Estimates for differential operators of vector analysis involving L^1-norm

Vladimir Maz'ya*

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL
and
Department of Mathematics, Linköping University, Linköping, SE-581 83
e-mail: vlmaz@mai.liu.se

Abstract. New Hardy and Sobolev type inequalities involving L^1-norms of scalar and vector-valued functions in $\mathbb{R}^n$ are obtained. The work is related to some problems stated in the recent paper by Bourgain and Brezis [BB2].

Mathematics Subject Classification (2000): 42B20, 42B25, 46E35

Key words: Newtonian potential, Hardy-type inequality, divergence free fields, div − curl inequalities

1 Introduction

Starting with the pioneering paper by Bourgain and Brezis [BB1], much interest arose in various L^1-estimates for vector fields (see [BB2], [BB3], [BV1], [BV2], [LS], [VS1]–[VS4], [Ma2], [MS] *et al*). The present article belongs to the same area and it was inspired by a question Haïm Brezis asked me at a recent conference in Rome. The question concerns the validity of the Hardy-type inequality

$$\int_{\mathbb{R}^n} |D\mathbf{u}(x)| \frac{dx}{|x|} \leq const. \int_{\mathbb{R}^n} |\Delta \mathbf{u}(x)|\, dx \tag{1}$$

in the case of divergence free $\Delta\mathbf{u}$ and, in a modified form, is included in *Open Problem* 1 formulated in [BB2] on p. 295.

In this paper a positive answer to Brezis' question is given (Theorem 2) and some related results are obtained. For instance, by Theorem 1, the inequality

$$\Big| \int_{\mathbb{R}^n} \sum_{1\leq j\leq n} a_j \Big| \frac{\partial u}{\partial x_j} \Big| \frac{dx}{|x|} \Big| \leq const. \int_{\mathbb{R}^n} |\Delta u|\, dx, \tag{2}$$

where a_j are real constants, holds for all real valued scalar functions $u \in C_0^\infty$ if and only if

$$\sum_{1\leq j\leq n} a_j = 0.$$

*The author was partially supported by the USA National Science Foundation grant DMS 0500029 and by the UK Engineering and Physical Sciences Research Council grant EP/F005563/1.

At the end of the paper certain inequalities for vector valued functions involving Hilbert-Sobolev spaces $\mathcal{H}^{-s}(\mathbb{R}^n)$ of negative order are collected. For example, by Theorem 3 (*iii*), the estimate

$$\big|\|\mathbf{g}\|^2_{\mathcal{H}^{-n/2}} - n\|\mathrm{div}\,\mathbf{g}\|^2_{\mathcal{H}^{-1-n/2}}\big| \leq \frac{(2\sqrt{\pi})^{-n}}{\Gamma(n/2)}\|\mathbf{g}\|^2_{L^1} \tag{3}$$

holds for all $\mathbf{g} \in L^1$ with $\mathrm{div}\,\mathbf{g} \in \mathcal{H}^{-1-n/2}$. An assertion dual to (3) replies in affirmative to *Open Problem* 2 on p. 297 in [BB2] for the particular case $l = 1$, $p = 2$, $s = n/2$.

We make no difference in notations between spaces of scalar and vector-valued functions. If the domain of integration is not indicated, the integral is taken over $\mathbb{R}^n$. We never mention $\mathbb{R}^n$ in notations of function spaces.

2 Inequality for scalar functions

Theorem 1. *Let f and Φ denote scalar real-valued functions defined on $\mathbb{R}^n$. Assume that $f \in L^1$ and*

$$\int f(x)dx = 0. \tag{4}$$

Furthermore, let Φ be Lipschitz on the unit sphere S^{n-1} and positively homogeneous of degree $q \in [1, \frac{n}{n-1})$. By u we mean the Newtonian (logarithmic for $n = 2$) potential of f:

$$u(x) = \int \Gamma(x-y)\, f(y)\, dy,$$

where $\Gamma(x)$ is the fundamental solution of $-\Delta$.

A necessary and sufficient condition for the inequality

$$\sup_{R>0}\Big| \int_{|x|<R} \Phi\big(\nabla u(x)\big)\, |x|^{n(q-1)-q} dx \,\Big| \leq C\Big(\int |f(x)| dx\Big)^q \tag{5}$$

to hold for all f is

$$\int_{S^{n-1}} \Phi(x)\, d\omega_x = 0. \tag{6}$$

The constant C in (5) depends only on Φ, q, and n.

Here and elsewhere $d\omega_x$ is the area element of the unit sphere S^{n-1} at the point $x/|x|$.

Conjecture. It seems plausible that the inequality (5) holds also for the critical value $q = n/(n-1)$. The following simple assertion obtained in [MS] speaks in favour. The inequality

$$\Big|\int_{\mathbb{R}^2} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx\Big| \leq C\Big(\int_{\mathbb{R}^2} |\Delta u|\, dx\Big)^2$$

with $a_{ij} = const$ holds for all $u \in C_0^\infty$ if and only if $a_{11} + a_{22} = 0$.

Proof of Theorem 1. The necessity of (6) can be derived by putting a sequence of radial mollifications of the Dirac function in place of f in (5).

Let us prove the sufficiency of (6). We write $\nabla u(x)$ in the form

$$\nabla u(x) = \sum_{j=1}^{4} A_j(x),$$

where

$$\begin{aligned}
A_1(x) &= \frac{1}{|S^{n-1}|}\int_{|y|<|x|/2}\Big(\frac{y-x}{|y-x|^n}+\frac{x}{|x|^n}\Big)f(y)\,dy,\\
A_2(x) &= \frac{1}{|S^{n-1}|}\int_{|x|/2<|y|<2|x|}\frac{y-x}{|y-x|^n}f(y)\,dy,\\
A_3(x) &= \frac{1}{|S^{n-1}|}\int_{|y|>2|x|}\frac{y-x}{|y-x|^n}f(y)\,dy,
\end{aligned}$$

and

$$A_4(x)=\frac{-1}{|S^{n-1}|}\frac{x}{|x|^n}\int_{|y|<|x|/2}f(y)\,dy.$$

By (6), for all $R>0$,

$$\int_{|x|<R}\Phi\big(A_4(x)\big)|x|^{n(q-1)-q}dx=0. \tag{7}$$

We check directly that

$$\begin{aligned}
|A_1(x)| &\le \frac{c}{|x|^n}\int_{|y|<|x|/2}|f(y)|\,|y|\,dy, &(8)\\
|A_2(x)| &\le c\int_{|x|/2<|y|<2|x|}\frac{|f(y)|}{|y-x|^{n-1}}dy, &(9)\\
|A_3(x)| &\le c\int_{|y|>2|x|}|f(y)|\frac{dy}{|y|^{n-1}}. &(10)
\end{aligned}$$

(Here and elsewhere, by c we denote constants depending only on n and q.) Hence

$$\begin{aligned}
&\int\sum_{j=1}^{3}|A_j(x)|\frac{dx}{|x|}\\
&\le c\int|f(y)|\Big(|y|\int_{|x|>2|y|}\frac{dx}{|x|^{n+1}}+\int_{|y|/2<|x|<2|y|}\frac{dx}{|x||x-y|^{n-1}}\\
&+\frac{1}{|y|^{n-1}}\int_{|x|<|y|/2}\frac{dx}{|x|}\Big)dy\le c\int|f(y)|\,dy
\end{aligned} \tag{11}$$

Since Φ is Lipschitz on S^{n-1} and positively homogeneous of degree q, we have

$$|\Phi(a+b)-\Phi(a)|\le C_\Phi(|a|^{q-1}|b|+|b|^q)$$

for all a and b in $\mathbb{R}^n$. Now, we deduce from (7) that the left-hand side of (5) does not exceed

$$c\,C_\Phi\Big(\int\sum_{j=1}^{3}|A_j(x)|\,|A_4(x)|^{q-1}|x|^{n(q-1)-q}dx+\int\sum_{j=1}^{3}|A_j(x)|^q|x|^{n(q-1)-q}dx\Big). \tag{12}$$

