Dedicated to
Solomon Grigorievich Mikhlin

When a problem about partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our new knowledge is, in fact, essentially contained in such inequalities. But the abstract theory is not only a tool, it is also a guide to general and fruitful problems.

Lars Gårding

Translator’s Preface

The history of this book is somewhat unusual. Originally it was written in Leningrad in Russian, at the end of the seventies. Submitted to the Soviet publishing house “Nauka”, it was rejected for political reasons and never appeared in Russian. The manuscript was smuggled to East Germany by friends of the authors, translated into German, and published in 1981 under the title *“Abschätzungen für Differentialoperatoren im Halbraum”* by Akademie-Verlag Berlin. In 1982, the German edition was reprinted by Birkhäuser Verlag. Now, almost forty years later, the English-speaking readers can also become familiar with this interesting book.

The book is devoted to a detailed study of numerous inequalities for differential operators with constant coefficients in a half-space. Although many years have passed since the book was written, the presented results are not outdated. They are in definitive form, without any restriction on the type of differential operators. Moreover, the presented inequalities open perspectives for possible generalizations to operators in other domains and operators with variable coefficients, as well as to pseudo-differential operators.

D. Apushkinskaya

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Preface to the German Edition

Inequalities for differential operators of the kind considered in this book play a fundamental role in the modern theory of partial differential equations. Among the numerous applications of such inequalities are existence and uniqueness theorems, error estimates for numerical approximation of solutions and for residual terms in asymptotic formulas, as well as results on the structure of the spectrum. The inequalities arise in a wide range of topics and differ by the choice of differential operators and boundary conditions, by requirements on the boundaries of domains, and by the norms in the relevant function spaces.

For general differential operators with constant coefficients considered in this book, estimates in L^2 for functions with compact support in a domain have been extensively studied in [H55].

Estimates up to the boundary are much less studied. Estimates of such type can be found in the papers of Aronszajn [Aro54], Agmon [Agm58] (coercivity of differential operators and integro-differential forms), Schechter [Sch63], [Sch64], [Sch64a] (sufficient conditions for dominance in a half-space) and in other publications that will be discussed in the bibliographical notes at the end of each chapter.

The subject of this book is estimates for differential operators with constant coefficients in a half-space. There are no a priori restrictions on the type of considered differential operators.

The right-hand sides of the studied integral inequalities involve matrices of differential operators or scalar differential operators in a half-space as well as boundary operators. Conditions under which the above-mentioned system of operators "dominates" an individual differential operator in a half-space or on its boundary are completely described. Applications of these results to the theory of well-posed boundary value problems in a half-space are given.

The domains of the relevant maximal operators are investigated in detail. In particular, the maximal operators weaker than the given one are described and a complete characterization of boundary values of functions from the specified domain is presented.

The results are complete. To a large extent, they are necessary and sufficient conditions. From these, more evident sufficient conditions are derived. General criteria are systematically applied to certain types of operators, in particular, to classical equations and systems of mathematical physics (Lamé's system of static elasticity theory, the linearized Navier–Stokes system, Cauchy–Riemann operators, Schrödinger operators, and so on).

The known results of Aronszajn, Agmon–Douglis–Nirenberg, Schechter fall into the general scheme and are sometimes strengthened.

This monograph does not overlap with the content of other books on linear differential operators and results presented have so far only been published in journal papers. The book summarizes the joint work of the authors on this topic during the period 1972–1977.

The authors hope that the book will be interesting and useful to a wide audience. It is intended for specialists and graduate students specializing in the theory of differential equations.

The reader is expected to be familiar with elements of the theory of ordinary differential equations, functional analysis, the theory of partial differential equations, and basics of linear algebra.

The content of the book is detailed in the introductions to each of its four chapters.

The authors thank the translator and the publisher for high quality of translation.

I. V. Gel′man
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Chapter 1
Estimates for matrix operators

1.0 Introduction

1.0.1 Description of results

The main result of this chapter is a theorem providing necessary and sufficient conditions for the validity of the estimate

$$\|\sum_{j=1}^{m} R_j(D)u_j\|^2_{B^{1/2}} \leqslant C\left(\sum_{k=1}^{m}\|\sum_{j=1}^{m} P_{kj}(D)u_j\|^2 + \sum_{\alpha=1}^{N}\langle\!\langle \sum_{j=1}^{m} Q_{\alpha j}(D)u_j\rangle\!\rangle^2\right), \tag{1.0.1}$$

where $u = (u_1(x;t), \ldots, u_m(x;t))$[1] denotes an arbitrary vector function belonging to $\mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

We will write the estimate (1.0.1) in the matrix form

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2\right). \tag{1.0.2}$$

Here $R(\xi;\tau) = \{R_j(\xi;\tau)\}$, $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$, and $Q(\xi;\tau) = \{Q_{\alpha j}(\xi;\tau)\}$ are $1 \times m$, $m \times m$, and $N \times m$ matrices, respectively. The elements of these matrices are polynomials of the variable $\tau \in \mathbb{R}^1$ with complex measurable locally bounded in $\mathbb{R}^{n-1}$ coefficients growing no faster as some power of $|\xi|$ as $|\xi| \to \infty$. The vector functions $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ with m components are regarded as $m \times 1$ matrices.

In Section 1.2 of this chapter, we will formulate some necessary and sufficient conditions for the validity of the estimate (1.0.2).

For a matrix operator $P(D)$ such that $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau) \not\equiv 0$ and $\operatorname{ord}\mathscr{P}(\xi;\tau) = J \geqslant 1$, we introduce the following matrices and polynomials:

$P^c = \{P^{jk}\}$ is the adjugate matrix of the $m \times m$ matrix P, i.e., the $m \times m$ matrix whose (j,k) entry is the (k,j) cofactor of P;

$$S = \{S_k\} = RP^c; \qquad T = \{T_{\alpha k}\} = QP^c;$$

$\mathscr{P}_+$ is the polynomial of the variable τ whose roots (counting multiplicities) coincide with all the roots of $\mathscr{P}$ in the half-space $\operatorname{Im}\zeta \geqslant 0$ ($\zeta = \tau + i\sigma$); we shall assume that the leading coefficient[2] of the polynomial $\mathscr{P}_+$ is equal to 1;

$\mathscr{P}_- = \mathscr{P}/(\mathscr{P}_+ p_0)$, where $p_0(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}$;

[1] Vectors and one-column matrices are explicitly given as row vectors. In formulas, they appear as column vectors without transpose sign. The reader will be able to easily recognize this from the context.

[2] The coefficient of the highest degree term of a polynomial is called the leading coefficient.

$\mathscr{M}$ is the greatest common divisor of the polynomials $\mathscr{P}_+, S_1, \ldots, S_m$ with the leading coefficient equals to 1.

We assume that $p_0(\xi) \neq 0$, $\operatorname{ord} S_k \leqslant J$, $\operatorname{ord} T_{\alpha k} \leqslant J-1$ ($k = 1, \ldots, m$; $\alpha = 1, \ldots, N$), and $\operatorname{ord}(\mathscr{P}_+/\mathscr{M}) = N \geqslant 1$ for almost all $\xi \in \mathbb{R}^{n-1}$.

Under the above assumptions, we consider the matrices $S_\pm$ and $T_\pm$, defined by the partial fraction decompositions with respect to τ

$$\frac{S}{\mathscr{P}} = c(\xi) + \frac{S_+}{\mathscr{P}_+} + \frac{S_-}{\mathscr{P}_-}, \qquad \frac{T}{\mathscr{P}} = \frac{T_+}{\mathscr{P}_+} + \frac{T_-}{\mathscr{P}_-},$$

and the matrix Γ, defined by the formula

$$\Gamma(\xi;\tau,\eta) = \frac{1}{\eta-\tau}\left[\mathscr{P}_+(\xi;\eta)S_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)S_+(\xi;\eta)\right],$$

where $\xi \in \mathbb{R}^{n-1}$; $\tau, \eta \in \mathbb{R}^1$.

In Theorem 1.2.2 it is asserted that the estimate (1.0.2) holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\tau, \eta \in \mathbb{R}^1$ the following conditions are satisfied:

$$B^{1/2}(\xi)|S(\xi;\tau)| \leqslant \text{const}\,|\mathscr{P}(\xi;\tau)|; \tag{1.0.3}$$

$$T(\xi;\tau) \equiv 0 \ (\operatorname{mod} \mathscr{M}(\xi;\tau))^{3}; \tag{1.0.4}$$

the rows of the matrix T are linearly independent modulo $\mathscr{P}_+$; (1.0.5)

there exists a uniquely determined $1 \times N$ matrix $G(\xi;\tau) = \{G_\alpha(\xi;\tau)\}$, whose elements are polynomials (in τ) such that $\max_\alpha \operatorname{ord} G_\alpha \leqslant N - 1 + \operatorname{ord} \mathscr{M}$ and $G(\xi;\tau) \equiv 0 \ (\operatorname{mod} \mathscr{M}(\xi;\tau))$ and $G(\xi;\tau)T_+(\xi;\eta) = \Gamma(\xi;\tau,\eta)$; (1.0.6)

$$B(\xi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)T_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\tau)}\right|^2 d\tau\, d\eta \leqslant \text{const}; \tag{1.0.7}$$

$$B(\xi)\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 d\tau \leqslant \text{const}. \tag{1.0.8}$$

The estimate (1.0.2) holds also for $N = 0$. (In this case, the matrix Q is omitted on the right-hand side of (1.0.2)). The criterion for the validity of such an estimate consists of condition (1.0.3) and the congruence $S(\xi;\tau) \equiv 0 \ (\operatorname{mod} \mathscr{P}_+(\xi;\tau))$ (see Theorem 1.2.3).

[3] This means that each element of the matrix T satisfies (1.0.4).

In Section 1.2 it is also shown that relations (1.0.3)–(1.0.7) are necessary and sufficient for the validity of the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C_0 \|P(D)u\|^2 \tag{1.0.9}$$

for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ satisfying the equation $Q(D)u(x;0) = 0$ (see Theorem 1.2.5).

All these criteria follow from upper and lower bounds for the sharp constants in inequalities of type (1.0.2) and (1.0.9) for ordinary differential operators on the semi-axis $t \geqslant 0$ (see Section 1.1). This allows us to get also the inequalities obtained from (1.0.2) and (1.0.9) by replacing the norm $\|\cdot\|$ by the norm $\|\cdot\|_\nu$, and the norm $\langle\!\langle\,\cdot\,\rangle\!\rangle$ by the norm $\langle\!\langle\,\cdot\,\rangle\!\rangle_\mu$, respectively (see Corollaries 1.2.13 and 1.2.14)[4].

Some sufficient conditions for the validity of the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2_{\mathfrak{M}} \right) \tag{1.0.10}$$

are established in Section 1.3 (in inequality (1.0.10) the matrices R, P, and Q are the same as in Section 1.2, while

$$\langle\!\langle Q(D)u \rangle\!\rangle^2_{\mathfrak{M}} = \int\limits_{\mathbb{R}^{n-1}} \left| \mathfrak{M}(\xi) Q\,(\xi; -\mathrm{i}\, d/dt)\,\widehat{u}(\xi;t)|_{t=0} \right|^2 d\xi,$$

where $\mathfrak{M}(\xi)$ is an arbitrary measurable $N \times N$ matrix, regular a.e. in $\mathbb{R}^{n-1}$).

In particular, Theorem 1.3.3 states that if conditions (1.0.3)–(1.0.6) are fulfilled and the inequality

$$B(\xi)\mathrm{tr}\left(\mathscr{T}_+^{-1}\mathscr{T}_-\right) \left| \frac{S_+}{\mathscr{P}_+} \right|^2 \leqslant \text{const} \tag{1.0.11}$$

with

$$\mathscr{T}_\pm(\xi) = \int\limits_{-\infty}^{\infty} \frac{T_\pm(\xi;\eta) T^*_\pm(\xi;\eta)}{|\mathscr{P}_\pm(\xi;\eta)|^2} d\eta \tag{1.0.12}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$, then the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2_{\mathscr{T}_+^{-1/2}} \right) \tag{1.0.13}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Conditions (1.0.3)–(1.0.6) are also necessary in this case.

The next assertion (Theorem 1.3.6) is a simple consequence of Section 1.2 and the arguments used in the proof of Theorem 1.3.3.

Let $Q(\xi;\tau) = \{Q_{\alpha_j}(\xi;\tau)\}$ be an $N \times m$ matrix, let the polynomial $\mathscr{P}_+(\xi;\tau)$ have no real τ-roots, and let the relation ord $\mathscr{P}_+(\xi;\tau) = N$ hold for almost all

[4] $\|\cdot\|_\nu$ and $\langle\!\langle\,\cdot\,\rangle\!\rangle_\mu$ are the norms in vector spaces $\mathscr{H}_\nu(\mathbb{R}^n_+)$ and $\mathscr{H}_\mu(\partial\mathbb{R}^n_+)$, respectively.

$\xi \in \mathbb{R}^{n-1}$. Let the $N \times N$ matrices $\mathscr{T}_\pm$ be defined by equation (1.0.12). If for almost all $\xi \in \mathbb{R}^{n-1}$ the rows of the matrix $T(\xi;\tau)$ are linearly independent modulo $\mathscr{P}_+(\xi;\tau)$ and the matrices $\mathscr{T}_\pm$ satisfy the condition

$$\mathscr{T}_- \mathscr{T}_+^{-1} \mathscr{T}_- = \text{const}\, \mathscr{T}_-, \tag{1.0.14}$$

then the estimate (1.0.13) holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ and any operator $R(D)$ satisfying condition (1.0.3).

A direct proof of this theorem was given by M. Schechter in [Sch64a].

We just mention, without precise formulation, two other sufficient conditions from Section 1.3.

In Theorem 1.3.9, sufficient conditions for the validity of the estimate (1.0.10) are established for the case where $\mathfrak{M}(\xi)$ is a diagonal matrix with eigenvalues $(1 + |\xi|^2)^{\mu_\beta/2}$ ($\beta = 1, \ldots, N$; $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{R}^N$).

Sufficient conditions for the validity of the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2 + \|u\|^2 \right), \tag{1.0.15}$$

which differs from the estimate (1.0.2) by an additional term on the right-hand side, are formulated in Proposition 1.3.12. It is obvious that conditions (1.0.3)–(1.0.8) ensure the validity of (1.0.15) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$. Proposition 3.1.12 is a strengthening of this assertion in the case when the leading coefficient $p_0(\xi)$ of the polynomial $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$ is uniformly bounded from below in some ball in $\mathbb{R}^{n-1}$.

Section 1.4 contains several examples of the estimates for operators of concrete types. The validity (or impossibility) of these estimates follows from the theorems proved in Sections 1.2 and 1.3.

Proposition 1.4.1 and Corollary 1.4.2 concern inequalities of the type (1.0.2) for generalized-homogeneous quasi-elliptic matrices P. As a special case, we get the corresponding estimates for general elliptic and parabolic systems.

Some applications of the results of Section 1.2 to concrete elliptic systems (the Lamé system of stationary elasticity theory, the Cauchy–Riemann system, the stationary linearized Navier–Stokes system) are considered in Subsections 1.4.2–1.4.4. For example, for the Lamé system, it is proved the validity of the “nonelliptic” estimate (1.4.12), which fails if the boundary operators are replaced by their principal parts.

Hyperbolic systems are treated in Subsection 1.4.5. It turns out that for the homogeneous hyperbolic operators $P(D)$ it is reasonable to examine only estimates corresponding to the case $N = 0$ (i.e., estimates without boundary operators). Necessary and sufficient conditions for the validity of such estimates are provided in Proposition 1.4.6. On the other hand, we provide several examples showing that for a nonhomogeneous hyperbolic operator $P(D)$ the trivial case $N = 0$ is not necessary.

Finally (in Subsections 1.4.6–1.4.7), we give several examples of estimates for scalar operators of the first and second order w.r.t. t “without a type”.

In Section 1.5, results from Section 1.2 are used to study conditions ensuring the well-posedness of boundary value problems in a half-space.

1.0.2 Outline of the proof of the main result

To give an idea about the method of proving the main result of this chapter (Theorem 1.2.2) and to understand how conditions (1.0.3)–(1.0.8) arise, we consider the estimate

$$\|\mathscr{R}(D)u\|^2_{B^{1/2}} \leqslant C\left(\|\mathscr{P}(D)u\|^2 + \langle\!\langle \mathcal{Q}(D)u\rangle\!\rangle^2\right) \tag{1.0.16}$$

for scalar operators ($m = 1$). Here $\mathscr{R}(\xi;\tau)$, $\mathscr{P}(\xi;\tau)$ are polynomials, and $\mathcal{Q}(\xi;\tau) = \{\mathcal{Q}_1(\xi;\tau),\dots,\mathcal{Q}_N(\xi;\tau)\}$ is a polynomial $N \times 1$ matrix.

All a priori assumptions, expressed in Subsection 1.0.1, remain valid; the only difference is that S, $S_\pm$, T, $T_\pm$ are replaced by $\mathscr{R}$, $\mathscr{R}_\pm$, $\mathcal{Q}$, $\mathcal{Q}_\pm$, respectively.

The proof of the equivalence between the estimate (1.0.16) and conditions (1.0.3)–(1.0.8) is based on the following simple observation: The estimate (1.0.16) holds for all $u \in C_0^\infty(\mathbb{R}^n_+)$ if and only if

$$\begin{aligned}&\int_0^\infty |\mathscr{R}(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt\\ &\leqslant \Lambda(\xi)\left[\int_0^\infty |\mathscr{P}(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt + \sum_{\alpha=1}^N |\mathcal{Q}_\alpha(\xi;-\mathrm{i}\,d/dt)\,v|_{t=0}|^2\right]\end{aligned} \tag{1.0.17}$$

for all $v \in C_0^\infty(\mathbb{R}^1_+)$ and almost all $\xi \in \mathbb{R}^{n-1}$, and the sharp constant $\Lambda(\xi)$ satisfies the condition

$$B(\xi)\Lambda(\xi) \leqslant C.\text{[5]} \tag{1.0.18}$$

First, we explain how conditions (1.0.3)–(1.0.6) follows from inequalities (1.0.17)–(1.0.18).

Condition (1.0.3). For $m = 1$ it takes the form

$$B^{1/2}(\xi)|\mathscr{R}(\xi;\tau)| \leqslant \mathrm{const}\,|\mathscr{P}(\xi;\tau)|. \tag{1.0.19}$$

The necessity of (1.0.19) follows from (1.0.18) and the estimate

$$\sup_{\tau\in\mathbb{R}^1} |\mathscr{R}(\xi;\tau)/\mathscr{P}(\xi;\tau)|^2 \leqslant \Lambda(\xi), \tag{1.0.20}$$

Notice that (1.0.20) is easily obtained by substituting in (1.0.17) a smooth function $v(t)$ vanishing near $t = 0$ (Lemma 1.1.5).

Condition (1.0.4). For $m = 1$ this condition reads

$$\mathcal{Q}_\alpha(\xi;\tau) \equiv 0 \ (\mathrm{mod}\,\mathscr{M}(\xi;\tau)), \qquad \alpha = 1,2,\dots,N, \tag{1.0.21}$$

where $\mathscr{M}(\xi;\tau)$ is the greatest common divisor of $\mathscr{R}$ and $\mathscr{P}_+$. The necessity of condition (1.0.4) is proved in Section 1.2 (see Theorem 1.2.2) with the help of Lemma 1.1.9.

[5] The necessity of (1.0.17) is proved by applying to (1.0.16) the localization method w.r.t. ξ; the sufficiency is checked by substituting in (1.0.17) the function $v = v_\xi(t) = \widehat{u}(\xi;t)$ (see Subsection 1.2.2).

The following example gives an idea of how inequality (1.0.17) implies congruences (1.0.21).

Example. Let $\operatorname{Im}\zeta_j(\xi) > 0\,(j = 1, 2)$, let

$$\mathscr{P}(\xi;\tau) = \mathscr{P}_+(\xi;\tau) = (\tau-\zeta_1(\xi))(\tau-\zeta_2(\xi)),\ \zeta_1(\xi) \neq \zeta_2(\xi),\ \zeta_3(\xi) \neq \zeta_2(\xi),$$

and let $\mathscr{R}(\xi;\tau) = (\tau-\zeta_1(\xi))(\tau-\zeta_2(\xi))$. Then $\mathscr{M}(\xi;\tau) = \tau-\zeta_1(\xi)$, $\mathscr{P}_+/\mathscr{M} = \tau-\zeta_2(\xi)$, and, consequently, $N = 1$. Let $\mathcal{Q}(\xi;\tau)$ be an arbitrary linear polynomial of τ such that $\mathcal{Q}(\xi;\zeta_2(\xi)) \neq 0$. Now substitute the function

$$v_\xi(t) = \exp(\mathrm{i}\zeta_1(\xi)t) - \frac{\mathcal{Q}(\xi;\zeta_1(\xi))}{\mathcal{Q}(\xi;\zeta_2(\xi))}\exp(\mathrm{i}\zeta_2(\xi)t)$$

in (1.0.17). Obviously,

$$\mathscr{P}(\xi;-\mathrm{i}\,d/dt)\,v_\xi(t) = 0, \qquad \mathcal{Q}(\xi;-\mathrm{i}\,d/dt)\,v_\xi(t)\big|_{t=0} = 0$$

and

$$\mathscr{R}(\xi;-\mathrm{i}\,d/dt)\,v_\xi(t) = \frac{\mathcal{Q}(\xi;\zeta_1(\xi))}{\mathcal{Q}(\xi;\zeta_2(\xi))}(\zeta_2(\xi)-\zeta_1(\xi))(\zeta_2(\xi)-\zeta_3(\xi))\exp(\mathrm{i}\zeta_2(\xi)t).$$

In view of the assumptions $\zeta_1(\xi) \neq \zeta_2(\xi)$, $\zeta_3(\xi) \neq \zeta_2(\xi)$, it follows from (1.0.17) that $\mathcal{Q}(\xi;\zeta_1(\xi)) = 0$, i.e., $\mathcal{Q} \equiv 0 \pmod{\mathscr{M}}$.

