On eigenfunctions of the Fourier transform

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Abstract. In [6] we considered a nontrivial example of eigenfunction in the sense of distribution for the planar Fourier transform. Here a method to obtain other eigenfunctions is proposed. Moreover we consider positive homogeneous eigenfunctions of order $n/2$. We show that $F(\omega)|\mathbf{x}|^{-n/2}$, $|\omega| = 1$, is an eigenfunction in the sense of distribution of the Fourier transform if and only if $F(\omega)$ is an eigenfunction of a certain singular integral operator on the unit sphere of $\mathbb{R}^n$. Since $Y^{(k)}_{m,n}(\omega)|\mathbf{x}|^{-n/2}$ are eigenfunctions of the Fourier transform, we deduce that $Y^{(k)}_{m,n}$ are eigenfunctions of the above mentioned singular integral operator. Here $Y^{(k)}_{m,n}$ denote the spherical functions of order m in $\mathbb{R}^n$. In the planar case, we give a description of all eigenfunctions of the Fourier transform of the form $F(\omega)|\mathbf{x}|^{-1}$ by means of the Fourier coefficients of $F(\omega)$.

1 Introduction

The Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ is defined as

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{-i(\mathbf{x},\xi)} d\mathbf{x}\,,$$

$(\mathbf{x}, \xi) = x_1\xi_1 + ... + x_n\xi_n$ denoting the standard inner product of $\mathbf{x}$ and ξ in $\mathbb{R}^n$. The inverse of the Fourier transform is given by

$$\mathcal{F}^{-1}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{e}^{i(\mathbf{x},\xi)} d\mathbf{x}\,.$$

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The Fourier transform is an isomorphism of the space $L^2(\mathbb{R}^n)$ and, for every $f, g \in L^2(\mathbb{R}^n)$, we have Parseval formula

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x})d\mathbf{x}\,.$$

In particular,

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}\,.$$

Here $\|\cdot\|_{L^2}$ denotes the norm in the space $L^2(\mathbb{R}^n)$. As a consequence the linear map $\mathcal{F}$ defines a unitary operator on $L^2(\mathbb{R}^n)$, so its spectrum lies on the unit circle in $\mathbb{C}$. Since $\mathcal{F}^4 f = f$, if $\lambda \in \mathbb{C}$ is an eigenvalue then $\lambda^4 = 1$. So $\lambda \in \{1, -1, i, -i\}$. Each of these values is an eigenvalue of infinite multiplicity. In dimension 1 a complete orthonormal set of eigenfunctions is given by the Hermite functions

$$\Phi_m(x) = \frac{1}{(\sqrt{\pi}2^m m!)^{1/2}} H_m(x)\,\mathrm{e}^{-x^2/2}, \qquad m \geq 0$$

with the Hermite polynomial

$$H_m(x) = (-1)^m \mathrm{e}^{x^2} \frac{d^m}{dx^m} \mathrm{e}^{-x^2}\,.$$

They satisfy $\mathcal{F}(\Phi_m) = (-i)^m \Phi_m$ (see [9]). In higher dimensions, the eigenfunctions of the Fourier transform can be obtained by taking tensor products of Hermite functions, one in each coordinate variable. That is

$$\Phi_{\mathbf{m}}(\mathbf{x}) = \prod_{j=1}^{n} \Phi_{m_j}(x_j)$$

are eigenfunctions corresponding to the eigenvalues $(-i)^{m_1+\ldots+m_n}$.

In this paper we are interested in non standard eigenfunctions i.e. eigenfunctions in the sense of distributions. In general such eigenfunctions do not belong to $L^2(\mathbb{R}^n)$. An interesting example is provided by $1/|\mathbf{x}|^{n/2}$ ([5, p.363]). In [6] we have showed that

$$\frac{\sqrt{x_1^2 + x_2^2}}{x_1 x_2}$$

is a eigenfunction in the sense of distribution in $\mathbb{R}^2$. In section 2 we propose a method to obtain other eigenfunctions (theorems 2.2 and 2.4). We find, for example, that

$$\frac{8x_1x_2}{(x_1^2 - x_2^2)^2}\sqrt{x_1^2 + x_2^2} \quad \text{or} \quad \frac{x_2^2 - x_1^2}{x_1^2 x_2^2}\sqrt{x_1^2 + x_2^2}$$

are eigenfunctions. In section 3 we consider positive homogeneous distributions of order $n/2$ that is

$$\frac{F(\omega)}{|\mathbf{x}|^{n/2}}, \quad \omega = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad \mathbf{x} \in \mathbb{R}^n \,. \tag{1.1}$$

We show that (1.1) is an eigenfunction of the Fourier transform if and only if $F(\omega)$ is an eigenfunction of a certain singular integral operator on the unit sphere of $\mathbb{R}^n$ (theorem 3.3). Since $Y_{m,n}^{(k)}(\omega)|\mathbf{x}|^{-n/2}$ are eigenfunctions of the Fourier transform (theorem 3.1), we deduce that $Y_{m,n}^{(k)}$ are eigenfunctions of the above mentioned singular integral operator. Here $Y_{m,n}^{(k)}$ denote the spherical functions of order m in $\mathbb{R}^n$. In section 4 we give a description of all eigenfunctions of the planar Fourier transform of the form (1.1) by means of the Fourier coefficients of $F(\omega)$ (theorem 4.1).

2 Fourier transform of distributions

Let $\mathcal{D}$ be the space $C_0^\infty(\mathbb{R}^n)$ of all real functions with continuous derivative of all order and with compact support. A sequence $\{\Phi_k\} \in \mathcal{D}$ converges to $\Phi \in \mathcal{D}$ if there is a compact $K \subset \mathbb{R}^n$ with $\mathrm{supp}\Phi_k \subseteq K$, $\mathrm{supp}\Phi \subseteq K$ and $\partial^\alpha \Phi_k$ converges uniformly to $\partial^\alpha \Phi$ on K for all $\alpha = (\alpha_1, ..., \alpha_n)$ multi-index. We shall denote by $\mathcal{D}'$ the set of all continuous linear functionals on $\mathcal{D}$. The elements of $\mathcal{D}'$ are called generalized functions or distributions. We write the action of a distribution f on a test function Φ as (f, Φ) with the property that if Φ_k converges to Φ in $\mathcal{D}$ then

$$\lim_{k \to +\infty} (f, \Phi_k) = (f, \Phi) \,.$$

If $f \in L^1_{loc}(\mathbb{R}^n)$ the functional

$$(f, \Phi) = \int_{\mathbb{R}^n} f(\mathbf{x}) \Phi(\mathbf{x}) d\mathbf{x} \tag{2.1}$$

belongs to the space $\mathcal{D}'$ and distributions of the form (2.1) are called regular distributions. If $f \notin L^1_{loc}(\mathbb{R}^n)$, if the integral exists in the Cauchy sense, then (2.1) still defines a distribution called Cauchy principal value distribution (cf., e.g., [5, p.10,p.46]).