Because of (11), the first integral in (12) is dominated by

$$c\,\|f\|_{L^1}^{q-1}\int\sum_{j=1}^{3}|A_j(x)|\,\frac{dx}{|x|}\le c\,C_\Phi\|f\|_{L^1}^{q}. \tag{13}$$

Let us turn to the second integral in (12). We deduce from (8) and Minkowski's inequality that

$$\|A_1\|_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |y|\,|f(y)| \Big(\int_{|x|>2|y|} \frac{dx}{|x|^{n+q}} \Big)^{1/q} dy. \tag{14}$$

Similarly, by (9)

$$\|A_2\|_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |f(y)| \Big(\int_{2|y|>|x|>|y|/2} \frac{|x|^{n(q-1)-q}dx}{|y-x|^{(n-1)q}} \Big)^{1/q} dy \tag{15}$$

and by (10)

$$\|A_3\|_{L^q(|x|^{n(q-1)-q}dx)} \le c \int |f(y)| \Big(\int_{|x|<|y|/2} |x|^{n(q-1)-q} dx \Big)^{1/q} \frac{dy}{|y|^{n-1}}. \tag{16}$$

Every right-hand side in (14) - (16) is majorized by $c\,\|f\|_{L^1}$. Therefore

$$\sum_{k=1}^{3} \|A_k\|_{L^q(|x|^{n(q-1)-q}dx)} \le c\,\|f\|_{L^1}. \tag{17}$$

The proof is complete.

3 Inequalities for vector functions

We turn to a generalization of the inequality (1).

Theorem 2. *Let* $\mathbf{f}$ *be an* n*-dimensional vector-valued function in* L^1 *subject to*

$$\operatorname{div} \mathbf{f} = 0. \tag{18}$$

Also, let $\mathbf{u}$ *denote the solution of* $-\Delta \mathbf{u} = \mathbf{f}$ *in* $\mathbb{R}^n$ *represented in the form*

$$\mathbf{u}(x) = \int \Gamma(x-y)\,\mathbf{f}(y)\,dy.$$

Then there is a constant c *depending on* n *and* $q \in [1, \frac{n}{n-1})$ *such that*

$$\Big(\int |D\mathbf{u}(x)|^q |x|^{n(q-1)-q} dx \Big)^{1/q} \le c \int |\mathbf{f}(x)|\,dx, \tag{19}$$

where $D\mathbf{u}$ is the Jacobi matrix $(\partial u_i/\partial x_j)_{i,j=1}^n$.

Remark 1. The case $q \in (1, n/(n-1))$ in Theorem 2 is a consequence of the marginal cases $q=1$ and $q = n/(n-1)$ because of the Hölder inequality

$$\|\varphi\|_{L^q(|x|^{n(q-1)-q}dx)} \le \|\varphi\|^{1-n(1-1/q)}_{L^1(|x|^{-1}dx)}\, \|\varphi\|^{n(1-1/q)}_{L^{n/(n-1)}}.$$

However, we prefer to deal with all values of q on the interval $[1, n/(n-1))$ simultaneously and independently of the deeper case $q = n/(n-1)$ treated in [BB2]).

Proof of Theorem 2. It follows from $\mathbf{f} \in L^1$ that the Fourier transform $\hat{\mathbf{f}}$ is continuous. Since $\xi \cdot \hat{\mathbf{f}}(\xi) = 0$ by (18), we have $|\xi|^{-1}\xi \cdot \hat{\mathbf{f}}(0) = 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$, which is equivalent to

$$\int \mathbf{f}(y)\,dy = 0. \tag{20}$$

(The implication (18) $\Longrightarrow$ (20) was noted in [BV1]).

By the integral representation $\mathbf{u} = (-\Delta)^{-1}\mathbf{f}$ we have

$$\Big|\frac{\partial \mathbf{u}}{\partial x_k}(x)\Big| \le \frac{1}{|S^{n-1}|}\Big|\int \frac{y_k - x_k}{|y-x|^n}\mathbf{f}(y)\,dy\Big|.$$

Obviously,

$$\Big|\frac{\partial \mathbf{u}}{\partial x_k}(x)\Big| \le \frac{1}{|S^{n-1}|}\sum_{k=1}^{4}\mathcal{A}_k(x), \tag{21}$$

where

$$\begin{aligned}
\mathcal{A}_1(x) &= \Big|\int_{|y|<|x|/2}\Big(\frac{y_k - x_k}{|y-x|^n} + \frac{x_k}{|x|^n}\Big)\mathbf{f}(y)\,dy\Big|,\\
\mathcal{A}_2(x) &= \Big|\int_{|x|/2<|y|<2|x|}\frac{y_k - x_k}{|y-x|^n}\mathbf{f}(y)\,dy\Big|,\\
\mathcal{A}_3(x) &= \Big|\int_{|y|>2|x|}\frac{y_k - x_k}{|y-x|^n}\mathbf{f}(y)\,dy\Big|,
\end{aligned}$$

and

$$\mathcal{A}_4(x) = \frac{1}{|x|^{n-1}}\Big|\int_{|y|<|x|/2}\mathbf{f}(y)\,dy\Big|. \tag{22}$$

Clearly, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ satisfy (14)-(16) with f replaced by $\mathbf{f}$. Therefore, by Minkowski's inequality (see the proof of (17)), we have

$$\sum_{k=1}^{3}\|\mathcal{A}_k\|_{L^q(|x|^{n(q-1)-q}dx)} \le c\,\|\mathbf{f}\|_{L^1}. \tag{23}$$

Let the $n\times n$ skew-symmetric matrix $\mathcal{F}$ be defined by

$$\mathcal{F} := \operatorname{curl}\mathbf{u} := \Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big)_{i,j=1}^{n},$$

i.e.,

$$\mathcal{F} := \operatorname{curl}(-\Delta)^{-1}\mathbf{f}, \tag{24}$$

where $(-\Delta)^{-1}$ stands for the Newtonian (logarithmic for $n=2$) potential. Let $\mathcal{F} = (F_{ij})_{i,j=1}^{n}$ and $\mathbf{F}_j = (F_{1j},\dots,F_{nj})^t$ with t indicating the transposition of a matrix. We need the row divergence of the matrix $\mathcal{F}$:

$$\operatorname{Div}\mathcal{F} = (\operatorname{div}\mathbf{F}_1,\dots,\operatorname{div}\mathbf{F}_n).$$

Since

$$(\operatorname{Div}\operatorname{curl})^t = \nabla\operatorname{div} - \Delta \qquad\text{and}\qquad \operatorname{div}\mathbf{f} = 0,$$

we have

$$\operatorname{Div}\mathcal{F} = \operatorname{Div}\operatorname{curl}(-\Delta)^{-1}\mathbf{f} = \mathbf{f}^t. \tag{25}$$

We turn to $\mathcal{A}_4(x)$ defined in (22). By (25), we obtain from Green's formula that

$$\mathcal{A}_4(x) = \frac{1}{|x|^{n-1}}\Big|\int_{|y|<|x|/2}\operatorname{Div}\mathcal{F}(y)\,dy\Big| \le c\int_{|y|=|x|/2}|\mathcal{F}(y)|d\omega_y, \tag{26}$$

where $|\mathcal{F}|$ is the matrix norm. The result will follow from (23), (26), and the next lemma.

Lemma. *Let $\mathcal{F}$ be the same skew-symmetric matrix field as in Theorem 2. Then*

$$\Big(\int |\mathcal{F}(x)|^q |x|^{n(q-1)-q} dx\Big)^{1/q} \le c \int |\mathrm{Div}\, \mathcal{F}(x)| dx, \tag{27}$$

where $q \in [1, n/(n-1))$ and c depends only on n and q.

