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BOUNDARY CHARACTERISTIC POINT REGULARITY FOR SEMILINEAR REACTION-DIFFUSION EQUATIONS: TOWARDS AN ODE CRITERION

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The classical problem of regularity of boundary characteristic points for semilinear heat equations with homogeneous Dirichlet conditions is considered. The Petrovskii ($2\sqrt{\log\log}$) *criterion* (1934) *of the boundary regularity for the heat equation can be adapted to classes of semilinear parabolic equations of reaction–diffusion type and takes the form of an ordinary differential equation* (ODE) *regularity criterion. Namely, after a special matching with a boundary layer, the regularity problem reduces to a one-dimensional perturbed nonlinear dynamical system for the first Fourier-like coefficient of the solution in an inner region. A similar ODE criterion, with an analogous matching procedures, is shown formally to exist for semilinear fourth order biharmonic equations of reaction-diffusion type. Extensions to regularity problems of backward paraboloid vertices in* $\mathbb{R}^N$ *are discussed. Bibliography*: 54 *titles. Illustrations:* 1 *figure.*

1 Introduction

1.1 Semilinear reaction-diffusion PDEs near parabola vertices

The present paper is devoted to a systematic study of the regularity of the origin $(0,0)$ as a boundary point for semilinear heat (reaction-diffusion) equations, which we first consider in 1D:

$$\begin{aligned} &u_t = u_{xx} + f(x,t,u) \quad \text{in} \;\; Q_0 \subset \mathbb{R}\times[-1,0), \\ &u = 0 \quad \text{on} \;\; \partial Q_0, \\ &u(x,-1) = u_0(x), \end{aligned} \tag{1.1}$$

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where Q_0 is a sufficiently smooth domain such that $(0,0) \in \overline{\partial Q_0}$ (∂Q_0 denotes the lateral boundary of Q_0) is its only *characteristic* boundary point, i.e., in the $\{x,t\}$-plane,

(i) the straight line $\{t=0\}$ is tangent to ∂Q_0 at this point, and

(ii) no such points exist on ∂Q_0 for $t \in [-1,0)$.

According to (1.1), we impose the zero Dirichlet condition on the lateral boundary ∂Q_0 and prescribe arbitrary bounded initial data $u_0(x)$ at $t=-1$ in $Q_0 \cap \{t=-1\}$.

We assume that the nonlinearities $f(x,t,u)$ in (1.1) satisfy necessary regularity and growth in u assumptions that guarantee the existence and uniqueness of a smooth classical solution $u \in C^{2,1}_{x,t}(Q_0) \cap C(\overline{Q}_0)$[1)] of (1.1) in Q_0 by the classical parabolic theory (cf., for example, the well-known monographs [1]–[4]). A standard regularity issue in the general theory of partial differential equations is then as follows: how to control the solution at the vertex, i.e.,

$$u(0,0^-)=?$$

Note that this does not *a priori* exclude *blow-up* at the vertex (regardless zero Dirichlet conditions on the lateral boundary), where $|u(0,0^-)|=+\infty$, in the sense of lim sup.

More precisely, as customary in the regularity theory, the goal is to derive conditions *showing how given smooth nonlinear perturbations* $f(\cdot)$ *can affect the regularity of the vertex* $(0,0)$ *of such a backward parabola* ∂Q_0. The regularity of $(0,0)$ (in the Wiener sense) means:

$$u(0,0^-)=0 \quad \text{for any initial data } u_0. \tag{1.2}$$

As is well known, for nonlinearities $f(\cdot)\equiv 0$, i.e., for the pure *heat equation*

$$\begin{aligned} &u_t = u_{xx} \quad \text{in } \ Q_0, \\ &u = 0 \quad \text{on } \ \partial Q_0, \\ &u(x,-1) = u_0(x), \end{aligned} \tag{1.3}$$

this regularity problem was solved by Petrovskii [5, 6] in 1934, who introduced his celebrated *Petrovskii regularity criterion* (the so-called $2\sqrt{\log\log}$–criterion).

Indeed, many and often strong and delicate boundary regularity and related asymptotic results are now known for a number of quasilinear parabolic equations, including even a few for degenerate porous medium operators. Nevertheless, some difficult questions remain open even for the second order parabolic equations with order-preserving semigroups. We refer to the results and surveys in [7]–[11] as a guide to a full history and already existing interesting extensions of these important results. Concerning further developing of Wiener's ideas in linear parabolic equations we refer to the bibliography and results in [12]–[15]. However, a more systematic study of those regularity issues for equations such as (1.1) with rather general nonlinear perturbations $f(\cdot)$ was not done properly still. In fact, it turned out that, for such arbitrary $f(\cdot)$, the classical barrier methods hardly applied and another asymptotic approach was necessary. We propose this in the present paper for a wide class of semilinear parabolic partial differential equations.

1) As customary in parabolic partial differential equations [1], the closure $\overline{Q}_0$ does not include the "upper lid," which is the vertex $(0,0)$ only.

1.2 Layout of the paper: key models, nonlinearities, and extensions

Section 2 contains some preliminary discussions and results. In Sections 3–5, the main goal is to show how a general "nonlinear perturbation" $f(\cdot)$ in (1.1) affects the regularity conditions by deriving sharp asymptotics of solutions near characteristic points. For this purpose, we apply the method of a matched asymptotic (blow-up) expansion, where the *boundary layer* behavior close to the lateral boundary ∂Q_0 (cf. Section 4) is matched, as $t \to 0^-$, with a *center subspace behavior* in an *inner region* (cf. Section 5). This leads to a nonlinear dynamical system for the first Fourier coefficient in the eigenfunction expansion via standard Hermite polynomials as eigenfunctions of the linear Hermite operator obtained after blow-up scaling near the vertex. Overall, the vertex regularity is shown to be governed by an *ODE criterion*, which principally does not admit any simply integral (Osgood–Dini type) treatment as in Petrovskii's one.

Indeed, such an approach falls into the scope of typical ideas of the asymptotic PDE theory, which got a full mathematical justification for many problems of interest. In particular, we refer to a recent general asymptotic analysis performed in [16]. According to its classification, our matched blow-up approach corresponds to perturbed ONE-DIMENSIONAL dynamical systems, i.e., to a rather elementary case being however a constructive one that detects a number of new asymptotic/regularity results.

In particular, to show a typical "interaction" between the linear Laplacian and the nonlinear perturbations in (1.1), we initially concentrate on the simplest case with

$$f(\cdot) = \frac{1}{(-t)}\, \kappa(u)u \quad \text{for} \quad t \in [-1, 0), \tag{1.4}$$

where $\kappa(u)$ is a smooth enough function satisfying

$$\begin{aligned} &\kappa(u) \to 0 \quad \text{as} \quad u \to 0, \\ &|\kappa(u)| \leqslant 1, \quad \kappa(u) \ne 0 \quad \text{for} \quad u \ne 0. \end{aligned} \tag{1.5}$$

We show that the nonlinear perturbation (1.4) will then affect the Petrovskii $2\sqrt{\log \log}$-time-factor starting from some awkward looking functions such as

$$\kappa(u) \sim |\ln|u||^{\frac{1}{3}} \mathrm{e}^{-(3\sqrt{\pi}|\ln|u||)^{2/3}} \to 0 \quad \text{as} \quad u \to 0. \tag{1.6}$$

For more general nonlinearities we derive the so-called *ODE regularity criterion* of the vertex $(0, 0)$, meaning that a special nonlinear ordinary differential equation for the first Fourier coefficients of rescaled solutions takes responsibility for the vertex regularity/irregularity.

The present research has been inspired by the regularity study of quasilinear elliptic equations with quadratic gradient-dependent nonlinearities [17], where, in 2D, new asymptotics of solutions near conical points were discovered. We also refer to the monographs [18]–[22] and [23]–[26] as an update guide to elliptic regularity theory including higher order equations. Sharp asymptotics of solutions of the heat equation in domains with conical points were derived in [27]–[30]. Higher order parabolic equations were treated in [31, 32]. It turned out that, unlike the present study, such asymptotics are of a self-similar form. See also [33] for a good short survey including compressible/incompressible Stokes and Navier–Stokes problems.

Therefore, as a next key model regularity problem, we briefly reflect the main differences and difficulties, which occur by studying the regularity issues for parabolic equations with a typical quadratic gradient dependence in the nonlinear term:

$$
\begin{aligned}
& u_t = u_{xx} + \kappa(u) u \, (u_x)^2 \quad \text{in} \;\; Q_0, \\
& u = 0 \quad \text{on} \;\; \partial Q_0, \\
& u(x,-1) = u_0(x).
\end{aligned}
\tag{1.7}
$$

Then the ODE regularity criterion is expressed in terms of another 1D dynamical system, with a weaker nonlinearity. We then convincingly show that, for any $\kappa(u)$ in (1.7) satisfying (1.5), the Petrovskii linear regularity criterion takes place, i.e., it remains the same as for the heat equation (1.3).

We also pay some attention to extensions to similar regularity problems in domains $Q_0 \subset \mathbb{R}^N \times [-1,1)$, with ∂Q_0 having a *backward paraboloid shape* and the vertex $(0,0)$ being their characteristic point. In Section 3, we thus discuss the semilinear problems

$$
\begin{aligned}
& u_t = \Delta u + \begin{cases} \dfrac{\kappa(u)u}{(-t)} \\ \kappa(u) u |\nabla u|^2 \end{cases} \quad \text{in} \;\; Q_0, \\
& u = 0 \quad \text{on} \;\; \partial Q_0.
\end{aligned}
\tag{1.8}
$$

Finally, in Appendix B (Appendix A is devoted to the corresponding spectral theory of rescaled operators), we show how our approach can be extended to higher order partial differential equations, for example, for the *semilinear biharmonic equations* having similar nonlinearities, with also zero Dirichlet conditions on ∂Q_0 and bounded initial data u_0 in $Q_0 \cap \{t=-1\}$. The mathematical analysis becomes much more difficult, and we do not justify rigorously all its main steps such as the boundary layer and matching with the inner region asymptotics. Moreover, the 1D dynamical system for the first Fourier coefficients becomes also more delicate and does not admit such a complete analysis, although some definite conclusions are possible. We must admit that this part of our study is formal, although some steps are expected to admit a full justification, which nevertheless can be rather time-consuming.

2 Petrovskii's $2\sqrt{\log\log}$–Criterion of 1934 and Some Extensions

We need to explain some details of Petrovskii's classical regularity analysis for the heat equation performed in 1934-35. Following his study, we consider the one-dimensional case $N=1$, where the analysis becomes more clear. Moreover, our further extensions to biharmonic operators (Appendix B) will be also performed for $N=1$, in view of rather complicated asymptotics occurred, so we are not interested in involving extra technicalities.

After Wiener's pioneering regularity criterion [34] for the *Laplace equation* in 1924, I. G. Petrovskii [5, 6] was the first who completed the study of the regularity question for the 1D and 2D *heat equation* in a noncylindrical domain. We formulate his result in a *blow-up manner*, which in fact was already used by Petrovskii [5] in 1934.

Petrovskii considered the question on an *irregular* or *regular* vertex $(x,t) = (0,0)$ in the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} \quad \text{in} \quad Q_0 = \{|x| < R(t), \ \ -1 < t < 0\}, \\ R(t) \to 0^+ \quad \text{as} \quad t \to 0^-, \\ \text{with bounded smooth data } u(x,0) = u_0(x) \text{ on } [-R(-1), R(-1)]. \end{cases} \tag{2.1}$$

Here, the lateral boundary $\{x = \pm R(t), \ t \in [-1,0)\}$ is given by a function $R(t)$ that is assumed to be positive, strictly monotone, C^1-smooth for all $-1 \leqslant t < 0$ (with $R'(t) > 0$), and is allowed to have a singularity of $R'(t)$ at $t = 0^-$ only. The regularity analysis then detects the value of $u(x,t)$ at the end "blow-up" *characteristic* point $(0,0^-)$, to which the domain Q_0 "shrinks" as $t \to 0^-$.

Remark 2.1 (on the first parabolic regularity results for $m = 1$ and $m \geqslant 2$)**.** It is well-known that, for the heat equation, the first existence of a classical solution (i.e., continuous at $(0,0)$) was obtained by Gevrey [35] in 1913–1914 (cf. Petrovskii's references in [5, p. 55] and [6, p. 425]), which assumed that the Hölder exponent of $R(t)$ is larger than $1/2$. In our setting, at $t = 0^-$, this comprises all types of boundaries given by the functions

$$R(t) = (-t)^\nu \quad \text{with any } \nu > 1/2 \text{ are regular} \quad \text{([Gevrey, 1913–1914].} \tag{2.2}$$

For the $2m$th order parabolic polyharmonic equations such as (A.3) below, a similar result

$$\text{for} \quad R(t) = (-t)^{\frac{1}{2m}} \quad \text{the problem is uniquely solvable} \tag{2.3}$$

was proved [2)] by Mikhailov [36] almost sixty years later and fifty years ago.