We denote by $\mathcal{S}$ the Schwartz space of functions $\Phi \in C^\infty(\mathbb{R}^n)$ rapidly decaying at infinity that is, for any multi-indeces α and β,

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}^\alpha \partial^\beta \Phi(\mathbf{x})| = 0 \,.$$

A sequence $\{\Phi_k\} \in \mathcal{S}$ converges to $\Phi \in \mathcal{S}$ if, for any multi-indeces α and β,

$$\lim_{k\to\infty} \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha D^\beta \Phi_k(\mathbf{x}) - \mathbf{x}^\alpha D^\beta \Phi(\mathbf{x})| = 0\,.$$

We denote by $\mathcal{S}'$ the class of continuous linear functionals $f : \mathcal{S} \to \mathbb{C}$. Elements of $\mathcal{S}'$ are called tempered distributions. Obviously $\mathcal{D}' \subset \mathcal{S}'$, moreover $\mathcal{D}'$ is dense in $\mathcal{S}'$. Let $f \in \mathcal{S}'$. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be an $n-$tuple of nonnegative integers and $|\alpha| = \alpha_1 + ... + \alpha_n$. The distribution $\partial^\alpha f$ is defined by

$$(\partial^\alpha f, \Phi) = (-1)^{|\alpha|}(f, \partial^\alpha \Phi), \qquad \Phi \in \mathcal{S}\,.$$

To introduce some notation, we define the reflection in the origin

$$(r_0\Phi)(\mathbf{x}) = \Phi(-\mathbf{x}), \qquad \Phi \in \mathcal{S}\,,$$

and the translation through the vector $\mathbf{h} \in \mathbb{R}^n$

$$\tau_{\mathbf{h}}\Phi(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{h}), \qquad \Phi \in \mathcal{S}\,.$$

The reflection of a distribution $f \in \mathcal{S}'$ is a distribution defined by

$$(r_0 f, \Phi) = (f, (r_0\Phi)), \qquad \Phi \in \mathcal{S}$$

and the translation of $f \in \mathcal{S}'$ is a distribution defined by duality

$$(\tau_{\mathbf{h}} f, \Phi) = (f, \tau_{-\mathbf{h}}\Phi), \qquad \Phi \in \mathcal{S}\,.$$

The Fourier transform is a continuous isomorphism of $\mathcal{S}$ onto $\mathcal{S}$. This allows to define the Fourier transform of tempered distributions.

Definition 2.1. *Let $f \in \mathcal{S}'$. Its Fourier transform $\widehat{f}$ (or $\mathcal{F}(f)$) is the tempered distribution*

$$(\widehat{f}, \Phi) = (f, \widehat{\Phi}), \qquad \Phi \in \mathcal{S}\,.$$

The formulas for the derivatives or for the translation of the Fourier transforms are preserved also for distributions. If α is a multi-index and $\mathbf{h} \in \mathbb{R}^n$ then the Fourier transform of $f \in \mathcal{S}'$ has the following properties, in the sense of distribution,

$$\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha \mathcal{F}(f),$$
$$\mathcal{F}(\mathbf{x}^\alpha f) = i^{|\alpha|}\partial^\alpha \mathcal{F}(f),$$
$$\mathcal{F}(\tau_{\mathbf{h}} f)(\xi) = \mathrm{e}^{-i\mathbf{h}\cdot\xi}\mathcal{F}(f)(\xi),$$
$$\mathcal{F}(\mathrm{e}^{i\mathbf{h}\cdot\mathbf{x}} f) = \tau_{\mathbf{h}}(\mathcal{F}(f))\,.$$

Definition 2.2. *The distribution $f \in \mathcal{S}'$ is an eigenfunction of the Fourier transform corresponding to the eigenvalue λ if*

$$(\widehat{f}, \Phi) = \lambda(f, \Phi), \quad \forall \Phi \in \mathcal{S}\,. \tag{2.2}$$

An example of eigenfunction understood in the sense of distribution, with eigenvalue 1, is provided by the generalized function $1/|\mathbf{x}|^{n/2}$ (see [3, p.71] and [5]). $1/|\mathbf{x}|^{n/2}$ does not define a regular distribution because it has a nonsommable singularity at the origin and the integral (2.1) can be defined by analytic continuation (see [5, p.71]). In [6] we proposed an example of non standard eigenfunction of the planar Fourier transform. The distribution

$$f(x_1, x_2) = \frac{|\mathbf{x}|}{x_1 x_2}, \qquad |\mathbf{x}| = \sqrt{x_1^2 + x_2^2} \tag{2.3}$$

defines a Cauchy principal value distribution in $\mathcal{S}$. We define the action on a test function $\Phi \in \mathcal{S}$ as

$$(f, \Phi) = \lim_{\epsilon \to 0} \int_{\substack{|x_1|>\epsilon \\ |x_2|>\epsilon}} \frac{|\mathbf{x}|}{x_1 x_2} \Phi(\mathbf{x}) d\mathbf{x} = \lim_{\epsilon \to 0} \int_{\substack{x_1>\epsilon \\ x_2>\epsilon}} \Psi(x_1, x_2) dx_1 dx_2 = \iint_{\mathbb{R}_+^2} \Psi(\mathbf{x}) d\mathbf{x}$$

where

$$\begin{aligned}\Psi(x_1, x_2) &= |\mathbf{x}| \frac{\Phi(x_1, x_2) - \Phi(x_1, -x_2) - \Phi(-x_1, x_2) + \Phi(-x_1, -x_2)}{x_1 x_2} \\ &= \frac{|\mathbf{x}|}{x_1 x_2} \int_{-x_2}^{x_2} \int_{-x_1}^{x_1} \Phi_{\xi\eta}(\xi, \eta) d\xi d\eta = |\mathbf{x}| \int_{-1}^{1} \int_{-1}^{1} \Phi_{\xi\eta}(t x_1, s x_2) dt ds\,.\end{aligned}$$

The eigenfunction (2.3) gives rise to a one parametric family of eigenfunctions. Indeed, denote by

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

the rotation matrix where α is the rotation angle in the counterclockwise direction. Then $(\mathcal{F}\Phi(R\cdot))(\xi) = (\mathcal{F}\Phi(\cdot))(R\xi)$. If $f \in \mathcal{S}'$ is an eigenfunction, then also the rotation f_R defined by

$$(f_R, \Phi) = (f, \Phi(R\cdot)), \qquad \Phi \in \mathcal{S}$$

is an eigenfunction. As a consequence

$$f(\alpha, x_1, x_2) = \frac{\sqrt{x_1^2 + x_2^2}}{(x_1 \cos\alpha + x_2 \sin\alpha)(-x_1 \sin\alpha + x_2 \cos\alpha)} \tag{2.4}$$

is an eigenfunction for the planar Fourier transform for any value of the parameter α. For example, if we choose $\alpha = \pi/4$ we obtain

$$\frac{\sqrt{x_1^2 + x_2^2}}{x_1^2 - x_2^2}\,.$$

With the change of variable $a = \tan(\alpha)$, we obtain the family of eigenfunctions

$$F(a, x_1, x_2) = \frac{\sqrt{x_1^2 + x_2^2}}{(x_1 + a x_2)(-x_1 a + x_2)}\,. \qquad (2.5)$$

If $a \neq 0$, putting $b = (a^2 - 1)/a$ and denoting by $P_b(x_1, x_2) = x_1^2 - x_2^2 + b x_1 x_2$ the homogeneous harmonic polynomial of degree 2 we get the following family of eigenfunctions

$$\phi(b, x_1, x_2) = \frac{\sqrt{x_1^2 + x_2^2}}{P_b(x_1, x_2)} = \frac{|\mathbf{x}|}{P_b(\mathbf{x})}\,. \qquad (2.6)$$

Definition 2.3. *The distribution $f \in \mathcal{S}'$ is an eigenfunction of the continuous spectrum for $\mathcal{F}$ if there is a sequence $\{\Phi_k\} \in \mathcal{S}$ such that the following conditions are satisfied:*

$$\lim_{k\to\infty} (\mathcal{F}(\Phi_k) - \lambda\Phi_k, \Phi) = 0, \qquad \forall \Phi \in \mathcal{S};$$

$$\lim_{k\to\infty} (\Phi_k, \Phi) = (f, \Phi), \qquad \forall \Phi \in \mathcal{S}\,.$$

The next theorem shows the relation between eigenfunctions in the sense of distribution and eigenfunctions of the continuous spectrum.