Proof. Using (24) and (25), we have

$$\begin{aligned} \mathcal{F}(x) &= \big(\mathrm{curl}(-\Delta)^{-1}(\mathrm{Div}\, \mathcal{F})^t\big)(x) \\ &= \Big(\int_{E_1} + \int_{E_2} + \int_{E_3}\Big) \mathrm{curl}_x \big((\Gamma(x-y) - \Gamma(x)\big)(\mathrm{Div}\, \mathcal{F}(y))^t dy, \end{aligned} \tag{28}$$

where

$$E_1 = \{y : |y| \le |x|/2\},\ E_2 = \{y : |x|/2 < |y| < 2|x|\},\ \text{and } E_3 = \{y : |y| \ge 2|x|\}.$$

Obviously, the norm of the part of the matrix integral (28) taken over E_1 does not exceed

$$\frac{c}{|x|^{n-1}} \int_{|y|<|x|/2} |\mathrm{Div}\, \mathcal{F}(y)| dy \tag{29}$$

and the norm of the integral over E_2 is dominated by

$$c \int_{|x|/2<|y|<2|x|} |\mathrm{Div}\, \mathcal{F}(y)| \frac{dy}{|x-y|^{n-1}}. \tag{30}$$

We write the part of the integral (28) extended over E_3 as

$$\int_{E_3} \mathrm{curl}_x \big(\Gamma(x-y)(\mathrm{Div}\mathcal{F}(y))^t\big) dy + \int_{E_3} \mathrm{curl}_x \big(-\Gamma(x)(\mathrm{Div}\mathcal{F}(y))^t\big) dy. \tag{31}$$

The matrix norm of the first term in (31) does not exceed

$$c \int_{|y|>2|x|} |\mathrm{Div}\mathcal{F}(y)| \frac{dy}{|y|^{n-1}}. \tag{32}$$

Let us denote the second integral in (31) by $\mathcal{G}(x)$ and let us put

$$\mathcal{G} = (\mathbf{G}_1, \dots, \mathbf{G}_n), \qquad \text{where} \qquad \mathbf{G}_j = (G_{1j}, \dots, G_{nj})^t.$$

Estimating the $L^q(|x|^{n(q-1)-q} dx)$-norms of the majorants (29), (30), and (32) by Minkowski's inequality, in the same way as we did for $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in the proof of Theorem 2, we obtain

$$\|\mathcal{F} - \mathcal{G}\|_{L^q(|x|^{n(q-1)-q} dx)} \le c \|\mathrm{div} \mathcal{F}\|_{L^1}. \tag{33}$$

By definitions of curl and Div,

$$\begin{aligned} G_{ij}(x) &= \frac{\partial \Gamma}{\partial x_j}(x) \int_{E_3} \mathrm{div}\, \mathbf{F}_i(y)\, dy - \frac{\partial \Gamma}{\partial x_i}(x) \int_{E_3} \mathrm{div}\, \mathbf{F}_j(y)\, dy \\ &= |S^{n-1}|^{-1} |x|^{1-n} \Big(\frac{x_i}{|x|} \int_{E_3} \mathrm{div} \mathbf{F}_j(y)\, dy - \frac{x_j}{|x|} \int_{E_3} \mathrm{div}\, \mathbf{F}_j(y)\, dy \Big) \end{aligned}$$

and by Green's formula,

$$G_{ij}(x) = \frac{2^{n-1}}{|S^{n-1}|}\Big(\frac{x_i}{|x|}\int_{|y|=2|x|}\Big(\frac{y}{|y|}, \mathbf{F}_j(y)\Big)d\omega_y - \frac{x_j}{|x|}\int_{|y|=2|x|}\Big(\frac{y}{|y|}, \mathbf{F}_i(y)\Big)d\omega_y\Big), \quad (34)$$

where $(\cdot\,,\cdot)$ stands for the inner product in $\mathbb{R}^n$. Obviously,

$$\begin{aligned}\int_{|z|=|x|} G_{ij}(z)\frac{z_i}{|z|}d\omega_z \quad &= \quad \frac{2^{n-1}}{|S^{n-1}|}\Big(|S^{n-1}|\int_{|y|=2|x|}\Big(\frac{y}{|y|}, \mathbf{F}_j(y)\Big)d\omega_y \\ &\quad - \int_{S^{n-1}}\frac{z_i z_j}{|z|^2}d\omega_z\int_{|y|=2|x|}\Big(\frac{y}{|y|}, \mathbf{F}_i(y)\Big)d\omega_y\Big)\end{aligned}$$

and since

$$\int_{S^{n-1}}\frac{z_i z_j}{|z|^2}d\omega_z = \frac{\delta_i^j}{n}|S^{n-1}|,$$

we obtain

$$\int_{|z|=|x|}\Big(\frac{z}{|z|}, \mathbf{G}_j(z)\Big)d\omega_z = 2^{n-1}\frac{n-1}{n}\int_{|y|=2|x|}\Big(\frac{y}{|y|}, \mathbf{F}_j(y)\Big)d\omega_y. \quad (35)$$

For an arbitrary $r > 0$ and a vector function $\mathbf{v}$ we set

$$\mathcal{P}(\mathbf{v}; r) := \int_{|y|=r}\frac{y}{|y|}\mathbf{v}(y)d\omega_y.$$

Now, using the majorants (29), (30), and (32), we deduce from (28) and the definition of $\mathcal{G}$ that

$$\begin{aligned}&\Big|\mathcal{P}(\mathbf{F}_j; |x|) - \mathcal{P}(\mathbf{G}_j; |x|)\Big| \\ &\le c\Big(\frac{1}{|x|^n}\int_{E_1}|\mathrm{Div}\,\mathcal{F}(y)|dy + \int_{E_2}|\mathrm{Div}\,\mathcal{F}(y)|\frac{dy}{|x-y|^{n-1}} + \int_{E_3}|\mathrm{Div}\,\mathcal{F}(y)|\frac{dy}{|y|^{n-1}}\Big).\end{aligned}$$

By (35) the left-hand side can be written in the form

$$\Big|\mathcal{P}(\mathbf{F}_j; |x|) - 2^{n-1}\frac{n-1}{n}\mathcal{P}(\mathbf{F}_j; 2|x|)\Big|.$$

Using the same argument as at the end of the proof of Theorem 1, we arrive at

$$\Big(\int\Big|\mathcal{P}(\mathbf{F}_j; |x|) - 2^{n-1}\frac{n-1}{n}\mathcal{P}(\mathbf{F}_j; 2|x|)\Big|^q |x|^{n(q-1)-q}dx\Big)^{1/q} \le c_0\int|\mathrm{div}\mathcal{F}(x)|dx$$

which yields

$$\begin{aligned}&\Big|\Big(\int\big|\mathcal{P}(\mathbf{F}_j; |x|)\big|^q |x|^{n(q-1)-q}dx\Big)^{1/q} - 2^{n-1}\frac{n-1}{n}\Big(\int|\mathcal{P}(\mathbf{F}_j; 2|x|)|^q |x|^{n(q-1)-q}dx\Big)^{1/q}\Big| \\ &\qquad \le c_0\int|\mathrm{div}\mathcal{F}(x)|dx.\end{aligned}$$

Replacing $2x$ by x in the second integral of the last inequality, we can simplify this inequality to the form

$$\Big(\int\big|\mathcal{P}(\mathbf{F}_j; |x|)\big|^q |x|^{n(q-1)-q}dx\Big)^{1/q} \le n\,c_0\int|\mathrm{div}\,\mathcal{F}(x)|dx. \quad (36)$$

By (34) and (36),

$$\|\mathbf{G}_j\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c\Big(\int \big|\mathcal{P}(\mathbf{F}_j;|x|)\big|^q |x|^{n(q-1)-q}dx\Big)^{1/q} \leq c\int |\mathrm{div}\,\mathcal{F}(x)|dx$$

which together with (33) completes the proof.

4 Generalization of Theorem 2

In this section we show that Theorem 2 can be extended to the vector fields $\mathbf{f}$, which are not necessarily divergence free.

