n-1) + 1 - \gamma}{\beta(n-1)}.$$

Then

$$T(q, \mu, \Omega) = (\mathrm{mes}_{n-1}(\omega))^{\frac{1-q}{q}} (\beta(n-1) + 1 - \gamma)^{-1/q}.$$

References

[1] D.R. Adams, *Traces of potentials arising from translation invariant operators*, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), 203–217.

[2] C.J. Amick, *Some remarks on Rellich's theorem and the Poincaré inequality*, J. London Math. Soc. (2) 18 (1978), no. 1, 81–93.

[3] G.A. Bliss, *An integral inequality*, J. London Math. Soc. **5** (1930), 40–46.

[4] D.E. Edmunds and W.D. Evans, *Spectral theory and embeddings of Sobolev spaces*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 120, 431–453.

[5] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, The Clarendon Press, Oxford University Press, New York, 1987.

[6] D.E. Edmunds and W.D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[7] W.D. Evans and D.J. Harris, *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc. (3) **54** (1987), no. 1, 141–175.

[8] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, 1st. ed., Cambridge Univ. Press 1934.

[9] V.I. Kondrashov, *On some properties of functions from the space L_p*, Dokl. Akad. Nauk SSSR **48** (1945) 563–566 (Russian).

[10] J.L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems and Applications, vol. I* , Springer-Verlag, New York, Heidelberg, 1972.

[11] V. Maz'ya, *Classes of regions and imbedding theorems for function spaces*, Dokl. Akad. Nauk SSSR **133** (1960) 527–530 (Russian). English translation: Soviet Math. Dokl. 1 (1960) 882–885.

[12] V. Maz'ya, *The p-conductivity and theorems on imbedding certain function spaces into C-space*, Dokl. Akad. Nauk SSSR, 140 (1961) 299–302 (Russian). English translation: Soviet Math. Dokl. 2 (1961) 1200–1203.

[13] V. Maz'ya, *The negative spectrum of the n-dimensional Schrödinger operator*, Dokl. Akad. Nauk SSSR, 144 (1962) 721–722 (Russian). English translation: Soviet Math. Dokl. 3 (1962) 808–810.

[14] V. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[15] V. Maz'ya and S.V. Poborchi, *Diferentiable Functions on Bad Domains*, World Scientific Publishing, 1997.

[16] V. Maz'ya, *Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces*, Contemp. Math. 338, 307–340 (2003).

[17] V. Maz'ya, *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev type imbeddings*, J. Funct. Anal. 224 (2005), no. 2, 408–430.

[18] F. Rellich, *Ein Satz über mittlere Konvergenz*, Math. Nachr. **31** (1930) 30–35.

[19] N.A. Yerzakova, *The measure of noncompactness of Sobolev embeddings*, Integral Equations Operator Theory 19 (1994), no. 3, 349–359.

[20] N.A. Yerzakova, *On measures of non-compactness and applications to embeddings*, Nonlinear Anal. 30 (1997), no. 1, 535–540.

Jan Lang
Department of Mathematics, The Ohio State University,
231 West 18th Avenue, Columbus, OH 43210, USA
e-mail: lang@math.ohio-state.edu

Vladimir Maz'ya
Department of Mathematics, The Ohio State University,
231 West 18th Avenue, Columbus, OH 43210, USA
e-mail: vlmaz@math.ohio-state.edu

Department of Mathematical Sciences, University of Liverpool
M&O Building, Liverpool L69 3BX, UK

Department of Mathematics, Linköping University
Linköping, SE-581 83, Sweden