Definition 2.1. (i) As usual in potential theory, the point $(x,t) = (0,0)$ is called *regular* (in the Wiener sense, cf. [25]) if any value of the solution $u(x,t)$ can be prescribed there by continuity as a standard boundary value on ∂Q_0. In particular, as a convenient and key for us evolution illustration, $(0,0)$ *is regular if the continuity holds for any initial data* $u_0(x)$ *in the following sense*:

$$u = 0 \quad \text{at the lateral boundary} \ \ \{|x| = R(t), \ \ -1 \leqslant t < 0\} \quad \Longrightarrow \quad u(0,0^-) = 0. \tag{2.4}$$

(ii) Otherwise, the point $(0,0)$ is *irregular* if the value $u(0,0^-)$ is not fixed by boundary conditions, i.e., $u(0,0) \neq 0$ for some data u_0, and hence is given by a "blow-up evolution" as $t \to 0^-$. Hence, formally, $(0,0)$ does not belong to the parabolic boundary of Q_0.

[2)] However, in the Slobodetskii–Sobolev classes, i.e., the continuity at $(0,0)$ is not understood in the above Wiener classical sense. In fact, for $m = 2$ Wiener's one (2.4) fails for the parabola with $R(t) = 5(-t)^{\frac{1}{4}}$ in (2.3), while the one with $R(t) = 4(-t)^{\frac{1}{4}}$ remains regular (cf. [8, § 4.3]).

Petrovskii's $2\sqrt{\log\log}$–criterion

Using novel barriers as upper and lower solutions of (2.1), Petrovskii [5, 6] established the following $2\sqrt{\log\log}$-criterion:

$$\boxed{\begin{array}{l} \text{(i)} \quad R(t) = 2\sqrt{-t}\,\sqrt{\ln|\ln(-t)|} \quad \Longrightarrow \quad (0,0) \text{ is regular,} \quad \text{and} \\[2mm] \text{(ii)} \quad R(t) = 2(1+\varepsilon)\sqrt{-t}\,\sqrt{\ln|\ln(-t)|}, \quad \varepsilon > 0 \quad \Longrightarrow \quad (0,0) \text{ is irregular.} \end{array}} \tag{2.5}$$

More precisely, he also showed that, for the curve expressed in terms of a positive function

$$\rho(h) \to 0^+ \quad \text{as } h \to 0^+$$

$$\Big(\rho(h) \sim \frac{1}{|\ln h|} \quad \text{is about right}\Big)$$

as follows:

$$R(t) = 2\sqrt{-t}\,\sqrt{-\ln \rho(-t)}, \tag{2.6}$$

the *sharp regularity criterion* holds (in Petrovskii's original notation):

$$\boxed{\int\limits_0 \frac{\rho(h)\sqrt{|\ln \rho(h)|}}{h}\,\mathrm{d}h < (=) +\infty \quad \Longrightarrow \quad (0,0^-) \text{ is irregular (regular).}} \tag{2.7}$$

Both converging (irregularity) and diverging (regularity) integrals in (2.7) as Dini–Osgood type regularity criteria already appeared in the first Petrovskii paper [5, p. 56] of 1934. Further historical and mathematical comments concerning Petrovskii's analysis including earlier (1933) Khinchin's criterion [37] in a probability representation can be found in a survey in [8].

Petrovskii's integral criterion of the Dini–Osgood type given in (2.7) is true in the N-dimensional radial case with (cf. [7] for a more recent updating)

$$\sqrt{|\ln p(h)|} \quad \text{replaced by} \quad |\ln p(h)|^{\frac{N}{2}}. \tag{2.8}$$

It is worth mentioning that, as far as we know, (2.5) is the first clear appearance of the "magic" $\sqrt{\log\log}$ in the theory of partial differential equations, currently associated with the "blow-up behavior" of the domain Q_0 and corresponding solutions. Concerning other classes of nonlinear partial differential equations generating blow-up $\sqrt{\log\log}$ in other settings, we refer the reader to [38].

Thus, since the 1930s, the Petrovskii regularity $\sqrt{\log\log}$–factor entered the parabolic theory and generated new types of asymptotic blow-up problems, which have been solved for a large class of parabolic equations with variable coefficients, as well as for some quasilinear ones. Nevertheless, such asymptotic problems were very delicate and some of them of Petrovskii type remained open even in the second order case, i.e., for (1.1), to be solved in the present paper for the first time.

3 Preliminaries of Matched Asymptotic Expansion

3.1 The basic initial-boundary value problem

Thus, we consider the semilinear parabolic equation (1.1), with a simple, "basic" nonlinear perturbation, which we take in the separable form (1.4). The eventual ODE regularity criterion will then also include the behavior of the nonlinear coefficient $\kappa(u)$ as $u \to 0$.

Hence our *basic second order initial-boundary value problem* takes the form

$$\begin{cases} u_t = u_{xx} + \frac{1}{(-t)}\,\kappa(u)u \quad \text{in} \quad Q_0 = \{|x| < R(t), \quad -1 < t < 0\}, \\ u = 0 \quad \text{at} \quad x = \pm R(t), \quad -1 \leqslant t < 0, \\ u(x,0) = u_0(x) \quad \text{on} \quad [-R(-1), R(-1)], \end{cases} \tag{3.1}$$

where $u_0(x)$ is a bounded and smooth function, $u_0(\pm R(-1)) = 0$. We then apply Definition 2.1 to the problem (3.1).

3.2 Slow growing factor $\varphi(\tau)$

According to (2.5), we need to assume that

$$\begin{aligned} & R(t) = (-t)^{\frac{1}{2}}\,\varphi(\tau), \\ & \text{where} \;\; \tau = -\ln(-t) \to +\infty \quad \text{as} \;\; t \to 0^-. \end{aligned} \tag{3.2}$$

Here, $\varphi(\tau) > 0$ is a smooth monotone increasing function satisfying $\varphi'(\tau) > 0$,

$$\varphi(\tau) \to +\infty, \quad \varphi'(\tau) \to 0^+, \quad \text{and} \quad \frac{\varphi'(\tau)}{\varphi(\tau)} \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{3.3}$$

Moreover, as a sharper characterization of the above class of *slow growing functions*, we use the following criterion:

$$\Big(\frac{\varphi(\tau)}{\varphi'(\tau)}\Big)' \to \infty \quad \text{as} \quad \tau \to +\infty \quad (\varphi'(\tau) \neq 0). \tag{3.4}$$

This is a typical condition in blow-up analysis, which distinguishes classes of exponential (the limit in (3.4) is 0), power-like (a constant $\neq 0$), and slow-growing functions (cf. [39, Lemma 1, pp. 390-400], where extra properties of slow-growing functions (3.4) are proved). For instance, one can use a comparison of such a $\varphi(\tau)$ with any power:

$$\text{for any } \alpha > 0 \quad \varphi(\tau) \ll \tau^{\alpha} \quad \text{and} \quad \varphi'(\tau) \ll \tau^{\alpha-1} \quad \text{for} \quad \tau \gg 1. \tag{3.5}$$

Such estimates are useful in evaluating perturbation terms in the rescaled equations.

Thus, the monotone positive function $\varphi(\tau)$ in (3.2) is assumed to determine a sharp behavior of the boundary of Q_0 near the shrinking point $(0,0)$ to guarantee its regularity. In the Petrovskii criterion (2.5), the almost optimal function satisfying (3.3) and (3.4) is

$$\varphi_*(\tau) = 2\sqrt{\ln \tau} \quad \text{as} \quad \tau \to +\infty. \tag{3.6}$$

3.3 First kernel scaling and two region expansion

By (3.2), we perform the similarity scaling

$$u(x,t) = v(y,\tau), \quad \text{where} \;\; y = \frac{x}{(-t)^{1/2}}. \tag{3.7}$$

Then the rescaled function $v(y,\tau)$ now solves the rescaled initial-boundary value problem

$$\begin{cases} v_\tau = \mathbf{B}^* v + \kappa(v)v \equiv v_{yy} - \frac{1}{2}\, y v_y + \kappa(v)v \quad \text{in} \;\; Q_0 = \{|y| < \varphi(\tau), \;\; \tau > 0\}, \\ v = 0 \quad \text{at} \;\; y = \pm\varphi(\tau), \quad \tau \geqslant 0, \\ v(0,y) = v_0(y) \equiv u_0(y) \quad \text{on} \;\; [-R(-1), R(-1)]. \end{cases} \tag{3.8}$$

The rescaled equation in (3.8), for the first time, shows how the classical Hermite operator

$$\mathbf{B}^* = D_y^2 - \frac{1}{2}\, y D_y \tag{3.9}$$

occurs after blow-up scaling (3.7). By the divergence (3.3) of $\varphi(\tau) \to +\infty$ as $\tau \to +\infty$, it follows that sharp asymptotics of solutions will essentially depend on the spectral properties of the linear operator $\mathbf{B}^*$ on the whole line $\mathbb{R}$ (cf. Appendix A), as well as on the nonlinearity $\kappa(v)v$, so that such an asymptotic "interaction" between linear and nonlinear operators therein eventually determines regularity of the vertex.

Studying asymptotics for the rescaled problem (3.8), as usual in asymptotic analysis, this blow-up problem is solved by *matching of expansions in two regions*:

(i) in the *inner region* which includes arbitrary compact subsets in y containing the origin $y = 0$, and

(ii) in the *boundary region* close to the boundaries $y = \pm\varphi(\tau)$, where a *boundary layer* occurs.

Actually, such a two-region structure, with the asymptotics specified below, defines the class of generic solutions under consideration. We begin with the simpler analysis in the boundary region (ii).

4 Boundary Layer (BL) Theory

4.1 BL-scaling and a perturbed parabolic equation

Sufficiently close to the lateral boundary of Q_0, it is natural to introduce the variables

$$z = \frac{y}{\varphi(\tau)} \;\text{ and } v(y,\tau) = w(z,\tau) \quad \Longrightarrow \quad w_\tau = \frac{1}{\varphi^2}\, w_{zz} - \frac{1}{2}\, z w_z + \frac{\varphi'}{\varphi}\, z w_z + \kappa(w)w. \tag{4.1}$$

We next introduce the BL-variables

$$\begin{aligned} &\xi = \varphi^2(\tau)(1 - z) \equiv \varphi(\varphi - y), \quad \varphi^2(\tau)\mathrm{d}\tau = \mathrm{d}s, \\ &w(z,\tau) = \rho(s) g(\xi, s), \end{aligned} \tag{4.2}$$

where $\rho(s) > 0$ for $s \gg 1$ is an unknown scaling time-factor depending on the function $\varphi(\tau)$. As usual, this ρ-scaling is chosen to get uniformly bounded rescaled solutions, i.e., for nonnegative solutions

$$\sup_{\xi} g(\xi, s) = 1 \quad \text{for all} \quad s \gg 1 \tag{4.3}$$

(for solutions which remain of changing sign for $s \gg 1$, one takes $|g(\xi, s)|$ in (4.3)). By the strong maximum principle (the Sturm theorem on zero sets, cf. [40]), $v_y(y, \tau)$ has a finite number of zeros in y for any $\tau > 0$ (possible supremum points), and a standard argument ensures that the normalization (4.3) implies that such a $\rho(s)$ can be treated as sufficiently smooth for $s \gg 1$[3)]. This describes the class of solutions under consideration. For instance, by the maximum principle, it is particular easier to work out, when:

$$\text{(4.3) holds for all nonnegative solutions } u(x, t) \not\equiv 0. \tag{4.4}$$

Respectively, for nonpositive solutions one can use -1 as the normalization in (4.3).

On substitution into the partial differential equation (4.1), we obtain the following small nonlinear perturbation of a linear uniformly parabolic equation:

$$g_s = \mathbf{A} g - \frac{1}{2} \frac{1}{\varphi^2} \xi g_\xi - \frac{\varphi'_\tau}{\varphi} \Big(1 - \frac{\xi}{\varphi^2}\Big) g_\xi - 2 \frac{\varphi'_\tau}{\varphi^3} \xi g_\xi - \frac{\rho'_s}{\rho} g + \frac{1}{\varphi^2} \kappa(\rho g) g, \tag{4.5}$$

where

$$\mathbf{A} g = g'' + \frac{1}{2} g'.$$

As usual in the boundary layer theory, this means that we then are looking for a generic pattern of the behavior described by (4.5) on compact subsets near the lateral boundary,

$$|\xi| = o\big(\varphi^{-2}(\tau)\big) \quad \Longrightarrow \quad |z - 1| = o\big(\varphi^{-4}(\tau)\big) \quad \text{as} \quad \tau \to +\infty. \tag{4.6}$$

On these space-time compact subsets, the second term on the right-hand side of (4.5) becomes asymptotically small, while all the other linear ones are much smaller in view of the slow growth/decay assumptions such as (3.4) for $\varphi(\tau)$ and $\rho(s)$.