First, let us collect some notation and known facts to be used in the sequel. Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. The mean value of the integral with respect to a finite measure will be denoted by the integral with bar. By $\hat{\varphi}$ we denote the Fourier transform of the distribution φ (see Sect. 7.1 in [H]).

The space of distributions φ with $\nabla\varphi \in L^1$ will be denoted by L^1_1. This space is endowed with the seminorm $\|\nabla\varphi\|_{L^1}$. It is well known and can be easily proved that the finite limit

$$\varphi_\infty := \lim_{R\to\infty} \fint_{|x|=R} \varphi(x)d\omega_x$$

exists for every $\varphi \in L^1_1$. Furthermore, $\varphi_\infty = 0$ is equivalent to the inclusion of φ in the closure $\mathring{L}^1_1$ of C_0^∞ in L^1_1.

The weighted Sobolev-type inequality for all $\varphi \in \mathring{L}^1_1$

$$\|\varphi\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c\,\|\nabla\varphi\|_{L^1} \tag{37}$$

with $q \in [1, \frac{n}{n-1})$ can be found, for example, in Corollary 2.1.6 [Ma1].

We formulate and prove a result concerning the case $q > 1$.

Proposition 1. *Let $q \in (1, \frac{n}{n-1})$ and let $\mathbf{u} = (-\Delta)^{-1}\mathbf{f}$, where $\mathbf{f}$ is a vector field in L^1 subject to (20). Also let*

$$h := \mathrm{div}\,\mathbf{f} \quad \text{and} \quad \nabla(-\Delta)^{-1}h \in L^1.$$

Then

$$\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c\big(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}\big). \tag{38}$$

Proof. Note that the vector function $-\xi|\xi|^{-2}(\hat{\mathbf{f}}(\xi),\xi)$ is the Fourier transform of $\nabla(-\Delta)^{-1}h$ and that it is equal to zero at the point $\xi = 0$ since $\hat{\mathbf{f}}(0) = 0$. Hence

$$\int \nabla(-\Delta)^{-1}h(y)\,dy = 0.$$

We see that the vector field $\mathbf{f}+\nabla(-\Delta)^{-1}h$ is divergence free and integrable. Therefore, by Theorem 2,

$$\|D(-\Delta)^{-1}\big(\mathbf{f} + \nabla(-\Delta)^{-1}h\big)\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c\big(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}\big), \tag{39}$$

which implies

$$\begin{aligned}\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \quad &\leq \quad c\Big(\|\mathbf{f}\|_{L^1} + \|D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h\|_{L^q(|x|^{n(q-1)-q}dx)} \\ &+ \quad \|\nabla(-\Delta)^{-1}h\|_{L^1}\Big). \end{aligned} \tag{40}$$

Since the singular integral operator $D(-\Delta)^{-1}\nabla$ is continuous in $L^q(|x|^{n(q-1)-q}dx)$ for $q \in (1, \frac{n}{n-1})$ (see [St1]), we derive from (40) that

$$\begin{aligned} \|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} &\leq c\Big(\|\mathbf{f}\|_{L^1} + \|(-\Delta)^{-1}h\|_{L^q(|x|^{n(q-1)-q}dx)} \\ &+ \|\nabla(-\Delta)^{-1}h\|_{L^1}\Big). \end{aligned} \tag{41}$$

Recalling that $h = \operatorname{div}\mathbf{f}$, we have

$$\begin{aligned} &\fint_{B_{2R}\setminus B_R} |(-\Delta)^{-1}h(x)|dx \leq c\,R^{-n}\int |\mathbf{f}(y)| \int_{B_{2R}\setminus B_R} \frac{dx}{|x-y|^{n-1}}dy \\ &\leq c\Big(R^{1-n}\int_{B_R} |\mathbf{f}(y)|\,dy + \int_{\mathbb{R}^n\setminus B_R} |\mathbf{f}(y)|\frac{dy}{|y|^{n-1}}\Big). \end{aligned}$$

Hence

$$\lim_{R\to\infty} \fint_{B_{2R}\setminus B_R} |(-\Delta)^{-1}h(x)|dx = 0$$

and by $(-\Delta)^{-1}h \in L_1^1$ we see that

$$\lim_{R\to\infty} \fint_{|x|=R} (-\Delta)^{-1}h(x)\,d\omega_x = 0,$$

i.e., $(-\Delta)^{-1}h \in \overset{\circ}{L}{}_1^1$.

Using (37), we remove the second norm on the right-hand side of (41) by changing the value of the factor c. The result follows. $\square$

We turn to the case $q = 1$ which is more technical being based on properties of the Riesz transform in the Hardy space $\mathbf{H}$.

By definition, the space $\mathbf{H}$ consists of all integrable functions orthogonal to 1 and is endowed with the norm

$$\|\varphi\|_{\mathbf{H}} = \|\varphi\|_{L^1} + \|\nabla(-\Delta)^{-1/2}\varphi\|_{L^1}. \tag{42}$$

This space can be introduced also as the completion in the norm (42) of the set of functions φ such that $\hat{\varphi} \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ (see [St2], Sect. 3).

The result concerning $q = 1$ which is analogous to Proposition 1 is stated as follows.

Proposition 2. *Let $\mathbf{u} = (-\Delta)^{-1}\mathbf{f}$, where $\mathbf{f}$ is a vector field in L^1 subject to (20). Also let*

$$h := \operatorname{div}\mathbf{f} \quad \text{and} \quad (-\Delta)^{-1/2}h \in \mathbf{H}.$$

Then

$$\|D\mathbf{u}\|_{L^1(|x|^{-1}dx)} \leq c\big(\|\mathbf{f}\|_{L^1} + \|(-\Delta)^{-1/2}h\|_{\mathbf{H}}\big). \tag{43}$$

Proof. Let us show that

$$D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h \in L_1^1 \tag{44}$$

if $(-\Delta)^{-1/2}h \in \mathbf{H}$. The Fourier transform of

$$\frac{\partial^2}{\partial x_i \partial x_j}(-\Delta)^{-1}\frac{\partial}{\partial x_k}(-\Delta)^{-1}h$$

equals

$$\xi_i\xi_j|\xi|^{-2}\xi_k|\xi|^{-2}(\hat{\mathbf{f}}(\xi),\xi)$$

and vanishes at $\xi = 0$ because $\hat{\mathbf{f}}(0) = 0$. Furthermore, by definition of $\mathbf{H}$ and the continuity of $\nabla(-\Delta)^{-1/2}$ in $\mathbf{H}$ (see [St2], Sect. 3.4), we obtain

$$\begin{aligned}\|\partial_{x_i} D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h\|_{L^1} &\leq \|D(-\Delta)^{-1/2}\nabla(-\Delta)^{-1/2}(-\Delta)^{-1/2}h\|_{\mathbf{H}} \\ &\leq c\,\|(-\Delta)^{-1/2}h\|_{\mathbf{H}}, \end{aligned} \tag{45}$$

i.e. (44) holds. Next we check that the mean value of $D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h$ on the sphere ∂B_R tends to zero as $R \to \infty$. Since $h = \operatorname{div}\mathbf{f}$, it follows that pointwise

$$|D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h| \leq c\,(-\Delta)^{-1/2}|\mathbf{f}|.$$

Therefore,

$$\fint_{B_{2R}\setminus B_R} |D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h(x)|dx \leq c\,R^{-n}\int |\mathbf{f}(y)| \int_{B_{2R}\setminus B_R} \frac{dx}{|x-y|^{n-1}}dy$$

$$\leq c\Big(R^{1-n}\int_{B_R} |\mathbf{f}(y)|\,dy + \int_{\mathbb{R}^n\setminus B_R} |\mathbf{f}(y)|\frac{dy}{|y|^{n-1}}\Big).$$

Hence

$$\lim_{R\to\infty} \fint_{B_{2R}\setminus B_R} |D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h(x)|dx = 0,$$

which ensures that the mean value just mentioned tends to zero. Now we can conclude that

$$D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h \in \overset{\circ}{L}{}^1_1.$$

This inclusion, together with (37) for $q = 1$ and (45), shows that the second norm on the right-hand side of (40) does not exceed

$$c\,\|D(-\Delta)^{-1}\nabla(-\Delta)^{-1}h\|_{L^1_1} \leq c_1\,\|(-\Delta)^{-1/2}h\|_{\mathbf{H}}.$$

As for the third norm, it has the majorant $c_2\,\|(-\Delta)^{-1/2}h\|_{\mathbf{H}}$ by definition of $\mathbf{H}$. The result follows.