Theorem 2.1. *Eigenfunctions in the sense of distribution are eigenfunctions of the continuous spectrum.*

Proof. We consider a function $\rho(\mathbf{x}) \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int_{\mathbb{R}^n} \rho(\mathbf{x}) d\mathbf{x} = 1$. For $k \geq 1$ we consider the regularizing family of functions

$$\rho_k(\mathbf{x}) = k^n \rho\,(k\,\mathbf{x})\,.$$

For all $\Phi \in \mathcal{S}$, the convolution $\rho_k * \Phi$ tends to Φ in $\mathcal{S}$ as k tends to $+\infty$. The convolution of ρ_k and $f \in \mathcal{S}'$ is given by the formula

$$(\rho_k * f)(\mathbf{x}) = (f(\mathbf{y}), \rho_k(\mathbf{x} - \mathbf{y}))\,.$$

$\rho_k * f$ belongs to $C^\infty(\mathbb{R}^n)$ and every derivative has at most polynomial growth. Moreover $\rho_k * f$ converges to f in $\mathcal{S}'$ when $k \to \infty$. Indeed, since $\rho_k * \Phi$ tends to Φ in $\mathcal{S}$, if we replace Φ by the reflected $\Psi = r_0\Phi$ we can write

$$\lim_{k\to\infty}(\rho_k * f, \Phi) = \lim_{k\to\infty}(\rho_k * f, r_0\Psi) = \lim_{k\to\infty}(f, r_0(\rho_k * \Psi)) = (f, \Phi), \quad \forall \Phi \in \mathcal{S}\,.$$

Since f satisfies (2.2) we have, for any $\Phi \in \mathcal{S}$

$$\begin{aligned}\lim_{k\to\infty}(\mathcal{F}(\rho_k * f) - \lambda(\rho_k * f), \Phi) &= \lim_{k\to\infty}(\rho_k * f, \widehat{\Phi}) - \lambda(\rho_k * f), \Phi) \\ &= (f, \widehat{\Phi}) - \lambda(f, \Phi) = 0\,.\end{aligned}$$

□

Let $f, g \in \mathcal{D}'$ and suppose that at least one has bounded support. We define the convolutional distribution

$$(f * g, \Phi) = (f(\mathbf{x}) \times g(\mathbf{y}), \Phi(\mathbf{x} + \mathbf{y})), \qquad \forall \Phi \in \mathcal{D}$$

(cf. e.g. [5, p.103] or [1, p.89]). If $f, g, h \in \mathcal{D}'$ and at least two of them have bounded support, then $f * g * h \in \mathcal{D}'$ and the convolution product is associative

$$f * g * h = f * (g * h) = (f * g) * h\,.$$

The next theorem shows that if we convolve (with respect to the parameter) a parametric family of eigenfunctions in the sense of distribution and a distribution we get again an eigenfunction.

Theorem 2.2. *Let $f(a, \mathbf{x})$ be a family of distributions in $\mathcal{S}'$ depending on a parameter $a \in \mathbb{R}$. Suppose that*
i. for fixed $a \in \mathbb{R}$, $f(a, \cdot) \in \mathcal{S}'$ is an eigenfunction in the sense of distribution;
ii. for fixed $\mathbf{x} \in \mathbb{R}^n$, $f(\cdot, \mathbf{x}) \in \mathcal{D}'$ is a distribution on the real line.
Let $g(a) \in \mathcal{D}'$ be a distribution on the real line with bounded support. For fixed $\mathbf{x} \in \mathbb{R}^n$ consider the convolution $g(a) *_a f(a, \mathbf{x})$ defined as

$$(g(a) *_a f(a, \mathbf{x}), \psi) = (g(a) \times f(b, \mathbf{x}), \psi(a + b)), \quad \forall \psi \in C_0^\infty(\mathbb{R})\,.$$

Then, for fixed $a \in \mathbb{R}$, $g(a) *_a f(a, \mathbf{x}) \in \mathcal{S}'$ and it is an eigenfunction in the sense of distribution of the Fourier transform.

Proof. $g(a) *_a f(a, \mathbf{x})$ is well defined as an element of $\mathcal{S}'$. By hypothesis

$$(f(a, \cdot), \widehat{\Phi}) = \lambda(f(a, \cdot), \Phi), \qquad \forall \Phi \in \mathcal{S}\,.$$

If we convolve both terms by g we get

$$\begin{aligned}&(g(a) *_a f(a,\mathbf{x}), \widehat{\Phi}(\mathbf{x})) = ((g(a) \times f(b,\mathbf{x}), \psi(a+b)), \widehat{\Phi}(\mathbf{x}))\\ &= ((g(a) \times (f(b,\mathbf{x}), \widehat{\Phi}(\mathbf{x})), \psi(a+b)) = \lambda((g(a) \times (f(b,\mathbf{x}), \Phi(\mathbf{x})), \psi(a+b))\\ &= \lambda(g(a) \times f(b,\mathbf{x}), \psi(a+b)), \Phi(\mathbf{x})) = \lambda(g(a) *_a f(a,x), \Phi)\end{aligned}$$

which proves the theorem. □

Example 2.1. *The convolution $D\delta *_a f(a,\mathbf{x})$ is well defined, where D is any differential operator and δ is the delta function*

$$(\delta, \psi) = \psi(0), \qquad \forall \psi \in C_0^\infty(\mathbb{R})\,.$$

This follows from the fact that $D\delta$ is concentrated in one point. By definition

$$\begin{aligned}(D\delta *_a f(\cdot,\mathbf{x}), \psi) &= (D\delta(b) \times f(a,\mathbf{x}), \psi(a+b))\\ &= (f(a,\mathbf{x}), (D\delta(b), \psi(a+b)) = (f(a,\mathbf{x}), D\psi(a))\\ &= (D_a f(a,\mathbf{x}), \psi(a)) = (D_a f(\cdot,\mathbf{x}), \psi)\,.\end{aligned}$$

Thus we have

$$D\delta *_a f(a,\mathbf{x}) = D_a f(a,\mathbf{x})\,. \tag{2.7}$$

Let us apply theorem 2.2 and (2.7) to the family of eigenfunctions (2.4). If we derive with respect to the parameter α we obtain the family of eigenfunctions

$$\frac{\cos(2\alpha)\left(x_1^2 - x_2^2\right) + 2x_1x_2\sin(2\alpha)}{(x_2\cos(\alpha) - x_1\sin(\alpha))^2(x_1\cos(\alpha) + x_2\sin(\alpha))^2}\sqrt{x_1^2 + x_2^2}\,.$$