4.2 Passing to the limit and convergence to a BL-profile

Thus, we arrive at a uniformly parabolic equation (4.5) perturbed by a number of linear and nonlinear terms being, under a given assumption, *asymptotically small* perturbations of the stationary elliptic operator $\mathbf{A}$. In particular, the last nonlinear term in (4.5) is clearly asymptotically small by the assumptions (1.5) and (3.3), so that for uniformly bounded g

$$\frac{1}{\varphi^2(\tau)} g \, \kappa(\rho(s) g) \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{4.7}$$

[3)] On the other hand, one can normalize in (4.2) by the smooth function $\rho(s) \equiv v(0, \tau)$, which also can be regarded as positive (negative) for $\tau \gg 1$ (infinitely many sign changes of $v(0, \tau)$ for $\tau \gg 1$ also mean that $v(y, \tau)$ has infinitely many zeros in y that is impossible for the heat equation [40]). This leads to some slight technical differences, although makes the normalization (4.8) below more straightforward.

By rescaling and (4.3), the BL-representation (4.2) naturally leads to the following asymptotic behavior at infinity:

$$\lim_{s\to+\infty} g(\xi,s) \to 1 \quad \text{as} \quad \xi \to +\infty, \tag{4.8}$$

where all the derivatives also vanish. Then we arrive at the problem of passing to the limit as $s \to +\infty$ in the problem (4.5), (4.8). Since the rescaled orbit $\{g(s),\ s>0\}$ is uniformly bounded by the definition in (4.2) (cf. [4, 3, 41]), one can pass to the limit in (4.5) along a subsequence $\{s_k\} \to +\infty$ by using the classical parabolic interior regularity theory. Namely, uniformly on compact subsets defined in (4.6), as $k \to \infty$,

$$\begin{aligned} & g(s_k+s) \to h(s), \quad \text{where} \quad h_s = \mathbf{A}h, \\ & h = 0 \quad \text{at} \quad \xi = 0, \\ & h\big|_{\xi=+\infty} = 1. \end{aligned} \tag{4.9}$$

Consider this *limit* (at $s=+\infty$) *equation* obtained from (4.5):

$$\begin{aligned} & h_s = \mathbf{A}h \equiv h_{\xi\xi} + \frac{1}{2} h_\xi \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+, \\ & h(0,s) = 0, \\ & h(+\infty,s) = 1. \end{aligned} \tag{4.10}$$

It is a linear parabolic partial differential equation in the unbounded domain $\mathbb{R}_+$, governed by the operator $\mathbf{A}$ admitting a standard symmetric representation in a weighted space. Namely, we have the following assertion.

Proposition 4.1. (i) (4.10) *is a gradient system in a weighted L^2-space, and*

(ii) *for bounded orbits the ω-limit set Ω_0 of* (4.10) *consists of a unique stationary profile*

$$g_0(\xi) = 1 - \mathrm{e}^{-\xi/2}, \tag{4.11}$$

and Ω_0 is uniformly stable in the Lyapunov sense in a weighted L^2-space.

Proof. As a second order equation, (4.10) is written in a symmetric form

$$\mathrm{e}^{\xi/2} h_s = (\mathrm{e}^{\xi/2} h_\xi)_\xi \tag{4.12}$$

and hence admits the multiplication by h_s in L^2 that yields the monotone Lyapunov function

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} \int \mathrm{e}^{\xi/2}(h_\xi)^2\,\mathrm{d}\xi = -\int \mathrm{e}^{\xi/2}(h_s)^2\,\mathrm{d}\xi \leqslant 0. \tag{4.13}$$

Note that, in (4.12), the derivatives h_ξ and h_s have to have an exponential decay at infinity in order the seminorms involved to make sense. It is essential that the limit profile (4.11) perfectly suits both.

Thus, the problem (4.5) is a perturbed *gradient system*, that allows to pass to the limit $s \to +\infty$ by using power tools of gradient system theory (cf., for example, [42]).

(ii) For a given bounded orbit $\{h(s)\}$ we denote $h(s) = g_0 + w(s)$, so that $w(s)$ solves the same equation (4.12). Multiplying by $w(s)$ in L^2 yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int \mathrm{e}^{\xi/2} w^2 \,\mathrm{d}\xi = -\int \mathrm{e}^{\xi/2}(w_\xi)^2 \,\mathrm{d}\xi < 0 \tag{4.14}$$

for any nontrivial solutions, whence the uniform stability (contractivity) property. □

Finally, we state the main stabilization result in the boundary layer, which establishes the actual class of generic solutions we are dealing with.

Theorem 4.1. (i) *There exists a class of solutions of the perturbed equation* (4.5) *for which, in a weighted L^2-space and uniformly on compact subsets,*

$$g(\xi, s) \to g_0(\xi) \quad \textit{as} \quad s \to +\infty. \tag{4.15}$$

(ii) (4.15) *is particularly true for all nontrivial nonnegative solutions.*

Proof. (i) Under given assumptions, the uniform stability result in (ii) of Proposition 4.1 implies [43, Chapter 1] that the ω-limit set of the asymptotically perturbed equation (4.5) is contained in that for the limit one (4.10), which consists of the unique profile (4.11).

(ii) This follows from the construction since then $\rho(s)$ in (4.2) can be chosen always positive. Then in the limit we are guaranteed to arrive at the gradient problem (4.9) admitting the unique uniformly stable stationary point (4.11). □

5 Inner Region Expansion: Towards an ODE Regularity Criterion

5.1 The Cauchy problem setting, eigenfunction expansion, and matching

In the inner region, we deal with the original rescaled problem (3.8). Without loss of generality, again for simplicity of final, rather technical and involved calculations, we consider even solutions defined for $y > 0$ by assuming the symmetry condition at the origin

$$v_y = 0 \quad \text{at} \quad y = 0. \tag{5.1}$$

As customary in the classical PDEs and potential theory (cf., for example, [44, § 6]), we extend $v(y, \tau)$ by 0 beyond the boundary points, i.e., for $y > \varphi(\tau)$:

$$\widehat{v}(y,\tau) = v(y,\tau)H(\varphi(\tau) - y) = \begin{cases} v(y,\tau), & 0 \leqslant y < \varphi(\tau), \\ 0, & y \geqslant \varphi(\tau), \end{cases} \tag{5.2}$$

where H is the Heaviside function. Since $v = 0$ on the lateral boundary $\{y = \varphi(\tau)\}$, one can check that, in the sense of the theory of distributions,

$$\begin{aligned} \widehat{v}_\tau &= v_\tau H, \\ \widehat{v}_y &= v_y H, \\ \widehat{v}_{yy} &= v_{yy} H - v_y\big|_{y=\varphi}\delta(y - \varphi). \end{aligned} \tag{5.3}$$

Therefore, $\widehat{v}$ satisfies the Cauchy problem

$$\widehat{v}_\tau = \mathbf{B}^* \widehat{v} + v_y \big|_{y=\varphi(\tau)} \delta(y - \varphi(\tau)) + \kappa(\widehat{v})\widehat{v} \quad \text{in} \;\; \mathbb{R} \times \mathbb{R}_+. \tag{5.4}$$

Since the extended solution (5.2) is uniformly bounded in $L^2_{\rho^*}(\mathbb{R})$ by construction, we can use the converging in the mean (and uniformly on compact subsets in y) the eigenfunction expansion via the standard Hermite polynomials given in (A.22) for $m = 1$:

$$\widehat{v}(y, \tau) = \sum_{(k \geqslant 0)} a_k(\tau) \psi_k^*(y). \tag{5.5}$$

Actually, as follows from the BL-theory in Section 4 (cf. Theorem 4.1), the only possible solutions admitting matching with (4.15) possess a constant in y behavior on compact subsets in y, i.e.,

$$\widehat{v}(y, \tau) = a_0(\tau) \cdot 1(1 + o(1)) \quad \text{as} \quad \tau \to +\infty. \tag{5.6}$$

Indeed, this "1" well corresponds to the first Hermite polynomial $\psi_0^*(y) \equiv 1$ in (5.5). Since $\lambda_0 = 0$ for this "polynomial," the behavior (5.6) can be referred as to a "center subspace" one for the operator $\mathbf{B}^*$ in (3.9), although we do not use this fact at all.

Thus, by the boundary layer theory establishing the boundary behavior (4.2) for $\tau \gg 1$, which we state again: in the rescaled sense, on the given compact subsets,

$$\widehat{v}(y, \tau) = \rho(s) g_0 \Big(\varphi^2(\tau)(1 - \frac{y}{\varphi(\tau)}) \Big) (1 + o(1)). \tag{5.7}$$

Overall, in the class of generic solutions satisfying the BL-expansion, we concentrate on the first Fourier pattern associated with

$$k = 0: \quad \lambda_0 = 0 \quad \text{and} \quad \psi_0^*(y) \equiv 1 \quad \big(\psi_0(y) \equiv F(y), \quad \text{the Gaussian (A.13)}\big). \tag{5.8}$$

The corresponding normalization condition is key for further projections:

$$\langle \psi_0, \psi_0^* \rangle \equiv \int F = 1. \tag{5.9}$$

Proposition 5.1. *Under the given assumptions*:

(i) *for solutions in Theorem* 4.1(i) *the relation* (5.6) *holds with* $a_0(\tau) > 0$ *for* $\tau \gg 1$, *and then the matching with the boundary layer behavior in* (4.2) *requires*

$$\frac{a_0(\tau)}{\rho(s)} \to 1 \quad \textit{as} \quad \tau \to +\infty \quad \Longrightarrow \quad \rho(s) = a_0(\tau)(1 + o(1)); \tag{5.10}$$

(ii) *in particular, these assertions are true for nontrivial nonnegative solutions.*

Proof. (i) follows from the construction of the boundary layer. (ii) follows from Theorem 4.1(ii). □

Thus, projecting Equation (5.4) onto the center subspace of $\mathbf{B}^*$ (i.e., multiplying in L^2 by $\psi_0(y) = F(y)$) yields, for the leading mode $a_0(\tau)$, the following "ordinary differential equation":

$$a_0' = v_y\big|_{y=\varphi(\tau)}\psi_0(\varphi(\tau)) + \langle \kappa(\widehat{v})\widehat{v}, \psi_0\rangle. \tag{5.11}$$

The convergence (5.7), which by a standard parabolic regularity is also true for the spatial derivatives, yields, as $\tau \to +\infty$,

$$\begin{aligned} &v_y\big|_{y=\varphi(\tau)} = \rho(s)\varphi(\tau)\gamma_1(1+o(1)) = a_0(\tau)\varphi(\tau)\gamma_1(1+o(1)), \\ &\gamma_1 = g_0'(0) = \frac{1}{2}. \end{aligned} \tag{5.12}$$

Finally, we need to estimate the last term in (5.11). By (5.6), using that $\kappa(a_0(\tau)) \neq 0$ for any $a_0(\tau) \neq 0$ via (1.5), we have

$$\langle \kappa(\widehat{v})\widehat{v}, \psi_0\rangle = \langle \kappa(a_0)a_0, F\rangle(1+o(1)) = \kappa(a_0)a_0(1+o(1)). \tag{5.13}$$

Indeed, since $\int F = 1$ for the Gaussian (A.13), in the last estimate, we have

$$\int\limits_0^{\varphi} F(y)\,\mathrm{d}y \equiv \frac{1}{2} - \int\limits_{\varphi}^{\infty} F(y)\,\mathrm{d}y = \frac{1}{2} - O\Big(\frac{1}{\varphi}\,\mathrm{e}^{-\varphi^2/4}\Big) \quad \text{as} \quad \varphi = \varphi(\tau) \to +\infty.$$

Thus, bearing in mind all above assumptions and estimates for generic patterns including (5.6), (5.10), (5.7), and (5.13), we obtain the following asymptotic ordinary differential equation for the first expansion coefficient $a_0(\tau) \neq 0$: as $\tau \to +\infty$,

$$\boxed{\frac{a_0'(\tau)}{a_0(\tau)} = -\frac{1}{4\sqrt{\pi}}\,\varphi(\tau)\,\mathrm{e}^{-\varphi^2(\tau)/4}(1+o(1)) + \kappa(a_0(\tau))(1+o(1))\,.} \tag{5.14}$$

One can see that, by the assumptions (1.5), all the solutions of the nonautonomous ordinary differential equation (5.14) are well defined for $\tau \in [0, +\infty)$. Moreover, by the classical comparison/monotonicity results for ordinary differential equations (Chaplygin's theorem [45] of 1920s), it follows that, under the above assumptions, solutions of (5.14) satisfy

$$a_0(0) \neq 0 \quad \Longrightarrow \quad a_0(\tau) \neq 0 \quad \text{for all} \quad \tau > 0. \tag{5.15}$$

Therefore, we can always consider positive orbits:

$$a_0(\tau) > 0 \quad \text{for all} \quad \tau \geqslant 0. \tag{5.16}$$

This makes our further asymptotic analysis easier. In particular, in view of (5.15) and (1.5), we can always omit all higher order terms appeared via the above asymptotics.