5 Inequalities involving L^2 Sobolev norms of negative order

In the sequel, the notation $\mathcal{H}^l$ will be used for the space of distributions h with finite norm

$$\|h\|_{\mathcal{H}^l} := \Big(\int |\hat{h}(\xi)|^2|\xi|^{2l}\,d\xi\Big)^{1/2},$$

where $l \in \mathbb{R}^1$.

By $|\cdot|$ and $(\cdot,\cdot)$ the norm and the inner product in the complex Euclidean space will be denoted.

Theorem 3. *Let* $\mathbf{g} \in C_0^\infty$ *and*

$$\mathbf{g}_\varepsilon(x) := \mathbf{g}(x) - (2\pi)^{-n/2}\varepsilon^n e^{-|\varepsilon x|^2/2}\int \mathbf{g}(y)\,dy. \tag{46}$$

Then

(*i*) *The following limit exists and satisfies the inequality*

$$\Big|\lim_{\varepsilon\to 0_+}\big(\|\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-n/2}} - n\|\mathrm{div}\,\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-1-n/2}}\big)\Big| \le \frac{(n-1)(2\sqrt{\pi})^{-n}}{\Gamma(1+n/2)}\|\mathbf{g}\|^2_{L^1}. \tag{47}$$

(*ii*) *The inequality*

$$\limsup_{\varepsilon\to 0_+}\big|\|\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-n/2}} - c_1\,\|\mathrm{div}\,\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-1-n/2}}\big| \le c_2\,\|\mathbf{g}\|^2_{L^1} \tag{48}$$

with certain constants c_1 *and* c_2 *implies* $c_1 = n$. *The constant* c_2 *satisfies*

$$c_2 \ge \frac{(n-1)(2\sqrt{\pi})^{-n}}{\Gamma(1+n/2)}, \tag{49}$$

i.e. (47) is sharp.

(*iii*) *If*

$$\int \mathbf{g}(y)\,dy = 0, \tag{50}$$

then

$$\Big|\|\mathbf{g}\|^2_{\mathcal{H}^{-n/2}} - n\,\|\mathrm{div}\,\mathbf{g}\|^2_{\mathcal{H}^{-1-n/2}}\Big| \le \frac{(2\sqrt{\pi})^{-n}}{\Gamma(n/2)}\|\mathbf{g}\|^2_{L^1}. \tag{51}$$

Proof. (*i*) The expression in parentheses on the left-hand side of (47) can be written as

$$(2\pi)^{-n}\Big(\int |\hat{\mathbf{g}}_\varepsilon(\xi)|^2\frac{d\xi}{|\xi|^n} - n\int |\,(\hat{\mathbf{g}}_\varepsilon(\xi),\xi)|^2\frac{d\xi}{|\xi|^{2+n}}\Big)$$
$$= (2\pi)^{-n}\Big(\sum_{1\le j,k\le n}\int \frac{\delta^k_j|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}\hat{g}_{\varepsilon,j}(\xi)\overline{\hat{g}_{\varepsilon,k}(\xi)}\,d\xi\Big), \tag{52}$$

where all integrals are absolutely convergent. By (46),

$$\hat{\mathbf{g}}_\varepsilon(\xi) = \hat{\mathbf{g}}(\xi) - e^{-|\xi|^2/2\varepsilon^2}\,\hat{\mathbf{g}}(0).$$

We note that for any $t > 0$

$$\int_{|\xi|>t}\frac{\delta^k_j|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}e^{-|\xi|^2/2\varepsilon^2}\,d\xi = 0$$

and

$$\int_{|\xi|>t}\frac{\delta^k_j|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}e^{-|\xi|^2/\varepsilon^2}\,\hat{g}_j(0)\,\overline{\hat{g}_k(0)}\,d\xi = 0.$$

Therefore

$$\int_{|\xi|>t}\frac{\delta^k_j|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}\,\hat{g}_{\varepsilon,j}(\xi)\,\overline{\hat{g}_{\varepsilon,k}(\xi)}\,d\xi$$
$$= \int_{|\xi|>t}\frac{\delta^k_j|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}\,\hat{g}_j(\xi)\,\overline{\hat{g}_k(\xi)}\,d\xi + O(\varepsilon)$$

uniformly with respect to t. Hence the value (52) tends to

$$(2\pi)^{-n}\Big(\sum_{1\le j,k\le n}\int \frac{\delta_j^k|\xi|^2-n\,\xi_j\xi_k}{|\xi|^{n+2}}\,\hat g_j(\xi)\,\overline{\hat g_k(\xi)}d\xi\Big) \tag{53}$$

as $\varepsilon\to 0_+$, where the integral is understood as the Cauchy value.

Note that for $n>2$

$$\big(|\xi|^2-n\,\xi_k^2\big)|\xi|^{-2-n}=(2-n)^{-1}\frac{\partial^2}{\partial\xi_k^2}\frac{1}{|\xi|^{n-2}}-\frac{|S^{n-1}|}{n}\delta(\xi)$$

and

$$\xi_j\xi_k\,|\xi|^{-2-n}=n^{-1}(n-2)^{-1}\frac{\partial^2}{\partial\xi_j\,\partial\xi_k}\frac{1}{|\xi|^{n-2}},\qquad \text{for } j\neq k.$$

Analogously, for $n=2$,

$$\big(|\xi|^2-2\,\xi_k^2\big)|\xi|^{-4}=-\frac{\partial}{\partial\xi_k}\frac{\xi_k}{|\xi|^2}-\pi\,\delta(\xi), \tag{54}$$

and

$$\xi_j\xi_k\,|\xi|^{-4}=-\frac{1}{2}\frac{\partial}{\partial\xi_j}\frac{\xi_k}{|\xi|^2}\qquad \text{for } j\neq k. \tag{55}$$

Therefore, in the case $n>2$, we express (53) as

$$(2\pi)^{-n}\sum_{1\le k\le n}\int\left(\frac{1}{2-n}\frac{\partial^2}{\partial\xi_k^2}\frac{1}{|\xi|^{n-2}}-\frac{|S^{n-1}|}{n}\,\delta(\xi)\right)\hat{g_k}(\xi)\,\bar{\hat{g_k}}(\xi)\,d\xi$$

$$-(2\pi)^{-n}n\sum_{j\neq k}\int\frac{1}{n(n-2)}\Big(\frac{\partial^2}{\partial\xi_j\partial\xi_k}\,\frac{1}{|\xi|^{n-2}}\Big)\hat{g_k}(\xi)\bar{\hat{g_j}}(\xi)\,d\xi.$$

Using Parseval's formula once more, we write the limit of the right-hand side in (52) as $\varepsilon\to 0_+$ in the form

$$\sum_{1\le k\le n}\int g_k(x)\,\mathcal{F}^{-1}_{\xi\to x}\left(\left(\frac{1}{2-n}\,\frac{\partial^2}{\partial\xi_k^2}\,\frac{1}{|\xi|^{n-2}}-\frac{|S^{n-1}|}{n}\,\delta(\xi)\right)\hat{g_k}(\xi)\right)\,dx$$

$$-\sum_{j\neq k}\int\frac{1}{n-2}\,g_j(x)\,\mathcal{F}^{-1}_{\xi\to x}\left(\left(\frac{\partial^2}{\partial\xi_j\partial\xi_k}\,\frac{1}{|\xi|^{n-2}}\right)\hat{g_k}(\xi)\right)\,dx, \tag{56}$$

where $\mathcal{F}^{-1}$ means the inverse Fourier transform (see formula (7.1.4) in [H]). Since $\mathcal{F}^{-1}(\hat u\,\hat v)=u*v$, where $*$ denotes the convolution, we have

$$\mathcal{F}^{-1}_{\xi\to x}\left(\left(\frac{\partial^2}{\partial\xi_j\partial\xi_k}\,\frac{1}{|\xi|^{n-2}}\right)\hat h(\xi)\right)=-\left(x_j\,x_k\Big(\mathcal{F}^{-1}_{\xi\to x}\frac{1}{|\xi|^{n-2}}\Big)\right)*h$$

$$=-(2\pi)^{-n}(n-2)|S^{n-1}|\frac{x_j\,x_k}{|x|^2}*h \tag{57}$$

for $1\le j,k\le n$.