Condition (1.0.5). Its necessity follows from Lemma 1.1.5. In the case $m = 1$, this condition is formulated as follows:

$$\text{The polynomials } \mathcal{Q}_\alpha \text{ are linearly independent modulo } \mathscr{P}_+ \text{ for almost all } \xi \in \mathbb{R}^{n-1}. \tag{1.0.22}$$

We show how (1.0.22) can be derived from (1.0.17). For simplicity, we assume that $\mathscr{M}(\xi;\tau) = 1$ and the τ-roots $\zeta_1(\xi), \dots, \zeta_N(\xi)$ of the polynomial $\mathscr{P}_+$ are pairwise distinct a.e. in $\mathbb{R}^{n-1}$. It follows from (1.0.20) that $\operatorname{Im}\zeta_j(\xi) > 0\,(j = 1, \dots, N)$. Therefore (cf. Remark 1.1.8), the solution

$$v_\xi(t) = \sum_{j=1}^{N} c_j(\xi)\exp(i\zeta_j(\xi)t)$$

to the equation $\mathscr{P}_+(\xi;-\mathrm{i}\,d/dt)\,v = 0$ satisfies inequality (1.0.17). Now let the coefficients $c_j(\xi)$ satisfy the conditions

$$\sum_{j=1}^{N} c_j(\xi)\mathcal{Q}_\alpha(\xi;\zeta_j(\xi)) = \mathcal{Q}_\alpha(\xi;-\mathrm{i}\,d/dt)\,v_\xi(t)\big|_{t=0} = 0 \qquad (\alpha = 1, \dots, N).$$

Then the estimate (1.0.17) yields

$$\mathscr{R}\,(\xi; -\mathrm{i}\,d/dt)\,v_\xi(t) = \sum_{j=1}^{N} c_j(\xi)\mathscr{R}(\xi;\zeta_j(\xi))\exp\left(\mathrm{i}\zeta_j(\xi)t\right) = 0.$$

Since the polynomials $\mathscr{R}$ and $\mathscr{P}_+$ are relatively prime, the last equality implies that $c_j(\xi) = 0\,(j = 1,\ldots,N)$.

Hence, the $N \times N$ matrix

$$\mathfrak{D}(\xi) = \{\mathcal{Q}_\alpha(\xi;\zeta_j(\xi))\} \tag{1.0.23}$$

is nondegenerate, which is equivalent to (1.0.22).

Condition (1.0.6). We show that for $m = 1$ this condition follows from (1.0.3)–(1.0.5). For simplicity, we assume $\mathscr{M}(\xi;\tau) = 1$.

If $m = 1$, then relation (1.0.6) can be written as

$$\sum_{\alpha=1}^{N} G_\alpha(\xi;\tau)\mathcal{Q}_{\alpha+}(\xi;\eta) = \frac{[\mathscr{P}_+(\xi;\eta)\mathscr{R}_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)\mathscr{R}_+(\xi;\eta)]}{(\eta-\tau)}. \tag{1.0.24}$$

Denote the right-hand side of (1.0.24) by $\Gamma(\xi;\tau,\eta)$. Clearly, $\partial^N\Gamma/\partial\eta^N = 0$ for all $\tau,\eta \in \mathbb{R}^1$. In addition, it follows from (1.0.22) that the polynomials $\mathcal{Q}_{\alpha+}(\xi;\eta)$ are linearly independent. Since ord $\mathcal{Q}_{\alpha+}(\xi;\eta) \leqslant N-1$, the coefficients $G_\alpha(\xi;\tau)$ are uniquely determined by (1.0.24). On the other hand, $\partial^N\Gamma/\partial\tau^N = 0$ and, consequently,

$$\sum_{\alpha=1}^{N} \frac{\partial^N G_\alpha(\xi;\tau)}{\partial\tau^N}\mathcal{Q}_{\alpha+}(\xi;\eta) = 0$$

for all $\tau,\eta \in \mathbb{R}^1$. Using again the linear independence of the polynomials $\mathcal{Q}_{\alpha+}(\xi;\eta)$, we conclude that $\partial^N G_\alpha(\xi;\tau)/\partial\tau^N = 0$ for all $\tau \in \mathbb{R}^1$ $(\alpha = 1,2,\ldots,N)$. This means that $G(\xi;\tau) = \{G_\alpha(\xi;\tau)\}$ is an $1 \times N$ matrix of polynomials (in τ) and $\max_\alpha$ ord $G_\alpha(\xi;\tau) \leqslant N-1$.

The polynomials $G_\alpha(\xi;\tau)$ can be expressed explicitly in different ways in terms of the polynomials $\mathscr{R}$, $\mathscr{P}$ and $\mathcal{Q}_\alpha$. We give one such representation, which will be used later.

Let, for simplicity, the τ-roots $\zeta_j(\xi)$ $(j = 1,\ldots,N)$ of the polynomial $\mathscr{P}_+(\xi;\tau)$ be pairwise distinct a.e. in $\mathbb{R}^{n-1}$, and let $\mathscr{M}(\xi;\tau) = 1$. We show that

$$G(\xi;\tau) = \{G_j(\xi;\tau)\} = H(\xi;\tau)\mathfrak{D}^{-1}(\xi), \tag{1.0.25}$$

where $H(\xi;\tau)$ is the $1 \times N$ matrix defined by

$$H(\xi;\tau) = \left\{\mathscr{R}(\xi;\zeta_j(\xi))\frac{\mathscr{P}_+(\xi;\tau)}{\tau-\zeta_j(\xi)}\right\}, \tag{1.0.26}$$

and $\mathfrak{D}^{-1}(\xi)$ is the inverse of the matrix (1.0.23).

Indeed, setting in (1.0.24) $\eta = \zeta_j(\xi)$ $(j = 1, \dots, N)$ and taking into account the equalities

$$\mathcal{Q} = \mathcal{Q}_+ \mathscr{P}_- + \mathcal{Q}_- \mathscr{P}_+, \qquad \mathscr{R} = c(\xi)\mathscr{P} + \mathscr{R}_+ \mathscr{P}_- + \mathscr{R}_- \mathscr{P}_+, \tag{1.0.27}$$

which follow from definitions of the polynomials $\mathscr{R}_\pm$ and the matrices $\mathcal{Q}_\pm$, we find that

$$\sum_{\alpha=1}^{N} G_\alpha(\xi;\tau)\mathcal{Q}_\alpha(\xi;\zeta_j(\xi)) = \mathscr{R}(\xi;\zeta_j(\xi))\frac{\mathscr{P}_+(\xi;\tau)}{\tau - \zeta_j(\xi)},$$

or, equivalently,

$$G(\xi;\tau)\mathfrak{D}(\xi) = H(\xi;\tau). \tag{1.0.28}$$

It remains only to observe that (1.0.28) implies (1.0.25).

Integral representation. The proof of the sufficiency of conditions (1.0.3)–(1.0.8) and the necessity of conditions (1.0.7)–(1.0.8) is based on the integral representation (1.0.29) given below. For simplicity, we assume that $\mathscr{M}(\xi;\tau) = 1$ and the τ-roots $\zeta_j(\xi)$ of the polynomial $\mathscr{P}_+(\xi;\tau)$ are pairwise distinct a.e. in $\mathbb{R}^{n-1}$. Furthermore, suppose that conditions (1.0.19) and (1.0.22) are fulfilled, and $\operatorname{Im}\zeta_j(\xi) > 0$ $(j = 1, \dots, N)$.

We show that for $t \geqslant 0$ the function $\mathscr{R}(\xi; -\mathrm{i}\,d/dt)\,v$ $(v \in C_0^\infty(\mathbb{R}^1_+))$ can be represented as

$$\begin{aligned}\mathscr{R}(\xi;-\mathrm{i}\,d/dt)\,v = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}t\tau}\Bigg[\frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}(\xi;\tau)}(F_{t\to\tau}f) + \frac{\mathrm{i}}{\sqrt{2\pi}}\frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \\ \times\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathcal{Q}_-(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)}(F_{t\to\eta}f)\,d\eta - \mathcal{Q}(\xi;-\mathrm{i}\,d/dt)\,v|_{t=0}\right)\Bigg]d\tau,\end{aligned} \tag{1.0.29}$$

where $f(\xi;t) = \mathscr{P}(\xi;-\mathrm{i}\,d/dt)\,v$ for $t \geqslant 0$ and $f = 0$ for $t < 0$; $\mathcal{Q}(\xi;\tau)$ is the given $N \times 1$ matrix of boundary polynomials $\mathcal{Q}_\alpha(\xi;\tau)$, $G(\xi;\tau)$ is the matrix (1.0.25), and $\mathcal{Q}_-(\xi;\tau)$ is the matrix satisfying the first of the relations (1.0.27); $F_{t\to\tau}f$ denotes the Fourier transform of the function $f(\xi;t)$ w.r.t. t. The inverse Fourier transform is denoted by $F^{-1}_{t\to\tau}$.

To derive representation (1.0.29), we express $v(t) \in C_0^\infty(\mathbb{R}^1_+)$ as

$$v(t) = w_\xi(t) + \sum_{j=1}^{N} c_j(\xi)\exp(\mathrm{i}\zeta_j(\xi)t), \qquad t \geqslant 0, \tag{1.0.30}$$

where

$$w_\xi(t) = F^{-1}_{t\to\tau}(F_{t\to\tau}f/\mathscr{P}(\xi;\tau)).$$ [6]

[6] The representation (1.0.30) holds, and the coefficients $c_j(\xi)$ are uniquely determined by it. It is obvious that $\mathscr{P}(\xi;-\mathrm{i}\,d/dt)\,w_\xi = f$ for $t \geqslant 0$ and, therefore, $\mathscr{P}_+(\xi;-\mathrm{i}\,d/dt)\,[v(t) - w_\xi(t)] = 0$.

From the definition of $f(\xi;t)$ it follows that the function $F_{t\to\tau} f$ can be continued analytically to the half-plane $\operatorname{Im}\zeta < 0$ $(\zeta = \tau + i\sigma)$, and hence

$$\int_{-\infty}^{\infty} \mathcal{Q}(\xi;\tau)\frac{F_{t\to\tau} f}{\mathcal{P}(\xi;\tau)} d\tau = \int_{-\infty}^{\infty} \mathcal{Q}_{-}(\xi;\tau)\frac{F_{t\to\tau} f}{\mathcal{P}_{-}(\xi;\tau)} d\tau.$$

Therefore,

$$\begin{aligned} \mathcal{Q}(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{Q}_{-}(\xi;\eta)\frac{F_{t\to\eta} f}{\mathcal{P}_{-}(\xi;\eta)} d\eta \\ + \sum_{j=1}^{N} c_j(\xi)\mathcal{Q}(\xi;\zeta_j(\xi)). \end{aligned} \tag{1.0.31}$$

The condition (1.0.22), as we have already noted, is equivalent to the nondegeneracy of the matrix (1.0.23). Hence system (1.0.31) is solvable with respect to $c_j(\xi)$ and its solution takes the form

$$\begin{aligned} \mathfrak{C} &= \{c_j(\xi)\} \\ &= \mathfrak{D}^{-1}(\xi)\left[\mathcal{Q}(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{Q}_{-}(\xi;\eta)\frac{F_{t\to\eta} f}{\mathcal{P}_{-}(\xi;\eta)} d\eta \right]. \end{aligned} \tag{1.0.32}$$

From (1.0.30) and (1.0.32) we get

$$\begin{aligned} \mathcal{R}(\xi; -\mathrm{i}\, d/dt)(v(t) - w_\xi(t)) = \mathfrak{r}(\xi;t)\mathfrak{D}^{-1}(\xi)\Bigg[\mathcal{Q}(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} \\ - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{Q}_{-}(\xi;\eta)\frac{F_{t\to\eta} f}{\mathcal{P}_{-}(\xi;\eta)} d\eta \Bigg], \end{aligned} \tag{1.0.33}$$

where the $1 \times N$ matrix $\mathfrak{r}(\xi;t)$ satisfies

$$\mathfrak{r}(\xi,\tau) = \{\mathcal{R}(\xi;\zeta_j(\xi)) \exp(\mathrm{i}\zeta_i(\xi)t)\}.$$

Notice that $F_{t\to\tau}\mathfrak{r}(\xi;t) = \mathrm{i}^{-1}(2\pi)^{-1/2} H(\xi;\tau)\mathcal{P}_{+}(\xi;\tau)$, where $H(\xi;\tau)$ is the matrix (1.0.26). Therefore, representation (1.0.29) follows from (1.0.33) and the obvious equality

$$\mathcal{R}(\xi; -\mathrm{i}\, d/dt)\, w_\xi(t) = F^{-1}_{\tau\to t}\left[\mathcal{R}(\xi;\tau)\frac{F_{t\to\tau} f}{\mathcal{P}(\xi;\tau)}\right].$$

Sufficiency of conditions (1.0.3)–(1.0.8). As it was already shown, conditions (1.0.3)–(1.0.6) imply representation (1.0.29) for $t \geqslant 0$. Considering $\xi \in \mathbb{R}^{n-1}$ as a fixed parameter value, we extend the function $\mathscr{R}(\xi; -\mathrm{i}\, d/dt)\, v$ to the whole $\mathbb{R}^1$ by setting it equal to 0 for $t < 0$. Then, it follows from Parseval's identity and (1.0.29) that

$$\int_0^\infty |\mathscr{R}(\xi; -\mathrm{i}\, d/dt)\, v|^2\, dt \leqslant C \left\{ \left[\sup_{\tau \in \mathbb{R}^1} \left| \frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \right. \right.$$
$$\left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathscr{Q}_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)} \right|^2 d\tau\, d\eta \right] \int_0^\infty |\mathscr{P}(\xi; -\mathrm{i}\, d/dt)\, v|^2\, dt \qquad (1.0.34)$$
$$\left. + \int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau\, |\mathscr{Q}(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0}|^2 \right\}.$$

The estimate (1.0.34) can be treated as an inequality of the type (1.0.17). Thus, the sharp constant $\Lambda(\xi)$ in (1.0.17) admits the upper bound

$$\Lambda(\xi) \leqslant \text{const} \left[\sup_{\tau \in \mathbb{R}^1} \left| \frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \right.$$
$$\left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathscr{Q}_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)} \right|^2 d\tau\, d\eta + \int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau \right], \qquad (1.0.35)$$

and inequality (1.0.18) follows from conditions (1.0.3), (1.0.7) and (1.0.8) (for $m = 1$).

Necessity of condition (1.0.8). Let v_0 be a solution of the equation $\mathscr{P}_+(\xi; -\mathrm{i}\, d/dt)\, v = 0$. Substituting v_0 into (1.0.17), we obtain

$$\int_0^\infty |\mathscr{R}(\xi; -\mathrm{i}\, d/dt)\, v_0|^2\, dt \leqslant \Lambda(\xi)\, |\mathscr{Q}(\xi; -\mathrm{i}\, d/dt)\, v_0|_{t=0}|^2. \qquad (1.0.36)$$

On the other hand, representation (1.0.29) for v_0 has the form

$$\mathscr{R}(\xi; -\mathrm{i}\, d/dt)\, v_0 = -\frac{\mathrm{i}}{2\pi} F_{\tau \to t} \left[\frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\, \mathscr{Q}(\xi; -\mathrm{i}\, d/dt)\, v_0|_{t=0} \right]. \qquad (1.0.37)$$

Since the $1 \times N$ matrix $G(\xi;\tau)/\mathscr{P}_+(\xi;\tau)$ admits an analytic continuation to the half-plane $\operatorname{Im} \xi < 0$ ($\xi = \tau + \mathrm{i}\sigma$) for any $\xi \in \mathbb{R}^{n-1}$, it follows from (1.0.36) and (1.0.37) that

$$\Lambda(\xi) \geqslant \text{const} \int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau. \qquad (1.0.38)$$

Therefore, (1.0.18) implies condition (1.0.8).

We also note that (1.0.18) and the following estimate of $\Lambda(\xi)$ from below,

$$\Lambda(\xi) \geqslant \text{const} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathcal{Q}_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)} \right|^2 d\tau d\eta \qquad (1.0.39)$$

imply the necessity of condition (1.0.7). The estimate (1.0.39) for arbitrary $m \geqslant 1$ is proved in Section 1.1. The proof is quite involved. Since the restriction to $m = 1$ does not lead to any substantial simplification, we do not provide it in this short presentation.

Case $m > 1$. The sketch of the proof of the main result of this chapter given above for the case m = 1 already contains the most essential arguments of the proof for the general case.

However, there are some additional special features for $m > 1$. Thus, instead of the matrices $\mathscr{R}$ and $\mathcal{Q}$, which enter into the formulation of all the conditions for $m = 1$, in the case $m > 1$ we have to consider the matrices $S = RP^{\mathrm{c}}$ and $T = QP^{\mathrm{c}}$. The transition to the matrices S and T is accompanied by a diagonalization of the matrix P. So, the problem for ordinary differential operators reduces to the study of estimates that are equivalent to the initial ones, but have the simpler form (1.1.6). The mathematical apparatus necessary for these investigations is constructed in Lemmas 1.1.5–1.1.18.

A further feature of the case $m > 1$ is that the existence of the matrix G satisfying identity (1.0.6) occurs now as an independent condition, while for $m = 1$ it follows from other conditions of the criterion for the validity of estimate (1.0.2). We establish (see Propositions 1.2.6 and 1.2.7) necessary and sufficient conditions and more simply formulated sufficient conditions ensuring that in the case $m \neq 1$ the existence of the matrix G with the above-mentioned properties follows from (1.0.3)–(1.0.5). In particular, this is true if $\mathscr{M}(\xi,\tau) = 1$ for all $\tau \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$, or if the τ-roots of the polynomial $\mathscr{P}_+(\xi;\tau)$ are pairwise distinct a.e. in $\mathbb{R}^{n-1}$. On the other hand, if the polynomials $\mathscr{M}(\xi;\tau)$ and $\mathscr{P}_+(\xi;\tau)/\mathscr{M}(\xi;\tau)$ are not relatively prime (in τ), then counterexamples exist (see Section 1.1, Example 1.1.20)[7].

Remark on the notation. Along with notations already introduced above, we will use the following designations:

In the expressions $\sup|S(\xi;\tau)/\mathscr{P}(\xi;\tau)|$, $\sup|S_+(\xi;\tau)/\mathscr{P}_+(\xi;\tau)|$ and the similar ones, the upper limit is taken over all $\tau \in \mathbb{R}^1$;

If $f(t) = (f_1(t),\dots,f_m(t))$ and $\psi(\tau) = (\psi_1(\tau),\dots,\psi_m(\tau))$, then we set

$$F_{\tau\to t} f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}t\tau} f(t)dt, \qquad F^{-1}_{\tau\to t}\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}t\tau}\psi(\tau)d\tau.$$

We denote by $C, c_1, C_2, \dots$ various positive constants which do not depend on the polynomials P_{kj}, R_j, and $Q_{\alpha j}$.

[7] This implies that, in the dominance problems for general differential operators, the sharp constants in the estimates are not always continuous functions of the coefficients of differential polynomials entering these estimates.

1.1 Estimates for systems of ordinary differential operators on a semi-axis

Let $R(\tau) = \{R_j(\tau)\}$, $P(\tau) = \{P_{kj}(\tau)\}$, $Q(\tau) = \{Q_{\alpha j}(\tau)\}$ be matrices of size $(1\times m)$, $(m\times m)$, and $(N\times m)$, respectively. Suppose that the entries of these matrices are polynomials of the variable $\tau \in \mathbb{R}^1$ with complex coefficients. In this section we establish necessary and sufficient conditions for the validity of the inequality

$$\int_0^\infty |R(-\mathrm{i}\, d/dt)\, u|^2\, dt \leqslant \Lambda \left[\int_0^\infty |P(-\mathrm{i}\, d/dt)\, u|^2\, dt + |Q(-\mathrm{i}\, d/dt)\, u|_{t=0}|^2 \right] \tag{1.1.1}$$

for all $u = (u_1(t), \dots, u_m(t)) \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$, and give the upper and lower bounds for the sharp constant Λ figuring in (1.1.1).