5.2 ODE regularity criterion

From (5.14) it follows that a natural way to formulate a regularity criterion for the parabolic partial differential equation (3.1) is to use the "ordinary differential equation language"[4)].

[4)] In fact, this is quite natural and unavoidable: for semilinear PDEs the characteristic point regularity depends on asymptotic properties of ODEs, i.e., the regularity issues for infinite-dimensional dynamical systems are characterized by 1D ones. This reveals a sufficient and successful reduction of dimensions.

Theorem 5.1 (ODE regularity criterion)**.** *In the parabolic problem* (3.1)*, the origin* $(0,0)$ *is regular if and only if* 0 *is globally asymptotically stable for the ordinary differential equation* (5.14)*, i.e., any solution of* (5.14) *is global and satisfies*

$$a_0(\tau) \to 0 \quad \textit{as} \quad \tau \to +\infty, \quad \textit{i.e.,} \quad \ln|a_0(\tau)| \to -\infty. \tag{5.17}$$

Proof. (i) *Necessity.* Given any classical solution $u(x,t)$ (3.1), one can always construct positive and negative *barrier* solutions $u_{\pm}(x,t)$ such that

$$u_-(x,t) \leqslant u(x,t) \leqslant u_+(x,t) \quad \text{in} \quad Q_0 \tag{5.18}$$

by standard comparison (maximum principle) arguments [1]. Since, by Theorem 4.1(ii) and Proposition 5.1, such non sign-changing solutions $u_{\pm}(x,t)$ do obey our matched asymptotics, their positive (negative) first Fourier coefficients satisfy the asymptotic ordinary differential equation (5.14) for $\tau \gg 1$. Hence, by the BL-construction, (5.17) implies that $u_{\pm}(x,t) \to 0$ as $t \to 0^-$ uniformly, so, by comparison (5.18), the same does an arbitrary $u(x,t)$.

(ii) *Sufficiency by contradiction.* Let there exist a solution $\{\overline{a}_0(\tau)\}$ of (5.17) (by (5.15), we may assume it to be positive) such that

$$\limsup_{\tau \to +\infty} \overline{a}_0(\tau) > 0. \tag{5.19}$$

Then, by the ODE comparison, the same is true for solutions of (5.17) with arbitrarily large Cauchy data at $\tau = 0$, i.e., for any

$$a_0(0) > \overline{a}_0(0). \tag{5.20}$$

Therefore, there exists a sufficiently large positive solution $u_+(x,t)$ of (3.1), whose first Fourier coefficient satisfies (5.14) and (5.20), so the regularity is violated by (5.19). □

For the heat equation (1.3), with $\kappa = 0$, integrating (5.14) immediately yields

$$\boxed{\kappa = 0: \quad (0,0) \text{ is regular} \iff \int^{\infty} \varphi(\tau)\, \mathrm{e}^{-\frac{\varphi^2(\tau)}{4}}\, \mathrm{d}\tau = +\infty,} \tag{5.21}$$

which is indeed another equivalent form of the Petrovskii criterion (2.7) (in the Khinchin form).

5.3 Applications: further regularity results

We now present a few corollaries of Theorem 5.1, with simpler and more traditional conditions of regularity/irregularity.

First of all, from the ordinary differential equation (5.14) it follows (and actually is true by comparison) that *negative* coefficients $\kappa(v)$ can "improve" the regularity of $(0,0)$. Moreover, in this simpler case, we find a condition, under which *any backward parabola has a regular vertex.*

Proposition 5.2. *Let* $\kappa(u)$ *satisfy* (1.5)*, and let*

$$\kappa(u) < 0 \quad \textit{for} \quad u > 0. \tag{5.22}$$

Then for any backward parabola ∂Q_0 *with arbitrary* φ *in* (3.2) *and* (3.3) *the vertex* $(0,0)$ *is regular.*

Proof. From (5.14) it follows that for $\tau \gg 1$

$$\frac{a_0'}{a_0} \leqslant -|\kappa(a_0)|(1+o(1)) \leqslant -\frac{1}{2}\,|\kappa(a_0)|. \tag{5.23}$$

Then, on integration, assuming without loss of generality that $a_0(0) = 1$ and checking an Osgood–Dini type condition

$$\int\limits_{0^+} \frac{\mathrm{d}z}{z|\kappa(z)|} = \infty, \tag{5.24}$$

which obviously holds for the coefficients (1.5), we have

$$\int\limits_{a_0(\tau)}^{1} \frac{\mathrm{d}z}{z|\kappa(z)|} \geqslant \frac{\tau}{2} \to +\infty \quad \text{as} \quad \tau \to \infty. \tag{5.25}$$

Hence, (5.24) reinforces (5.17) to hold. $\square$

Second, for *positive* coefficients κ the regularity can be destroyed. We first state the result establishing the conditions on monotone $\kappa(v) > 0$, under which the nonlinear term changes regularity for the pure heat equation into the irregularity.

Proposition 5.3. *Let $\kappa(u)$ satisfy* (1.5)*, and let*

$$\kappa(u) > 0 \quad \textit{be increasing for} \;\; u > 0. \tag{5.26}$$

Let (5.21) *be valid, i.e.,* $(0,0)$ *is regular for the heat equation* (1.3) *for* $N = 1$. *Denote by* $\widehat{a}_0(\tau) \to 0$ *as* $\tau \to \infty$ *the corresponding Fourier coefficient satisfying* (5.14) *for* $\kappa = 0$:

$$\widehat{a}_0(\tau) = \widehat{a}_0(0)\,\exp\left\{-\frac{1}{4\sqrt{\pi}}\int\limits_0^{\tau} \varphi(s)\,\mathrm{e}^{-\varphi^2(s)/4}\,\mathrm{d}s\right\} \quad \textit{for} \;\; \tau \gg 1. \tag{5.27}$$

Then the linear regularity criterion (5.21) *fails for the semilinear problem* (3.1) *and* $(0,0)$ *becomes irregular provided that the nonlinearity* κ *is such that*

$$\int^{+\infty}\left[-\frac{1}{4\sqrt{\pi}}\,\varphi(\tau)\mathrm{e}^{-\frac{\varphi^2(\tau)}{4}} + \kappa(\widehat{a}_0(\tau))\right]\mathrm{d}\tau > -\infty. \tag{5.28}$$

Proof. One can see that, in the present proof of a sharp estimate, one can omit both $o(1)$-terms in (5.14), meaning that one can replace those by $1+\varepsilon$ and $1-\varepsilon$ with an $\varepsilon > 0$ respectively and pass to the limit $\varepsilon \to 0^+$.

As the first iteration of the full ordinary differential equation (5.14), we have for $\tau \gg 1$

$$\frac{a_0'}{a_0} \geqslant -\frac{1}{4\sqrt{\pi}}\,\varphi(\tau)\,\mathrm{e}^{-\frac{\varphi^2(\tau)}{4}} \quad \Longrightarrow \quad a_0(\tau) \geqslant \widehat{a}_0(\tau). \tag{5.29}$$

In view of (5.26) and (5.28), we then obtain via the second iteration of (5.14):

$$\frac{a_0'}{a_0} \geqslant -\frac{1}{4\sqrt{\pi}}\,\varphi(\tau)\mathrm{e}^{-\frac{\varphi^2(\tau)}{4}} + \kappa(\widehat{a}_0(\tau)) \quad \text{for} \;\; \tau \gg 1. \tag{5.30}$$

Integrating this yields, by (5.14), that $(0,0)$ is no more regular for such nonlinear coefficients $\kappa(v)$. $\square$

Corollary 5.1. *Under the assumptions of Proposition* 5.3, *the Petrovskii backward parabola with the* $2\sqrt{\log\log}$*-factor* (3.6) *is no more a regular vertex of* Q_0 *for the semilinear problem* (3.1) *provided that*

$$\kappa(v) \gg |\ln|v||^{\frac{1}{3}}\ \mathrm{e}^{-(3\sqrt{\pi}|\ln|v||)^{2/3}} \quad \textit{as} \quad v \to 0. \tag{5.31}$$

Thus, (5.31) is the estimate, where the function in (1.6) comes from.

Proof of Corollary 5.1. From (5.14) with $\kappa = 0$ it follows that the function (5.27) reads

$$\widehat{a}_0(\tau) \sim \mathrm{e}^{-\frac{1}{3\sqrt{\pi}}(\ln\tau)^{3/2}} \quad \text{as} \quad \tau \to \infty. \tag{5.32}$$

Substituting (5.32) into (5.28) and changing the variable $\widehat{a}_0(\tau) = v$ yields (5.31). $\square$

Further iterating inequalities such as (5.30), one can obtain other sufficient conditions of the origin irregularity. For instance, if the integral in (5.28) still diverges to $-\infty$, integrating (5.30) gives the next iteration estimate: for $\tau \gg 1$

$$a_0(\tau) \geqslant \widehat{a}_0^{(1)}(\tau) \equiv a_0(0)\ \exp\Big\{ \int\limits_0^\tau \Big[-\frac{1}{4\sqrt{\pi}}\,\varphi(\eta)\mathrm{e}^{-\varphi^2(\eta)/4} + \kappa\Big(C_1\ \exp\Big\{ -\frac{1}{4\sqrt{\pi}} \int\limits_0^\eta \varphi(s)\mathrm{e}^{-\varphi^2(s)/4}\,\mathrm{d}s \Big\}\Big)\Big]\,\mathrm{d}\eta \Big\}, \tag{5.33}$$

where $C_1 > 0$ is some constant. Then, the next iteration leads to an awkward looking inequality:

$$\frac{a_0'}{a_0} \geqslant -\frac{1}{4\sqrt{\pi}}\,\varphi(\tau)\mathrm{e}^{-\varphi^2(\tau)/4} + \kappa\Big(a_0(0)\ \exp\Big\{ \int\limits_0^\tau \Big[-\frac{1}{4\sqrt{\pi}}\,\varphi(\eta)\mathrm{e}^{-\varphi^2(\eta)/4} + \kappa\Big(C_1\ \exp\Big\{ -\frac{1}{4\sqrt{\pi}} \int\limits_0^\eta \varphi(s)\mathrm{e}^{-\varphi^2(s)/4}\,\mathrm{d}s \Big\}\Big)\Big]\,\mathrm{d}\eta \Big\}\Big). \tag{5.34}$$

Integrating it gives an estimate of $a_0(\tau) \geqslant \widehat{a}_0^{(2)}(\tau)$ for $\tau \gg 1$ from below to be used also for the purpose of the irregularity, if $a_0^{(2)}(\tau) \not\to 0$ as $\tau \to +\infty$. If this fails, we then apply the third iteration of the ordinary differential equation (5.14) again leading to a sharper estimate from below for the regularity etc.

Since the number of such iterations can increase without bound (and hence the same do the numbers of exponents and corresponding integrals in the argument of $\kappa(\cdot)$ in (5.34) etc.), it seems inevitable that a single and a simply finite integral criterion of irregularity, similar to the Petrovskii one (5.21), cannot be derived for the nonlinear dynamical system (5.14) in the maximal generality. In other words, the ODE criterion of Theorem 5.1 is a right way to regularity issues and is even optimal.

In more general cases of equations in (3.1), where, in our notation,

$$\kappa = \kappa(x, t, u, u_x), \tag{5.35}$$

the derivation of matched asymptotics remains the same. The only difference is that, in accurate estimating of the integral in the last nonlinear term in (5.11), we should take into account that $v_y \approx 0$ in the whole inner region due to the "center subspace expansion" (5.6), so actually we integrate there $\kappa(\cdot, 0)$. But this term must also include integrals over the boundary layers close to $y = \pm\varphi(\tau)$, where the solution v and its derivative v_y is sharply given by (5.7) with the matching condition (5.10). We do not perform these general and, at the same time, rather straightforward and not that principal computations here, and restrict our attention to a particular model.