For $\alpha = \pi/4$ we get the eigenfunction

$$\frac{8x_1x_2}{(x_1^2 - x_2^2)^2}\sqrt{x_1^2 + x_2^2}$$

whereas, for $\alpha = 0$,

$$\frac{x_1^2 - x_2^2}{x_1^2x_2^2}\sqrt{x_1^2 + x_2^2}\,.$$

If we take the second and third derivative at $\alpha = 0$ we obtain, respectively,

$$\frac{2\left(x_1^4 + x_2^4\right)}{x_1^3x_2^3}\sqrt{x_1^2 + x_2^2}, \qquad \left(6\frac{x_1^6 - x_2^6}{x_1^4x_2^4} + 2\frac{x_1^4x_2^2 - x_1^2x_2^4}{x_1^4x_2^4}\right)\sqrt{x_1^2 + x_2^2}$$

and, at $\alpha = \pi/4$ *,*

$$-\frac{8\left(x_1^4+6x_1^2x_2^2+x_2^4\right)}{\left(x_1^2-x_2^2\right)^3}\sqrt{x_1^2+x_2^2}, \qquad \frac{32\left(5x_1^5x_2+14x_1^3x_2^3+5x_1x_2^5\right)}{\left(x_1^2-x_2^2\right)^4}\sqrt{x_1^2+x_2^2}.$$

In this way we can obtain many eigenfunctions for the planar Fourier Transform starting from (2.4)*.*

Example 2.2. *The integration of an eigenfunction* $f(a, \mathbf{x})$ *with respect to any Borel measure* $\mu(a)$ *on* $\mathbb{R}$ *gives rise to new eigenfunctions. Indeed*

$$(\mu, \psi) = \int_{\mathbb{R}} \psi(a) d\mu(a)$$

defines a distribution on $C_0^\infty(\mathbb{R})$*. If* $\mu(a)$ *has bounded support then*

$$\begin{aligned}(\mu *_a f(\cdot, \mathbf{x}), \psi) &= (\mu(a) \times f(b, \mathbf{x}), \psi(a+b)) = (\mu(a), (f(b, \mathbf{x}), \psi(a+b))) \\ &= (\mu(a), (f(c-a, \mathbf{x}), \psi(c)) = ((\mu(a), f(c-a, \mathbf{x})), \psi(c))\,.\end{aligned}$$

Then, for any fixed $c \in \mathbb{R}$

$$\mu *_a f(\cdot, \mathbf{x}) = \int_{\mathbb{R}} f(c-a, \mathbf{x}) d\mu(a)$$

is an eigenfunction.
As an example, consider the family of eigenfunctions (2.6)*. New eigenfunctions are given by*

$$\mu(b) *_b \phi(b, \mathbf{x}) = \int_0^b \phi(y, \mathbf{x}) dy = \frac{\sqrt{x_1^2+x_2^2}}{x_1x_2} \log\left|bx_1x_2 + x_1^2 - x_2^2\right|, \quad b > 0\,.$$

Similarly, if we integrate the family of eigenfunctions (2.5) *with respect to the parameter, we get the eigenfunctions*

$$\int_0^a F(y, x_1, x_2) dy = \frac{1}{|\mathbf{x}|}\left(\log|ax_2 + x_1| - \log|ax_1 - x_2|\right)\,.$$

Let us consider now the planar case. We introduce polar coordinates (R, φ) and (r, ϑ) such that $R = |\mathbf{x}|$, $\mathrm{e}^{i\varphi} = \mathbf{x}/R$ and $r = |\mathbf{y}|$, $\mathrm{e}^{i\vartheta} = \mathbf{y}/r$. We use the notation $f(\mathbf{x}) = f(R, \varphi)$ and $g(\mathbf{y}) = g(r, \vartheta)$. It is obvious that f and g are $2\pi-$periodic functions with respect to the angle. We write the 2D Fourier transform in polar coordinates

$$\widehat{f}(R, \varphi) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} f(r, \vartheta)\mathrm{e}^{-irR\cos(\varphi-\vartheta)} r dr d\vartheta\,. \tag{2.8}$$

Let $f \in \mathcal{S}$ and $g \in \mathcal{P}$, where $\mathcal{P}$ is the set of all $C^\infty(\mathbb{R})$ functions which are 2π-periodic. We define the angular (or circular) convolution $*_\varphi$ as follows

$$(g *_\varphi f)(r,\varphi) = \int_0^{2\pi} g(\omega) f(r, \varphi - \omega) d\omega \,.$$

We have

$$(g *_\varphi \mathcal{F}f) = \mathcal{F}(g *_\varphi f)\,.$$

Indeed,

$$\begin{aligned}
&(g *_\varphi \mathcal{F}f)(R,\varphi) = \int_0^{2\pi} g(\omega)(\mathcal{F}f)(R,\varphi-\omega)d\omega \\
&= \frac{1}{2\pi}\int_0^\infty rdr \int_0^{2\pi} g(\omega)d\omega \int_0^{2\pi} f(r,\vartheta)\mathrm{e}^{-irR\cos(\varphi-\omega-\vartheta)}d\vartheta \\
&= \frac{1}{2\pi}\int_0^\infty rdr \int_0^{2\pi} \left(\int_0^{2\pi} g(\omega) f(r,\chi-\omega)d\omega\right)\mathrm{e}^{-irR\cos(\varphi-\chi)}d\chi \\
&= \frac{1}{2\pi}\int_0^\infty rdr \int_0^{2\pi} (g *_\chi f)(r,\chi)\mathrm{e}^{-irR\cos(\varphi-\chi)}d\chi = \mathcal{F}(g *_\varphi f)(R,\varphi)\,.
\end{aligned}$$

It follows that, if $f \in \mathcal{S}$ is an eigenfunction of the Fourier transform, then the angular convolution $g *_\varphi f$ is still an eigenfunction. This is valid also if g is a distribution supported on the unit circle and $f \in \mathcal{S}'$.

We denote by $\mathcal{P}'$ the topological dual of $\mathcal{P}$. A seguence $\{g_k\} \in \mathcal{P}$ converges to g in $\mathcal{P}$ if $g_k^{(s)}$ tends to $g^{(s)}$ uniformly in $\mathbb{R}$. Elements of $\mathcal{P}'$ are called periodic distributions. The action of $g \in \mathcal{P}'$ on a test function $\psi \in \mathcal{P}$ is denoted by $\langle g, \psi\rangle$ and the translation $\tau_\alpha g$ is defined by $\langle \tau_\alpha g, \psi\rangle = \langle g, \tau_{-\alpha}\psi\rangle$ with $(\tau_{-\alpha}\psi)(\vartheta) = \psi(\vartheta + \alpha)$.

Definition 2.4. *If $g \in \mathcal{P}'$ and $f \in \mathcal{S}'$, we define the angular convolution*

$$(g *_\vartheta f, \Phi) = \langle g, \psi\rangle \quad \text{with} \quad \psi(\omega) = ((f(R,\vartheta), (\tau_{-\omega}\Phi)(R,\vartheta)) \quad \Phi \in \mathcal{S}$$

where $(\tau_{-\omega}\Phi)(R,\vartheta) = \Phi(R, \vartheta + \omega)$.