Now let $n = 2$. By (54) and Parseval's formula, we present (53) in the form analogous to (56)

$$
\begin{aligned}
&(2\pi)^{-2} \sum_{1\le k\le 2} \int \Big(-\frac{\partial}{\partial \xi_k}\,\frac{\xi_k}{|\xi|^2} - \pi\,\delta(\xi)\Big)\hat{g_k}(\xi)\,\bar{\hat{g_k}}(\xi)\,d\xi \\
&+ (2\pi)^{-2} \sum_{j\ne k} \int \Big(\frac{\partial}{\partial \xi_j}\,\frac{\xi_k}{|\xi|^2}\Big)\hat{g_k}(\xi)\,\bar{\hat{g_j}}(\xi)\,d\xi \\
&= \sum_{1\le k\le 2} \int \mathcal{F}^{-1}_{\xi\to x}\left(\big(-\frac{\partial}{\partial \xi_k}\,\frac{\xi_k}{|\xi|^2} - \pi\,\delta(\xi)\big)\hat{g_k}(\xi)\right) g_k(x)\,dx \\
&+ \sum_{j\ne k} \int \mathcal{F}^{-1}_{\xi\to x}\left(\big(\frac{\partial}{\partial \xi_j}\,\frac{\xi_k}{|\xi|^2}\big)\hat{g_k}(\xi)\right) g_j(x)\,dx. \qquad (58)
\end{aligned}
$$

We check directly that

$$
\mathcal{F}^{-1}_{\xi\to x}\left(\big(\frac{\partial}{\partial \xi_j}\,\frac{\xi_k}{|\xi|^2}\big)\hat{h}(\xi)\right) = i\,x_j \mathcal{F}^{-1}_{\xi\to x}\frac{\xi_k}{|\xi|^2}\hat{h}(\xi) = -(2\pi)^{-2}\frac{x_j\,x_k}{|x|^2} * h.
$$

Combining this with (57), we deduce from (56) and (58) that for every $n \ge 2$ the limit of the expression (52) as $\varepsilon \to 0_+$ is equal to

$$
\begin{aligned}
&\Big(\sum_{1\le k\le n} \int \Big(\Big(\frac{|S^{n-1}|}{(2\pi)^n}\,\frac{x_k^2}{|x|^2} - \frac{|S^{n-1}|}{(2\pi)^n n}\Big) * g_k\Big)\, g_k dx + \sum_{j\ne k} \frac{|S^{n-1}|}{(2\pi)^n}\Big(\frac{x_j\,x_k}{|x|^2} * g_k\Big) g_j dx\Big) \\
&= \frac{|S^{n-1}|}{(2\pi)^n}\Big(\int \sum_{1\le j,k\le n}\Big(\frac{x_j\,x_k}{|x|^2} * g_j\Big) g_k\,dx - \frac{1}{n}\sum_{k=1}^{n}\Big(\int g_k\,dx\Big)^2\Big) \\
&= \frac{|S^{n-1}|}{(2\pi)^n}\int\!\!\int \Big(\mathcal{M}\Big(\frac{x-y}{|x-y|}\Big)\mathbf{g}(x), \mathbf{g}(y)\Big)dx\,dy, \qquad (59)
\end{aligned}
$$

where $\mathcal{M}(\omega)$ is the $(n\times n)$-matrix given by

$$
\mathcal{M}(\omega) = (\omega_j\,\omega_k - n^{-1}\delta_j^k)_{j,k=1}^n. \qquad (60)
$$

Since the norm of $\mathcal{M}(\omega)$ does not exceed $(n-1)\,n^{-1}$, it follows that the absolute value of the last double integral is not greater than

$$
\frac{|S^{n-1}|(n-1)}{(2\pi)^n\, n}\Big(\int |\mathbf{g}(x)|\,dx\Big)^2. \qquad (61)
$$

Hence (61) is a majorant for the left-hand side of (47). It remains to recall that $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$.

(ii) By (48) and (i),

$$
\begin{aligned}
\frac{|n-c_1|}{n}\limsup_{\varepsilon\to 0_+}\|\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-n/2}} \;&\le\; \frac{c_1}{n}\limsup_{\varepsilon\to 0_+}\Big|\|\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-n/2}} - n\|\mathrm{div}\,\mathbf{g}_\varepsilon\|^2_{\mathcal{H}^{-1-n/2}}\Big| \\
&+\; c_2\,\|\mathbf{g}\|^2_{L^1} \le c_3\,\|\mathbf{g}\|^2_{L^1}. \qquad (62)
\end{aligned}
$$

Since L^1 is not embedded into $\mathcal{H}^{-n/2}$, we have $c_1 = n$.

Suppose that (48) holds. Then $c_1 = n$ and by (52) and (53) the inequality

$$
(2\pi)^{-n}\Big|\sum_{1\le j,k\le n}\int \frac{\delta_j^k|\xi|^2 - n\,\xi_j\xi_k}{|\xi|^{n+2}}\hat{g}_j(\xi)\overline{\hat{g}_k(\xi)}\,d\xi\Big| \le c_2\,\|\mathbf{g}\|^2_{L^1} \qquad (63)
$$

holds for $\mathbf{g} \in C_0^\infty$ with the integral understood as the Cauchy value. It was shown in the proof of part (i) that (53) is equal to (59). Thus (63) can be written as the inequality

$$\frac{|S^{n-1}|}{(2\pi)^n}\Big|\int\int\Big(\mathcal{M}\Big(\frac{x-y}{|x-y|}\Big)\mathbf{g}(x), \mathbf{g}(y)\Big)dx\,dy\Big| \leq c_2\,\|\mathbf{g}\|^2_{L^1}, \tag{64}$$

where the matrix $\mathcal{M}$ is defined by (60). Let θ denote the north pole of S^{n-1}, i.e. $\theta = (0, \dots, 0, 1)$. We choose the vector function $\mathbf{g}$ in (64) as $(0, \dots, \eta(|x|)\varphi(x/|x|))$, where $\eta \in C_0^\infty([0,\infty))$, $\eta \geq 0$, and φ is a regularization of the δ-function on S^{n-1} concentrated at θ. Then (64) implies

$$\frac{|S^{n-1}|}{(2\pi)^n}\Big|\int_0^\infty\int_0^\infty m_{nn}\Big(\frac{\rho-r}{|\rho-r|}\theta\Big)\eta(r)\; r^{n-1}\,\eta(\rho)\,\rho^{n-1}drd\rho\Big|$$
$$\leq c_2\Big|\int_0^\infty \eta(r)r^{n-1}dt\Big|^2,$$

and since $m_{nn}(\pm\theta) = 1 - 1/n$, we obtain $c_2 \geq (1-1/n)(2\pi)^{-n}|S^{n-1}|$.