5.4 Equations with a gradient-dependent nonlinearity

Let us very briefly consider Equation (1.7). The first rescaling (3.7) gives the equation

$$v_\tau = \mathbf{B}^* v + \kappa(v) v (v_y)^2. \tag{5.36}$$

It is easy to check that the BL-analysis yields the same asymptotics as in (5.7), with a similar proof. However, the eventual derivation of the 1D dynamical system for the first Fourier coefficient $a_0(\tau)$ is now different: the nonlinear term is much weaker since $v_y \approx 0$ on the center subspace patterns, except a $(1/\varphi(\tau))$-neighborhood of the boundary point $y = \varphi(\tau)$. Overall, the nonlinear perturbation in (5.14) is estimated as follows:

$$\begin{aligned} J(a_0) &= \langle \kappa(a_0) a_0^2 \big[g_0'\big(\varphi^2\big(1 - \tfrac{y}{\varphi}\big)\big)\big]^2 (-\varphi)^2,\ \psi_0(y) \rangle \\ &= \kappa(a_0) a_0^2 \varphi^2 \, \frac{1}{2\sqrt{\pi}} \int_0^{\varphi} \big[g_0'\big(\varphi^2\big(1 - \tfrac{y}{\varphi}\big)\big)\big]^2 \mathrm{e}^{-y^2/4} \, \mathrm{d}y, \end{aligned} \tag{5.37}$$

where $\psi_0 = F$ given by (A.13). Using the BL-profile (4.11) and setting $z = y/\varphi$ yields

$$\begin{aligned} J(a_0) &= \kappa(a_0) a_0^2 \varphi^3 \frac{1}{8\sqrt{\pi}} \int_0^1 \mathrm{e}^{-\varphi^2(1-z)} \mathrm{e}^{-\varphi^2 z^2/4} \, \mathrm{d}z \\ &= \kappa(a_0) a_0^2 \varphi^3 \frac{1}{8\sqrt{\pi}} \, \mathrm{e}^{-\varphi^2} \int_0^1 \mathrm{e}^{\varphi^2 z (1 - \frac{z}{4})} \, \mathrm{d}z. \end{aligned} \tag{5.38}$$

Estimating roughly the last integral as follows:

$$\int_0^1 \mathrm{e}^{\varphi^2 z (1 - \frac{z}{4})} \, \mathrm{d}z \leqslant \mathrm{e}^{\frac{3}{4}\varphi^2},$$

we obtain the following approximate dynamical system for $a_0(\tau) > 0$:

$$\frac{a_0'}{a_0} \leqslant -\frac{1}{4\sqrt{\pi}} \, \varphi(\tau) \, \mathrm{e}^{-\varphi^2(\tau)/4} + \frac{1}{8\sqrt{\pi}} \, \kappa(a_0) a_0^2 \varphi^3(\tau) \mathrm{e}^{-\varphi^2(\tau)/4} + \ldots. \tag{5.39}$$

This is enough for us to prove that the nonlinear perturbation is now much weaker than that in (5.14).

Proposition 5.4. *For* (5.39), *the Petrovskii double log-function* (3.6) *forms a regular vertex* $(0, 0)$ *for any function* $\kappa(u)$ *satisfying* (1.5).

Proof. Assuming that the linear term is dominant that creates the behavior (5.32), one can check that, on this $\widehat{a}_0(\tau)$, the nonlinear term in (5.39) is always negligible, so (5.30) follows. $\square$

5.5 Backward paraboloid in $\mathbb{R}^N$

More carefully, aspects of checking regularity of the vertex of a backward paraboloid in $\mathbb{R}^N$ was done in [46], where the authors applied matching techniques to the Navier–Stokes equations in $\mathbb{R}^3$. Now, we present a few comments.

For the N-dimensional case (1.8), the lateral boundary of the domain Q_0 in $\mathbb{R}^{N+1}$ is given by a *backward paraboloid* of the form

$$\partial Q_0 : \quad \sqrt{\sum_{i=1}^{N} a_i |x_i|^2} = \sqrt{-t}\, \varphi(\tau), \quad \tau = -\ln(-t), \quad a_i > 0, \quad \sum a_i^2 = 1. \tag{5.40}$$

Then a boundary layer close to the rescaled (via (4.1)) boundary

$$\partial \widehat{Q}_0 : \quad \sum a_i |z_i|^2 = 1, \tag{5.41}$$

leads to a linear elliptic problem, which can be solved. Moreover, in the direction of the unit inward normal $\mathbf{n}$ to $\partial \widehat{Q}_0$, the boundary layer profile $g_0(\xi)$ remains one-dimensional depending on the single variable

$$\eta = \xi \cdot \mathbf{n}, \tag{5.42}$$

so that $g_0 = g_0(\eta)$ is still given by (4.11). Therefore, in the expanding domain with the boundary

$$\partial \widetilde{Q}_0(\tau) : \quad \sum a_i |y_i|^2 = \varphi(\tau) \to +\infty \quad \text{as} \quad \tau \to +\infty, \tag{5.43}$$

the BL-profile is expressed in terms of the distance function:

$$g_0(y,\tau) = 1 - \mathrm{e}^{-\frac{1}{2}\varphi(\tau)\, \mathrm{dist}\, \{y, \partial \widetilde{Q}_0(\tau)\}}. \tag{5.44}$$

This allows us to apply the same blow-up scaling and matching techniques.

The final ordinary differential equation for $a_0(\tau)$ takes a similar to (5.14) form, with φ in the first term replaced by φ^N, in a full accordance to (2.8). However, the computations get more involved and further coefficients of this asymptotic ODE will essentially depend on the geometric shape of the backward paraboloid (5.40) in a neighborhood of its characteristic vertex $(0,0)$. However, final regularity conclusions remain approximately the same as for $N = 1$, including both cases of nonlinearities in (1.8).

Appendix A.
Hermitian Spectral Theory for Operator Pair $\{\mathbf{B}, \mathbf{B}^*\}$

For the maximal generality and further applications, we describe the necessary spectral properties of the linear $2m$th order differential operator in $\mathbb{R}^N$

$$\mathbf{B}^* = (-1)^{m+1} \Delta_y^m - \frac{1}{2m}\, y \cdot \nabla_y, \tag{A.1}$$

and of its adjoint $\mathbf{B}$ in the standard L^2-metric given by

$$\mathbf{B} = (-1)^{m+1}\Delta_y^m + \frac{1}{2m}\, y \cdot \nabla_y + \frac{N}{2m}\, I \quad (I \text{ denotes the identity}). \tag{A.2}$$

Both operators occur after global and blow-up scaling respectively of solutions of the *polyharmonic equation*

$$u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+. \tag{A.3}$$

Of course, for $m = 1$ the operators (A.1) and (A.2) are classical Hermite selfadjoint operators with completely known spectral properties (cf., for example, [47, pp. 44-48]). However, for any $m \geqslant 2$ both operators (A.1) and (A.2), although looking very similar to those for $m = 1$, *are not symmetric* and do not admit a selfadjoint extension, so we follow more recent paper [48] in presenting necessary spectral results. In what follows, we mainly must concentrate on the less known case $m \geqslant 2$, naturally assuming that, for the classical selfadjoint case $m = 1$, we can borrow any result from several textbooks and/or monographs.

A.1 Fundamental solution, rescaled kernel, and first estimates

We begin with the necessary fundamental solution $b(x,t)$ of the linear polyharmonic parabolic equation (A.3), which is of standard similarity form and satisfies, in the sense of bounded measures:

$$b(x,t) = t^{-\frac{N}{2m}} F(y), \quad y = x/t^{\frac{1}{2m}} \quad \text{such that} \quad b(x, 0^+) = \delta(x), \tag{A.4}$$

where $\delta(x)$ is the Dirac delta. The rescaled kernel $F = F(|y|)$ is then the unique radially symmetric solution of the elliptic equation with the operator (A.2), i.e.,

$$\mathbf{B}F \equiv -(-\Delta)^m F + \frac{1}{2m}\, y \cdot \nabla F + \frac{N}{2m}\, F = 0 \quad \text{in } \mathbb{R}^N, \quad \text{with} \quad \int F = 1. \tag{A.5}$$

In the case $m = 1$, F is the classical positive Gaussian

$$F(y) = \frac{1}{(4\pi)^{N/2}}\, \mathrm{e}^{-|y|^2/4} > 0 \quad \text{in} \quad \mathbb{R}^N. \tag{A.6}$$

For any $m \geqslant 2$ the rescaled kernel function $F(|y|)$ is oscillatory as $|y| \to \infty$ and satisfies the estimate (for $m = 1$ this is trivial, with $\alpha = 2$) [3, 49]

$$|F(y)| < D\, \mathrm{e}^{-d_0 |y|^\alpha} \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad \alpha = \frac{2m}{2m-1} \in (1,2), \tag{A.7}$$

for some positive constants D and d_0 depending on m and N.

A.2 Sharp estimates in one dimension

For further use in our regularity study, we need some sharp estimates of the rescaled kernel, which we present for $N = 1$, where the regularity analysis gets also rather involved. Taking the Fourier transform in (A.5) leads to the expression

$$F(y) = \alpha_0 \int_0^\infty \mathrm{e}^{-s^{2m}} \cos(sy)\, \mathrm{d}s, \tag{A.8}$$

where $\alpha_0 > 0$ is the normalization constant, and, more precisely [41],

$$F(y) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \mathrm{e}^{-s^{2m}} \sqrt{s|y|}\, J_{-\frac{1}{2}}(s|y|)\, \mathrm{d}s \quad \text{in } \mathbb{R}, \tag{A.9}$$

where J_ν denotes the Bessel function. The rescaled kernel $F(y)$ satisfies (A.7), where d_0 admits an explicit expression (cf. below). Such optimal exponential estimates of the fundamental solutions of higher order parabolic equations are well known and were first obtained by Evgrafov–Postnikov (1970) and Tintarev (1982) (cf. [50, 51] for key references and results).

As a crucial issue for the further boundary point regularity study, we will need a sharper, than given by (A.7), asymptotic behavior of the rescaled kernel $F(y)$ as $y \to +\infty$. To get that, we rewrite Equation (A.5) on integration once as

$$(-1)^{m+1} F^{(2m-1)} + \frac{1}{2m}\, yF = 0 \quad \text{in } \mathbb{R}. \tag{A.10}$$

Using standard classical WKBJ asymptotics, we substitute into (A.10) the function

$$F(y) \sim y^{-\delta_0}\, \mathrm{e}^{a y^\alpha} \quad \text{as } y \to +\infty, \tag{A.11}$$

exhibiting two scales. This gives the algebraic equation for a:

$$(-1)^m (\alpha a)^{2m-1} = \frac{1}{2m} \quad \text{and} \quad \delta_0 = \frac{m-1}{2m-1} > 0\,. \tag{A.12}$$

Note that the slow algebraically decaying factor $y^{-\delta_0}$ in (A.11) is available for any $m \geqslant 2$. For $m = 1$ this algebraic factor is absent for the exponential positive Gaussian profile

$$F(y) = \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{-y^2/4} \quad (m = N = 1). \tag{A.13}$$

By construction, one needs to get the root a of (A.12) with the maximal $\mathrm{Re}\, a < 0$. This yields (cf., for example, [50, 51] and [52, p. 141])

$$a = \frac{2m-1}{(2m)^\alpha} \left[-\sin\left(\frac{\pi}{2(2m-1)}\right) + \mathrm{i}\, \cos\left(\frac{\pi}{2(2m-1)}\right) \right] \equiv -d_0 + \mathrm{i}\, b_0 \quad (d_0 > 0). \tag{A.14}$$

Finally, this gives the following double-scale asymptotic of the kernel:

$$F(y) = y^{-\delta_0}\, \mathrm{e}^{-d_0 y^\alpha} \left[C_1 \sin(b_0 y^\alpha) + C_2 \cos(b_0 y^\alpha) \right] + \ldots \quad \text{as } y \to +\infty, \tag{A.15}$$

where $C_{1,2}$ are real constants, $|C_1| + |C_2| \neq 0$. In (A.15), we present the first two leading terms from the m-dimensional bundle of exponentially decaying asymptotics.

In particular, for the linear biharmonic operator in (B.1) ($N = 1$), we have

$$m = 2: \quad \alpha = \frac{4}{3}, \quad d_0 = 3 \cdot 2^{-\frac{11}{3}}, \quad b_0 = 3^{\frac{3}{2}} \cdot 2^{-\frac{11}{3}}, \quad \text{and} \quad \delta_0 = \frac{1}{3}. \tag{A.16}$$

A.3 The discrete real spectrum and eigenfunctions of B

Both linear operators $\mathbf{B}$ and the corresponding adjoint operator $\mathbf{B}^*$ should be considered in weighted L^2-spaces with the weight functions induced by the exponential estimate of the rescaled kernel (A.7). We again more concentrate on the non-selfadjoint case $m \geqslant 2$, and refer to [47] for the classical case $m = 1$.