We can easily check directly that

$$\tau_{-\omega}(\mathcal{F}(\Phi))(R,\vartheta) = \mathcal{F}(\tau_{-\omega}\Phi)(R,\vartheta), \qquad \Phi \in \mathcal{S}\,. \tag{2.9}$$

Proposition 2.3. *For $f \in \mathcal{S}'$ and $g \in \mathcal{P}'$ we have*

$$(\mathcal{F}(g *_\vartheta f), \Phi) = (g *_\vartheta \mathcal{F}(f), \Phi), \qquad \Phi \in \mathcal{S}\,.$$

Proof. Indeed, by definition of Fourier transform and angular convolution we have

$$(\mathcal{F}(g *_\vartheta f), \Phi) = (g *_\vartheta f, \mathcal{F}(\Phi)) = \langle g(\omega), (f(R,\vartheta), \tau_{-\omega}(\mathcal{F}(\Phi))(R,\vartheta)\rangle \,.$$

Then, keeping in mind (2.9),

$$\begin{aligned}(\mathcal{F}(g *_\vartheta f), \Phi) &= \langle g(\omega), (f(R,\vartheta), \mathcal{F}(\tau_{-\omega}\Phi)(R,\vartheta))\rangle \\ &= \langle g(\omega), (\mathcal{F}(f)(R,\vartheta), (\tau_{-\omega}\Phi)(R,\vartheta))\rangle = (g *_\vartheta \mathcal{F}(f), \Phi)\,.\end{aligned}$$

□

Theorem 2.4. *Let f be an eigenfunction in the sense of distribution for $\mathcal{F}$ and $g \in \mathcal{P}'$. Then the angular convolution $g *_\vartheta f$ is an eigenfunction in the sense of distribution for $\mathcal{F}$.*

Proof. Indeed, if $f \in \mathcal{S}'$ satisfies (2.2), then

$$\begin{aligned}(\mathcal{F}(g *_\vartheta f), \Phi) &= (g *_\vartheta f, \mathcal{F}(\Phi)) = \langle g(\omega), (f(R,\vartheta), \mathcal{F}(\tau_{-\omega}\Phi)(R,\vartheta))\rangle \\ &= \lambda\langle g(\omega), (f(R,\vartheta), (\tau_{-\omega}\Phi)(R,\vartheta))\rangle = \lambda(g *_\vartheta f, \Phi)\,.\end{aligned}$$

Hence also $g *_\vartheta f$ satisfies (2.2). □

Corollary 2.5. *Suppose that $f \in \mathcal{S}'$ is an eigenfunction for $\mathcal{F}$. Then*
i. the translation of f with respect to the angle, $\tau_\alpha f$, defined by

$$(\tau_\alpha f, \Phi) = (f, \tau_{-\alpha}\Phi), \qquad \Phi \in \mathcal{S}$$

is an eigenfunction for $\mathcal{F}$;
ii. the derivative in the sense of distribution of f with respect to the angle $\frac{\partial^s}{\partial^s\vartheta} f, s \geq 1$ defined by

$$(\frac{\partial^s}{\partial^s\vartheta} f, \Phi) = (-1)^s (f, \frac{\partial^s}{\partial^s\vartheta}\Phi), \qquad \Phi \in \mathcal{S}$$

is an eigenfunction for $\mathcal{F}$.

Proof. i. Let us denote by $\delta_{(\alpha)}$ the delta function at the point α that is

$$\langle \delta_{(\alpha)}, \psi\rangle = \psi(\alpha), \qquad \psi \in \mathcal{P}\,.$$

We have $\delta_{(\alpha)} \in \mathcal{P}'$ and for any $f \in \mathcal{S}'$

$$\tau_\alpha f = \delta_{(\alpha)} *_\vartheta f\,.$$

Indeed, by definition,

$$\begin{aligned}(\tau_\alpha f, \Phi) &= (f(R,\vartheta), \Phi(R,\vartheta+\alpha)) = (f(R,\vartheta), \langle \delta_{(\alpha)}(\omega), \Phi(R,\vartheta+\omega)\rangle) \\ &= \langle \delta_{(\alpha)}(\omega), (f(R,\vartheta), \Phi(R,\vartheta+\omega))\rangle = (\delta_{(\alpha)} *_\vartheta f, \Phi), \quad \Phi \in \mathcal{S}\end{aligned}$$

where we have used that

$$\Phi(R,\vartheta+\alpha) = (\delta_{(\alpha)}(\omega), \Phi(R,\vartheta+\omega))\,.$$

Then we can apply theorem 2.4.

ii. Let $\delta^{(s)}$ be the derivative of the delta function, defined as

$$\langle \delta^{(s)}, \psi\rangle = (-1)^s \psi^{(s)}(0), \qquad \psi \in \mathcal{P}\,.$$

Hence

$$\langle \delta^{(s)}, \tau_{-\vartheta}\psi\rangle = (-1)^s \psi^{(s)}(\vartheta)\,.$$

Then

$$\begin{aligned}&(\delta^{(s)} *_\vartheta f, \Phi) = \langle \delta^{(s)}(\omega), (f(R,\vartheta), \Phi(R,\vartheta+\omega))\rangle \\ &= (f(R,\vartheta), \langle \delta^{(s)}(\omega), \Phi(R,\vartheta+\omega)\rangle) = (-1)^s (f(R,\vartheta), \frac{d^s}{d\vartheta}\Phi(R,\vartheta)) \\ &= (\frac{d^s}{d\vartheta} f(R,\vartheta), \Phi(R,\vartheta))\,.\end{aligned}$$

Thus we have

$$\delta^{(s)} *_\vartheta f = \frac{d^s}{d\vartheta} f(R,\vartheta)$$

and we can apply theorem 2.4. □

Remark 2.6. *The translation in angle is equivalent to rotation therefore statement i. states that rotated eigenfunctions are still eigenfunctions. Theorem 2.4 can be viewed as a particular case of theorem 2.2. Indeed, if the parameter a in theorem 2.2 is the angle of rotation, that is $f(a, \mathbf{x})$ in polar coordinates is $f(R, \vartheta - a)$, then the convolution in a gives the same result of the convolution in the angle.*

Example 2.3. *Let us write the eigenfunction* (2.3) *in polar coordinates*

$$f(R,\vartheta) = \frac{\Phi(\vartheta)}{R} \qquad \text{with} \qquad \Phi(\vartheta) = \frac{1}{\sin(\vartheta)\cos(\vartheta)}\,.$$

By virtue of Corollary 2.5 differentiation of any order with respect to the angle produces new eigenfunctions. If we consider first and second derivatives we get the eigenfunctions

$$\frac{\Phi'(\vartheta)}{R} = \frac{1}{R}\left(\frac{1}{\cos^2(\vartheta)} - \frac{1}{\sin^2(\vartheta)}\right) = \frac{x_2^2 - x_1^2}{x_1^2 x_2^2}\sqrt{x_1^2 + x_2^2}\,,$$

$$\frac{\Phi''(\vartheta)}{R} = \frac{2}{R}\left(\frac{\cos(\vartheta)}{\sin^3(\vartheta)} + \frac{\sin(\vartheta)}{\cos^3(\vartheta)}\right) = 2\left(\frac{x_1}{x_2^3} + \frac{x_2}{x_1^3}\right)\sqrt{x_1^2 + x_2^2}\,.$$

3 A characterization of eigenfunctions

We denote by $Y_{m,n}^{(k)}(\omega)$ the spherical functions of order m in the n dimensional space, ω is a point of the unit sphere S. The upper index k numbers the linearly independent spherical functions of the same order m and it varies between the bounds

$$1 \le k \le k_{m,n} = (2m + n - 2)\frac{(m + n - 3)!}{(n - 2)!m!}\,.$$

Theorem 3.1. *The functions*

$$\frac{Y_{m,n}^{(k)}(\omega)}{|\mathbf{x}|^{n/2}}, \qquad \omega = \frac{\mathbf{x}}{|\mathbf{x}|}, \qquad k = 1, ..., k_{m,n}, \quad m \ge 0$$

are eigenfunctions of the Fourier transform and we have

$$\mathcal{F}\left(\frac{Y_{m,n}^{(k)}(\cdot)}{|\cdot|^{n/2}}\right)(\xi) = (-i)^m\,\frac{Y_{m,n}^{(k)}(\Lambda)}{|\xi|^{n/2}}, \qquad \Lambda = \frac{\xi}{|\xi|}\,.$$