1.1.1 Some assumptions and notation

First, we formulate assumptions on the matrices R, P, and Q under which the estimate (1.1.1) is studied. By $P^{\mathrm{c}}(\tau) = \{P_{jk}(\tau)\}$ we denote the $m \times m$ matrix whose rows are composed of the algebraic complements of the column elements of the matrix $P(\tau)$. We set

$$S(\tau) = \{S_k(\tau)\} = R(\tau)P^{\mathrm{c}}(\tau), \quad T(\tau) = \{T_{\alpha k}(\tau)\} = Q(\tau)P^{\mathrm{c}}(\tau). \tag{1.1.2}$$

Let $\mathscr{P}(\tau) = \det P(\tau)$, let $\mathscr{P}_+(\tau)$ be a polynomial whose roots (with multiplicities taken into account) coincide with the all roots of $\mathscr{P}(\tau)$ in the half-plane $\operatorname{Im}\zeta \geqslant 0$ $(\zeta = \tau + \mathrm{i}\sigma)$, and let $\mathscr{P}_-(\tau) = \mathscr{P}(\tau)/\mathscr{P}_+(\tau)$. Finally, let $\mathscr{M}(\tau)$ be the greatest common divisor of the polynomials $\mathscr{P}_+(\tau)$, $S_1(\tau), \dots, S_m(\tau)$, and let $\dot{\mathscr{P}}_+(\tau) = \mathscr{P}_+(\tau)/\mathscr{M}(\tau)$.

We assume that

1. $\mathscr{P}(\tau) \not\equiv 0$.
2. The leading coefficients of the polynomials $\mathscr{P}(\tau)$, $\mathscr{P}_+(\tau)$ and $\mathscr{M}(\tau)$ are equal to 1.
3. $\operatorname{ord} \dot{\mathscr{P}}_+(\tau) = N \geqslant 1$, where N is the number of rows of the matrix $Q(\tau)$.
4. $\max\limits_k \operatorname{ord} S_k(\tau) \leqslant \operatorname{ord} \mathscr{P}(\tau)$ and $\max\limits_{\alpha,k} \operatorname{ord} T_{\alpha k}(\tau) = \operatorname{ord} \mathscr{P}(\tau) - 1$.

Based on the last of these assumptions, we define the $1 \times m$ matrices $S_\pm(\tau) = \{S_{k\pm}(\tau)\}$ and the $N \times m$ matrices $T_\pm(\tau) = \{T_{\alpha k\pm}(\tau)\}$ by the identities

$$\frac{S}{\mathscr{P}} = c + \frac{S_+}{\mathscr{P}_+} + \frac{S_-}{\mathscr{P}_-}, \qquad \frac{T}{\mathscr{P}} = \frac{T_+}{\mathscr{P}_+} + \frac{T_-}{\mathscr{P}_-}, \tag{1.1.3}$$

where the relations

$$\max_k \operatorname{ord} S_{k+},\ \max_{\alpha,k} \operatorname{ord} T_{\alpha k+} < \operatorname{ord} \mathscr{P}_+,$$
$$\max_k \operatorname{ord} S_{k-},\ \max_{\alpha,k} \operatorname{ord} T_{\alpha k-} < \operatorname{ord} \mathscr{P}_-$$

are valid, and $c = \{c_k\}$ is a $1 \times m$ matrix with constant entries.

1.1.2 Transformation of the basic inequality

In the case $m \neq 1$, an essential difference between the estimates (1.1.1) and (1.0.17) lies in the fact that P entering on the right-hand side of (1.1.1) is no longer a polynomial, but an arbitrary $m \times m$ matrix of polynomials. Now we present a simple method that allows to replace the estimate (1.1.1) by the estimate (1.1.6) (and even by inequality (1.1.8)), where the matrix $P\,(-\mathrm{i}\,d/dt)$ on the right-hand side is replaced by the *diagonal matrix* $\mathscr{P}\,(-\mathrm{i}\,d/dt)\,I$ (by the matrix $\dfrac{\mathscr{P}}{\mathscr{M}}\,(-\mathrm{i}\,d/dt)\,I$, respectively). This enable us to follow the plan of proving the estimate (1.0.17) outlined in Subsection 1.0.2.

Lemma 1.1.1. *For any vector function $g = (g_1, \dots, g_m) \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ there exists a solution $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ of the system of equations*

$$\mathscr{P}\,(-\mathrm{i}\,d/dt)\,I\varphi = g. \tag{1.1.4}$$

Proof. Let $J = \operatorname{ord} \mathscr{P}(\tau)$, let $v^1(t), \dots, v^J(t)$ be a system of linearly independent solutions of the equation $\mathscr{P}(-\mathrm{i}\,d/dt)v = 0$, let $W(v^1, \dots, v^J)$ be the Wronskian of this system, and let $W_k(v^1, \dots, v^J)$ be the determinant obtained from $W(v^1, \dots, v^J)$ by replacing its k-th column $(1 \leqslant k \leqslant J)$ by $(0, \dots, 0, 1)$. Then the vector function

$$\varphi(t) = -\sum_{k=1}^{J} v^k(t) \int_t^{+\infty} \frac{W_k(v^1, \dots, v^J)(\tau)}{W(v^1, \dots, v^J)(\tau)} g(\tau)\,d\tau$$

is the solution of system (1.1.4) in the space $\mathbf{C}_0^\infty(\mathbb{R}^1_+)$. □

Lemma 1.1.2. *For any $u \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ there exists a solution $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ of the system of equations*

$$P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi = u. \tag{1.1.5}$$

Proof. We set $g = P\,(-\mathrm{i}\,d/dt)\,u$ and observe that $g \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$. Then by Lemma 1.1.1 there exists a vector function $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ such that $\mathscr{P}\,(-\mathrm{i}\,d/dt)\,I\varphi = g$. We show that φ is also a solution of the system (1.1.5). Indeed, from the definition of g and the equality $P^{\mathrm{c}}(\tau)P(\tau) = \mathscr{P}(\tau)I$, which is obviously true for the matrix P^{c}, it follows that

$$\mathscr{P}\,(-\mathrm{i}\,d/dt)\,I\,[u - P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi] = \mathscr{P}\,(-\mathrm{i}\,d/dt)\,I u - P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,g = 0.$$

Since $u - P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$, we finally get $u = P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi$. □

From the definition of the matrices S and T, and this lemma, the following assertions can be immediately obtained.

Lemma 1.1.3. *The inequality* (1.1.1) *with some* $\Lambda < \infty$ *holds for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, *if and only if*

$$\int_0^\infty |S(-\mathrm{i}\, d/dt)\,\varphi|^2\, dt \leqslant \Lambda \left[\int_0^\infty |\mathscr{P}(-\mathrm{i}\, d/dt)\, I\varphi|^2\, dt + |T(-\mathrm{i}\, d/dt)\,\varphi|_{t=0}|^2\right] \tag{1.1.6}$$

for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

Lemma 1.1.4. *Let the matrix* $T(\tau)$ *satisfy the congruence*

$$T(\tau) \equiv 0\ (\mathrm{mod}\, \mathscr{M}(\tau)). \tag{1.1.7}$$

We set $\dot{S}(\tau) = S(\tau)/\mathscr{M}(\tau)$, $\dot{\mathscr{P}}(\tau) = \mathscr{P}(\tau)/\mathscr{M}(\tau)$, *and* $\dot{T}(\tau) = T(\tau)/\mathscr{M}(\tau)$.

The inequality (1.1.1) *with some* $\Lambda < \infty$ *is true for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, *iff the estimate*

$$\int_0^\infty \left|\dot{S}(-\mathrm{i}\, d/dt)\,\psi\right|^2 dt \leqslant \Lambda \left[\int_0^\infty \left|\dot{\mathscr{P}}(-\mathrm{i}\, d/dt)\, I\psi\right|^2 dt + \left|\dot{T}(-\mathrm{i}\, d/dt)\,\psi|_{t=0}\right|^2\right] \tag{1.1.8}$$

holds for all $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

1.1.3 The simplest lower bound of the constant Λ

In this subsection we obtain the lower bound (1.1.9) for the constants Λ as a very simple corollary of inequality (1.1.6). It will be used in the proof of Theorem 1.1.19. The estimate (1.1.9) can be regarded as the first natural restriction on the class of operators R that obey inequality (1.1.1).

Lemma 1.1.5. *If for some* $\Lambda < \infty$ *inequality* (1.1.6) *holds for all* $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, *then*

$$\sup |S(\tau)/\mathscr{P}(\tau)|^2 \leqslant \Lambda. \tag{1.1.9}$$

Proof. We substitute in (1.1.6) the vector function $\varphi(t) = v(t+a)$, where $v(t) \in \mathbf{C}_0^\infty(\mathbb{R}^1)$ and $a \in \mathbb{R}^1$ are chosen such that $\operatorname{supp} v_k(t) \cap (-\infty, a) = \emptyset$ $(k = 1, \dots, m)$. It is evident that $\varphi(t) \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ and $\varphi^{(j)}(0) = 0$ $(j = 0, 1, 2, \dots)$. This means that $T(-\mathrm{i}\, d/dt)\,\varphi|_{t=0} = 0$.

Thus, the estimate

$$\int_{-\infty}^{\infty} |S(-\mathrm{i}\, d/dt)\, v|^2\, dt \leqslant \Lambda \int_{-\infty}^{\infty} |\mathscr{P}(-\mathrm{i}\, d/dt)\, I v|^2\, dt.$$

holds true for all vector functions $v \in \mathbf{C}_0^\infty(\mathbb{R}^1)$. Now, applying the Fourier transform and using the standard arguments, we get (1.1.9). □

Remark 1.1.6. It follows from inequality (1.1.9) that the polynomial $\dot{\mathscr{P}}(\tau)$ has no real roots.

1.1.4 On solutions of the system $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,I\varphi = 0$

In Section 1.0, we repeatedly substituted solutions of the equation $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,v = 0$ in inequality (1.0.17). However, these solutions do not belong to the space $\mathbf{C}_0^\infty(\mathbb{R}_+^1)$ (with respect to t), so this procedure requires a justification. In this subsection, we show that the validity of inequality (1.1.6) for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ establishes its correctness for solutions of the system $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,I\varphi = 0$ as well. Thus, we complete a gap in the arguments of Section 1.0 and, at the same time, provide a necessary basis for further studying of the inequalities (1.1.6) and (1.1.8).

We begin with some remarks on solutions of the system of equations $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,I\varphi = 0$.

Let ζ_ϱ be the root of the polynomial $\dot{\mathscr{P}}_+(\tau) = \mathscr{P}_+(\tau)/\mathscr{M}(\tau)$ of multiplicity k_ϱ, so that $\dot{\mathscr{P}}_+(\tau) = \prod_{\varrho=1}^{l}(\tau-\zeta_\varrho)^{k_\varrho}$ $(k_1+\dots+k_l = N)$. The roots of the polynomial $\mathscr{P}_+(\tau)$ will also be denoted by ζ_ϱ and their multiplicities by $\varkappa_\varrho$. Then we have $\mathscr{P}_+(\tau) = \prod_{\varrho=1}^{l}(\tau-\zeta_\varrho)^{\varkappa_\varrho}$. It is clear that $l_1 \geqslant l$ and $\varkappa_\varrho \geqslant k_\varrho$ provided $1 \leqslant \varrho \leqslant l$. Let l_2 $(0 < l_2 \leqslant l)$ be an integer such that $\mathscr{M}(\zeta_\varrho) = 0$ for $1+l_2 \leqslant \varrho \leqslant l_1$ and $\mathscr{M}(\zeta_\varrho) \neq 0$ for $\varrho \leqslant l_2$.[8] Then for $\varrho \leqslant l_2$ we have $\varkappa_\varrho = k_\varrho$.

From the definition of $\mathscr{M}(\tau)$ it follows that the $1 \times m$ matrix $S(\tau)$ satisfies the following conditions:

(a) $S(\zeta_\varrho) \neq 0$, if $1 \leqslant \varrho \leqslant l_2$;

(b) $S^{(\alpha)}(\zeta_\varrho) = 0$, if

1. $l_2+1 \leqslant \varrho \leqslant l$ and $0 \leqslant \alpha \leqslant \varkappa_\varrho - k_\varrho - 1$,
2. $l+1 \leqslant \varrho \leqslant l_1$ and $0 \leqslant \alpha \leqslant \varkappa_\varrho - 1$;

(c) $S^{(\varkappa_\varrho - k_\varrho)}(\zeta_\varrho) \neq 0$, if $l_2+1 \leqslant \varrho \leqslant l$.

Let $\operatorname{ord}\mathscr{M}(\tau) = N_1$. We introduce two $N \times mn$ matrices $\mathfrak{S}$ and $\mathfrak{T}$, and the $N \times mN_1$ matrix $\mathfrak{T}(\mathscr{M})$ as follows:

Matrix $\mathfrak{S}$. Let $S^k = \{s^{\varrho\sigma(\varrho)k}_{\nu\beta(\nu)}\}$ be the $N \times N$ matrices whose rows are labeled by the indices ν, $\beta(\nu)$, and whose columns are labeled by the indices ϱ, $\sigma(\varrho)$, respectively.

[8] If $\mathscr{M}(\zeta_\varrho) = 0$ for all $\varrho = 1,\dots,l_1$, then we set $l_2 = 0$. We restrict ourselves to the case $l_2 > 0$ and leave it to the reader to make the obvious modifications in all subsequent arguments and formulas for $l_2 = 0$.

These indices take the values $\nu, \varrho = 1, \dots, l$; $\beta(\nu) = 0, \dots, k_\nu - 1$ and $\sigma(\varrho) = 0, \dots, k_\varrho - 1$, while the index k runs through the numbers $1, \dots, m$. We set

$$\left.\begin{aligned}
&\mathfrak{S} = \{S^1, \dots, S^m\}; \qquad s^{\varrho\sigma(\varrho)k}_{\nu\beta(\nu)} = 0, \quad \text{if} \quad \nu \neq \varrho;\\
&s^{\varrho\sigma(\varrho)k}_{\varrho\beta(\varrho)} = \begin{cases} 0, & \text{if} \quad \sigma < \beta,\\ \dfrac{\sigma!}{(\sigma-\beta)!} S_k^{(\sigma-\beta)}(\zeta_\varrho), & \text{if} \quad \sigma \geqslant \beta \end{cases}\\
&\qquad\qquad (1 \leqslant \rho \leqslant l_2, \sigma = \sigma(\varrho), \beta = \beta(\varrho));\\
&s^{\varrho\sigma(\varrho)k}_{\varrho\beta(\varrho)} = \begin{cases} 0, & \text{if} \quad \sigma + \varkappa_\varrho - k_\varrho < \beta,\\ \dfrac{(\sigma + \varkappa_\varrho - k_\varrho)!}{(\sigma + \varkappa_\varrho - k_\varrho - \beta)!} S_k^{(\sigma+\varkappa_\varrho-k_\varrho-\beta)}(\zeta_\varrho), & \text{if} \, \sigma + \varkappa_\varrho - k_\varrho \geqslant \beta \end{cases}\\
&\qquad\qquad (l_2 + 1 \leqslant \rho \leqslant l, \sigma = \sigma(\varrho), \beta = \beta(\varrho)),
\end{aligned}\right\} \tag{1.1.10}$$

where $S_k(\tau)$ $(k = 1, \dots, m)$ are the entries of the matrix $S(\tau)$ defined by (1.1.2).

It follows from conditions (a) and (c) that $\operatorname{rg} \mathfrak{S} = N$.

Matrix $\mathfrak{T}$. Let $T^k = \{t^{\varrho\sigma(\varrho)k}_\alpha\}$ be the $N \times N$ matrices whose rows are labeled by the index α, and whose columns are labeled by the indices ϱ, $\sigma(\varrho)$, respectively. These indices take the values $\alpha = 1, \dots, N$; $\varrho = 1, \dots, l$; $\sigma(\varrho) = 0, \dots, k_\varrho - 1$ in the case $1 \leqslant \varrho \leqslant l_2$, and $\sigma(\varrho) = \varkappa_\varrho - k_\varrho, \dots, \varkappa_\varrho - 1$ in the case $l_2 + 1 \leqslant \varrho \leqslant l$, respectively. The index k runs through the numbers $1, \dots, m$. We set

$$\mathfrak{T} = \{T^1, \dots, T^m\}, \qquad t^{\varrho\sigma(\varrho)k}_\alpha = T^{(\sigma)}_{\alpha k}(\zeta_\varrho), \qquad \sigma = \sigma(\varrho), \tag{1.1.11}$$

where $T_{\alpha k}(\tau)$ are the entries of the matrix $T(\tau)$ defined by (1.1.2).

Matrix $\mathfrak{T}(\mathscr{M})$. Let $T^k(\mathscr{M}) = \{t^{\varrho\sigma(\varrho)k}_\alpha(\mathscr{M})\}$ be the $N \times N_1$ matrices whose rows are labeled by the index α, $(1 \leqslant \alpha \leqslant N)$, and whose columns are labeled by the indices ϱ, $\sigma(\varrho)$, respectively. These indices satisfy the conditions $l_2 + 1 \leqslant \varrho \leqslant l_1$ and $0 \leqslant \sigma(\varrho) \leqslant \varkappa_\varrho - k_\varrho - 1$ in the case $l_2 + 1 \leqslant \varrho \leqslant l$, while $0 \leqslant \sigma(\varrho) \leqslant \varkappa_\varrho - 1$ in the case $l + 1 \leqslant \varrho \leqslant l_1$. The index k runs through the numbers $1, \dots, m$. We set

$$\begin{gathered}\mathfrak{T}(\mathscr{M}) = \{T^1(\mathscr{M}), \dots, T^m(\mathscr{M})\}, \quad t^{\varrho\sigma(\varrho)k}_\alpha(\mathscr{M}) = T^{(\sigma)}_{\alpha k}(\zeta_\varrho),\\ \sigma = \sigma(\varrho).\end{gathered} \tag{1.1.12}$$

The solutions of the system $\mathscr{P}_+(-\mathrm{i}\, d/dt) I\varphi = 0$, which we need later, are constructed as linear combinations of the vector functions x^1, x^2, y^1, y^2 with the following components:

$$\begin{aligned}
x_k^1(t) &= \sum_{\varrho=1}^{l_2} \sum_{\sigma=0}^{k_\varrho-1} x_{\varrho\sigma k} z_{\varrho\sigma}(t), \\
x_k^2(t) &= \sum_{\varrho=l_2+1}^{l} \sum_{\sigma=\varkappa_\varrho-k_\varrho}^{\varkappa_\varrho-1} x_{\varrho\sigma k} z_{\varrho\sigma}(t), \\
y_k^1(t) &= \sum_{\varrho=l_2+1}^{l} \sum_{\sigma=0}^{\varkappa_\varrho-k_\varrho-1} y_{\varrho\sigma k} z_{\varrho\sigma}(t), \\
y_k^2(t) &= \sum_{\varrho=l+1}^{l_1} \sum_{\sigma=0}^{\varkappa_\varrho-1} y_{\varrho\sigma k} z_{\varrho\sigma}(t).
\end{aligned} \tag{1.1.13}$$

Here, $z_{\varrho\sigma}(t) = \exp(\mathrm{i}\zeta_\varrho t)(\mathrm{i}t)^\sigma$, $x_{\varrho\sigma k}$, and $y_{\varrho\sigma k}$ are arbitrary complex constants, and $k = 1, \dots, m$.

A direct verification shows that

1. $S(-\mathrm{i}\, d/dt)\, y^1 = S(-\mathrm{i}\, d/dt)\, y^2 = 0$ for any choice of the constants $y_{\varrho\sigma k}$.

2. $S(-\mathrm{i}\, d/dt)\,(x^1 + x^2) = 0$ if and only if the vector $\Xi \in \mathbb{C}^{mN}$ composed of the coefficients $x_{\varrho\sigma k}$ such that

$$\begin{gathered}
\Xi = (x_{\varrho\sigma k}) \\
(k = 1, \dots, m; \quad 0 \leqslant \sigma \leqslant k_\varrho - 1 \quad \text{if} \quad 1 \leqslant \varrho \leqslant l_2, \\
\text{and } \varkappa_\varrho - k_\varrho \leqslant \sigma = \varkappa_\varrho - 1 \quad \text{if} \quad l_2 + 1 \leqslant \varrho \leqslant l),
\end{gathered} \tag{1.1.14}$$

satisfies the condition

$$\mathfrak{G}\Xi = 0, \tag{1.1.15}$$

where $\mathfrak{G}$ is the matrix (1.1.10).

3. $T(-\mathrm{i}\, d/dt)\,(x^1 + x^2)\big|_{t=0} = 0$ if and only if vector (1.1.14) satisfies the condition

$$\mathfrak{T}\Xi = 0, \tag{1.1.16}$$

where $\mathfrak{T}$ is the matrix (1.1.11).