For $m \geqslant 2$, we consider $\mathbf{B}$ in the weighted space $L^2_\rho(\mathbb{R}^N)$ with the exponentially growing weight function

$$\rho(y) = \mathrm{e}^{a|y|^\alpha} > 0 \quad \text{in} \quad \mathbb{R}^N, \tag{A.17}$$

where $a \in (0, 2d_0)$ is a fixed constant. We next introduce a standard Hilbert (a weighted Sobolev) space of functions $H^{2m}_\rho(\mathbb{R}^N)$ with the inner product

$$\langle v, w \rangle_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} D_y^k v(y) \, \overline{D_y^k w(y)} \, \mathrm{d}y$$

and the induced norm

$$\|v\|^2_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} |D_y^k v(y)|^2 \, \mathrm{d}y.$$

Then

$$H^{2m}_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$$

and $\mathbf{B}$ is a bounded linear operator from $H^{2m}_\rho(\mathbb{R}^N)$ to $L^2_\rho(\mathbb{R}^N)$. The necessary spectral properties of the operator $\mathbf{B}$ are as follows [48].

Lemma A.1. (i) *The spectrum of* $\mathbf{B}$ *comprises real simple eigenvalues only,*

$$\sigma(\mathbf{B}) = \Big\{ \lambda_\beta = -\frac{k}{2m}, \; k = |\beta| = 0, 1, 2, \dots \Big\}. \tag{A.18}$$

(ii) *The eigenfunctions* $\psi_\beta(y)$ *are given by*

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \quad \text{for any} \quad |\beta| = k \tag{A.19}$$

and form a complete set in $L^2(\mathbb{R})$ *and in* $L^2_\rho(\mathbb{R})$.

(iii) *The resolvent* $(\mathbf{B} - \lambda I)^{-1}$ *for* $\lambda \notin \sigma(\mathbf{B})$ *is a compact integral operator in* $L^2_\rho(\mathbb{R}^N)$.

By Lemma A.1, the center and stable subspaces of $\mathbf{B}$ are given by

$$E^c = \mathrm{Span}\{\psi_0 = F\},$$
$$E^s = \mathrm{Span}\{\psi_\beta, \, |\beta| > 0\}.$$

Note also that the operators $\mathbf{B}$ has the zero Morse index, i.e., no eigenvalues have positive real part. In the classical Hermite case $m = 1$ (the only selfadjoint case), the spectrum is again given by (A.18) and the eigenfunction formula (A.19) with the rescaled kernel (A.13) generates standard Hermite polynomials (cf. [47, p. 48] for a full spectral account for the operator $\mathbf{B}$).

A.4 The polynomial eigenfunctions of the operator $\mathbf{B}^*$

We now consider the adjoint operator (A.1) in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$ ($\langle \cdot, \cdot \rangle_{\rho^*}$ is the inner product and $\| \cdot \|_{\rho^*}$ is the norm) with the "adjoint" exponentially decaying weight function

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0. \tag{A.20}$$

We ascribe to $\mathbf{B}^*$ the domain $H^{2m}_{\rho^*}(\mathbb{R}^N)$, which is dense in $L^2_{\rho^*}(\mathbb{R}^N)$, and then

$$\mathbf{B}^* : \; H^{2m}_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N)$$

is a bounded linear operator. The operator $\mathbf{B}$ is adjoint to $\mathbf{B}^*$ in the usual sense: denoting by $\langle \cdot, \cdot \rangle$ the inner product in the dual space $L^2(\mathbb{R}^N)$, we have

$$\langle \mathbf{B} v, w \rangle = \langle v, \mathbf{B}^* w \rangle \quad \text{for any} \;\; v \in H^{2m}_{\rho}(\mathbb{R}^N) \;\; \text{and} \;\; w \in H^{2m}_{\rho^*}(\mathbb{R}^N). \tag{A.21}$$

The eigenfunctions of $\mathbf{B}^*$ take a particularly simple polynomial form and are as follows.

Lemma A.2. (i) $\sigma(\mathbf{B}^*) = \sigma(\mathbf{B})$.

(ii) *The eigenfunctions $\psi^*_\beta(y)$ of $\mathbf{B}^*$ are polynomials in y of the degree $|\beta|$ given by*

$$\psi^*_\beta(y) = \frac{1}{\sqrt{\beta!}} \left[y^\beta + \sum_{j=1}^{[|\beta|/2m]} \frac{1}{j!} (-\Delta)^{mj} y^\beta \right] \tag{A.22}$$

and form a complete subset in $L^2_{\rho^*}(\mathbb{R}^N)$.

(iii) $\mathbf{B}^*$ *has a compact resolvent $(\mathbf{B}^* - \lambda I)^{-1}$ in $L^2_{\rho^*}(\mathbb{R}^N)$ for $\lambda \notin \sigma(\mathbf{B}^*)$.*

Of course, for $m = 1$, (A.22) yields standard Hermite polynomials, so, for $m \geqslant 2$, we call (A.22) *generalized Hermite polynomials*. The biorthonormality condition holds:

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta\gamma}. \tag{A.23}$$

Remark A.1 (on closure)**.** This is an important issue for using eigenfunction expansions of solutions. First, as is well known, for $m = 1$ the sets of eigenfunctions are complete and closed in the corresponding spaces (cf. [47]).

Second, for $m \geqslant 2$ one needs some extra speculations. Namely, using (A.23), we can introduce the subspaces of eigenfunction expansions and begin with the operator $\mathbf{B}$. We denote by $\widetilde{L}^2_\rho$ the subspace of eigenfunction expansions

$$v = \sum c_\beta \psi_\beta$$

with coefficients $c_\beta = \langle v, \psi^*_\beta \rangle$ defined as the closure of the finite sums

$$\Big\{ \sum_{|\beta| \leqslant M} c_\beta \psi_\beta \Big\}$$

in the L^2_ρ-norm. Similarly, for the adjoint operator $\mathbf{B}^*$ we define the subspace $\widetilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}$. Note that, since the operators are not selfadjoint and the eigenfunction subsets are not orthonormal, in general, these subspaces can be different from L^2_ρ and $L^2_{\rho^*}$, and the equality is guaranteed in the selfadjoint case $m = 1$, $a = 1/4$ only. For $m \geqslant 2$, in the above subspaces obtained via suitable closure, we can apply standard eigenfunction expansion techniques as in the classical selfadjoint case $m = 1$.

For $m = 2$ and $N = 1$ (this simpler case will be treated in greater detail) the first "adjoint" generalized Hermite polynomial eigenfunctions are as follows:

$$
\begin{aligned}
&\psi_0(y) = 1, \quad \psi_1(y) = y, \\
&\psi_2(y) = \frac{1}{\sqrt{2}}\, y^2, \quad \psi_3(y) = \frac{1}{\sqrt{6}}\, y^3, \\
&\psi_4(y) = \frac{1}{\sqrt{24}}\, (y^4 + 24), \quad , \psi_5(y) = \frac{1}{2\sqrt{30}}\, (y^5 + 120\, y), \\
&\psi_6(y) = \frac{1}{12\sqrt{5}}\, (y^6 + 360 y^2) \ \text{ etc.}
\end{aligned}
\tag{A.24}
$$

with the corresponding eigenvalues

$$
0, \quad -\frac{1}{4}, \quad -\frac{1}{2}, \quad -\frac{3}{4}, \quad -1, \quad -\frac{5}{4}, \quad -\frac{3}{2} \ \text{ etc.}
$$

Appendix B. Semilinear Biharmonic Equations

B.1 Regularity problem setting

Here, we show how our approach can be extended to higher order PDEs, for examples, for the *semilinear biharmonic equations* having similar nonlinearities, with also zero Dirichlet conditions on ∂Q_0 and bounded initial data u_0 in $Q_0 \cap \{t = -1\}$:

$$
\begin{aligned}
&u_t = -u_{xxxx} + \begin{cases} \dfrac{\kappa(u)u}{(-t)} \\[2mm] \kappa(u)u(u_x)^4 \end{cases} \quad \text{in } \ Q_0, \\
&u = u_x = 0 \quad \text{on } \ \partial Q_0, \\
&u(x, -1) = u_0(x).
\end{aligned}
\tag{B.1}
$$

Then, after a proper similar matching with a boundary layer, we again arrive a nonlinear dynamical system viewed as a "center subspace" approximation of solutions in the space of generalized Hermite polynomials as eigenfunctions of a rescaled non-selfadjoint operator. We also discuss the regularity problems for *backward paraboloids* ∂Q_0 in $\mathbb{R}^N \times [-1, 0)$, where the initial-boundary value problem reads

$$u_t = -\Delta^2 u + \begin{cases} \frac{\kappa(u)u}{(-t)} \\ \kappa(u)u|\nabla u|^4 \end{cases} \quad \text{in } \; Q_0, \tag{B.2}$$

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \; \partial Q_0,$$

where $\mathbf{n}$ is the unit inward normal to the smooth boundary of the domain $Q_0 \cap \{t\}$. Further extensions to $2m$th order parabolic partial differential equations are also discussed.

Thus, we now show that a similar sequence of mathematical transformations can be performed for the fourth order semilinear biharmonic equations (B.1).

B.2 Initial-boundary value problem

We again fix $N = 1$, i.e., consider (B.1) with the simplest nonlinearity (1.4), leading to the initial-boundary value problem

$$\begin{cases} u_t = -u_{xxxx} + \frac{1}{(-t)}\,\kappa(u)u \quad \text{in } \; Q_0 = \{|x| < R(t), \;\; -1 < t < 0\}, \\ u = u_x = 0 \quad \text{at } \; x = \pm R(t), \quad -1 \leqslant t < 0, \\ u(x,0) = u_0(x) \quad \text{on } \; [-R(-1), R(-1)], \end{cases} \tag{B.3}$$

where $u_0(x)$ is bounded and satisfies

$$u_0 = u_0' = 0 \quad \text{at } \; x = \pm R(-1).$$

B.3 Slow growing factor $\varphi(\tau)$

Similar to (2.5), we assume that

$$R(t) = (-t)^{\frac{1}{4}}\,\varphi(\tau), \quad \text{where } \; \tau = -\ln(-t) \to +\infty \quad \text{as } \; t \to 0^-. \tag{B.4}$$

Here, the main scaling factor $(-t)^{1/4}$ naturally comes from the biharmonic kernel variables (cf. (3.7) and (A.4)), and $\varphi(\tau) > 0$ is again a slow growing function satisfying (3.3). For "shrinking backward parabolas" with

$$\varphi(\tau), \quad \varphi'(\tau) \to 0 \quad \text{as } \; \tau \to \infty,$$

the regularity in the linear case $\kappa = 0$ was proved by Mikhailov [53, 54] in 1963; in a certain sense, this extended the Gevrey-like result (2.2) for $m = 1$ (cf. (2.3)).

B.4 First kernel scaling

By (3.2), we perform the similarity scaling

$$u(x,t) = v(y,\tau), \quad \text{where } \; y = \frac{x}{(-t)^{1/4}}. \tag{B.5}$$

The rescaled function $v(y,\tau)$ solves the rescaled initial-boundary value problem

$$\begin{cases} v_\tau = \mathbf{B}^* v + \kappa(v)v \equiv -v_{yyyy} - \frac{1}{4}\, y v_y + \kappa(v) v \quad \text{in} \;\; Q_0 = \{|y| < \varphi(\tau), \;\; \tau > 0\}, \\ v = v_y = 0 \quad \text{at} \;\; y = \pm\varphi(\tau), \quad \tau \geqslant 0, \\ v(0,y) = v_0(y) \equiv u_0(y) \quad \text{on} \;\; [-R(-1), R(-1)]. \end{cases} \tag{B.6}$$

B.5 Boundary layer

Sufficiently close to the lateral boundary of Q_0, we naturally introduce the variables

$$z = \frac{y}{\varphi(\tau)}, \;\; v(y,\tau) = w(z,\tau) \quad \Longrightarrow \quad w_\tau = -\frac{1}{\varphi^4}\, w_{zzzz} - \frac{1}{4}\, z w_z + \frac{\varphi'}{\varphi}\, z w_z + \kappa(w) w. \tag{B.7}$$

The BL-variables now read

$$\begin{aligned} &\xi = \varphi^{\frac{4}{3}}(\tau)(1-z), \\ &\varphi^{\frac{4}{3}}(\tau)\mathrm{d}\tau = \mathrm{d}s, \\ &w(z,\tau) = \rho(s) g(\xi,s), \end{aligned} \tag{B.8}$$

where $\rho(s)$ is a slow varying function for which eventually (5.10) will hold by matching.