Proof. We seek for the Fourier transform

$$\mathcal{F}(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{Y_{m,n}^{(k)}(\frac{\mathbf{x}}{|\mathbf{x}|})}{|\mathbf{x}|^{n/2}}\mathrm{e}^{-i(\mathbf{x},\xi)}d\mathbf{x}\,.$$

We substitute spherical coordinates $R = |\mathbf{x}|$, $\theta = \mathbf{x}/R$. Then $d\mathbf{x} = R^{n-1}dRd_\theta S$ where S denotes the unit sphere, and $(\mathbf{x}, \xi) = R|\xi|\cos\gamma$ with γ denoting the angle between the vectors ξ and $\mathbf{x}$. Hence

$$\mathcal{F}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_0^\infty R^{n/2-1}dR\int_S Y_{m,n}^{(k)}(\theta)\mathrm{e}^{-iR|\xi|\cos\gamma}d_\theta S\,.$$

In the integral herein we substitute $t = R|\xi|$ and we obtain

$$\mathcal{F}(\xi) = \frac{1}{(2\pi)^{n/2}|\xi|^{n/2}} \int_0^\infty t^{n/2-1} dt \int_S Y_{m,n}^{(k)}(\theta) \mathrm{e}^{-it\cos\gamma} d_\theta S\,.$$

We use the formula ([7, p.250]

$$\int_S Y_{m,n}^{(k)}(\theta) \mathrm{e}^{it\cos\gamma} d_\theta S = t^{1-n/2}(2\pi)^{n/2} i^m J_{n/2+m-1}(t) Y_{m,n}^{(k)}(\Lambda), \quad \Lambda = \frac{\xi}{|\xi|}$$

where $J_\mu(t)$ denotes the Bessel function of the first kind of order μ (cf.[10]). Hence

$$\begin{aligned}\mathcal{F}(\xi) &= i^m \frac{Y_{m,n}^{(k)}(\Lambda)}{|\xi|^{n/2}} (-1)^{1-n/2} \int_0^\infty J_{n/2+m-1}(-t) dt \\ &= (-i)^m \frac{Y_{m,n}^{(k)}(\Lambda)}{|\xi|^{n/2}} \int_0^\infty J_{n/2+m-1}(t) dt\,. \end{aligned} \tag{3.1}$$

Since (cf. [10, 13.24])

$$\int_0^\infty J_{n/2+m-1}(t) dt = 1$$

the theorem is proved. □

Remark 3.2. *Theorem 3.1 can be obtained as a particular case of the Bochner formula (cf.[2, Theorem 2]):*

$$\begin{aligned}&\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\mathbf{x}|^m Y_{m,n}^{(k)}(\frac{\mathbf{x}}{|\mathbf{x}|}) \varphi(|\mathbf{x}|) \mathrm{e}^{-i(\mathbf{x},\xi)} d\mathbf{x} \\ &\qquad = \frac{(-i)^m Y_{m,n}^{(k)}(\Lambda)}{|\xi|^{n+m}} \int_0^\infty \varphi\left(\frac{t}{|\xi|}\right) J_{\frac{n}{2}+m-1}(t) t^{\frac{n}{2}+m} dt\end{aligned}$$

where φ is measurable in $(0,\infty)$. Indeed, assuming $\varphi(|\mathbf{x}|) = |\mathbf{x}|^{-m-n/2}$ we get (3.1).

Let us consider homogeneous functions of degree $-n/2$ of the form

$$f(\mathbf{x}) = \frac{F(\omega)}{R^{n/2}}, \quad R = |\mathbf{x}|, \quad \omega = \frac{\mathbf{x}}{|\mathbf{x}|}\,. \tag{3.2}$$

Here $F(\omega)$ is defined on the unit sphere S. Following [5] we use the notation

$$(\sigma \pm i0)^\lambda = \sigma_+^\lambda + \mathrm{e}^{\pm i\lambda\pi} \sigma_-^\lambda$$

where σ_+^λ is equal to σ^λ for $\sigma > 0$ and to 0 if $\sigma \le 0$ and σ_-^λ is equal to $|\sigma|^\lambda$ for $\sigma < 0$ and to 0 for $\sigma \ge 0$.

Theorem 3.3. *Let $\mathcal{K}$ be the following singular integral operator on the $(n-1)-$dimensional unit sphere*

$$\mathcal{K}F(\Lambda) = \frac{1}{(2\pi)^{n/2}}\Gamma\left(\frac{n}{2}\right)\mathrm{e}^{-in\pi/4}\int_S (\omega\cdot\Lambda - i0)^{-n/2}\,F(\omega)dS_\omega\,. \tag{3.3}$$

The function (3.2) *is an eigenfunction of the Fourier transform corresponding to the eigenvalue λ if and only if F is an eigenfunction of* (3.3) *corresponding to the same eigenvalue, i.e.*

$$\mathcal{K}F = \lambda F\,. \tag{3.4}$$

Proof. The Fourier transform of f in spherical coordinates has the form

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_S dS_\omega \int_0^\infty f(R\omega)\mathrm{e}^{-iR\rho\cos\gamma}R^{n-1}\,dR$$

where we made use of the notations $\rho = |\xi|$, $\Lambda = \xi/\rho$; the angle between $\mathbf{x}$ and ξ is denoted by γ that is $\cos\gamma = \omega\cdot\Lambda$.
Hence an eigenfunction $f(R\omega)$ of the Fourier transform is defined by the equation

$$\frac{1}{(2\pi)^{n/2}}\int_S dS_\omega \int_0^\infty f(R\omega)\mathrm{e}^{-iR\rho\cos\gamma}R^{n-1}\,dR = \lambda f(\rho\Lambda), \quad \rho>0, \quad |\Lambda| = 1\,.$$

In the integral above we replace the function f in the form (3.2) and substitute $t = R\rho$, so that we obtain the following integral equation for F on the unit sphere

$$\frac{1}{(2\pi)^{n/2}}\int_S F(\omega)dS_\omega \int_0^\infty t^{n/2-1}\mathrm{e}^{-it\cos\gamma}\,dt = \lambda F(\Lambda)\,.$$

Now we make use of the following formula (cf. [5, pp.172-174])

$$\int_0^\infty t^{n/2-1}\mathrm{e}^{-it\sigma}dt = \Gamma\left(\frac{n}{2}\right)\mathrm{e}^{in\pi/4}(-\sigma+i0)^{-n/2} = \Gamma\left(\frac{n}{2}\right)\mathrm{e}^{-in\pi/4}(\sigma-i0)^{-n/2}\,.$$

(3.4) and (3.3) follow. □

Remark 3.4. *From theorems 3.1 and 3.3 we obtain that spherical functions $Y^{(k)}_{m,n}$ solve the singular integral equation*

$$\mathcal{K}Y^{(k)}_{m,n} = (-i)^m Y^{(k)}_{m,n}, \qquad 1\le k\le k_{m,n}\,.$$

Remark 3.5. *If $n=2$ then ([5, p.60])*

$$(\sigma - i0)^{-1} = \frac{1}{\sigma} + i\pi\delta(\sigma)$$

where δ denotes the delta function. Then (3.3) can be written as

$$\mathcal{K}F(\Lambda) = -\frac{i}{2\pi}\int_{|\omega|=1}\left(\frac{1}{\cos\gamma} + \pi i\delta(\cos\gamma)\right)F(\omega)d\omega\,.$$

The homogeneous harmonic polynomials on the unit circle are

$$Y^{(1)}_{m,2}(\vartheta) = \cos(m\vartheta), \qquad Y^{(2)}_{m,2}(\vartheta) = \sin(m\vartheta) \quad m\in\mathbb{Z}\,.$$

If we denote by $Y_m(\vartheta) = \mathrm{e}^{im\vartheta}$, we obtain that Y_m satisfies the singular integral equation on the unit circle

$$\mathcal{K}Y_m = (-i)^m Y_m, \qquad \forall m\in\mathbb{Z}\,.$$

4 Description of planar eigenfunctions via Fourier series

Every function $\Phi\in L^2([0,2\pi])$ admits an expansion into a series with respect to the spherical functions

$$\Phi(\vartheta) = \sum_{k=-\infty}^{\infty} c_k Y_k(\vartheta), \qquad Y_k(\vartheta) = \mathrm{e}^{ik\vartheta} \tag{4.1}$$

with the coefficients $c_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\Phi(\vartheta)\mathrm{e}^{-ik\vartheta}d\vartheta$. A similar result holds for periodic distributions.