4. $T(-\mathrm{i}\, d/dt)\,(x^1 + x^2 + y^1 + y^2)\big|_{t=0} = 0$ if and only if

$$\mathfrak{T}\,\Xi = -\mathfrak{T}(\mathscr{M})H. \tag{1.1.17}$$

Here, Ξ is the vector defined by (1.1.14), $\mathfrak{T}$ and $\mathfrak{T}(\mathscr{M})$ are the matrices defined by (1.1.11) and (1.1.12), respectively, and $H \in \mathbb{C}^{mN_1}$ is the vector composed of the coefficients $y_{\varrho\sigma k}$ such that

$$H = (y_{\varrho\sigma k}) \tag{1.1.18}$$
$$(k = 1, \dots, m; \quad 0 \leqslant \sigma \leqslant \varkappa_\varrho - k_\varrho - 1 \quad \text{if} \quad l_2 + 1 \leqslant \varrho \leqslant l,$$
$$\text{and } 0 \leqslant \sigma \leqslant \varkappa_\varrho - 1 \quad \text{if} \quad l + 1 \leqslant \varrho \leqslant l_1).$$

Lemma 1.1.7. *Let inequality (1.1.6) hold for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Then, for any solution $\varphi(t)$ of the system $\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi = 0$ there exists a sequence $\varphi^s \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ satisfying*

$$T\,(-\mathrm{i}\, d/dt)\, \varphi^s|_{t=0} = 0 \qquad (s = 1, 2, \dots), \tag{1.1.19}$$

and

$$\lim_{s\to\infty} \int_0^\infty |\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi^s|^2\, dt = 0,$$
$$\lim_{s\to\infty} \int_0^\infty |S\,(-\mathrm{i}\, d/dt)\,(\varphi - \varphi^s)|^2\, dt = 0. \tag{1.1.20}$$

Proof. Consider a cut-off function $\eta(t) \in C_0^\infty(\mathbb{R}_+^1)$ such that $\eta(t) = 1$ for $0 \leqslant t \leqslant 1$ and $\eta(t) = 0$ for $2 \leqslant t < \infty$. We set $\varphi^s(t) = \eta(t/s)\varphi(t)$ $(s = 1, 2, \dots)$. Obviously, $\varphi^s \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ and (1.1.19) follows from the definition of $\eta(t)$. Since $\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi = 0$, it suffices to show for the proof of (1.1.20) that

$$\lim_{s\to\infty} \int_0^\infty \left|\mathscr{P}_+(-\mathrm{i}\, d/dt)\,[\eta(t/s) z_{\varrho\sigma}(t)]\right|^2 dt = 0 \tag{1.1.21}$$

and

$$\lim_{s\to\infty} \int_0^\infty \left|S_k\,(-\mathrm{i}\, d/dt)\,[z_{\varrho\sigma}(t) - \eta(t/s) z_{\varrho\sigma}(t)]\right|^2 dt = 0 \tag{1.1.22}$$
$$(k = 1, 2, \dots, m),$$

where $z_{\varrho\sigma}(t) = (\mathrm{i}t)^\sigma \exp(\mathrm{i}\zeta_\varrho t)$, and ϱ and σ take the same values as in (1.1.13).

We now prove equalities (1.1.21) and (1.1.22). Since the estimate (1.1.6) holds true for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, the polynomial $\dot{\mathscr{P}}(\tau)$ has no real roots in accordance with Remark 1.1.6, Chapter 1. Therefore, for $\varrho = 1, \dots, l_2$, $\sigma = 0, \dots, k_\varrho - 1$, and $\varrho = l_2 + 1, \dots, l$, $\sigma = \varkappa_\varrho - k_\varrho, \dots, \varkappa_\varrho - 1$, the functions $z_{\varrho\sigma}(t)$ and their derivatives decrease exponentially as $t \to \infty$, so that (1.1.21) and (1.1.22) hold.

Now, let $\varrho = l_2 + 1, \dots, l$; $\sigma = 0, \dots, \varkappa_\varrho - k_\varrho - 1$ or $\varrho = l + 1, \dots, l_1$; $\sigma = 0, \dots, \varkappa_\varrho - 1$. It follows from property (b) of the matrix $S(\tau)$ that for these values of ϱ and σ, $S_k\,(-\mathrm{i}\, d/dt)\, z_{\varrho\sigma}(t) = 0$ $(k = 1, \dots, m)$.

On the other hand, it is obvious that

$$\mathscr{P}_+^{(\omega)}(-\mathrm{i}\, d/dt)\,[z_{\varrho\sigma}(t)]\,(-\mathrm{i}\, d/dt)^\omega\,[\eta(t/s)] \leqslant c s^{\sigma - \operatorname{ord} \mathscr{P}_+},$$

and, consequently,

$$\int_0^\infty \left|\mathscr{P}_+(-\mathrm{i}\, d/dt)\,[\eta(t/s) z_{\varrho\sigma}(t)]\right|^2 dt \leqslant c s^{1+2(\sigma - \operatorname{ord} \mathscr{P}_+)},$$

where $c > 0$ is a constant. The integrals on the left-hand side of (1.1.22) can be estimated in the same manner (with $\operatorname{ord} \mathscr{P}_+$ replaced by $\operatorname{ord} S_k$). To complete the proof, it remains to note that all the values of σ considered here do not exceed $\min\{\operatorname{ord} \mathscr{P}_+, \operatorname{ord} S_1, \dots, \operatorname{ord} S_m\} - 1$. □

Remark 1.1.8. It follows immediately from Lemma 1.1.7 that the vector functions $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ in inequality (1.1.6) can be replaced by the solutions of the system $\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi = 0$. From now on we will perform these (and similar) substitutions without further comments.

1.1.5 Properties of the matrix $T(\tau)$

In this subsection we specify the algebraic conditions that the matrix $T(\tau)$ figuring in the estimate (1.1.6) must satisfy. We show that if inequality (1.1.6) holds for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, then the greatest common divisor of the polynomials $\mathscr{P}_+(\tau), S_1(\tau), \dots, S_m(\tau)$ is a divisor of the matrix $T(\tau)$, and the rows of $T(\tau)$ are linearly independent modulo $\mathscr{P}_+(\tau)$.

Lemma 1.1.9. *If inequality* (1.1.6) *with some* $\Lambda < \infty$ *holds for all* $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, *then the matrix* $T(\tau)$, *defined by equation* (1.1.2), *satisfies relation* (1.1.7).

Proof. We show that (1.1.6) leads to the equation $\operatorname{rg} \mathfrak{T} = N$, where $\mathfrak{T}$ is the matrix (1.1.11), and to the equivalence of conditions (1.1.15) and (1.1.16).

We substitute in (1.1.6) the vector function $\varphi = x^1 + x^2$, where x^1 and x^2 are the vectors introduced in (1.1.13). Since $\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi = 0$, the implication (1.1.16) $\Rightarrow$ (1.1.15) follows from inequality (1.1.6) and assertions 2 and 3 (see page 17). Since $\operatorname{rg} \mathfrak{G} = N$, we have $\operatorname{rg} \mathfrak{T} = N$.

From the implication (1.1.16) $\Rightarrow$ (1.1.15) it follows that the rows of the matrix $\mathfrak{G}$ belong to the linear span of the rows of the matrix $\mathfrak{T}$. Since $\operatorname{rg} \mathfrak{T} = N$, then, conversely, the rows of $\mathfrak{T}$ belong to the linear span of the rows of $\mathfrak{G}$. Therefore, we have the equivalence (1.1.16) $\Leftrightarrow$ (1.1.15).

We proceed now to the proof of relation (1.1.7). Obviously, it suffices to prove that $\mathfrak{T}(\mathscr{M}) = 0$, where $\mathfrak{T}(\mathscr{M})$ is the matrix defined in (1.1.12).

Let Ξ and H be arbitrary vectors of the form (1.1.14) and (1.1.18), respectively, and let x^1, x^2, y^1, y^2 be the vector functions defined by (1.1.13). We substitute the vector function $\varphi^* = x^1 + x^2 + y^1 + y^2$ in (1.1.6). Since $\mathscr{P}_+(-\mathrm{i}\, d/dt)\, I\varphi^* = 0$, the implication (1.1.17) $\Rightarrow$ (1.1.15) follows from assertions 1, 2 and 4 (see page 18). Since $\operatorname{rg} \mathfrak{T} = N$, the system of equations (1.1.17) is solvable with respect to Ξ for any $H \in \mathbb{C}^{mN_1}$. For a fixed H it follows that every solution Ξ of (1.1.17) satisfies also (1.1.15). The equivalence (1.1.15) $\Leftrightarrow$ (1.1.16) has already been shown. Taking into account (1.1.17), we get $\mathfrak{T}(\mathscr{M})H = 0$ for all $H \in \mathbb{C}^{mN_1}$, and, consequently, $\mathfrak{T}(\mathscr{M}) = 0$. □

To prove the linear independence of the rows of the matrix T modulo $\mathscr{P}_+$ (Lemma 1.1.10), we again introduce two $N \times mN$ matrices $\dot{\mathfrak{S}}$ and $\dot{\mathfrak{T}}$.

Let the matrix $T(\tau)$ defined by (1.1.2) satisfy relation (1.1.7), and suppose that $\dot{S}(\tau) = S(\tau)/\mathscr{M}(\tau)$, $\dot{T}(\tau) = T(\tau)/\mathscr{M}(\tau)$, and $\dot{\mathscr{P}}_+(\tau) = \mathscr{P}_+(\tau)/\mathscr{M}(\tau)$.

Matrix $\dot{\mathfrak{S}}$. Let $\dot{S}^k = \{\dot{s}^{\varrho\sigma(\varrho)k}_{\nu\beta(\nu)}\}$ be the $N \times N$ matrices whose rows are labeled by the indices ν, $\beta(\nu)$, and whose columns are labeled by the indices ϱ, $\sigma(\varrho)$, respectively. These indices take the values $\nu, \varrho = 1, \dots, l$; $\beta(\nu) = 0, \dots, k_\nu - 1$, and $\sigma(\varrho) = 0, \dots, k_\varrho - 1$. The index k runs through the numbers $1, \dots, m$. We set

$$\left.\begin{aligned} &\dot{\mathfrak{S}} = \{\dot{S}^1, \dots, \dot{S}^m\}; \qquad \dot{s}^{\varrho\sigma(\varrho)k}_{\nu\beta(\nu)} = 0, \quad \text{if} \quad \nu \neq \varrho; \\ &\dot{s}^{\varrho\sigma(\varrho)k}_{\varrho\beta(\varrho)} = \begin{cases} 0, & \text{if} \quad \sigma < \beta, \\ \dfrac{\sigma!}{(\sigma-\beta)!}\dot{S}^{(\sigma-\beta)}_k(\zeta_\varrho), & \text{if} \quad \sigma \geqslant \beta \end{cases} \\ &\qquad\qquad\qquad\qquad\qquad (\sigma = \sigma(\varrho), \beta = \beta(\varrho)), \end{aligned}\right\} \tag{1.1.23}$$

where $\dot{S}_k(\tau)$ are the entries of the matrix $\dot{S}(\tau)$.

In accordance with the definition of the polynomial $\mathscr{M}(\tau)$, the greatest common divisor of $\dot{\mathscr{P}}_+(\tau), \dot{S}_1(\tau), \dots, \dot{S}_m(\tau)$ is equal to 1. Therefore, the matrix $\dot{S}(\tau)$ satisfies the condition $\dot{S}(\zeta_\varrho) \neq 0$, $(\varrho = 1, \dots, l)$. This condition and definition (1.1.23) yield to the equalities rg $\dot{\mathfrak{S}} =$ ord $\dot{\mathscr{P}}_+(\tau) = N$.

Matrix $\dot{\mathfrak{T}}$. Let $\dot{T}^k = \{\dot{t}^{\varrho\sigma(\varrho)k}_\alpha\}$ be the $N \times N$ matrices whose rows are labeled by the index α, and whose columns are labeled by the indices ϱ, $\sigma(\varrho)$, respectively. Here, $1 \leqslant \alpha \leqslant N$, $1 \leqslant \varrho \leqslant l$, and $0 \leqslant \sigma(\varrho) \leqslant k_\varrho - 1$. The index k runs through the numbers $1, \dots, m$. We set

$$\dot{\mathfrak{T}} = \{\dot{T}^1, \dots, \dot{T}^m\}, \qquad \dot{t}^{\varrho\sigma(\varrho)k}_\alpha = \dot{T}^{(\sigma)}_{\alpha k}(\zeta_\varrho), \qquad (\sigma = \sigma(\varrho)), \tag{1.1.24}$$

where $\dot{T}_{\alpha k}$ are the entries of the matrix $\dot{T}(\tau)$.

First, we highlight two properties of the matrices $\dot{\mathfrak{S}}$ and $\dot{\mathfrak{T}}$. These are similar to properties 2 and 3 of the matrices $\mathfrak{S}$ and $\mathfrak{T}$, which are defined by (1.1.10) and (1.1.11), respectively.

We introduce the vector

$$\begin{aligned} &\psi(t) = (\psi_1(t), \dots, \psi_m(t)), \\ &\psi_k(t) = \sum_{\varrho=1}^{l}\sum_{\sigma=0}^{k_\varrho-1} \psi_{\varrho\sigma k}(it)^\sigma \exp(i\zeta_\varrho t) \qquad (k = 1, \dots, m), \end{aligned} \tag{1.1.25}$$

where $\psi_{\varrho\sigma k}$ are arbitrary complex constants. Consider the vector $\Psi \in \mathbb{C}^{mN}$ composed by the coefficients $\psi_{\varrho\sigma k}$ as follows:

$$\Psi = (\psi_{\varrho\sigma k}) \qquad (k = 1, \dots, m;\ \varrho = 1, \dots, l;\ \sigma = 0, \dots, k_\varrho - 1). \tag{1.1.26}$$

A direct verification shows that

1. $\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi(t) = 0$ if and only if vector (1.1.26) satisfies the condition
$$\dot{\mathfrak{G}}\Psi = 0, \tag{1.1.27}$$
where $\dot{\mathfrak{G}}$ is the matrix (1.1.23).
2. $\dot{T}\;(-\mathrm{i}\,d/dt)\,\psi(t)|_{t=0} = 0$ if and only if vector (1.1.26) satisfies the condition
$$\dot{\mathfrak{T}}\Psi = 0, \tag{1.1.28}$$
where $\dot{\mathfrak{T}}$ is the matrix (1.1.24).

Lemma 1.1.10. *Let the matrix $T(\tau)$ satisfy (1.1.7). If for some $\Lambda < \infty$ inequality (1.1.8) holds for all $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, then the rows of the matrix $T(\tau)$ are linearly independent modulo $\mathscr{P}_+(\tau)$.*

Proof. We substitute the vector function (1.1.25) into inequality (1.1.8). Since $\dot{\mathscr{P}}_+\,(-\mathrm{i}\,d/dt)\,I\,\psi(t) = 0$, the implication (1.1.28) ⇒ (1.1.27) follows from (1.1.8) and assertions 1 and 2. Observe also that rg $\dot{\mathfrak{G}} = N$, and, consequently, rg $\dot{\mathfrak{T}} = N$. In accordance with definition (1.1.24), this last equality is equivalent to the linear independence of the rows of the matrix $\dot{T}(\tau)$ modulo $\dot{\mathscr{P}}_+(\tau)$ or, what is the same, to the linear independence of the rows of the matrix $T(\tau)$ modulo $\mathscr{P}_+(\tau)$. □

Remark 1.1.11. The estimate (1.1.8) implies not only the inclusion $\ker\dot{\mathfrak{T}} \subset \ker\dot{\mathfrak{G}}$ (or, what is the same, the implication (1.1.28) ⇒ (1.1.27)), but also the equality $\ker\dot{\mathfrak{T}} = \ker\dot{\mathfrak{G}}$. Indeed, if we assume that (1.1.28) ⇒ (1.1.27), then the rows of the matrix $\dot{\mathfrak{G}}$ belong to the linear span of the rows of the matrix $\dot{\mathfrak{T}}$. Since rg $\dot{\mathfrak{G}} = N$ (see the definition of the matrix $\dot{\mathfrak{G}}$), we conclude that the rows of $\dot{\mathfrak{G}}$ form a basis of the linear span of the rows of $\dot{\mathfrak{T}}$. Therefore, it follows from the implication (1.1.28) ⇒ (1.1.27) that the inverse implication (1.1.27) ⇒ (1.1.28) also holds. Thus, $\ker\,\dot{\mathfrak{T}} = \ker\,\dot{\mathfrak{G}}$[9].

1.1.6 Integral representation for $\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi$

In this subsection, we derive for vector functions $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ the formula (1.1.31), which gives an integral representation of $\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi$ in terms of $\dot{\mathscr{P}}\,(-\mathrm{i}\,d/dt)\,I\,\psi$ and $\dot{T}\;(-\mathrm{i}\,d/dt)\,\psi|_{t=0}$. Representation (1.1.31) will be frequently used in the sequel. In particular, it provides the estimate (1.1.8) as a direct corollary.

If $m = 1$ and the roots of the polynomial $\dot{\mathscr{P}}_+$ are pairwise distinct, then the formula (1.1.31) is already proved in Subsection 1.0.2 (see (1.0.29)). In the general case, an essential role is played by the $N \times mN$ matrices $\dot{\mathfrak{G}}$ and $\dot{\mathfrak{T}}$, which are defined by (1.1.23) and (1.1.24), respectively. For $m > 1$, both these matrices have nontrivial kernels, and, in accordance with Remark 1.1.11, the condition
$$\ker\,\dot{\mathfrak{T}} = \ker\dot{\mathfrak{G}} \tag{1.1.29}$$

[9] The reader will note the similarity of this result with the assertion (1.1.15) ⇔ (1.1.16) established in the proof of Lemma 1.1.9.

is necessary for the validity of (1.1.8). Assuming $\ker \dot{\mathfrak{T}} \subset \ker \dot{\mathfrak{G}}$, we construct the matrix $\dot{\mathfrak{G}}(\tau)$ (its existence in the scalar case follows from other conditions necessary for the validity of (1.1.8))[10] and obtain representation (1.1.31), according to the plan outlined for the scalar case.

Lemma 1.1.12. *Let the matrix $T(\tau)$, defined by (1.1.2), satisfy relation (1.1.7), let its rows be linearly independent modulo $\mathscr{P}_+(\tau)$, and let the polynomial $\dot{\mathscr{P}}(\tau)$ have no real roots. Suppose also that*

$$\ker \dot{\mathfrak{T}} \subset \ker \dot{\mathfrak{G}}, \tag{1.1.30}$$

where $\dot{\mathfrak{G}}$ and $\dot{\mathfrak{T}}$ are the matices defined in (1.1.23) and (1.1.24), respectively. Then there exists a $1 \times N$ matrix $\dot{G}(\tau) = \{\dot{G}_\alpha\}(\tau)$ with polynomial entries $\dot{G}_\alpha$ satisfying $\max_\alpha \operatorname{ord} \dot{G}_\alpha(\tau) \leqslant N-1$, such that for all vector-functions $\psi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ the representation

$$\begin{aligned}\dot{S}(-\mathrm{i}\,d/dt)\,\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}t\tau} \bigg\{ &\frac{\dot{S}(\tau)}{\dot{\mathscr{P}}(\tau)}(F_{t\to\tau} f) \\ &+ \frac{\mathrm{i}}{\sqrt{2\pi}} \frac{\dot{G}(\tau)}{\dot{\mathscr{P}}(\tau)} \bigg[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta} f)\, d\eta \\ &- \dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} \bigg] \bigg\} d\tau\end{aligned} \tag{1.1.31}$$

holds. Here, $f(t) = \dot{\mathscr{P}}\,(-\mathrm{i}\,d/dt)\,I\psi$ for $t \geqslant 0$ and $f(t) = 0$ for $t < 0$, while T_- is the matrix defined by (1.1.3).

Proof. Set $v(t) = F^{-1}_{\tau\to t}(F_{t\to\tau} f/\dot{\mathscr{P}}(\tau))$. Since the polynomial $\dot{\mathscr{P}}(\tau)$ does not have real roots, the components of the vector function $F_{t\to\tau} f/\dot{\mathscr{P}}(\tau)$ belong to space $L^2(-\infty,\infty)$. Since $\dot{\mathscr{P}}\,(-\mathrm{i}\,d/dt)\,I v = f$ for $t \geqslant 0$, we have for $t \geqslant 0$ the representation

$$\left.\begin{aligned} \psi - v = \psi^0 &= (\psi_1^0, \dots, \psi_m^0), \\ \psi_k^0 &= \sum_{\varrho=1}^{l} \sum_{\sigma=0}^{k_\varrho - 1} \psi^0_{\varrho\sigma k} (it)^\sigma e^{i\zeta_\varrho t} \quad (k = 1, \dots, m), \end{aligned}\right\} \tag{1.1.32}$$

where $\Psi^0 = (\psi^0_{\varrho\sigma k}) \in \mathbb{C}^{mN}$.

Further, as the vector function $F_{t\to\eta} f$ can be continued analytically into the half-plane $\operatorname{Im}\zeta < 0$, we get

$$\int_{-\infty}^{\infty} \frac{\dot{T}(\eta)}{\dot{\mathscr{P}}(\eta)}(F_{t\to\eta} f)\, d\eta = \int_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta} f)\, d\eta,$$

[10] See Subsection 1.0.2.

which in conjunction with (1.1.32) yields

$$\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta} f)d\eta + \dot{\mathfrak{T}}\Psi^0, \tag{1.1.33}$$

where $\dot{\mathfrak{T}}$ is the matrix (1.1.24).