Substituting into (B.7) yields the perturbed equation

$$\begin{aligned} &g_s = \mathbf{A} g - \frac{1}{4}\,\frac{1}{\varphi^{4/3}}\,\xi g_\xi - \frac{\varphi'_\tau}{\varphi}\,\big(1 - \frac{\xi}{\varphi^{4/3}}\big) g_\xi - \frac{4}{3}\,\frac{\varphi'_\tau}{\varphi^{1/3}}\,\xi g_\xi - \frac{\rho'_s}{\rho}\, g + \frac{1}{\varphi^{4/3}}\,\kappa(\rho g)\, g, \\ &\text{where} \;\; \mathbf{A} g = -g^{(4)} + \frac{1}{4}\, g'. \end{aligned} \tag{B.9}$$

In this boundary layer, we are looking for a generic pattern of the behavior described by (B.9) on compact subsets near the lateral boundary,

$$|\xi| = o\big(\varphi^{-\frac{4}{3}}(\tau)\big) \quad \Longrightarrow \quad |z-1| = o\big(\varphi^{-\frac{8}{3}}(\tau)\big) \quad \text{as} \;\; \tau \to +\infty. \tag{B.10}$$

We next pose the same asymptotic behavior (4.8) at infinity. Assuming that, by (B.8), the rescaled orbit $\{g(s), \;\; s > 0\}$ is uniformly bounded, by the parabolic theory [3], we can again pass to the limit in (B.9) in the asymptotically small perturbations, along a subsequence $\{s_k\} \to +\infty$. Therefore, uniformly on compact subsets defined in (B.10), as $k \to \infty$,

$$\begin{aligned} &g(s_k + s) \to h(s), \\ &\text{where} \;\; A h_s = \mathbf{A} h, \quad h = h_\xi = 0 \;\; \text{at} \;\; \xi = 0, \quad h\big|_{\xi=+\infty} = 1. \end{aligned} \tag{B.11}$$

The limit equation obtained from (B.9),

$$h_s = \mathbf{A} h \equiv -h_{\xi\xi\xi\xi} + \frac{1}{4}\, h_\xi \tag{B.12}$$

is again a standard linear parabolic partial differential equation in the unbounded domain $\mathbb{R}_+$, although now it is governed by a non-selfadjoint operator $\mathbf{A}$. Actually, we need to show that, in

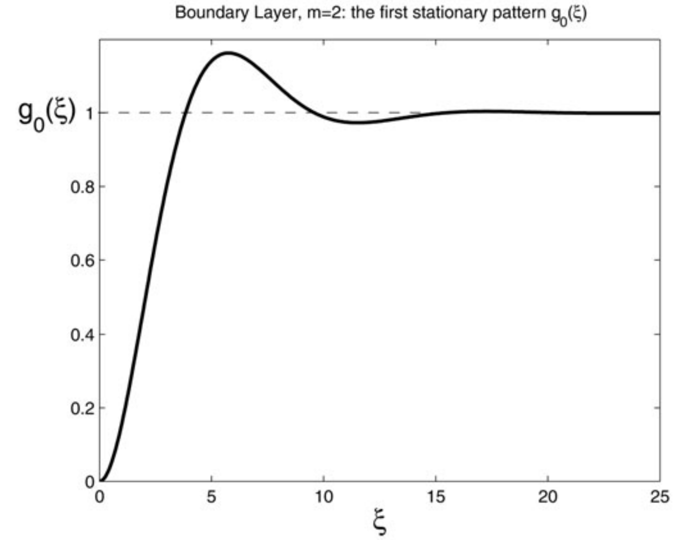

Figure 1: [8] The unique stationary solution $g_0(\xi)$ of the problem (B.13).

an appropriate weighted L^2-space if necessary and under the assumption (4.8), the stabilization holds, i.e., the ω-limit set of the orbit $\{h(s)\}_{s>0}$ consists of a single equilibrium: as $s \to +\infty$,

$$\begin{cases} h(\xi, s) \to g_0(\xi), \quad \text{where} \quad \mathbf{A} g_0 = 0 \quad \text{for} \quad \xi > 0, \\ g_0 = g_0' = 0 \quad \text{at} \quad \xi = 0, \quad g_0(+\infty) = 1. \end{cases} \tag{B.13}$$

The characteristic equation for the linear operator $\mathbf{A}$ yields

$$-\lambda^4 + \frac{1}{4}\,\lambda = 0 \quad \Longrightarrow \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_{2,3} = \frac{1}{4^{1/3}}\Big(-\frac{1}{2} \pm \mathrm{i}\,\frac{\sqrt{3}}{2} \Big). \tag{B.14}$$

This gives the unique solution of (B.13), shown in Figure 1,

$$g_0(\xi) = 1 - \mathrm{e}^{-\frac{\xi}{2^{5/3}}} \Big[\cos\Big(\frac{\sqrt{3}\,\xi}{2^{5/3}} \Big) + \frac{1}{\sqrt{3}} \sin\Big(\frac{\sqrt{3}\,\xi}{2^{5/3}} \Big) \Big]. \tag{B.15}$$

It turns out that the limit problem (B.12) possesses a number of strong gradient and contractivity properties. Namely setting by linearization

$$h(s) = g_0 + w(s) \quad \Longrightarrow \quad w_s = \mathbf{A} w \equiv -w_{\xi\xi\xi\xi} + \frac{1}{4}\, w_\xi, \quad w = w_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{B.16}$$

we arrive at the following assertion (cf. Proposition 4.1 for $m = 1$).

Proposition B.5. (i) (B.16) *is a gradient system in* L^2, *and*

(ii) *in the given class of solutions, the* ω*-limit set* Ω_0 *of* (B.16) *consists of the origin only and it is uniformly stable.*

Proof. (i) One can see that (B.16) admits a monotone Lyapunov function obtained by multiplying $w_{\xi\xi}$ in L^2:

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}s} \int (w_\xi)^2 = -\int (w_{\xi\xi\xi})^2 \leqslant 0. \tag{B.17}$$

Hence (ii) also follows. □

Thus, quite similar to the second order case, under given assumptions, we can pass to the limit $s \to +\infty$ along any sequence in the perturbed gradient system (B.9). Then, again similarly to $m = 1$, the uniform stability of the stationary point g_0 in the limit autonomous system (B.12) in a suitable metric guarantees that the asymptotically small perturbations do not affect the omega-limit set (cf. [43, Chapter 1]). However, at this moment, we cannot avoid the following convention, which for $m = 2$ is much more key than for $m = 1$, where the maximum principle makes this part of the analysis simpler, at least, for nonnegative or nonpositive solutions (but for others of changing sign, this remains necessary). Actually, the convergence (B.11) and (B.13) for the perturbed dynamical system (B.9) should be considered as the main hypothesis *characterizing the class of generic patterns* under consideration (and then (4.8) is its partial consequence). Since the positivity (negativity) is not an invariant property for biharmonic equations, a more clear characterization of this class of generic patterns is difficult. It seems that a correct language of doing this (in fact, for both cases $m = 1$ and $m \geqslant 2$) is to reinforce a "center subspace behavior" as in (5.6), rather than other (possibly, "stable") ones. Or, equivalently (and even more solidly mathematically), to impose the BL-behavior (B.13).

Finally, we summarize these conclusions as follows.

Proposition B.6. *Under the given assumptions and conditions, the problem* (B.9) *admits a family of solutions* (*called generic*) *satisfying* (B.13).

Such a definition of generic patterns looks rather nonconstructive, which is unavoidable for higher order parabolic partial differential equations without positivity and order-preserving features. However, we expect that (B.13) occurs for "almost all" solutions.

B.6 Inner region analysis: towards the dynamical system

As usual, in the inner region, we treat the original rescaled problem (B.6). For simplicity of calculations, we again consider symmetric solutions defined for $y > 0$ by assuming the symmetry at the origin:

$$v_y = v_{yyy} = 0 \quad \text{at} \quad y = 0. \tag{B.18}$$

We next extend $v(y, \tau)$ by 0 for $y > \varphi(\tau)$ and use the change (5.2). Since $v = v_y = 0$ on the lateral boundary $\{y = \varphi(\tau)\}$, one can check that, in the sense of the theory of distributions,

$$\begin{aligned} &\widehat{v}_\tau = v_\tau H, \quad \widehat{v}_y = v_y H, \quad \widehat{v}_{yy} = v_{yy} H, \\ &\widehat{v}_{yyy} = v_{yyy} H - v_{yy}\big|_{y=\varphi} \delta(y - \varphi), \\ &\widehat{v}_{yyyy} = v_{yyyy} H - v_{yyy}\big|_{y=\varphi} \delta(y - \varphi) - v_{yy}\big|_{y=\varphi} \delta'(y - \varphi). \end{aligned} \tag{B.19}$$

Therefore, $\widehat{v}$ satisfies the following equation:

$$\widehat{v}_\tau = \mathbf{B}^* \widehat{v} - v_{yyy}\big|_{y=\varphi} \delta(y - \varphi) - v_{yy}\big|_{y=\varphi} \delta'(y - \varphi) + \kappa(\widehat{v})\widehat{v} \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+. \tag{B.20}$$

Since such an extended solution orbit (5.2) is uniformly bounded in $L^2_{\rho^*}(\mathbb{R})$, we use the eigenfunction expansion via the generalized Hermite polynomials (A.22):

$$\widehat{v}(y, \tau) = \sum_{(k \geqslant 0)} a_k(\tau) \psi_k^*(y). \tag{B.21}$$

Substituting (B.21) into (B.20) and using the biorthonormality property (A.23) yields a dynamical system: for $k = 0, 1, 2, \dots$,

$$a_k' = \lambda_k a_k - v_{yyy}\big|_{y=\varphi(\tau)} \langle \delta(y-\varphi(\tau)), \psi_k \rangle - v_{yy}\big|_{y=\varphi(\tau)} \langle \delta'(y-\varphi), \psi_k \rangle + \langle \kappa(\widehat{v})\widehat{v}, \psi_k \rangle, \qquad \text{(B.22)}$$

where $\lambda_k = -k/4$ by (A.18). Here, $\lambda_k < 0$ for all $k \geqslant 1$. More importantly, the corresponding eigenfunctions $\psi_k^*(y)$ are unbounded polynomials and are not monotone for $k \geqslant 1$ according to (A.24). Therefore, regardless proper asymptotics given by (B.22), these inner patterns cannot be matched with the BL-behavior such as (4.8), and demand other matching theory. However, these are not generic, so we skip them.

Thus, we concentrate on the "maximal" first Fourier generic pattern associated with

$$k = 0: \quad \lambda_0 = 0 \quad \text{and} \quad \psi_0^*(y) \equiv 1 \quad \big(\psi_0(y) = F(y)\big), \qquad \text{(B.23)}$$

which corresponds to a "center subspace behavior" (5.6) for the equation (B.22), which can be treated as another characterization of our class of generic patterns. The equation for $a_0(\tau)$ is:

$$a_0' = -v_{yyy}\big|_{y=\varphi(\tau)} \psi_0(\varphi(\tau)) + v_{yy}\big|_{y=\varphi(\tau)} \psi_0'(\varphi(\tau)) + \langle \kappa(a_0)a_0, \psi_0 \rangle + \dots \qquad \text{(B.24)}$$

Next, we use the boundary behavior (B.8), (B.13) for $\tau \gg 1$, which for convenience we state again: in the rescaled sense, on the given compact subsets,

$$v(y,\tau) = \rho(s) g_0 \Big(\varphi^{\frac{4}{3}}(\tau)(1 - \frac{y}{\varphi(\tau)}) \Big) + \dots , \qquad \text{(B.25)}$$

where g_0 is as in (B.15). Then, by the matching of both regions for such generic patterns, (5.10) must remain valid. Therefore, by (B.25), which by a standard parabolic regularity is also true for the spatial derivatives, we have that, as $\tau \to +\infty$,

$$\begin{aligned} v_{yy}\big|_{y=\varphi(\tau)} &\to \rho(s)\varphi^{\frac{2}{3}}(\tau)\gamma_1 \to a_0(\tau)\varphi^{\frac{2}{3}}(\tau)\gamma_1, \quad \text{where} \;\; \gamma_1 = g_0''(0) = 2^{-\frac{4}{3}}, \\ v_{yyy}\big|_{y=\varphi(\tau)} &\to -\rho(s)\varphi(\tau)\gamma_2 \to -a_0(\tau)\varphi(\tau)\gamma_2, \quad \text{where} \;\; \gamma_2 = g_0'''(0) = -\frac{1}{4}. \end{aligned} \qquad \text{(B.26)}$$

Finally, for such generic patterns, we arrive at the asymptotic ordinary differential equation for the first Fourier coefficient:

$$\boxed{\frac{a_0'}{a_0} = G_4(\varphi(\tau), \kappa) \equiv \gamma_2 \varphi(\tau) \psi_0(\varphi(\tau)) + \gamma_1 \varphi^{\frac{2}{3}}(\tau) \psi_0'(\varphi(\tau)) + \kappa(a_0) + \dots \quad \text{for} \;\; \tau \gg 1,} \qquad \text{(B.27)}$$

where, as usual, we omit higher order terms relative all those remaining. Note that, for this ordinary differential equation, the properties (5.15) and (5.16) remain valid by comparison.