(iii) By (50), we change n^{-1} in (59) for $1/2$ and notice that the norm of the matrix $(\omega_j\omega_k - \delta_j^k/2)^n_{j,k=1}$ equals $1/2$. Inequality (51) follows. □

As an immediate consequence of Theorem 3 (iii), we derive

Corollary. *Let u be a scalar function in C_0^∞. Then*

$$\|u\|_{\mathcal{H}^{1-n/2}} \leq \Big(\frac{(2\sqrt{\pi})^{-n}}{\Gamma(n/2)(n-1)}\Big)^{1/2}\|\nabla u\|_{L^1}. \tag{65}$$

Proof. It suffices to put $\mathbf{g} = \nabla u$ in (51) and note that

$$\|\mathbf{g}\|_{\mathcal{H}^{-n/2}} = \|\nabla u\|_{\mathcal{H}^{-n/2}} = \|u\|_{\mathcal{H}^{1-n/2}}$$

and

$$\|\mathrm{div}\,\mathbf{g}\|_{\mathcal{H}^{-1-n/2}} = \|\Delta u\|_{\mathcal{H}^{-1-n/2}} = \|u\|_{\mathcal{H}^{1-n/2}}. \qquad \square$$

Remark 2. Passing from quadratic to sesquilinear forms in the proof of Theorem 3 (i) leads to the identity

$$(-\Delta)^{-n/2}\big(\mathbf{g} + n(-\Delta)^{-1}\nabla\mathrm{div}\,\mathbf{g}\big)(x) = \frac{2^{1-n}\pi^{-n/2}}{\Gamma(n/2)}\int\mathcal{N}\Big(\frac{x-y}{|x-y|}\Big)\mathbf{g}(y)\,dy \tag{66}$$

for all $\mathbf{g} \in C_0^\infty$ orthogonal to 1. The kernel $\mathcal{N}(\omega)$ is the matrix function $(\omega_j\omega_k)^n_{j,k=1}$. Needless to say, if additionally $\mathbf{g}$ is divergence free, we have the representation

$$(-\Delta)^{-n/2}\mathbf{g}(x) = \frac{2^{1-n}\pi^{-n/2}}{\Gamma(n/2)}\int\mathcal{N}\Big(\frac{x-y}{|x-y|}\Big)\mathbf{g}(y)\,dy. \tag{67}$$

Another consequence of the identity (66) is obtained by putting $\mathbf{g} = \nabla u$ in it, where u is a scalar function in C_0^∞. Then

$$(-\Delta)^{-n/2}\nabla u(x) = \frac{2^{1-n}\pi^{-n/2}}{(1-n)\Gamma(n/2)}\int\mathcal{N}\Big(\frac{x-y}{|x-y|}\Big)\nabla u(y)\,dy. \tag{68}$$

Remark 3. If $\operatorname{div}\mathbf{g} \in \mathcal{H}^{-1-n/2}$ and $\mathbf{g} \in L^1$, then $\mathbf{g}$ is orthogonal to one (see the beginning of the proof of Theorem 2). On the other hand, even if $\mathbf{g} \in C_0^\infty$ but

$$\int \mathbf{g}(y)\,dy \neq 0,$$

both norms $\|\operatorname{div}\mathbf{g}\|_{\mathcal{H}^{-1-n/2}}$ and $\|\mathbf{g}\|_{\mathcal{H}^{-n/2}}$ are infinite. The estimate (47) shows that the formal expression

$$\|\mathbf{g}\|^2_{\mathcal{H}^{-n/2}} - n\|\operatorname{div}\mathbf{g}\|^2_{\mathcal{H}^{-1-n/2}} \tag{69}$$

can be given sense as the finite limit $\varepsilon \to 0_+$on the left-hand side of (47). One can see that the limit does not change if (46) is replaced by

$$\mathbf{g}_\varepsilon(x) = \mathbf{g}(x) - \varepsilon^n\,\eta(\varepsilon x)\int \mathbf{g}(y)\,dy,$$

where η is an arbitrary function in the Schwartz space $\mathcal{S}$ normalized by

$$\int \eta(y)\,dy = 1.$$

By Theorem 3 (iii) and a duality argument, similar to that used in [BB3], one can arrive at the following existence result which is supplied with a proof for reader's convenience.

Proposition 3. *For any vector function* $\mathbf{u} \in \mathcal{H}^{n/2}$ *there exists a vector function* $\mathbf{v} \in L^\infty$ *and a scalar function* $\varphi \in \mathcal{H}^{1+n/2}$ *satisfying* $\mathbf{u} = \mathbf{v} + \operatorname{grad}\varphi$.

Proof. By $\mathcal{B}$ we denote the Banach space of the pairs $\{\mathbf{g}, k\} \in L^1 \times \mathcal{H}^{-1-n/2}$ endowed with the norm

$$\|\{\mathbf{g}, k\}\|_{\mathcal{B}} = \|\mathbf{g}\|_{L^1} + \|k\|_{\mathcal{H}^{-1-n/2}}.$$

Representing $\{\mathbf{g}, k\}$ as $\{\mathbf{g}, 0\} + \{0, k\}$, we see that an arbitrary linear functional on $\mathcal{B}$ can be given by

$$\int (\mathbf{v},\,\mathbf{g})\,dx + \int \varphi\,k\,dx, \tag{70}$$

where $\mathbf{v} \in L^\infty$ and $\varphi \in \mathcal{H}^{1+n/2}$. The range of the operator

$$L^1 \cap \mathcal{H}^{-n/2} \ni \mathbf{g} \to \{\mathbf{g}, -\operatorname{div}\mathbf{g}\},$$

which is a closed subspace of $\mathcal{B}$, will be denoted by S.

Any vector-valued function $\mathbf{u} \in \mathcal{H}^{n/2}$ generates the continuous functional

$$f(\mathbf{g}) = \int (\mathbf{u},\,\mathbf{g})\,dx \tag{71}$$

on the space $\mathcal{H}^{-n/2}$. By (51),

$$|f(\mathbf{g})| \leq c_n\,\|\mathbf{u}\|_{\mathcal{H}^{n/2}}\big(\|\mathbf{g}\|_{L^1} + \|\operatorname{div}\mathbf{g}\|_{\mathcal{H}^{-1-n/2}}\big). \tag{72}$$

We introduce the functional Φ by

$$\Phi(\{\mathbf{g}, k\}) := f(\mathbf{g}) \quad \text{for} \quad k = -\operatorname{div}\mathbf{g},$$

i.e. Φ is defined on S. Being prescribed on a closed subspace of $\mathcal{B}$, this functional is bounded in the norm of $\mathcal{B}$ because of (72). By the Hahn-Banach theorem, Φ can be extended with preservation of the norm onto the whole space $\mathcal{B}$. Using (70) and (71), we see that there exist $\mathbf{v} \in L^\infty$ and $\varphi \in \mathcal{H}^{1+n/2}$ such that, for all $\mathbf{g} \in L^1 \cap \mathcal{H}^{-n/2}$,

$$\int (\mathbf{u}, \mathbf{g})\, dx = \int \big((\mathbf{v}, \mathbf{g}) - \varphi \operatorname{div} \mathbf{g}\big) dx.$$

The result follows. □

The next assertion guarantees the existence of the solution $\mathbf{u} \in \mathcal{H}^{2-n/2}$ to the equation $-\Delta\mathbf{u} = \mathbf{f}$ provided that $\mathbf{f}$ *is a vector field in* L^1 *subject to* $\operatorname{div} \mathbf{f} \in \mathcal{H}^{-1-n/2}$.

Proposition 4. *Under the condition on* $\mathbf{f}$ *just mentioned, the inequality*

$$\Big| \|(-\Delta)^{-1}\mathbf{f}\|^2_{\mathcal{H}^{2-n/2}} - n\|\operatorname{div} \mathbf{f}\|^2_{\mathcal{H}^{-1-n/2}} \Big| \le \frac{(2\sqrt{\pi})^{-n}}{\Gamma(n/2)} \|\mathbf{f}\|^2_{L^1} \tag{73}$$

holds.

Proof. It suffices to replace $\mathbf{g}$ by $\mathbf{f}$ in (51). □

In the forthcoming Theorem 4 we obtain an estimate which leads by duality to the following existence result. Its proof is quite similar to that of Proposition 3 and is omitted.