Let $\dot{\mathfrak{T}}_R^{-1}$ be the right inverse[11] of the matrix $\dot{\mathfrak{T}}$, and let

$$\Psi = (\psi_{\varrho\sigma k}) = \dot{\mathfrak{T}}_R^{-1}\left[\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} - \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta} f)d\eta\right] \tag{1.1.34}$$

be a solution of system (1.1.33). If Ψ^0 is any other solution of this system, then the vector $\Psi^0 - \Psi$ satisfies condition (1.1.28). In accordance with (1.1.30), this vector satisfies also condition (1.1.27).

For $t \geqslant 0$, set

$$h_{\varrho\sigma k}(t) = \sum_{\beta=0}^{\sigma} \frac{1}{\beta!} \dot{s}_{\varrho\beta}^{\varrho\sigma k} (it)^{\sigma-\beta} e^{i\zeta_\varrho t} \tag{1.1.35}$$
$$(\varrho = 1,\dots,l;\ \sigma = 0,\dots,k_\varrho - 1;\ k = 1,\dots,m),$$

where $\dot{s}_{\varrho\beta}^{\varrho\sigma k}$ with $\beta = \beta(\varrho)$ and $\sigma = \sigma(\varrho)$ are the entries of the matrix (1.1.23).

A direct verification shows that representation (1.1.32) implies the equality

$$\dot{S}\,(-\mathrm{i}\,d/dt)\,(\psi - v) = \sum_{k=1}^{m}\sum_{\varrho=1}^{l}\sum_{\sigma=0}^{k_\varrho-1} \psi^0_{\varrho\sigma k} h_{\varrho\sigma k}(t), \qquad t \geqslant 0. \tag{1.1.36}$$

Taking into account (1.1.36) and the fact that the vector $\Psi^0 - \Psi$ satisfies condition (1.1.27), we can replace the coefficients $\psi^0_{\varrho\sigma k}$ in (1.1.36) by the coefficients $\psi_{\varrho\sigma k}$, which are defined by (1.1.34). Then representation (1.1.36) takes the form

$$\begin{aligned} \dot{S}\,(-\mathrm{i}\,d/dt)\,(\psi - v) = h(t)\dot{\mathfrak{T}}_R^{-1}\Big[&\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} \\ &- \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta} f)d\eta\Big], \end{aligned} \tag{1.1.37}$$

where $h(t) = \{h_{\varrho\sigma k}(t)\}$ is the $1 \times mN$ matrix with entries defined by (1.1.35) for $t \geqslant 0$ and $h(t) = 0$ for $t < 0$. We also point out that the matrix $\dot{\mathfrak{T}}_R^{-1}$ in (1.1.37) can

[11] The existence of the right inverse matrices follows from the assumptions about the matrix $T(\tau)$ formulated in the lemma to be proved, since under these assumptions we have rg $\dot{\mathfrak{T}} = N$.

be replaced by any right inverse of the matrix $\dot{\mathfrak{T}}$. (The latter is equivalent to replacing $\psi^0_{\varrho\sigma k}$ in (1.1.36) by another solution of system (1.1.28). This is legitimate by virtue of the implication (1.1.28) $\Rightarrow$ (1.1.27)). Since the vector function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ is arbitrarily chosen, it follows that the product $h(t)\dot{\mathfrak{T}}_R^{-1}$ does not depend on the choice of the matrix $\dot{\mathfrak{T}}_R^{-1}$.

Let $H(\tau) = i(2\pi)^{1/2}\dot{\mathscr{P}}_+(\tau)F_{t\to\tau}h(t)$ or, what is the same,

$$H(\tau) = \{H_{\varrho\sigma k}(\tau)\}, \quad H_{\varrho\sigma k}(\tau) = \sum_{\beta=0}^{\sigma} \frac{\sigma!}{\beta!}\dot{S}_k^{(\beta)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_+(\tau)}{(\tau-\zeta_\varrho)^{\sigma+1-\beta}} \tag{1.1.38}$$
$$(1 \leqslant \varrho \leqslant l;\ 0 \leqslant \sigma \leqslant k_\varrho - 1;\ 1 \leqslant k \leqslant m).$$

We set

$$\dot{G}(\tau) = H(\tau)\dot{\mathfrak{T}}_R^{-1}. \tag{1.1.39}$$

It follows from (1.1.38) and (1.1.39) that entries of the $1 \times N$ matrix $\dot{G}(\tau) = \{\dot{G}_1(\tau), \dots, \dot{G}_N(\tau)\}$ are polynomials and $\max_\alpha \operatorname{ord} \dot{G}_\alpha(\tau) \leqslant N-1$. Taking into account (1.1.37), we arrive at (1.1.31). □

Remark 1.1.13. If the conditions of Lemma 1.1.12 are satisfied, then the matrix $\dot{G}$ does not depend on choice of the matrix $\dot{\mathfrak{T}}_R^{-1}$. This follows immediately from the analogous property of the matrix $h(t)\dot{\mathfrak{T}}_R^{-1}$ and equations (1.1.38) and (1.1.39).

Remark 1.1.14. The representation (1.1.31) remains valid when ψ is a solution of the system $\dot{\mathscr{P}}_+(-\mathrm{i}\,d/dt)\,I\psi = 0$. Indeed, setting $f = 0$ and $v = 0$, we can repeat the remaining part of the proof of Lemma 1.1.12 without any changes.

1.1.7 Properties of the matrix $G(\tau)$

In this subsection, we study some connections between the integral representation (1.1.31) and the identity (1.1.40). Corresponding results are contained in Lemmas 1.1.15 and 1.1.17, respectively.

Lemma 1.1.15. *Let the assumptions of Lemma 1.1.12 be satisfied. Assume also that $G(\tau) = \mathscr{M}(\tau)\dot{G}(\tau)$, where $\dot{G}(\tau)$ is the $1 \times N$ matrix defined by (1.1.39). Then*

$$G(\tau)T_+(\eta) = (\eta-\tau)^{-1}\left[\mathscr{P}_+(\eta)S_+(\tau) - \mathscr{P}_+(\tau)S_+(\eta)\right] \tag{1.1.40}$$

for all $\tau, \eta \in \mathbb{R}^1$. Here T_+ and S_+ are the matrices defined by (1.1.3).

Proof. From (1.1.7) and (1.1.3) it follows that

$$T_+(\tau) \equiv 0 \,(\operatorname{mod} \mathscr{M}(\tau)) \quad \text{and} \quad S_+(\tau) \equiv 0 \,(\operatorname{mod} \mathscr{M}(\tau)).$$

Therefore, it suffices to verify that for all $\tau, \eta \in \mathbb{R}^1$ one has

$$\dot{G}(\tau)\dot{T}_+(\eta) = (\eta-\tau)^{-1}\left[\dot{\mathscr{P}}_+(\eta)\dot{S}_+(\tau) - \dot{\mathscr{P}}_+(\tau)\dot{S}_+(\eta)\right], \tag{1.1.41}$$

where $\dot{T}_+ = T_+/\mathscr{M}$, $\dot{S}_+ = S_+/\mathscr{M}$, and $\dot{G}$ is the matrix defined by (1.1.39).

We show that

$$\dot{G}(\tau)\dot{\mathfrak{T}}_+^{\varrho\sigma k} = \sum_{\beta=0}^{\sigma} \frac{\sigma!}{(\sigma-\beta)!}\dot{S}_+^{(\sigma-\beta)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_+(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}} \tag{1.1.42}$$

for all $\tau \in \mathbb{R}^1$ and for all ϱ and σ satisfying $1 \leqslant \varrho \leqslant l$ and $0 \leqslant \sigma \leqslant k_\varrho - 1$, respectively. Here,

$$\dot{\mathfrak{T}}_+^{\varrho\sigma k} = \{\dot{T}_{\alpha k_+}^{(\sigma)}(\zeta_\varrho)\}$$
$$(\alpha = 1,\dots,N;\ \varrho = 1,\dots,l;\ \sigma = 0,\dots,k_\varrho - 1;\ k = 1,\dots,m) \tag{1.1.43}$$

the $\varrho\sigma k$-th column of the $N \times mN$ matrix constructed from the matrix $\dot{T}_+$ in the same way as the matrix $\dot{\mathfrak{T}}$ was constructed from the matrix T (cf. (1.1.24)). For this purpose, we substitute into (1.1.31) the vector functions $\psi^k = (\psi_{1k},\dots,\psi_{mk})$[12], whose components are defined as follows:

$$\psi_{jk} \equiv 0 \quad \text{for} \quad j \neq k, \qquad \psi_{kk} = 0 \quad \text{for} \quad t < 0$$
$$\psi_{kk} = (it)^\gamma \exp(i\zeta_\varrho t) \quad \text{for} \quad t \geqslant 0$$
$$(k = 1,\dots,m;\ \varrho = 1,\dots,l;\ \gamma = 0,\dots,k_\varrho - 1).$$

Applying the Fourier transform, we find that for all $\tau \in \mathbb{R}^1$

$$\dot{G}(\tau)\dot{\mathfrak{T}}^{\varrho\gamma k} = \sum_{\beta=0}^{\gamma} \frac{\gamma!}{(\gamma-\beta)!}\dot{S}_k^{(\gamma-\beta)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_+(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}}$$
$$(1 \leqslant \varrho \leqslant l;\ 0 \leqslant \gamma \leqslant k_\varrho - 1). \tag{1.1.44}$$

Here $\dot{\mathfrak{T}}^{\varrho\gamma k}$ is the $\varrho\gamma k$-th column of the matrix (1.1.24).

It follows from identities (1.1.3) that

$$\dot{T}_+^{(\sigma)}(\zeta_\varrho) = \sum_{\gamma=0}^{\sigma} \frac{\sigma!}{\gamma!(\sigma-\gamma)!}\dot{T}^{(\gamma)}(\zeta_\varrho)\left(\frac{1}{\mathscr{P}_-}\right)^{(\sigma-\gamma)}(\zeta_\varrho)$$

and

$$\dot{S}_+^{(\sigma-\beta)}(\zeta_\varrho) = \sum_{\gamma_1=0}^{\sigma-\beta} \frac{(\sigma-\beta)!}{\gamma_1!(\sigma-\beta-\gamma_1)!}\dot{S}^{(\gamma_1)}(\zeta_\varrho)\left(\frac{1}{\mathscr{P}_-}\right)^{(\sigma-\beta-\gamma_1)}(\zeta_\varrho).$$

[12] The possibility of such substitution follows from Remark 1.1.14.

Using these relations and equations (1.1.44), we obtain

$$\begin{aligned}\dot{G}(\tau)\dot{\mathfrak{T}}_{+}^{\varrho\sigma k} &= \sum_{\beta=0}^{\sigma}\sum_{\gamma=\beta}^{\sigma}\frac{\sigma!}{(\gamma-\beta)!(\sigma-\gamma)!}\left(\frac{1}{\mathscr{P}_{-}}\right)^{(\sigma-\gamma)}(\zeta_\varrho)\dot{S}_k^{(\gamma-\beta)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_{+}(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}}\\ &= \sum_{\beta=0}^{\sigma}\sum_{\gamma_1=0}^{\sigma-\beta}\frac{\sigma!}{\gamma_1!(\sigma-\beta-\gamma_1)!}\left(\frac{1}{\mathscr{P}_{-}}\right)^{(\sigma-\beta-\gamma_1)}(\zeta_\varrho)\dot{S}_k^{(\gamma_1)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_{+}(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}}\\ &= \sum_{\beta=0}^{\sigma}\frac{\sigma!}{(\sigma-\beta)!}\dot{S}_{k+}^{(\sigma-\beta)}(\zeta_\varrho)\frac{\dot{\mathscr{P}}_{+}(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}},\end{aligned}$$

which is exactly equation (1.1.42).

Since the entries of the matrices $\dot{T}_{+}(\eta)$ and $\dot{S}_{+}(\eta)$ are polynomials of degree at most $N-1$, identity (1.1.41) follows from (1.1.42). □

Remark 1.1.16. If the matrix T satisfies (1.1.7) and its rows are linearly independent modulo $\mathscr{P}_{+}$, then the matrix $G(\tau) = \{G_1(\tau), \dots, G_N(\tau)\}$, which satisfies (1.1.40) and the relation $G(\tau) \equiv 0 \,(\mathrm{mod}\,\mathscr{M}(\tau))$ and consists of the polynomial entries such that $\max_\alpha \operatorname{ord} G_\alpha(\tau) \leqslant N-1+\operatorname{ord}\mathscr{M}(\tau)$, is uniquely determined. Indeed, the right-hand side of (1.1.41) is the $1\times m$ matrix, whose elements are polynomials w.r.t. η of degree at most $N-1$. Since the rows of the matrix $\dot{T}_{+}(\eta)$, consisting of polynomials of degree at most $N-1$, are linearly independent, the coefficients of the expansion of the right-hand side of (1.1.41) with respect to these rows (if such an expansion is possible) are uniquely determined.

Lemma 1.1.17. *Let the* $1\times N$ *matrix* $G(\tau) = \{G_\alpha(\tau)\}$ *with polynomial entries satisfy the conditions* $G(\tau) \equiv 0\,(\mathrm{mod}\,\mathscr{M}(\tau))$ *and* $\max_\alpha \operatorname{ord} G_\alpha(\tau) \leqslant N-1+\operatorname{ord}\mathscr{M}(\tau)$ *as well as identity* (1.1.40). *Then* $\ker\dot{\mathfrak{T}} \subset \ker\dot{\mathfrak{G}}$, *where* $\dot{\mathfrak{T}}$ *and* $\dot{\mathfrak{G}}$ *are the matrices* (1.1.23) *and* (1.1.24), *respectively. If, in addition, the other assumptions of Lemma* 1.1.12 *are in force, then the matrix* $\dot{G} = G/\mathscr{M}$ *admits representation* (1.1.39) *and, consequently, equality* (1.1.31) *holds for all vector-valued functions* $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_{+}^1)$.

Proof. Dividing both sides of identity (1.1.40) by $\mathscr{M}(\tau)\mathscr{M}(\eta)$, we see that the matrix $\dot{G}$ satisfies (1.1.41), and, consequently, conditions (1.1.42). Using relation (1.1.3) we can express $\dot{T}^{(\gamma)}(\zeta_\varrho)$ and $\dot{S}^{(\gamma_1)}(\zeta_\varrho)$ in terms of $\dot{T}_{+}^{(\sigma)}(\zeta_\varrho)$ and $\dot{S}_{+}^{(\sigma-\beta)}(\zeta_\varrho)$, respectively. The latter guarantees that $\dot{G}$ satisfies equations (1.1.44)[13].

Let $\Psi = (\psi_{\varrho\sigma k})$ be an arbitrary solution of system (1.1.28). We show that Ψ is also a solution of system (1.1.27). To this end, using (1.1.44) and definition (1.1.23),

[13]The computation justifying this is completely analogous to one in the proof of Lemma 1.1.15, where equations (1.1.42) are derived from (1.1.44). For this reason, we do not repeat this computation here.

we observe that for all $\tau \in \mathbb{R}^1$,

$$0 = \dot{G}(\tau)\dot{\mathfrak{T}}\Psi$$
$$= \sum_{k=1}^{m}\sum_{\varrho=1}^{l}\sum_{\gamma=0}^{k_\varrho-1}\sum_{\beta=0}^{\gamma}\frac{\gamma!}{(\gamma-\beta)!}\dot{S}_k^{(\gamma-\beta)}(\zeta_\varrho)\psi_{\varrho\gamma k}\frac{\dot{\mathscr{P}}_+(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}} = E(\tau)\dot{\mathfrak{G}}\Psi,$$

where $E(\tau) = \{E_{\varrho\beta}(\tau)\}$ is the $1\times N$ matrix whose entries are the polynomials given by

$$E_{\varrho\beta}(\tau) = \frac{\dot{\mathscr{P}}_+(\tau)}{(\tau-\zeta_\varrho)^{\beta+1}} \qquad (1 \leqslant \varrho \leqslant l;\ 0 \leqslant \beta \leqslant k_\varrho - 1). \tag{1.1.45}$$

Since polynomials (1.1.45) are obviously linearly independent, we have $\dot{\mathfrak{G}}\Psi = 0$, and, consequently, $\ker\dot{\mathfrak{T}} \subset \ker\dot{\mathfrak{G}}$.

Suppose now that the other conditions of Lemma 1.1.12 are satisfied. We show that the matrix $G/\mathscr{M}$ admits the representation (1.1.39). Let $H(\tau)$ be the matrix (1.1.38), and let $\dot{\mathfrak{T}}_R^{-1}$ be an arbitrary right-inverse matrix of $\dot{\mathfrak{T}}$. By Lemma 1.1.15, the matrix $\mathscr{M}(\tau)H(\tau)\dot{\mathfrak{T}}_R^{-1}$ satisfies identity (1.1.40). But then, in accordance with Remark 1.1.14, we get $G(\tau) = \mathscr{M}(\tau)H(\tau)\dot{\mathfrak{T}}_R^{-1}$. □

1.1.8 A quadratic functional

In this subsection we calculate the norm of a quadratic functional in the Hilbert space (Lemma 1.1.12). The obtained result is fundamental for the proof of the lower bound (1.1.49) for the sharp constant Λ in inequality (1.1.1) (see Subsection 1.1.9).

Lemma 1.1.18. *Let $[\cdot,\cdot]$ denote the scalar product in the Hilbert space $\mathscr{H}$, let g, $a_1,\dots,a_N$, $b_1,\dots,b_N \in \mathscr{H}$, and let*

$$\Phi(g) = \frac{1}{[g,g]}\left[\sum_{\alpha=1}^{N}[g,a_\alpha]b_\alpha, \sum_{\alpha=1}^{N}[g,a_\alpha]b_\alpha\right].$$

Then $\sup\{\Phi(g) : g \in \mathscr{H}\}$ is equal to the largest eigenvalue of the matrix $\mathfrak{A}\mathfrak{B}$, where the $N\times N$ matrices $\mathfrak{A}$ and $\mathfrak{B}$ are defined by $\mathfrak{A} = \{[a_\alpha,a_\beta]\}$ and $\mathfrak{B} = \{[b_\alpha,b_\beta]\}$, respectively.

Proof. Consider the operator $K : \mathscr{H} \to \mathscr{H}$ defined by the formula

$$Kg = \sum_{\alpha=1}^{N}[g,a_\alpha]b_\alpha, \qquad g \in \mathscr{H}.$$

Since this operator is finite-dimensional, $\sup\{\Phi(g) : g \in \mathscr{H}\}$ is attained at some $g_0 \in \mathscr{H}$. Let $\lambda = \Phi(g_0)$. Varying $\Phi(g)$ w.r.t. $g \in \mathscr{H}$, we get

$$\sum_{\alpha,\beta=1}^{N}[b_\beta,b_\alpha][g_0,a_\beta]a_\alpha = \lambda g_0,$$

and, consequently,

$$\sum_{\alpha,\beta=1}^{N} [a_j, a_\alpha][b_\alpha, b_\beta][a_\beta, g_0] = \lambda[a_j, g_0] \qquad (1 \leqslant j \leqslant N). \tag{1.1.46}$$

From the definitions of the matrices $\mathfrak{U}$, $\mathfrak{B}$ it follows that all the eigenvalues of the matrix $\mathfrak{U}\mathfrak{B}$ are nonnegative. Let λ_0 be the largest of these eigenvalues. Then, by (1.1.46), we have $\lambda \leqslant \lambda_0$.

Now let us show that $\lambda \geqslant \lambda_0$. For this purpose, we denote by $\gamma = (\gamma_1, \dots, \gamma_N)$ an arbitrary eigenvector of the matrix $(\mathfrak{U}\mathfrak{B})^*$ corresponding to λ_0 and set

$$g^* = \sum_{j=1}^{N} \bar{\gamma}_j a_j.$$

Then, on the one hand, we have

$$\begin{aligned}
\left[\sum_{\alpha=1}^{N}[g^*, a_\alpha], \sum_{\alpha=1}^{N}[g^*, a_\alpha]\right] &= \sum_{\alpha,\beta,k,j=1}^{N} \bar{\gamma}_j [a_j, a_\alpha]\gamma_k \overline{[a_k, a_\beta]}[b_\alpha, b_\beta] \\
&= \sum_{\alpha,j=1}^{N} \sum_{\beta,k=1}^{N} [b_\alpha, b_\beta][a_\beta, a_k]\gamma_k [a_j, a_\alpha]\bar{\gamma}_j \\
&= \lambda_0 \sum_{\alpha,j=1}^{N} [a_j, a_\alpha]\bar{\gamma}_j \gamma_\alpha,
\end{aligned}$$

while on the other hand,

$$[g^*, g^*] = \sum_{\alpha,j=1}^{N} [a_j, a_\alpha]\bar{\gamma}_j \gamma_\alpha.$$

Therefore, $\Phi(g^*) = \lambda_0$ and $\lambda \geqslant \lambda_0$. □

1.1.9 Necessary and sufficient conditions for the validity of inequality (1.1.1)

We now state the main result of Section 1.1.