B.7 ODE regularity criterion and further applications

In general, the criterion of regularity (5.17) remains the same. However, it now reads:

Theorem B.2 (ODE regularity criterion)**.** *In the fourth order parabolic problem* (B.3)*, the origin* (0, 0) *is regular in the class of generic solutions if an d only if any solution of the ordinary differential equation* (B.27) *satisfies* (5.17)*.*

Recall that by generic solutions we mean those that obey the boundary layer behavior (B.25) and hence, by matching with the inner region asymptotics, lead to the asymptotic ordinary differential equation (B.27). In this class, the proof of Theorem B.2 is straightforward.

However, we must admit that we do not have a constructive way of describing generic solutions. In fact, this is not that exciting and/or surprising since, even in the second order case, solutions of constant sign were attributed to generic ones *only* by using the maximum principle, which is not available for biharmonic operators. For both second- and fourth order parabolic equations conditions of attributing solutions of changing sign to generic patterns are not fully known.

Linear biharmonic equation: $\kappa = 0$. However, the integrals in (B.27) are, in general, *oscillatory*, so that a proper regularity analysis becomes not straightforward even in the linear case $\kappa = 0$ (cf. [8]). Then

$$\boxed{\kappa = 0: \quad \int^{\infty} G_4(\varphi(\tau), 0)\, \mathrm{d}\tau \quad \text{diverges to } -\infty \quad \Longleftrightarrow \quad a_0(\tau) \to 0 \quad \text{as} \;\; \tau \to +\infty.} \tag{B.28}$$

Using asymptotic expansions of the kernel (A.15) and the corresponding eigenfunctions, as well as sharp values of the parameters (A.16), yields a more practical condition:

$$\frac{a_0'}{a_0} = \varphi^{\frac{2}{3}}(\tau) C_3 \cos\big(b_0 \varphi^{\frac{4}{3}}(\tau) + C_4\big) \mathrm{e}^{-d_0 \varphi^{4/3}(\tau)} + \dots \quad \text{for} \;\; \tau \gg 1, \tag{B.29}$$

with some constants $C_{3,4}$ depending in an obvious way on $C_{1,2}$ in (A.15) and other parameters from (A.16). Integrating yields

$$\ln|a_0(\tau)| = \int^{\tau} \varphi^{\frac{2}{3}}(s) C_1 \cos\big(b_0 \varphi^{\frac{4}{3}}(s) + C_2\big) \mathrm{e}^{-d_0 \varphi^{4/3}(s)}\, \mathrm{d}s + \dots \quad \text{for} \;\; \tau \gg 1. \tag{B.30}$$

The regularity condition (B.28) is then reformulated according to (B.30). Namely, the "critical" backward parabola occurs for the function (cf. [8, § 7])

$$\varphi_*(\tau) = 3^{-\frac{3}{4}}\, 2^{\frac{11}{4}} \, \big(\ln \tau\big)^{\frac{3}{4}} + \dots \quad \text{for} \;\; \tau \gg 1, \tag{B.31}$$

although, to guarantee divergence to minus infinity in (B.28), a special "oscillatory cut-off" of the function $\varphi_*(\tau)$ may be necessary.

Semilinear equations. For $\kappa \neq 0$, instead of the linear (B.29), we deal with a nonlinear ordinary differential equation

$$\frac{a_0'}{a_0} = \widehat{\gamma} \varphi^{\frac{2}{3}}(\tau) C_3 \cos\big(b_0 \varphi^{\frac{4}{3}}(\tau) + C_4\big)\, \mathrm{e}^{-d_0 \varphi^{4/3}(\tau)} + \kappa(a_0) + \dots \quad \text{for} \;\; \tau \gg 1, \tag{B.32}$$

and the analysis becomes more difficult. However, some of the results from Section 5.3 can be extended. First, Proposition like 5.2 can be restored provided an oscillatory cut-off of $\varphi(\tau)$ is performed for the first integral on the right-hand side of (B.32) to be nonpositive (although this business could look too artificial). Second, a statement similar to Proposition 5.3 remains valid with the "linear" function (5.27) replaced by

$$\widehat{a}_0(\tau) = a_0(0)\, \exp\left\{ \widehat{\gamma} C_3 \int\limits_0^\tau \varphi^{\frac{2}{3}}(s) \cos\big(b_0 \varphi^{\frac{4}{3}}(s) + C_4\big) \mathrm{e}^{-d_0 \varphi^{4/3}(s)}\, \mathrm{d}s \right\} \tag{B.33}$$

and with the corresponding changes in the integrals in (5.27) and (5.28).

Let us briefly (and more formally) derive "critical" nonlinearities κ. It is easy to see that a somehow optimal and close to the critical dependence (B.31) is then achieved for the nonlinearity

$$\kappa(v) = -\frac{1}{|\ln v|} < 0 \quad \text{for} \quad v \approx 0^+. \tag{B.34}$$

Indeed, solving the corresponding ordinary differential equation without the linear term yields

$$\widetilde{a}_0' = \kappa(\widetilde{a}_0) \quad \Longrightarrow \quad \int\limits_{\widetilde{a}_0(\tau)}^1 \frac{\mathrm{d}z}{z|\kappa(z)|} = \tau, \quad \text{where} \;\; \widetilde{a}_0(\tau) \to 0^+ \quad \text{as} \;\; \tau \to \infty. \tag{B.35}$$

From (B.35) it follows that, for the nonlinear coefficient (B.34),

$$\widetilde{a}_0(\tau) = \mathrm{e}^{-\sqrt{2\tau}}, \tag{B.36}$$

so that, as is easy to see, the linear term in negligible on the asymptotics (B.36), i.e.,

$$\varphi^{\frac{2}{3}}(\tau)\, \mathrm{e}^{-d_0 \varphi^{4/3}(\tau)} = o\big(|\kappa(\widetilde{a}_0(\tau))|\big) \quad \text{as} \;\; \tau \to \infty, \tag{B.37}$$

provided that

$$\varphi(\tau) \gg (\ln \tau)^{\frac{3}{4}} \quad \text{for} \;\; \tau \gg 1 \quad (\text{cf. (B.31)}). \tag{B.38}$$

We thus arrive at a conclusion, which is similar to that in Proposition 5.2: for such negative κ's the vertices of arbitrarily "wide" backward parabolas ∂Q_0 are regular.

Nevertheless, there are some principal differences with the much simpler second order case. For instance, if the integral in (B.30) diverges and both linear and nonlinear terms on the right-hand side of (B.32) are sufficiently "balanced," i.e., both equally involved in the asymptotics of $a_0(\tau)$, the *actual checking regularity/irregularity of the origin becomes a principally nonsolvable problem.* It is curious that the most interesting "interactional case" (of linear and nonlinear terms in (B.32)) also begins at functions such as (1.6), where the explicit constant $3\sqrt{\pi}$ must be replaced by a more complicated one composed from those in (A.16) and $\gamma_{1,2}$ in (B.26) uniquely given by the BL-profile (B.15).

On the other hand, if the nonlinear term is asymptotically negligible on the "linear solutions" of (B.32), then the regularity and/or irregularity conditions remain practically the same as for the pure biharmonic flow. These are rather trivial results, which we do not intend to state and avoid such artificial "rigorous" theorems.

B.8 Gradient dependent nonlinear perturbation

For the second equation in (B.1), the rescaled equation in (B.6) takes the form

$$v_\tau = \mathbf{B}^* v + \kappa(v) v \, (v_y)^4, \tag{B.39}$$

so that using the same BL-profile (B.27) and the variables (B.8), we obtain a similar dynamical system as in (B.32), where the weaker nonlinear perturbation is estimates as in Section 5.4 (cf. (5.37)), where $\psi_0(y) \equiv F(y)$ is the oscillatory kernel (A.15). One can complete these computations; however, as before, such gradient dependent nonlinear terms do not affect the linear regularity criterion.

B.9 Backward paraboloid in $\mathbb{R}^N$

Again, in greater detail, regularity analysis in $\mathbb{R}^N$ for Burnett equations (with the bi-Laplacian rather than the pure Laplacian in the Navier–Stokes equations) is performed in [46, App. A], so we present here a brief notice only. For the equations (B.2), the lateral boundary of the domain Q_0 in $\mathbb{R}^{N+1}$ can be given by the corresponding *backward paraboloid*

$$\Big(\sum_{i=1}^{N} a_i |x_i|^{2m} \Big)^{\frac{1}{2m}} = (-t)^{\frac{1}{2m}} \, \varphi(\tau), \quad \tau = -\ln(-t), \quad a_i > 0, \quad \sum a_i^{2m} = 1. \tag{B.40}$$

Again, a boundary layer study close to the rescaled (via (4.1)) boundary

$$\partial \widehat{Q}_0 : \quad \sum a_i |z_i|^{2m} = 1, \tag{B.41}$$

leads to a linear elliptic problem, which in the orthogonal direction becomes "quasi" one-dimensional, so that $g_0(\xi)$ given in (B.15) depends on the single variable (5.42). Eventually, in the inner region, the BL-behavior leads to the stabilization to

$$g_0(y,\tau) = g_0\big(\varphi^{\frac{1}{3}}(\tau) \, \mathrm{dist}\, \{y, \partial \widetilde{Q}_0(\tau)\}\big), \tag{B.42}$$

where $g_0(\xi)$ is as in (B.15). This makes it possible to derive the asymptotic dynamical system for the first Fourier coefficient and hence an ODE regularity criterion for generic solutions.

The resulting asymptotic ordinary differential equation for $a_0(\tau)$ is similar to (B.27), with the extra multiplier φ^{N-1} in the first two terms on the right-hand side. Inevitably, the final ordinary differential equation will depend on the geometry of the backward paraboloid (B.40) in a neighborhood of its characteristic vertex $(0,0)$, which, in the most sensitive critical cases, makes it even less suitable for a definite regularity conclusion.

B.10 More on generalizations

Using the above approach, there is no much principle differences and difficulties to treat the asymptotics of characteristic points for $2m$th order polyharmonic equations

$$u_t = (-1)^{m+1} \Delta^m u + f(x,t,u,\nabla u, D^2 u, \ldots) \quad \text{in} \ \ Q_0, \tag{B.43}$$

with zero Dirichlet (or others homogeneous) boundary conditions on ∂Q_0. Although, of course, some involved technicalities occur indeed. Since the first rescaled variables are

$$u(x,t) = v(y,\tau), \quad y = \frac{x}{(-t)^{1/2m}}, \quad \tau = -\ln(-t), \tag{B.44}$$

most interesting nonlinear terms in (B.43) are now:

$$f(\cdot) = \frac{1}{(-t)}\,\kappa(u)u, \quad \kappa(u)u\,|\nabla u|^{2m}, \quad \kappa(u)u|D^2u|^m \quad \text{etc.}$$

Then scalings (B.44) lead to the rescaled parabolic equations

$$v_\tau = \mathbf{B}^* v + \begin{cases} \kappa(v)v, \\ \kappa(v)v\,|\nabla v|^{2m}, \quad \text{etc.}, \\ \kappa(v)v|D^2v|^m \end{cases} \tag{B.45}$$

where $\mathbf{B}^*$ is the linear adjoined operator (A.1). The corresponding dynamical systems for the expansion coefficients are obtained, as above, by (i) constructing a BL, and (ii) projecting the resulting PDEs (B.45), with the BL-approximation, onto generalized Hermite polynomials (A.22). The dynamical system, in general, becomes extremely oscillatory, and both linear and nonlinear terms can essentially affect regularity of the vertex $(0,0)$.

Finally, we again mention that here our main goal: to show how the evolution of the first Fourier coefficient of generic solutions of biharmonic partial differential equations leads to an ODE regularity criterion, has been We must admit however that, in some cases, this did not end up with constructive/deterministic regularity conclusions, which are not always possible and are even illusive in general for higher order nonlinear parabolic partial differential equations.

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