Let $\mathcal{P}$ be the set of all $C^\infty(\mathbb{R})$ functions with complex values that are $2\pi-$periodic. Any $u\in\mathcal{P}$ can be written as the Fourier series (4.1). Let $\mathcal{P}'$ be the set of all continuous linear functionals on $\mathcal{P}'$. The action of $\Phi\in\mathcal{P}'$ on a test function ψ is denoted by $\langle\Phi,\psi\rangle$. Any $\Phi\in\mathcal{P}'$ can be written as the Fourier series (4.1), which converges in the sense of distributions

$$\sum_{k=-\infty}^{\infty} c_k\langle\mathrm{e}^{ik\vartheta},\psi\rangle = \langle\Phi,\psi\rangle, \qquad \forall\psi\in\mathcal{P}$$

where the coefficients are defined by

$$c_k = \frac{1}{2\pi}\langle\Phi(\vartheta),\mathrm{e}^{-ik\vartheta}\rangle\,.$$

A complex sequence $\{c_k\}_{k\in\mathbb{Z}}$ is said to have polynomial growth if there exists an integer L and a positive constant C such that

$$|c_k| \le C|k|^L, \qquad k \in \mathbb{Z}\,. \tag{4.2}$$

Any series of the form $\sum_{k=-\infty}^{\infty} c_k e^{ik\vartheta}$ whose coefficients have polynomial growth converges in the sense of distributions to a distribution with the coefficients $\{c_k\}$ as its Fourier coefficients and, conversely, Fourier coefficients of any periodic distribution are a sequence of polynomial growth. (cf. [8, p.225], [4, p.33], [5, p.30]).

As a consequence of Theorem 3.1 we prove a characterization of positive homogeneous eigenfunctions of the form $\Phi(\vartheta)r^{-1}$ of the planar Fourier transform by means of their Fourier coefficients.

Theorem 4.1. *If the distribution $\Phi(\vartheta)r^{-1}$, with $\Phi \in \mathcal{P}'$, is an eigenfunction of the planar Fourier transform corresponding to the eigenvalue λ, then the coefficients c_k of the Fourier series* (4.1) *satisfy the conditions*

$$c_k\left((-i)^k - \lambda\right) = 0, \qquad \forall k \in \mathbb{Z}\,. \tag{4.3}$$

Conversely, let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\{c_k\}$ be a sequence with polynomial growth (4.2) *satisfying conditions* (4.3)*. Then $\Phi(\vartheta)r^{-1}$, with Φ defined in* (4.1)*, is an eigenfunction of the planar Fourier transform. The convergence of the series in the sense of distribution is in the space $W_2^{-\ell}((0, 2\pi))$, $\ell > L + 1/2$.*

Proof. Let $\Phi \in \mathcal{P}'$. $\Phi(\vartheta)r^{-1}$ is an eigenfunction corresponding to the eigenvalue λ if it satisfies

$$\mathcal{F}(\frac{\Phi(\varphi)}{R})(r,\vartheta) = \lambda\frac{\Phi(\vartheta)}{r}\,.$$

If we replace Φ by its Fourier series (4.1), the last equation can be rewritten as

$$\sum_{k=-\infty}^{\infty} c_k \mathcal{F}(\frac{Y_k(\varphi)}{R})(r,\vartheta) = \lambda \sum_{k=-\infty}^{\infty} c_k \frac{Y_k(\vartheta)}{r}\,. \tag{4.4}$$

According to theorem 3.1 we have

$$\mathcal{F}(\frac{Y_k(\varphi)}{R})(r,\vartheta) = (-i)^k \frac{Y_k(\vartheta)}{r}, \qquad \forall k \in \mathbb{Z}\,.$$

Hence (4.4) implies

$$\sum_{k=-\infty}^{\infty} c_k \left((-i)^k - \lambda\right) Y_k(\vartheta) = 0$$

which gives (4.3).

Conversely, suppose that $\{c_k\}$ is a sequence of polynomial growth, which satisfies (4.3). Then Φ defined in (4.1) belongs to $\mathcal{P}'$ and, due to theorem 3.1,

$$\begin{aligned}\mathcal{F}(\frac{\Phi(\varphi)}{R})(r,\vartheta) &= \sum_{k=-\infty}^{\infty} c_k \mathcal{F}(\frac{Y_k(\varphi)}{R})(r,\vartheta) = \sum_{k=-\infty}^{\infty} c_k (-i)^k \frac{Y_k(\vartheta)}{r} \\ &= \lambda \sum_{k=-\infty}^{\infty} c_k \frac{Y_k(\vartheta)}{r} = \lambda \frac{\Phi(\vartheta)}{r} \,.\end{aligned}$$

We obtain that $\Phi(\vartheta) r^{-1}$ is an eigenfunction of the planar Fourier transform. The series (4.4) can be obtained by ℓ term-by-term differentations of the series $\sum_{k=-\infty}^{\infty} (c_k/(ik)^\ell) \mathrm{e}^{ik\vartheta}$ which converges in $L^2((0,2\pi))$ if $\ell > L + 1/2$.

□

Corollary 4.2. $\Phi(\vartheta) r^{-1}$ *is an eigenfunction of the Fourier transform* (2.8) *corresponding to the eigenvalue* λ *if and only if* $\Phi \in \mathcal{P}'$, $\Phi \not\equiv 0$, *admits the following Fourier expansion*

$$\begin{aligned}
\Phi(\vartheta) &= \sum_{s=0}^{\infty} c_{4s} \mathrm{e}^{4is\vartheta} + \sum_{s=1}^{\infty} c_{-4s} \mathrm{e}^{-4is\vartheta} && \text{if} \quad \lambda = 1; \\
\Phi(\vartheta) &= \sum_{s=0}^{\infty} c_{4s+2} \mathrm{e}^{i(4s+2)\vartheta} + \sum_{s=0}^{\infty} c_{-(4s+2)} \mathrm{e}^{-i(4s+2)\vartheta} && \text{if} \quad \lambda = -1; \\
\Phi(\vartheta) &= \sum_{s=0}^{\infty} c_{4s+3} \mathrm{e}^{i(4s+3)\vartheta} + \sum_{s=0}^{\infty} c_{-(4s+1)} \mathrm{e}^{-i(4s+1)\vartheta} && \text{if} \quad \lambda = i; \\
\Phi(\vartheta) &= \sum_{s=0}^{\infty} c_{4s+1} \mathrm{e}^{i(4s+1)\vartheta} + \sum_{s=0}^{\infty} c_{-(4s+3)} \mathrm{e}^{-i(4s+3)\vartheta} && \text{if} \quad \lambda = -i \,.
\end{aligned} \tag{4.5}$$