Theorem 1.1.19. *Let* $N \geqslant 1$. *The estimate* (1.1.1) *holds true for some* $\Lambda < \infty$ *and all* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ *if and only if the following conditions are satisfied:*

1. *The matrix* $S(\tau)$, *defined by* (1.1.2), *satisfies the inequality* $\sup |S(\tau)/\mathscr{P}(\tau)| < \infty$.
2. *The matrix* $T(\tau)$, *defined by* (1.1.2), *satisfies relation* (1.1.7).
3. *The rows of the matrix* $T(\tau)$ *are linearly independent modulo* $\mathscr{P}_+(\tau)$.
4. *There exists a uniquely determined polynomial* $1 \times N$ *matrix* $G(\tau) = \{G_\alpha(\tau)\}$ *such that*
$$\max_\alpha \text{ ord } G_\alpha(\tau) \leqslant N - 1 + \text{ord } \mathscr{M}(\tau).$$
This matrix satisfies the congruence $G(\tau) \equiv 0 \ (\text{mod}\, \mathscr{M}(\tau))$, *and identity* (1.1.40) *holds for all* $\tau, \eta \in \mathbb{R}^1$.

Moreover, the sharp constant Λ *in* (1.1.1) *obeys the estimates:*

$$C_1\Lambda \leqslant \sup \left|\frac{S(\tau)}{\mathscr{P}(\tau)}\right|^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{G(\tau)T_-(\eta)}{\mathscr{P}_+(\tau)\mathscr{P}_-(\tau)}\right|^2 d\tau d\eta + \int_{-\infty}^{\infty} \left|\frac{G(\tau)}{\mathscr{P}_+(\tau)}\right|^2 d\tau \leqslant C_2\Lambda, \tag{1.1.47}$$

where T_- *and* G *are the matrices defined by the identities* (1.1.3) *and* (1.1.40), *respectively.*

Proof. Necessity. Suppose the estimate (1.1.1) holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Then, in accordance with Lemma 1.1.3, inequality (1.1.6) holds for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Therefore, the necessity of conditions 1 and 2 of our theorem follows from Lemmas 1.1.5 and 1.1.9, respectively. The necessity of condition 3 follows from Lemmas 1.1.4 and 1.1.10.

In the proof of Lemma 1.1.10 it was shown that the implication (1.1.27)⇒(1.1.28) or, what is the same, the inclusion (1.1.30), follows from the validity of (1.1.8). Taking this into account, we conclude that the necessity of condition 4 of our theorem follows from Remark 1.1.6, Lemmas 1.1.4, 1.1.12, 1.1.15, and Remark 1.1.16.

Sufficiency. Let $G(\tau)$ be a $1 \times N$ matrix satisfying condition 4 of our theorem. Then, conditions 1–3 of the theorem, Remark 1.1.6 and Lemma 1.1.17 guarantee that representation (1.1.31) holds for all vector functions $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Using Parseval's identity and condition 1, we observe that (1.1.31) yields the following estimate for all $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$:

$$\begin{aligned} \int_0^\infty \left|\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi\right|^2 dt \leqslant C\Bigg[\Bigg(&\sup\left|\frac{S(\tau)}{\mathscr{P}(\tau)}\right|^2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\tau)T_-(\eta)}{\mathscr{P}_+(\tau)\mathscr{P}_-(\eta)}\right|^2 d\tau d\eta\Bigg)\int_{-\infty}^{\infty}\left|\dot{\mathscr{P}}\,(-\mathrm{i}\,d/dt)\,I\,\psi\right|^2 dt \\ &+ \int_{-\infty}^{\infty}\left|\frac{G(\tau)}{\mathscr{P}_+(\tau)}\right|^2 d\tau\,\left|\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0}\right|^2\Bigg]. \end{aligned} \tag{1.1.48}$$

This estimate is obviously an inequality of the type (1.1.8).

Finally, applying Lemma 1.1.4, we get the estimate (1.1.1) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

Estimates for the sharp constant Λ. The upper bound for the sharp constant Λ in inequality (1.1.1) follows from (1.1.48) and Lemma 1.1.4.

Let us prove the lower bound in (1.1.47). Due to Lemmas 1.1.2 and 1.1.5, it suffices to check the inequalities

$$\Lambda \geqslant C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\tau)T_-(\eta)}{\mathscr{P}_+(\tau)\mathscr{P}_-(\eta)}\right|^2 d\tau d\eta \tag{1.1.49}$$

and

$$\Lambda \geqslant C\int_{-\infty}^{\infty}\left|\frac{G(\tau)}{\mathscr{P}_+(\tau)}\right|^2 d\tau. \tag{1.1.50}$$

We now prove the estimate (1.1.49). Substituting in (1.1.31) the vector-functions $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ satisfying the condition $\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0}$, we see that the integral representation takes the form

$$\begin{aligned} \dot{S}\,(-\mathrm{i}\,d/dt)\,\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}t\tau}\Bigg[&\frac{\dot{S}(\tau)}{\dot{\mathscr{P}}(\tau)}(F_{t\to\tau}f) \\ &+ \frac{\mathrm{i}}{\sqrt{2\pi}}\frac{\dot{G}(\tau)}{\dot{\mathscr{P}}_+(\tau)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{T_-(\eta)}{\mathscr{P}_-(\eta)}(F_{t\to\eta}f)d\eta\Bigg]d\tau. \end{aligned} \tag{1.1.51}$$

Since the entries of the $1 \times N$ matrix $\dot{G}(\tau)/\dot{\mathscr{P}}(\tau)$ can be continued analytically to the half-plane $\operatorname{Im}\zeta < 0$ ($\zeta = \tau + \mathrm{i}\sigma$), equation (1.1.51) implies the estimate

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \left| \frac{G(\tau)}{\mathscr{P}_+(\tau)} \int_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)} (F_{t\to\eta} f) d\eta \right|^2 d\tau \right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} \left| \frac{\dot{G}(\tau)}{\dot{\mathscr{P}}_+(\tau)} \int_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)} (F_{t\to\eta} f) d\eta \right|^2 d\tau \right)^{1/2} \\
&\leqslant \left(2\pi \int_0^{\infty} \left| \dot{S}\left(-\mathrm{i}\, d/dt\right) \psi \right|^2 dt \right)^{1/2} \\
&\quad + \left(2\pi \int_0^{\infty} \left| F^{-1}_{\tau\to t} \left[\frac{\dot{S}(\tau)}{\dot{\mathscr{P}}(\tau)} (F_{t\to\tau} f) \right] \right|^2 dt \right)^{1/2} \\
&\leqslant \sqrt{2\pi} \left[\Lambda^{1/2} + (\sup |S(\tau)/\mathscr{P}(\tau)|^2)^{1/2} \right] \int_{-\infty}^{\infty} |F_{t\to\tau} f|^2 d\tau \\
&\leqslant 2\sqrt{2\pi}\, \Lambda^{1/2} \int_{-\infty}^{\infty} |F_{t\to\tau} f|^2 d\tau,
\end{aligned} \tag{1.1.52}$$

where the last step used the inequality (1.1.9).

Consider the Hilbert space $\mathscr{H}$ of vector functions $g = (g_1, \dots, g_m)$, whose components belong to $L^2(\mathbb{R}^1)$ and admit analytic continuation to the half-plane $\operatorname{Im}\zeta < 0$ ($\zeta = \tau + \mathrm{i}\sigma$). We define a scalar product by

$$[g, h] = \sum_{k=0}^{m} \int_{-\infty}^{\infty} g_k(\tau) \overline{h_k(\tau)} d\tau, \qquad g, h \in \mathscr{H}.$$

The vector functions $g = F_{t\to\tau} f$, figuring in (1.1.52), are obviously dense in $\mathscr{H}$.

We consider also the vector functions $a_\alpha(\tau) = (a_{\alpha 1}(\tau), \dots, a_{\alpha m}(\tau))$ and $b_\alpha(\tau) = (b_{\alpha 1}(\tau), \dots, b_{\alpha m}(\tau)) \in \mathscr{H}$ ($\alpha = 1, \dots, N$), where $a_{\alpha k}(\tau) = \overline{T}_{\alpha k-}(\tau)/\overline{\mathscr{P}_-}(\tau)$ and $b_{\alpha k}(\tau) = G_\alpha(\tau)/\mathscr{P}_+(\tau)$ ($\alpha = 1, \dots, N; k = 1, \dots, m$). Here, $T_{\alpha k-}(\tau)$ and $G_\alpha(\tau)$ are the entries of the $N \times m$ matrix T_- and the $1 \times N$ matrix $G(\tau)$, respectively. Suppose that

$$\Phi(g) = \frac{1}{[g, g]} \left[\sum_{\alpha=1}^{N} [g, a_\alpha] b_\alpha, \sum_{\alpha=1}^{N} [g, a_\alpha] b_\alpha \right], \qquad g \in \mathscr{H}.$$

The inequality (1.1.52) can be recast as

$$\Phi(g) \leqslant 8\pi\Lambda m.$$

Then the estimate (1.1.49) follows directly from Lemma 1.1.18, if we observe that the double integral on the right-hand side is equal to $m^{-1}\mathrm{tr}(\mathfrak{U}\mathfrak{B})$.

It remains to prove inequality (1.1.50). Using Remark 1.1.14, we substitute in (1.1.51) a solution ψ^0 of the system $\dot{\mathscr{P}}_+(-\mathrm{i}\,d/dt)\,I\psi = 0$. Then representation (1.1.31) takes the form

$$\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi^0 = -\frac{i}{\sqrt{2\pi}} F^{-1}_{\tau\to t}\left(\frac{\dot{G}(\tau)}{\dot{\mathscr{P}}_+(\tau)}\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi^0\big|_{t=0}\right). \tag{1.1.53}$$

On the other hand, substituting ψ^0 in the estimate (1.1.8), we obtain

$$\int_0^\infty \left|\dot{S}\,(-\mathrm{i}\,d/dt)\,\psi^0\right|^2 dt \leqslant \Lambda \left|\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi^0\big|_{t=0}\right|^2. \tag{1.1.54}$$

Let $\dot{\mathfrak{T}}$ be the matrix (1.1.24). Since $\mathrm{rg}\,\dot{\mathfrak{T}} = N$ (see the proof of Lemma 1.1.10), the mapping $\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi^0\big|_{t=0}$ of the space of solutions of the system $\dot{\mathscr{P}}_+(-\mathrm{i}\,d/dt)\,I\psi = 0$ into the space $\mathbb{C}^N$ is surjective. Therefore, inequality (1.1.50) follows from (1.1.53) and (1.1.54). □

1.1.10 On condition 4 of Theorem 1.1.19

We have already noted in Subsection 1.0.2 that in the case $m = 1$ condition 4 of Theorem 1.1.19 follows from the other conditions of this theorem. This assertion was proved there under the additional assumption $\mathscr{M}(\xi;\tau) = 1$. To remove this restriction, it is sufficient to divide both sides of (1.0.24) by $\mathscr{M}(\xi;\tau)\mathscr{M}(\xi;\eta)$ and repeat for the resulting equality all the subsequent arguments.

If $m > 1$, then, in general, condition 4 of Theorem 1.1.19 does not follow from the other conditions of this theorem, as shown by the following example:

Example 1.1.20. Let $m = 2$, $R(\tau) = \{1,1\}$ and $P(\tau) = (\tau - \zeta)I$ with $\zeta \in \mathbb{C}^1$ and $\mathrm{Im}\,\zeta > 0$. Then we obviously have $S(\tau) = \{\tau - \zeta, \tau - \zeta\}$ and $\mathscr{P}_+(\tau) = \mathscr{P}(\tau) = (\tau - \zeta)^2$, and, consequently, $\mathscr{M}(\tau) = \dot{\mathscr{P}}_+(\tau) = \tau - \zeta$, $N = 1$ and $\dot{S}(\tau) = \{1,1\}$. We set $Q(\tau) = \{1,0\}$. Since $T(\tau) = \{\tau - \zeta, 0\}$, we get $\dot{T}(\tau) = \{1,0\}$. In view of definitions (1.1.23) and (1.1.24) we find: $\dot{\mathfrak{G}} = \{1,1\}$ and $\dot{\mathfrak{T}} = \{1,0\}$. However, the latter means that $\ker\dot{\mathfrak{T}}$ is not a subset of $\ker\dot{\mathfrak{G}}$, and, consequently (see Lemma 1.1.15), condition 4 of Theorem 1.1.19 is not fulfilled. At the same time, it is clear that all the other conditions of this theorem are satisfied.

The question of when condition 4 of Theorem 1.1.19 follows from other conditions of this theorem is completely answered by Lemmas 1.1.15 and 1.1.17. Indeed, these lemmas imply

Proposition 1.1.21. *Let $\dot{\mathfrak{G}}$ and $\dot{\mathfrak{T}}$ be the matrices defined by equations* (1.1.23) *and* (1.1.24)*, respectively. Then condition* 4 *of Theorem* 1.1.19 *follows from conditions* 1–3 *if and only if the rows of the matrix $\dot{\mathfrak{G}}$ belong to the linear span of the rows of the matrix $\dot{\mathfrak{T}}$.*

In the context of this proposition, it would be appropriate to identify some easily verifiable sufficient conditions for the validity of the inclusion $\ker \dot{\mathfrak{T}} \subset \ker \dot{\mathfrak{G}}$ in the case $m > 1$. A result of this type is Proposition 1.1.25, established below. Its proof is based on the following lemma.

Lemma 1.1.22. *Let $\mathcal{U}$ be the subspace of solutions of the system $P\,(-\mathrm{i}\,d/dt)\,u = 0$ defined by*

$$\mathcal{U} = \Big\{u : \ u = P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi, \dot{\mathscr{P}}_{+}\,(-\mathrm{i}\,d/dt)\,I\varphi = 0\Big\}. \tag{1.1.55}$$

If the polynomials $\mathscr{M}(\tau)$ and $\dot{\mathscr{P}}_{+}(\tau)$ are relatively prime, then $\dim \mathcal{U} = \operatorname{ord}\ \dot{\mathscr{P}}_{+}(\tau) = N$.

Proof. Since $P(\tau)$ is a polynomial $m \times m$ matrix, there exist polynomial $m \times m$ matrices $A(\tau)$ and $B(\tau)$ such that $\det A = \mathrm{const} \neq 0$, $\det B = \mathrm{const} \neq 0$, and the matrix $L = APB$ is diagonal. A direct verification shows that $P^{\mathrm{c}} = BKA$, where $K(\tau) = \mathscr{P}(\tau)L^{-1}(\tau)$.

Consider the subspace of vector-functions

$$\mathcal{V} = \Big\{\varphi : \dot{\mathscr{P}}_{+}\,(-\mathrm{i}\,d/dt)\,I\varphi = 0\Big\}. \tag{1.1.56}$$

In accordance with (1.1.55), we have

$$\mathcal{U} = B\,(-\mathrm{i}\,d/dt)\,K\,(-\mathrm{i}\,d/dt)\,A\,(-\mathrm{i}\,d/dt)\,\mathcal{V}.$$

Let $\mathcal{V}_1 = K\,(-\mathrm{i}\,d/dt)\,A\,(-\mathrm{i}\,d/dt)\,\mathcal{V}$. Since $\det B = \mathrm{const} \neq 0$, it follows that $\dim \mathcal{V}_1 = \dim \mathcal{U}$.

Now we show that $\dim \mathcal{V}_1 = \operatorname{ord}\ \dot{\mathscr{P}}_{+}(\tau)$. Because $\det A = \mathrm{const} \neq 0$, a basis of the subspace $A\,(-\mathrm{i}\,d/dt)\,\mathcal{V}$ is provided by the vector functions with components $\exp(\mathrm{i}\zeta_\varrho t)(\mathrm{i}t)^\sigma$, where $1 \leqslant \varrho \leqslant l$ and $0 \leqslant \sigma \leqslant k_\varrho - 1$ (here we use the notation introduced on page 16).

We denote by $K_j(\tau)$ the diagonal elements of the matrix K, and by $L_j(\tau)$ the diagonal elements of the matrix L, respectively. Let $\gamma_{j\varrho}$ denote the multiplicity of a root ζ_ϱ of the polynomial K_j. Since $K_j = \mathscr{P}/L_j$ and (as $\mathscr{M}$ and $\dot{\mathscr{P}}_{+}$ are relatively prime) ζ_ϱ is not a root of the polynomial $\mathscr{M}$, we have the relations $\gamma_{j\varrho} \leqslant k_\varrho - 1$ ($\varrho = 1, \dots, l$; $j = 1, \dots, m$). We also note that

$$\begin{aligned} &K_j\,(-\mathrm{i}\,d/dt)\big[(\mathrm{i}t)^\sigma \exp(\mathrm{i}\zeta_\varrho t)\big] \\ &= \begin{cases} 0, & \text{if } \sigma < \gamma_{j\varrho} \\ \displaystyle\sum_{\gamma=\gamma_{j\varrho}}^{\sigma} \frac{\sigma!}{\gamma!(\sigma-\gamma)!} K_j^{(\gamma)}(\zeta_\varrho)(\mathrm{i}t)^{\sigma-\gamma} \exp(\mathrm{i}\zeta_\varrho t), & \text{if } \sigma \geqslant \gamma_{j\varrho}. \end{cases} \end{aligned} \tag{1.1.57}$$

Therefore,

$$\dim \mathcal{V}_1 = \sum_{j=1}^{m}\sum_{\varrho=1}^{l}(k_\varrho - \gamma_{j\varrho}) = m \text{ ord } \dot{\mathscr{P}}_+ - \sum_{j=1}^{m}\sum_{\varrho=1}^{l}\gamma_{j\varrho}.$$

On the other hand, since the polynomials $\mathscr{M}$ and $\dot{\mathscr{P}}_+$ are relatively prime and since

$$\prod_{j=1}^{m} K_j(\tau) = \text{const} \det P^{\mathrm{c}}(\tau) = \text{const}\,[\mathscr{P}(\tau)]^{m-1}, \tag{1.1.58}$$

we arrive at

$$\sum_{j=1}^{m}\sum_{\varrho=1}^{l}\gamma_{j\varrho} = (m-1) \text{ ord } \dot{\mathscr{P}}_+.$$

Using the last equality, we get $\dim \mathcal{V}_1 = \text{ord } \dot{\mathscr{P}}_+$. □

Remark 1.1.23. The assumption that the polynomials $\mathscr{M}$ and $\dot{\mathscr{P}}_+$ are relatively prime is not necessary for the equality $\dim \mathcal{U} = \text{ord } \dot{\mathscr{P}}_+$, as the following example shows.

Example 1.1.24. Let $m = 3$ and $\zeta \in \mathbb{C}^1$ with $\operatorname{Im}\zeta > 0$. Consider the 1×3 matrix $R = \{0, 0, \tau-\zeta\}$ and suppose that P is a a diagonal matrix with the diagonal elements 1, 1, and $(\tau-\zeta)^2$. Then $\mathscr{P}(\tau) = \mathscr{P}_+(\tau) = (\tau-\zeta)^2$, the diagonal elements of the matrix P^{c} are equal to $(\tau-\zeta)^2$, $(\tau-\zeta)^2$, 1, and $S = \{0, 0, \tau-\zeta\}$. Therefore, we have $\mathscr{M}(\tau) = \dot{\mathscr{P}}_+(\tau) = \tau - \zeta$. Let $\mathcal{V}$ be the subspace (1.1.56). It is obvious that $\dim \mathcal{V} = 3$ and as a basis we can take the vector functions $(\exp(\mathrm{i}\zeta t), 0, 0)$, $(0, \exp(\mathrm{i}\zeta t), 0)$, $(0, 0, \exp(\mathrm{i}\zeta t))$. Thus, we obtain the relations

$$P^{\mathrm{c}}(-\mathrm{i}\, d/dt)\,(\exp(\mathrm{i}\zeta t), 0, 0) = (0, 0, 0), \qquad P^{\mathrm{c}}(0, \exp(\mathrm{i}\zeta t), 0) = (0, 0, 0),$$

and $P^{\mathrm{c}}(-\mathrm{i}\, d/dt)\,(0, 0 \exp(\mathrm{i}\zeta t)) = (0, 0, 0)$. Therefore,

$$\dim \mathcal{U} = \text{ord } \dot{\mathscr{P}}_+ = 1.$$

On the other hand, if the polynomials $\mathscr{M}(\tau)$ and $\dot{\mathscr{P}}(\tau)$ are not relatively prime, we can not guarantee that $\dim \mathcal{U} = \text{ord } \dot{\mathscr{P}}_+$. For instance, in Example 1.1.20 the matrix $P^{\mathrm{c}} = (\tau-\zeta)I$ coincides with the matrix $\dot{\mathscr{P}}_+(\tau)I$. Hence $\dim \mathcal{U} = 0$, while $\text{ord } \dot{\mathscr{P}}_+ = 1$.