Proposition 5. *Let* $\mathbf{f}$ *be a divergence free vector function in* $\mathbb{R}^3$ *from the space* $\mathcal{H}^{1/2}$. *Then the equation*

$$\operatorname{curl} \mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^3$$

has a solution in $\mathcal{H}^{3/2} \cap L^\infty$.

Theorem 4. *Let*

$$\operatorname{curl} \mathbf{w} = \mathbf{f} + \mathbf{g} \quad \text{in } \mathbb{R}^3, \tag{74}$$

where

$$\operatorname{div} \mathbf{w} = 0, \quad \mathbf{f} \in \mathcal{H}^{-3/2}(\mathbb{R}^3)$$

and

$$\mathbf{g} \in L^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \mathbf{g}(y) dy = 0.$$

Then

$$\Big| \|\Delta\mathbf{w} + \operatorname{curl} \mathbf{f}\|^2_{\mathcal{H}^{-5/2}} - 2\, \|\operatorname{div} \mathbf{f}\|^2_{\mathcal{H}^{-5/2}} \Big| \le \frac{1}{4\pi^2} \Big(\int_{\mathbb{R}^3} |\mathbf{g}(x)| dx \Big)^2. \tag{75}$$

Proof. Since $\operatorname{curl}^2 \mathbf{w} = -\Delta\mathbf{w}$, we have by (74) that $-\Delta\mathbf{w} = \operatorname{curl} \mathbf{f} + \operatorname{curl} \mathbf{g}$. Using the identity $\operatorname{div} \operatorname{curl} \mathbf{w} = 0$, we see that $\operatorname{div} \mathbf{f} + \operatorname{div} \mathbf{g} = 0$. Therefore,

$$\|\Delta\mathbf{w} + \operatorname{curl} \mathbf{f}\|^2_{\mathcal{H}^{-5/2}} - 2\, \|\operatorname{div} \mathbf{f}\|^2_{\mathcal{H}^{-5/2}} = \|\operatorname{curl} \mathbf{g}\|^2_{\mathcal{H}^{-5/2}} - 2\, \|\operatorname{div} \mathbf{g}\|^2_{\mathcal{H}^{-5/2}}.$$

The right-hand side can be written in the form

$$(2\pi)^{-3} \Big| \int_{\mathbb{R}^3} \Big(|\xi \times \hat{\mathbf{g}}|^2 - 2\, |(\xi, \hat{\mathbf{g}})|^2 \Big) \frac{d\xi}{|\xi|^5} \Big| = (2\pi)^{-3} \Big| \int_{\mathbb{R}^3} \Big(|\xi|^2 |\hat{\mathbf{g}}|^2 - 3\, |(\xi, \hat{\mathbf{g}})|^2 \Big) \frac{d\xi}{|\xi|^5} \Big|.$$

This value is a particular case of (52) for $n = 3$ and hence it does not exceed

$$\frac{1}{4\pi^2} \Big(\int_{\mathbb{R}^3} |\mathbf{g}(x)|\, dx \Big)^2$$

(see the proof of Theorem 3 (iii)). $\square$

Remark 5. It is natural to ask how the results of the present section change if the role of the homogeneous space $\mathcal{H}^l$ is played by the standard Sobolev space H^l endowed with the norm

$$\|\phi\|_{H^l} := \Big(\int |\hat{\phi}(\xi)|^2(|\xi|^2+1)^{l/2}d\xi\Big)^{1/2}.$$

Restricting ourselves to Theorem 3, we check directly that

$$\Big|\lim_{\varepsilon\to 0_+}\big(\|\mathbf{g}_\varepsilon\|^2_{H^{-n/2}} - n\,\|\operatorname{div}\mathbf{g}_\varepsilon\|^2_{H^{-1-n/2}}\big)\Big|$$

$$= (2\pi)^{-n}\Big|\sum_{1\le j,k\le n}\int \frac{\delta_j^k(|\xi|^2+1) - n\,\xi_j\xi_k}{(|\xi|^2+1)^{1+n/2}}\hat{g}_j(\xi)\overline{\hat{g}_k(\xi)}\,d\xi\Big|,$$

which in its turn is equal to

$$(2\pi)^{-n}(n-2)^{-1}\Big|\sum_{1\le j,k\le n}\int \frac{\partial^2}{\partial\xi_j\partial\xi_k}(|\xi|^2+1)^{(2-n)/2}\hat{g}_j(\xi)\overline{\hat{g}_k(\xi)}\,d\xi\Big|$$

$$= c\Big|\int \sum_{1\le j,k\le n}\frac{x_j-y_j}{|x-y|}\frac{x_k-y_k}{|x-y|}|x-y|K_1(|x-y|)g_j(x)g_k(y)\,dxdy\Big|,$$

where K_1 is the modified Bessel function of the third kind. Since the function $t\,K_1(t)$ is bounded, we obtain

$$\Big|\lim_{\varepsilon\to 0_+}\big(\|\mathbf{g}_\varepsilon\|^2_{H^{-n/2}} - n\,\|\operatorname{div}\mathbf{g}_\varepsilon\|^2_{H^{-1-n/2}}\big)\Big| \le c(n)\Big(\int |\mathbf{g}(x)|\,dx\Big)^2.$$

Needless to say, this inequality becomes

$$\Big|\|\mathbf{g}\|^2_{H^{-n/2}} - n\,\|\operatorname{div}\mathbf{g}\|^2_{H^{-1-n/2}}\Big| \le c(n)\Big(\int |\mathbf{g}(x)|\,dx\Big)^2$$

if the last norm of $\operatorname{div}\mathbf{g}$ is finite.

Acknowledgement

I cordially thank Haïm Brezis whose questions infused me with the idea to write this article. I gratefully acknowledge referee's poignant comments.

References

[BB1] J. Bourgain, H. Brezis, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16**:2 (2002), 393–426.

[BB2] J. Bourgain, H. Brezis, *New estimates for the Laplacian, the* div-curl*, and related Hodge systems*, C. R. Math. Acad. Sci. Paris, **338** (2004), 539–543.

[BB3] J. Bourgain, H. Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. **9** (2007), 277–315.

[BV1] H. Brezis, J. Van Schaftingen, *Boundary estimates for elliptic systems with L^1-data.*, Calc. Var. Partial Diff. Eq. **30**, no. 3 (2007), 369–388.

[BV2] H. Brezis, J. Van Schaftingen, *Circulation integrals and critical Sobolev spaces: problems of optimal constants*, to appear in Proc. Symp. in Pure Math., 2008, AMS.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.

[LS] L. Lanzani, E. Stein, *A note on* div-curl *inequalities*, Math. Res. Lett. **12**:1 (2005), 57–61.

[Ma1] V. Maz'ya, *Sobolev Spaces*, Springer, 1985.

[Ma2] V. Maz'ya, *Bourgain-Brezis type inequality with explicit constants*, Interpolation Theory and Applications, Contemporary Mathematics, vol. 445, pp. 247–264, 2007, AMS.

[MS] V. Maz'ya, T. Shaposhnikova, *Collection of sharp dilation invariant integral inequalities for differentiable functions*, to appear in the book: Sobolev Spaces in Mathematics I. Sobolev Type Inequalities, pp. 223–248, 2008, Springer.

[St1] E.M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. **8** (1957), 250–254.

[St2] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N. J., 1970.

[VS1] J. Van Schaftingen, *A simple proof of an inequality of Bourgain, Brezis and Mironescu*, C.R. Math. Acad. Sci. Paris **338** (2004), 23–26.

[VS2] J. Van Schaftingen, *Estimates for L^1 vector fields*, C. R. Math. Acad. Sci. Paris **339** (2004), 181–186.

[VS3] J. Van Schaftingen, *Estimates for L^1 vector fields with a second order condition*, Acad. Roy. Belg. Bull. Cl. Sci. **15** (2004), 103–112.

[VS4] J. Van Schaftingen, *Estimates for L^1 vector fields under higher-order differential conditions*, J. Eur. Math. Soc. (to appear).