Proof. We write the series (4.1) as follows

$$\Phi(\vartheta) = \sum_{k=0}^{\infty} c_k \mathrm{e}^{ik\vartheta} + \sum_{k=1}^{\infty} c_{-k} \mathrm{e}^{-ik\vartheta} \,.$$

Kepping in mind the obvious relations valid for $k \geq 0$

$$(-i)^k = \begin{cases} 1 & \text{if} \quad k \equiv 0 \mod 4 \\ -i & \text{if} \quad k \equiv 1 \mod 4 \\ -1 & \text{if} \quad k \equiv 2 \mod 4 \\ i & \text{if} \quad k \equiv 3 \mod 4 \end{cases}$$

from theorem 4.1 we deduce that $\Phi(\vartheta)r^{-1}$ is an eigenfunction with eigenvalue λ if and only the Fourier series of Φ has the form (4.5) $\square$

Example 4.1. *The eigenfunction* (2.3) *in polar coordinates has the form*

$$\frac{\Phi(\vartheta)}{r} \qquad \text{with} \qquad \Phi(\vartheta) = \frac{2}{\sin(2\vartheta)}\,.$$

Let us compute the Fourier coefficients of Φ. *It is clear that*

$$c_k = \frac{1}{\pi}\left(\int_0^\pi \frac{\mathrm{e}^{-ik\varphi}d\varphi}{\sin(2\varphi)} + \int_\pi^{2\pi} \frac{\mathrm{e}^{-ik\varphi}d\varphi}{\sin(2\varphi)}\right) = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{e}^{-ik\varphi}(1+(-1)^k)}{\sin(2\varphi)}d\varphi$$

Hence the coefficients are zero if k *is odd. Assume that* $k = 2s$. *Then*

$$c_{2s} = \frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{e}^{-i2s\varphi}(1-(-1)^s)}{\sin(2\varphi)}d\varphi = \begin{cases} 0 & \text{if} \qquad s = 2r \\ \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \frac{\mathrm{e}^{-i2(2r+1)\varphi}}{\sin(2\varphi)}d\varphi & \text{if} \qquad s = 2r+1 \end{cases}$$

It remains to compute

$$c_{4r+2} = \frac{4}{\pi}\left(\int_0^{\pi/2} \frac{\cos((4r+2)\varphi)}{\sin(2\varphi)}d\varphi - i\int_0^{\pi/2} \frac{\sin((4r+2)\varphi)}{\sin(2\varphi)}d\varphi\right)$$

The first integral is zero. Indeed,

$$\int_0^{\pi/2} \frac{\cos((4r+2)\varphi)}{\sin(2\varphi)}d\varphi = \int_0^{\pi/4} \frac{\cos((4r+2)\varphi)}{\sin(2\varphi)}d\varphi + \int_{\pi/4}^{\pi/2} \frac{\cos((4r+2)\varphi)}{\sin(2\varphi)}d\varphi = 0$$

We prove by induction that

$$I_r = \int_0^{\pi/2} \frac{\sin((4r+2)\varphi)}{\sin(2\varphi)}d\varphi = \frac{\pi}{2}, \qquad r \geq 0\,.$$

Clearly $I_0 = \pi/2$. *Suppose that* $I_r = \pi/2, r \geq 1$. *From the relation*

$$\sin((4r+6)\varphi) = 2\sin(2\varphi)\cos((4r+4)\varphi) + \sin((4r+2)\varphi)$$

we obtain

$$I_{r+1} = \int_0^{\pi/2} \frac{\sin((4r+6)\varphi)}{\sin(2\varphi)} d\varphi = I_r + 2\int_0^{\pi/2} \cos((4r+4)\varphi) d\varphi = I_r = \frac{\pi}{2}.$$

If $r < 0$

$$I_r = -I_{-r-1} = -\frac{\pi}{2}.$$

Hence

$$c_{4r+2} = \begin{cases} -2i & r \geq 0 \\ 2i & r < 0 \end{cases}$$

and

$$\frac{2}{\sin(2\theta)} = 2i \sum_{r=0}^{\infty} \left(\mathrm{e}^{-i(4r+2)\theta} - \mathrm{e}^{i(4r+2)\theta} \right) = 4 \sum_{r=0}^{\infty} \sin((4r+2)\theta).$$

Example 4.2. *Let us compute the Fourier coefficients of*

$$\Phi(\vartheta) = 2\frac{\cos(2\vartheta)}{\sin(2\vartheta)}.$$

We have

$$c_k = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos(2\varphi)}{\sin(2\varphi)} \mathrm{e}^{-ik\varphi} d\varphi = \frac{1}{\pi} \int_0^{\pi} \frac{\cos(2\varphi)}{\sin(2\varphi)} (1 + (-1)^k) \mathrm{e}^{-ik\varphi} d\varphi.$$

Hence the coefficients are zero if k *is odd. Assume that* $k = 2s$*. Then*

$$c_{2s} = \frac{2}{\pi} \int_0^{\pi} \frac{\cos(2\varphi)}{\sin(2\varphi)} \mathrm{e}^{-i2s\varphi} d\varphi = \begin{cases} 0 & \text{if} \quad s = 2r+1 \\ \dfrac{4}{\pi} \displaystyle\int_0^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \mathrm{e}^{-i4r\varphi} d\varphi & \text{if} \quad s = 2r \end{cases}$$

Let us compute

$$c_{4r} = \frac{4}{\pi} \left(\int_0^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \cos(4r\varphi) d\varphi - i \int_0^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \sin(4r\varphi) d\varphi \right).$$

We have

$$\begin{aligned} &\int_0^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \cos(4r\varphi) d\varphi \\ &= \int_0^{\pi/4} \frac{\cos(2\varphi)}{\sin(2\varphi)} \cos(4r\varphi) d\varphi + \int_{\pi/4}^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \cos(4r\varphi) d\varphi = 0 \end{aligned}$$

where we have made the substitution $\varphi = \pi/2 - \vartheta$ in the second integral. Since

$$2\cos(2\varphi)\sin(4r\varphi) = (\sin((4r+2)\varphi) + \sin((4r-2)\varphi))$$

we get, for $r \geq 1$,

$$J_r := \int_0^{\pi/2} \frac{\cos(2\varphi)}{\sin(2\varphi)} \sin(4r\varphi) d\varphi = \frac{1}{2}\left(I_r + I_{r-1}\right) = \frac{\pi}{2}, \qquad J_{-r} = -J_r = -\frac{\pi}{2}\,.$$

Hence

$$c_{4r} = \begin{cases} -2i & r \geq 1 \\ 0 & r = 0 \\ 2i & r < 1 \end{cases}$$

and

$$2\frac{\cos(2\vartheta)}{\sin(2\vartheta)} = -2i\sum_{r=1}^{\infty}\left(\mathrm{e}^{4ir\theta} - \mathrm{e}^{-4ir\theta}\right) = 4\sum_{r=1}^{\infty}\sin(4r\theta)\,.$$

From Corollary 4.2 we obtain that

$$2\frac{\cos(2\vartheta)}{\sin(2\vartheta)}\frac{1}{r} = \frac{x_1^2 - x_2^2}{x_1 x_2}\frac{1}{\sqrt{x_1^2 + x_2^2}}$$

is an eigenfunction of the planar Fourier transform corresponding to the eigenvalue $\lambda = 1$ that is a fixed point of $\mathcal{F}$.

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