Proposition 1.1.25. *If the polynomials $\mathscr{M}(\tau)$ and $\dot{\mathscr{P}}_+(\tau) = \mathscr{P}_+(\tau)/\mathscr{M}(\tau)$ are relatively prime, then condition* 4 *of Theorem* 1.1.19 *follows from conditions* 1–3 *of this theorem. In particular, this assertion holds true if all roots of the polynomial $\mathscr{P}_+(\tau)$ are simple or if $\mathscr{M}(\tau) = 1$.*

Proof. We consider the homomorphism $\mathcal{Q}$ which associates to each element u of the subspace (1.1.55) the vector

$$\mathcal{Q} = Q\,(-\mathrm{i}\,d/dt)\,u|_{t=0}$$

from space $\mathbb{C}^N$. Here $Q(\tau)$ is the $N \times m$ matrix figuring in (1.1.1). Proposition 1.1.25 follows from the facts that the homomorphism $\mathcal{Q}(u)$ is even an isomorphism, if the polynomials $\mathscr{M}(\tau)$ and $\dot{\mathscr{P}}_+(\tau)$ are relatively prime and conditions 2–3 of Theorem 1.1.19 are fulfilled. First, we prove the latter assertion.

Let $\mathcal{Q}$ be an arbitrary vector in $\mathbb{C}^N$, and let $\dot{\mathfrak{T}}$ be the matrix (1.1.24). Condition 3 of Theorem 1.1.19 means that $\operatorname{rg}\dot{\mathfrak{T}} = N$, and hence the equation $\dot{\mathfrak{T}}\Psi = \mathcal{Q}$ is solvable. We denote by Ψ an arbitrary solution of this equation. Considering $\Psi \in \mathbb{C}^{mN}$ as the vector (1.1.26), we define the vector function $\psi(t)$ by the equalities (1.1.25).

Since the polynomials $\mathscr{M}$ and $\dot{\mathscr{P}}_+$ are relatively prime, we can construct a solution φ of the system $\mathscr{M}\,(-\mathrm{i}\,d/dt)\,I\varphi = \psi$ such that $\dot{\mathscr{P}}_+\,(-\mathrm{i}\,d/dt)\,I\varphi = 0$ holds. We set $u = P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi$. In accordance with the definition of the matrix T and condition 2 of Theorem 1.1.19, we get

$$\mathcal{Q}(u) = T\,(-\mathrm{i}\,d/dt)\,\varphi|_{t=0} = \dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} = \dot{\mathfrak{T}}\Psi = \mathcal{Q}. \tag{1.1.59}$$

The latter means that $\mathcal{Q}(u) : \mathcal{U} \to \mathbb{C}^N$ is an epimorphism.

On the other hand, according to Lemma 1.1.22, we have $\dim\,\mathcal{U} = N$. Therefore, the mapping $\mathcal{Q}(u)$ is an isomorphism.

We proceed now to the proof of Proposition 1.1.25. Consider equalities (1.1.59) for $\mathcal{Q} = 0$. Since $\mathcal{Q}(u) : \mathcal{U} \to \mathbb{C}^N$, as shown above, is an isomorphism, we obtain $u = 0$. This means that

$$R\,(-\mathrm{i}\,d/dt)\,u = S\,(-\mathrm{i}\,d/dt)\,\varphi = \dot{S}\,(-\mathrm{i}\,d/dt)\,\psi = 0$$

or, equivalently (see Remark 1.1.11, p.22), $\dot{\mathfrak{S}}\Psi = 0$, where $\dot{\mathfrak{S}}$ is the matrix (1.1.23).

Thus, if the polynomials $\mathscr{M}(\tau)$ and $\dot{\mathscr{P}}_+(\tau)$ are relatively prime and conditions 2–3 of Theorem 1.1.19 are satisfied, then the implication (1.1.28)⇒(1.1.27) holds. In other words, $\ker\,\dot{\mathfrak{T}} \subset \ker\,\dot{\mathfrak{S}}$.

The validity of condition 4 of Theorem 1.1.19 can now be established by using Lemma 1.1.15 and Remarks 1.1.6 and 1.1.16. □

1.1.11 Matrix $G(\tau)$ for estimates with a "large" number of boundary operators

Up to now, we have discussed the estimate (1.1.1) with a matrix Q of boundary operators, the number of rows of which satisfies $N = \operatorname{ord}(\mathscr{P}_+/\mathscr{M})$. We note that the estimate (1.1.1) is, in general, not true for matrices Q with a smaller number of rows. Indeed, it follows from the definition of the matrix $\dot{\mathfrak{S}}$ (see (1.1.23)) that $\operatorname{rg}\dot{\mathfrak{S}} = \operatorname{ord}(\mathscr{P}_+/\mathscr{M})$. So, if $N < \operatorname{ord}(\mathscr{P}_+/\mathscr{M})$, then (1.1.28) does not (1.1.27) for every vector Ψ. Meanwhile, the validity of the implication (1.1.28)⇒(1.1.27) for an arbitrary N is necessary for the validity of (1.1.8)[14], which is equivalent to (1.1.1).

[14] See the proof of Lemma 1.1.10.

Along with the estimates containing such a minimal number of boundary operators, one can also study estimates of the type (1.1.1) where the number of rows of the matrix Q is greater than ord $(\mathscr{P}_+/\mathscr{M})$. For example, in Section 1.3 we prove Theorem 1.3.6 providing sufficient conditions for the validity of the estimate (1.3.9) in the case the number N of rows of the matrix of boundary operators is equal to ord $\mathscr{P}_+ >$ ord $(\mathscr{P}_+/\mathscr{M})$, and these rows are linearly independent modulo $\mathscr{P}_+$. (The last condition restricts naturally the number of rows of the matrix Q).

To pave the way for studying such estimates, we establish in this subsection a result (Proposition 1.1.26) which is similar to Lemmas 1.1.12 and 1.1.15. A special feature of this result is that the existence of the matrix G is independent of conditions of the type (1.1.30). The issues discussed in Subsection 1.1.10 do not arise here.

Proposition 1.1.26. *Suppose the roots of the polynomial $\mathscr{P}_+(\tau)$ are not real,* ord $\mathscr{P}_+(\tau) = N$, *and the rows of the $N \times m$ matrix $T(\tau) = Q(\tau)P^{\mathrm{c}}(\tau)$ are linearly independent modulo $\mathscr{P}_+(\tau)$. Then there exists a uniquely determined $1 \times N$ polynomial matrix $G(\tau) = \{G_1(\tau), \dots, G_N(\tau)\}$ such that* $\max_\alpha$ ord $G_\alpha(\tau) \leqslant N-1$ *and the following assertions hold true:*

1. *The representation*

$$S(-\mathrm{i}\,d/dt)\,\varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}t\tau} \left\{ \frac{S(\tau)}{\mathscr{P}(\tau)} (F_{t\to\tau} f) + \frac{\mathrm{i}}{\sqrt{2\pi}} \frac{G(\tau)}{\mathscr{P}_+(\tau)} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{T_-(\eta)}{\mathscr{P}_-(\eta)} (F_{t\to\eta} f)\, d\eta - T\,(-\mathrm{i}\,d/dt)\,\varphi|_{t=0} \right] \right\} d\tau \tag{1.1.60}$$

   is valid for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Here $f(t) = \mathscr{P}\,(-\mathrm{i}\,d/dt)\,I\varphi$ for $t \geqslant 0$, and $f(t) = 0$ for $t < 0$.

2. *The identity (1.1.40) is valid for all $\tau, \eta \in \mathbb{R}^1$.*

Proof. Consider

$$\mathcal{W} = \{u : u = P^{\mathrm{c}}\,(-\mathrm{i}\,d/dt)\,\varphi,\ \mathscr{P}_+\,(-\mathrm{i}\,d/dt)\,I\varphi = 0\}\,,$$

which is a subspace of solutions of the system $P\,(-\mathrm{i}\,d/dt)\,u = 0$. The definition of $\mathcal{W}$ is obtained from the definition of the subspace (1.1.55) by replacing $\dot{\mathscr{P}}_+$ by $\mathscr{P}_+$. Further, if in the proof of Lemma 1.1.22 we replace $\dot{\mathscr{P}}_+$ by $\mathscr{P}_+$, l by l_1, k_ϱ by $\varkappa_\varrho$, and take into account the fact that polynomials $\mathscr{P}_+(\tau)$ and $\mathscr{P}_-(\tau)$ are relatively prime, we obtain for the multiplicities $\gamma_{j\varrho}$ arising in the proof of this lemma the estimates $\gamma_{j\varrho} \leqslant \varkappa_\varrho - 1$ and the equality

$$\sum_{j=1}^{m} \sum_{\varrho=1}^{l_1} \gamma_{j\varrho} = (m-1) \text{ ord } \mathscr{P}_+.$$

Since $\sum_{j=1}^{m} \sum_{\varrho=1}^{l_1} \varkappa_\varrho = m \text{ ord } \mathscr{P}_+$, we get $\dim \mathcal{W} = \text{ord } \mathscr{P}_+ = N$.

Let $\mathcal{Q}(u)$ be the homomorphism of $\mathcal{W}$ into $\mathbb{C}^n$, defined by the relations $\mathcal{Q}(u) = Q\,(-\mathrm{i}\, d/dt)\, u|_{t=0} = T\,(-\mathrm{i}\, d/dt)\, \varphi|_{t=0}$. From the linear independence modulo $\mathscr{P}_+$ of the rows of the matrix $T(\tau)$ and the equality $\dim \mathcal{W} = N$ it follows that $\mathcal{Q}(u)$ is also an isomorphism. Therefore, we have $u = 0$ and $R\,(-\mathrm{i}\, d/dt)\, u = S\,(-\mathrm{i}\, d/dt)\, \varphi = 0$, provided that $T\,(-\mathrm{i}\, d/dt)\, \varphi|_{t=0} = 0$.

The last conclusion is an assertion of the type of implication (1.1.28)⇒(1.1.27). In combination with the facts that the roots of the polynomial $\mathscr{P}_+$ are not real and the rows of the matrix $T(\tau)$ are linearly independent modulo $\mathscr{P}_+$, it allows us to construct a polynomial $1 \times N$ matrix $G(\tau) = \{G_1(\tau), \ldots, G_N(\tau)\}$ such that $\max_\alpha \text{ord } G_\alpha(\tau) \leqslant N-1$ and representation (1.1.60) holds for all vector-valued functions $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. The arguments justifying this are a modification of the proof of Lemma 1.1.12, where $\dot{\mathscr{P}}_+$ is replaced by $\mathscr{P}_+$, $\dot{S}$ by S and $\dot{T}$ by T, respectively. The detailed justification is left to the reader.

The representation (1.1.60) is also valid for solutions φ of the system $\mathscr{P}_+\,(-\mathrm{i}\, d/dt)\, I\varphi = 0$ (cf. Remark 1.1.14). Therefore, a further development of the case discussed above leads to an assertion of the type of Lemma 1.1.15:

The matrix G in (1.1.60) satisfies identity (1.1.40). (Here again plays an essential role the fact that the roots of the polynomial $\mathscr{P}_+(\tau)$ are not real, cf. the proof of Lemma 1.1.15).

Finally, the uniqueness of the matrix G follows from the linear independence (mod $\mathscr{P}_+$) of the rows of the matrix T. The proof is complete. □

1.1.12 Explicit representations of the matrix $G(\tau)$

We provide now two explicit representations of the $1 \times N$ matrix $G(\tau)$ figuring in condition 4 of Theorem 1.1.19.

We have already obtained one such representation in Lemma 1.1.17:

$$G(\tau) = \mathscr{M}(\tau) H(\tau) \dot{\mathfrak{T}}_R^{-1}, \tag{1.1.61}$$

where $H(\tau)$ is the $1 \times mN$ matrix (1.1.38), and $\dot{\mathfrak{T}}_R^{-1}$ is any right-inverse to the matrix (1.1.24).

Another representation for $G(\tau)$ follows directly from identity (1.1.40).

Proposition 1.1.27. *Suppose that conditions* 1–4 *of Theorem* 1.1.19 *are satisfied. Define the* $N \times N$ *matrix* $\mathscr{T}_+$ *by*

$$\mathscr{T}_+ = \int_{-\infty}^{\infty} \frac{T_+(\eta)T_+^*(\eta)}{|\mathscr{P}_+(\eta)|^2} d\eta \tag{1.1.62}$$

Then the $1 \times N$ *matrix* $G(\tau)$, *satisfying condition* 4 *of Theorem* 1.1.19, *admits the representation*

$$G(\tau) = \int_{-\infty}^{\infty} \frac{\mathscr{P}_+(\eta)S_+(\tau) - \mathscr{P}_+(\tau)S_+(\eta)}{(\eta - \tau)|\mathscr{P}_+(\eta)|^2} T_+^*(\eta)d\eta\, \mathscr{T}_+^{-1}. \tag{1.1.63}$$

Proof. Thanks to the conditions 1–2 of Theorem 1.1.19 the integral on the right-hand side of (1.1.62) converges, while condition 3 ensures the invertibility of the matrix $\mathscr{T}_+$. Now multiply from the right both sides of identity (1.1.40) by the $m \times N$ matrix $T_+^*(\eta)/|\mathscr{P}_+(\eta)|^2$ and integrate over η. Multiplying both sides of resulting identity by $\mathscr{T}_+^{-1}$, we get representation (1.1.63). □

Remark 1.1.28. Obviously, representation (1.1.63) is also valid for the $1 \times N$ matrix $G(\tau)$ figuring in Proposition 1.1.26.

1.1.13 Estimates for vector functions satisfying homogeneous boundary conditions

In this subsection, we show that the necessary and sufficient criterion for the validity of (1.1.1) coincides with the necessary and sufficient criterion for the validity of (1.1.64) for vector functions satisfying the condition $Q\,(-\mathrm{i}\,d/dt)u|_{t=0} = 0$. However, the exact constants Λ and Λ_0 in (1.1.1) and (1.1.64) are estimated in different ways.

Theorem 1.1.29. *Let* $N \geqslant 1$. *The estimate*

$$\int_0^{\infty} |R\,(-\mathrm{i}\,d/dt)\,u|^2\,dt \leqslant \Lambda_0 \int_0^{\infty} |P\,(-\mathrm{i}\,d/dt)\,u|^2\,dt \tag{1.1.64}$$

holds for some $\Lambda_0 < \infty$ *and for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ *satisfying the condition* $Q\,(-\mathrm{i}\,d/dt)u|_{t=0} = 0$, *if and only if conditions* 1–4 *of Theorem* 1.1.19 *are satisfied. The sharp constant* Λ_0 *in* (1.1.64) *obeys the estimates*

$$C_1\Lambda_0 \leqslant \sup \left|\frac{S(\tau)}{\mathscr{P}(\tau)}\right|^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{G(\tau)T_-(\eta)}{\mathscr{P}_+(\tau)\mathscr{P}_-(\eta)}\right|^2 d\tau d\eta \leqslant C_2\Lambda_0, \tag{1.1.65}$$

where S, T_- *and* G *are the matrices appearing in inequality* (1.1.47).

Proof. The necessity of the conditions of this theorem has already been established in the proof of the necessity of conditions of Theorem 1.1.19. Indeed, that proof is based on the substitution of the vector functions $u(t)$ satisfying $Q\,(-\mathrm{i}\,d/dt)\,u|_{t=0} = 0$ in (1.1.1). The latter is equivalent to inserting the vector functions $\varphi(t)$ satisfying $T\,(-\mathrm{i}\,d/dt)\,\varphi|_{t=0} = 0$, and the vector functions $\psi(t)$ satisfying $\dot{T}\,(-\mathrm{i}\,d/dt)\,\psi|_{t=0} = 0$ in (1.1.6) and (1.1.8), respectively.

The sufficiency of the conditions of the theorem and the upper bound for the sharp constant Λ_0 follow from representation (1.1.51).

Finally, we note that in the proof of the lower bounds (1.1.9) and (1.1.49) for the sharp constant Λ, we considered only vector valued functions satisfying homogeneous boundary conditions. Hence, these estimates are also valid for the constant Λ_0. □

1.1.14 Estimates for vector functions without boundary conditions

Up to now we have assumed that the number N (the number of rows of the matrix Q of boundary operators in the estimate (1.1.1) or the number of rows of the matrix of homogeneous boundary conditions to which the vector functions are subjected in the estimate (1.1.64)) is at least 1. In this subsection we consider the case $N = 0$.

Here we will show (Theorem 1.1.30) that the congruence (1.1.67) is necessary and sufficient for the validity of (1.1.66). The left-hand side of the inequality $\sup |S(\tau)/\mathscr{P}(\tau)|^2 < \infty$, which follows from (1.1.67), is just the sharp constant Λ in the estimate (1.1.66).

Theorem 1.1.30. *The inequality*

$$\int_0^\infty |R\,(-\mathrm{i}\,d/dt)\,u|^2\,dt \leqslant \Lambda \int_0^\infty |P\,(-\mathrm{i}\,d/dt)\,u|^2\,dt \tag{1.1.66}$$

holds for some $\Lambda < \infty$ *and for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, *if and only if the* $1 \times m$ *matrix* $S(\tau)$, *defined by* (1.1.2), *satisfies the congruence*

$$S(\tau) \equiv 0 \ (\mathrm{mod}\ \mathscr{P}_+(\tau)). \tag{1.1.67}$$

The sharp constant Λ *in* (1.1.66) *is equal to* $\sup |S(\tau)/\mathscr{P}(\tau)|^2 < \infty$.

Proof. It follows from Lemma 1.1.3 that the inequality (1.1.66) is valid for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ if and only if the inequality

$$\int_0^\infty |S\,(-\mathrm{i}\,d/dt)\,\varphi|^2\,dt \leqslant \Lambda \int_0^\infty |\mathscr{P}\,(-\mathrm{i}\,d/dt)\,I\varphi|^2\,dt \tag{1.1.68}$$

is valid for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

On the other hand, it is clear that (1.1.68) holds for all vector functions $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ if and only if the inequalities (perhaps with a different constant Λ_1)

$$\int_0^\infty |S_k(-\mathrm{i}\,d/dt)\,\varphi_k|^2\,dt \leqslant \Lambda_1 \int_0^\infty |\mathscr{P}(-\mathrm{i}\,d/dt)\,\varphi_k|^2\,dt \quad (k = 1,\dots,m) \tag{1.1.69}$$

are satisfied for all their components φ_k. Substituting in (1.1.69) an arbitrary solution $z(t)$ of the equation $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,z = 0$ (see Remark 1.1.8) instead of $\varphi_k(t)$, we obtain

$$S_k(-\mathrm{i}\,d/dt)\,z(t) = 0 \qquad (k = 1,\dots,m),$$

which is equivalent to (1.1.67).

Conversely, let $S(\tau) \equiv 0 \pmod{\mathscr{P}_+(\tau)}$. Then $\mathscr{M}(\tau) = \mathscr{P}_+(\tau)$, $\dot{\mathscr{P}}_+(\tau) = 1$, and $\dot{\mathscr{P}}(\tau) = \mathscr{P}_-(\tau)$, and representation (1.1.31) is replaced by

$$\dot{S}(-\mathrm{i}\,d/dt)\,\psi = F^{-1}_{\tau\to t}\left(\frac{\dot{S}(\tau)}{\dot{\mathscr{P}}(\tau)}F_{t\to\tau}f\right), \qquad \psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1), \tag{1.1.70}$$

where $f(t) = \dot{\mathscr{P}}(-\mathrm{i}\,d/dt)\,I\psi$ for $t \geqslant 0$ and $f(t) = 0$ for $t < 0$.[15] It follows from (1.1.70) that

$$\int_0^\infty |S(-\mathrm{i}\,d/dt)\,\psi|^2\,dt \leqslant \sup\left|\frac{\dot{S}(\tau)}{\dot{\mathscr{P}}(\tau)}\right|^2 \int_0^\infty \left|\dot{\mathscr{P}}(-\mathrm{i}\,d/dt)\,I\psi\right|^2 dt, \tag{1.1.71}$$
$$\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1).$$

Hence, estimate (1.1.68) with the sharp constant $\Lambda < \sup|S(\tau)/\mathscr{P}(\tau)|^2$ holds for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

The opposite inequality for Λ is deduced from (1.1.68) along the lines of the proof of Lemma 1.1.5. □

1.2 Estimates in a half-space. Necessary and sufficient conditions

Suppose $R(\xi;\tau) = \{R_j(\xi;\tau)\}$, $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$, and $Q(\xi;\tau) = \{Q_{\alpha j}(\xi;\tau)\}$ are $1\times m$, $m\times m$ and $N\times m$ matrices, respectively, the entries of which are polynomials of the variable $\tau \in \mathbb{R}^1$ with measurable complex-valued coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$.

[15] Indeed, from the equality $\dot{\mathscr{P}}_+(\tau) = 1$ it follows that $\psi = F^{-1}_{t\to\tau}\left(F_{t\to\tau}f/\dot{\mathscr{P}}(\tau)\right)$ for $t \geqslant 0$.

In this section we establish necessary and sufficient conditions for the validity of the estimate

$$\|R(D)u\|_{B^{1/2}}^2 \leqslant C\left(\|P(D)u\|^2 + ⟪Q(D)u⟫^2\right) \tag{1.2.1}$$

for all $u = (u_1(x,t),\dots,u_m(x,t)) \in C_0^\infty(\mathbb{R}_+^n)$. Here $R(D)$, $P(D)$ and $Q(D)$ are pseudodifferential operators corresponding to the matrices $R(\xi;\tau)$, $P(\xi;\tau)$ and $Q(\xi;\tau)$, respectively, and $B(\xi)$ is a measurable function that is positive a.e. in $\mathbb{R}^{n-1}$.

1.2.1 Basic assumptions and notation

First, we formulate assumptions on the matrices R, P and Q entering in (1.2.1).

Let $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$. Assuming $\mathscr{P}(\xi;\tau) \not\equiv 0$, we rewrite this polynomial (of τ) in the form

$$\mathscr{P}(\xi;\tau) = \sum_{j=0}^{J} p_j(\xi)\tau^{J-j},$$

and set

$$\mathscr{Z} = \{\xi : \xi \in \mathbb{R}^{n-1},\ p_0(\xi) = 0\}.$$

We denote by $P^{\mathrm{c}}(\xi;\tau) = \{P^{jk}(\xi;\tau)\}$ the $m \times m$ matrix, whose rows are composed of the algebraic complements of the elements of colunms of the matrix P. We also define the $1 \times m$ and $N \times m$ matrices $S(\xi;\tau)$ and $T(\xi;\tau)$ by

$$\left.\begin{aligned} S(\xi;\tau) = \{S_k(\xi;\tau)\} = R(\xi;\tau)P^{\mathrm{c}}(\xi;\tau), \\ T(\xi;\tau) = \{T_{\alpha k}(\xi;\tau)\} = Q(\xi;\tau)P^{\mathrm{c}}(\xi;\tau) \end{aligned}\right\}. \tag{1.2.2}$$

Let $\mathring{\mathscr{P}}(\xi;\tau) = \mathscr{P}(\xi;\tau)/p_0(\xi)$ with $\xi \in \mathbb{R}^{n-1} \setminus \mathscr{Z}$. In addition, we consider the following polynomials (of τ):

- $\mathscr{P}_+(\xi;\tau)$ – the polynomial with leading coefficient 1, whose τ-roots (counting multiplicities) coincide with the τ-roots of $\mathscr{P}$ lying in the half-plane $\operatorname{Im}\zeta \geqslant 0$ $(\zeta = \tau + \mathrm{i}\sigma)$;
- $\mathscr{P}_-(\xi;\tau)$ = $\mathring{\mathscr{P}}(\xi;\tau)/\mathscr{P}_+(\xi;\tau)$;
- $\mathscr{M}(\xi;\tau)$ – the greatest common divisor of the polynomials $\mathscr{P}_+(\xi;\tau), S_1(\xi;\tau),\dots,S_m(\xi;\tau)$ with leading coefficients 1;
- $\dot{\mathscr{P}}_+(\xi;\tau)$ = $\mathscr{P}_+(\xi;\tau)/\mathscr{M}(\xi;\tau)$.

The basic assumptions are:

1. $J \geqslant 1$;
2. $\operatorname{mes}_{n-1}\mathscr{Z} = 0$;
3. $\operatorname{ord}\, \dot{\mathscr{P}}_+(\xi;\tau) = N$, $\operatorname{ord}\, S_k(\xi;\tau) \leqslant J$, $\operatorname{ord}\, T_{\alpha k}(\xi;\tau) \leqslant J-1$ $(k = 1,\dots,m;$ $\alpha = 1,\dots,N)$ on a full-measure set $X \subseteq \mathbb{R}^{n-1} \setminus \mathscr{Z}$.

We define on the set $X \times \mathbb{R}^1$ the $1 \times m$ matrices $S_\pm(\xi;\tau) = \{S_{k\pm}(\xi;\tau)\}$ and the $N \times m$ matrices $T_\pm(\xi;\tau) = \{T_{k\pm}(\xi;\tau)\}$ by the following partial fraction decompositions (w.r.t. τ):

$$\frac{S(\xi;\tau)}{\mathscr{P}(\xi;\tau)} = c(\xi) + \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} + \frac{S_-(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)} \tag{1.2.3}$$

and

$$\frac{T(\xi;\tau)}{\mathscr{P}(\xi;\tau)} = \frac{T_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} + \frac{T_-(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)}, \tag{1.2.4}$$

where the $1 \times m$ matrix $c(\xi)$ does not depend on τ and

$$\begin{aligned}
&\text{ord } S_{k+}(\xi;\tau),\ \text{ord } T_{\alpha k+}(\xi;\tau) < \text{ord } \mathscr{P}_+(\xi;\tau),\\
&\text{ord } S_{k-}(\xi;\tau),\ \text{ord } T_{\alpha k-}(\xi;\tau) < \text{ord } \mathscr{P}_-(\xi;\tau),\\
&(k = 1,\dots,m; \quad \alpha = 1,\dots,N).
\end{aligned}$$

Remark 1.2.1. The condition $\text{mes}_{n-1}\mathscr{Z} = 0$ is satisfied, for example, if $p_0(\xi)$ is a polynomial of the variable ξ.

This is a consequence of the following assertion:

Let $p(\xi)$ be a non-identically vanishing polynomial of the variable $\xi \in \mathbb{R}^{n-1}$ with complex coefficients. Then

$$\text{mes}_{n-1}\{\xi : \xi \in \mathbb{R}^{n-1},\ p(\xi) = 0\} = 0.$$

We prove this by an argument from [LouSim72] (see [LouSim72], pp. 11–12). It suffices to show that the assertion is true for polynomials with real coefficients.

The proof is done by induction w.r.t. $\deg p$. In the case $\deg p = 0$, the assertion is trivial.

Now, let $r \geqslant 0$ and suppose that the assertion is true for all polynomials of degree less than or equal r that are not identically equal to zero.

Consider an arbitrary polynomial $p(\xi)$ of degree $r + 1$ with real coefficients, and set

$$\begin{aligned}
\mathscr{N} &= \{\xi : \xi \in \mathbb{R}^{n-1},\ p(\xi) = 0\},\\
\mathscr{N}_1 &= \{\xi : \xi \in \mathbb{R}^{n-1},\ p(\xi) = 0,\ \text{grad}\, p(\xi) \neq 0\},\\
\mathscr{N}_2 &= \mathscr{N} \setminus \mathscr{N}_1.
\end{aligned}$$

We claim that

$$\text{mes}_{n-1}\mathscr{N}_1 = \text{mes}_{n-1}\mathscr{N}_2 = 0.$$

Indeed, for every integer $K > 0$ the set $\mathscr{N}_1 \cap \{\xi : \xi \in \mathbb{R}^{n-1},\ |\xi| \leqslant K\}$ can be represented as a union of pieces of regular hypersurfaces in $\mathbb{R}^{n-1}$. Therefore, its $(n-1)$-dimensional Lebesgue measure equals zero. Since $\mathscr{N}_1$ is a countable union of such zero-measure sets, we conclude that $\text{mes}_{n-1}\mathscr{N}_1 = 0$.

To prove that $\text{mes}_{n-1}\mathscr{N}_2 = 0$, we note that deg $\left(\partial p/\partial \xi_j\right) \leqslant r$, $(j = 1, \ldots, n-1)$. On the other hand, grad $p(\xi) \not\equiv 0$: otherwise we would have $p(\xi) \equiv \text{const} \neq 0$. The latter implies deg $p(\xi) \neq r + 1$ $(r \geqslant 0)$, which contradicts the induction hypothesis.

Applying the induction hypothesis to the polynomial $\partial p/\partial \xi_j$ $(j = 1, \ldots, n-1)$ we get $\text{mes}_{n-1}\mathscr{N}_2 = 0$.

1.2.2 Theorems on necessary and sufficient conditions for the validity of the estimates in a half-space

In this subsection we describe the main results of Section 1.2. Necessary and sufficient conditions for the validity of the estimates (1.2.1) and (1.2.12) for all vector-functions $u \in \mathbf{C}_0^\infty\left(\mathbb{R}^n_+\right)$ are established in Theorems 1.2.2 and 1.2.3, respectively. Necessary and sufficient criterion for the validity of (1.2.13) for vector functions that satisfy homogeneous boundary conditions is given in Theorem 1.2.5.

We show that the statements of these theorems are simple corollaries of the analogous statements about estimates for matrix ordinary differential operators on the semi-axis $t \geqslant 0$, obtained in Section 1.1.

Theorem 1.2.2. *Let $N \geqslant 1$. The estimate* (1.2.1) *is valid for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if the following conditions are satisfied:*

1. *The matrix $S(\xi;\tau)$, defined by* (1.2.2)*, satisfies the inequality*

$$B^{1/2}(\xi)|S(\xi;\tau)| \leqslant \text{const}|\mathscr{P}(\xi;\tau)| \tag{1.2.5}$$

for all $\tau \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$.

2. *The matrix $T(\xi;\tau)$, defined by* (1.2.2)*, satisfies the congruence*

$$T(\xi;\tau) \equiv 0 \; (\text{mod}\, \mathscr{M}(\xi;\tau)) \tag{1.2.6}$$

for almost all $\xi \in \mathbb{R}^{n-1}$.

3. *The rows of the matrix $T(\xi;\tau)$ are linearly independent modulo $\mathscr{P}_+(\xi;\tau)$ for almost all $\xi \in \mathbb{R}^{n-1}$.*
4. *There exists a uniquely determined $1 \times N$ matrix $G(\xi;\tau) = \{G_1(\xi;\tau), \ldots, G_N(\xi;\tau)\}$ with polynomial (of τ) entries such that*

$$\max_\alpha \text{ord } G_\alpha(\xi;\tau) \leqslant N - 1 + \text{ord } \mathscr{M}(\xi;\tau).$$

Moreover, the congruence

$$G(\xi;\tau) \equiv 0 \; (\text{mod}\, \mathscr{M}(\xi;\tau))$$

and the identity w.r.t. $\tau, \eta \in \mathbb{R}^1$

$$G(\xi;\tau)T_+(\xi;\eta) = \frac{1}{\eta - \tau}[\mathscr{P}_+(\xi;\eta)S_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)S_+(\xi;\eta)] \tag{1.2.7}$$

hold a.e. in $\mathbb{R}^{n-1}$. Here $S_+(\xi;\tau)$ and $T_+(\xi;\tau)$ are the matrices defined by (1.2.3) *and* (1.2.4)*, respectively.*

5. *The inequality*

$$B(\xi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)T_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)}\right|^2 d\tau d\eta \leqslant \text{const} \tag{1.2.8}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$. *Here* $G(\xi;\tau)$ *is the matrix satisfying condition* 4, *and* $T_-(\xi;\eta)$ *is the matrix defined by* (1.2.4).

6. *The inequality*

$$B(\xi)\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 d\tau \leqslant \text{const} \tag{1.2.9}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$.

Proof. Necessity. Consider for arbitrary $A > 0$ the "cut-off" function

$$B_A(\xi) = \begin{cases} B(\xi), & \text{if } B(\xi) \leqslant A, \\ A, & \text{if } B(\xi) > A. \end{cases}$$

In accordance with definition of the norm $\|\cdot\|_{B^{1/2}}$, it follows from (1.2.1) that the estimate

$$\|R(D)u\|^2_{B_A^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2\right). \tag{1.2.10}$$

holds for any $A > 0$ and for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Let $\xi \in \mathbb{R}^{n-1} \setminus \mathscr{Z}$, and let $p(\xi) = [p_0(\xi)]^{1/m}$ be the principal value of the m-th root of $p_0(\xi)$. We set $\mathring{R}(\xi;\tau) = R(\xi;\tau)/p(\xi)$, $\mathring{P}(\xi;\tau) = P(\xi;\tau)/p(\xi)$ and $\mathring{Q}(\xi;\tau) = Q(\xi;\tau)/p(\xi)$. We substitute in (1.2.10) the vector function

$$u(x;t) = h^{(1-n)/2}\varphi\left(\frac{x}{h}\right)e^{\mathrm{i}x\cdot\xi}v(t),$$

where $h > 0$ is a parameter, $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$, and $v(t) = (v_1(t),\dots,v_m(t)) \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$, and then let h to $+\infty$ and take into account that the "cut-off" function $B_A(\xi)$ is bounded and the coefficients of the polynomials $R_j(\xi;\tau)$, $P_{kj}(\xi;\tau)$, and $Q_{\alpha j}(\xi;\tau)$ are measurable, locally bounded functions growing not faster than some power of $|\xi|$ as $|\xi| \to \infty$. In this way we get a new inequality. Finally, reducing all its terms by the factor

$$|p(\xi)|^2\int_{\mathbb{R}^{n-1}}|\varphi(x)|^2dx$$

and letting $A \to +\infty$, we conclude that

$$\int_0^\infty\left|\mathring{R}(\xi;-\mathrm{i}\,d/dt)\,v(t)\right|^2dt \leqslant \frac{C}{B(\xi)}\left[\int_0^\infty\left|\mathring{P}(\xi;-\mathrm{i}\,d/dt)\,v(t)\right|^2dt + \left|\mathring{Q}(\xi;-\mathrm{i}\,d/dt)\,v(t)|_{t=0}\right|^2\right] \tag{1.2.11}$$

for almost all $\xi \in \mathbb{R}^{n-1}$ and all $v(t) \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Here $\det \mathring{P}(\xi;\tau) = \mathring{\mathscr{P}}(\xi;\tau)$ is a polynomial of degree J (in τ) with leading coefficient equal to 1.

Regarding for fixed $\xi \in \mathbb{R}^{n-1}$ inequality (1.2.11) as an estimate of the type (1.1.1), we observe that the necessity of the all assumptions of Theorem 1.2.2 follows from Theorem 1.1.19.

Sufficiency. The conditions 1–6 imply the estimate (1.2.11) for all $v \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Let $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$. We substitute in (1.2.11) the vector function $v_\xi(t) = \hat{u}(\xi;t)$. Multiplying both sides of the resulting inequality by $B(\xi)|p(\xi)|^2$ and integrating over $\mathbb{R}^{n-1}$ we find that $u(x;t)$ satisfies (1.2.1). □

Next, we formulate a result relating to the case $N = 0$.

Theorem 1.2.3. *The estimate*

$$\|R(D)u\|_{B^{1/2}}^2 \leqslant C\|P(D)u\|^2 \tag{1.2.12}$$

holds true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ if and only if the $1 \times m$ matrix $S(\xi;\tau)$, defined by (1.2.2)*, satisfies the following conditions:*

1. $S(\xi;\tau) \equiv 0 \pmod{\mathscr{P}_+(\xi;\tau)}$ *for almost all* $\xi \in \mathbb{R}^{n-1}$*;*
2. $B^{1/2}(\xi)|S(\xi;\tau)| \leqslant \text{const}\,|\mathscr{P}(\xi;\tau)|$ *for all* $\tau \in \mathbb{R}^1$ *and almost all* $\xi \in \mathbb{R}^{n-1}$.

This theorem is deduced from Theorem 1.1.30 in the same way as Theorem 1.2.2 is deduced from Theorem 1.1.19.

Remark 1.2.4. If for almost all $\xi \in \mathbb{R}^{n-1}$ all τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ lie in the half-plane $\operatorname{Im}\zeta < 0$, then condition 2 of Theorem 1.2.3 is evidently necessary and sufficient for the validity of (1.2.12). Indeed, in this case $\mathscr{P}_+(\xi;\tau) = 1$ and condition 1 of Theorem 1.2.3 is automatically satisfied.

Finally, for vector functions $u(x;t) \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ satisfying homogeneous boundary conditions, we have the following direct consequence of Theorem 1.1.29.

Theorem 1.2.5. *Let* $N \geqslant 1$. *The inequality*

$$\|R(D)u\|_{B^{1/2}}^2 \leqslant C_0\|P(D)u\|^2 \tag{1.2.13}$$

is valid for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ satisfying the equation $Q(D)u(x;0) = 0$ if and only if conditions 1–5 *of Theorem* 1.2.2 *are satisfied.*

1.2.3 Matrix $G(\xi;\tau)$ and its properties

Now, let us discuss some properties of the $1 \times N$ matrix $G(\xi;\tau)$ that appears in condition 4 of Theorem 1.2.2.

Recall that, for $m = 1$, the existence and uniqueness of the matrix $G(\xi;\tau)$ follow from conditions 1–3 of Theorem 1.2.2. However, for $m > 1$ this is, in general, not true (see Subsection 1.1.10). From this point of view, one can interpret failure of certain estimates of the type (1.2.1). For example, the estimate

$$\|u_1(x;t)+u_2(x;t)\|^2 \leqslant C\left(\sum_{k=1}^{2}\left\|\left(\frac{\partial}{\partial t}-\Delta+1\right)u_k(x;t)\right\|^2+\int_{\mathbb{R}^{n-1}}|u_1(x;0)|^2dx\right),$$

where $x=(x_1,\dots,x_{n-1})$ and $\Delta=\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_{n-1}^2}$, is not true for all $u=(u_1,u_2)\in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. Here the matrices $R(\xi;\tau)=\{1,1\}$, $P(\xi;\tau)=\mathrm{i}(\tau+\mathrm{i}(\xi^2+1))I$, and $R(\xi;\tau)=\{1,0\}$ are the same as in Example 1.1.20.

We formulate a criterion for condition 4 of Theorem 1.2.2 to follow from conditions 1–3 of the same theorem. Let

$$\dot{\mathscr{P}}_+(\xi;\tau)=\prod_{\varrho=1}^{l(\xi)}(\tau-\zeta_\varrho(\xi))^{\kappa_\varrho(\xi)} \qquad (\kappa_1(\xi)+\dots+\kappa_{l(\xi)}(\xi)=N),$$
$$\dot S(\xi;\tau)=S(\xi;\tau)/\mathscr{M}(\xi;\tau)=\{\dot S_k(\xi;\tau)\},$$
$$\dot T(\xi;\tau)=T(\xi;\tau)/\mathscr{M}(\xi;\tau)=\{\dot T_{\alpha k}(\xi;\tau)\}.$$

Let $\dot{\mathfrak{S}}(\xi)$ be the $N\times mN$ matrix obtained from the matrix (1.1.23) after we replace $\dot S_k^{(\sigma-\beta)}(\zeta_\varrho)$ by $\dot S_k^{(\sigma-\beta)}(\xi;\zeta_\varrho(\xi))$, and let $\dot{\mathfrak{T}}(\xi)$ be the $N\times mN$ matrix obtained from the matrix (1.1.24) after we replace $\dot T_{\alpha k}^{(\sigma)}(\zeta_\varrho)$ by $\dot T_{\alpha k}^{(\sigma)}(\xi;\zeta_\varrho(\xi))$. (In both cases we differentiate with respect to the variable τ).

Proposition 1.2.6. *The condition* 4 *of Theorem* 1.2.2 *follows from conditions* 1–3 *of the same theorem if and only if, for almost all* $\xi\in\mathbb{R}^{n-1}$*, the rows of the matrix* $\dot{\mathfrak{S}}(\xi)$ *belong to the linear span of the rows of the matrix* $\dot{\mathfrak{T}}$.

This proposition follows directly from Proposition 1.1.21.

We also give an easier stated sufficient condition, which follows from Proposition 1.1.25.

Proposition 1.2.7. *If the polynomials* $\mathscr{M}(\xi;\tau)$ *and* $\dot{\mathscr{P}}_+(\xi;\tau)=\frac{\mathscr{P}_+(\xi;\tau)}{\mathscr{M}(\xi;\tau)}$ *are relatively prime for almost all* $\xi\in\mathbb{R}^{n-1}$*, then condition* 4 *of Theorem* 1.2.2 *follows from conditions* 1–3 *of the same theorem. In particular, this assertion holds if the* τ*-roots of the polynomial* $\mathscr{P}_+(\xi;\tau)$ *are pairwise distinct a.e. in* $\mathbb{R}^{n-1}$*, or if* $\mathscr{M}(\xi;\tau)=1$ *for all* $\tau\in\mathbb{R}^1$ *and almost all* $\xi\in\mathbb{R}^{n-1}$.

Finally, we provide a result concerning estimates with a “large” number of boundary operators (cf. Subsection 1.1.11).

Proposition 1.2.8. *Let the τ-roots of the polynomial $\mathscr{P}_+(\xi;\tau)$ be nonreal for almost all $\xi \in \mathbb{R}^{n-1}$, and let the rows of the $N \times m$ matrix $T(\xi;\tau) = Q(\xi;\tau)P^{\mathrm{c}}(\xi;\tau)$ be linearly independent modulo $\mathscr{P}_+(\xi;\tau)$. Then there exists a uniquely determined $1 \times N$ matrix $G(\xi;\tau) = \{G_1(\xi;\tau), \dots, G_N(\xi;\tau)\}$ with polynomial in τ entries such that*

$$\max_{\alpha} \operatorname{ord} G_\alpha(\xi;\tau) \leqslant N-1$$

a.e. in $\mathbb{R}^{n-1}$, and the following conditions are satisfied:

1. *For all vector functions $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ and almost all $\xi \in \mathbb{R}^{n-1}$ one has the representation*

$$\begin{aligned} \overset{\circ}{S}(\xi;-\mathrm{i}\,d/dt)\varphi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}t\tau} \left\{ \frac{\overset{\circ}{S}(\xi;\tau)}{\overset{\circ}{\mathscr{P}}(\xi;\tau)} F_{t\to\tau} f + \frac{\mathrm{i}}{\sqrt{2\pi}} \frac{G(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right. \\ &\times \left. \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\overset{\circ}{T}_-(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)} (F_{t\to\eta} f)\,d\eta - \overset{\circ}{T}(\xi;-\mathrm{i}\,d/dt)\varphi|_{t=0} \right] \right\} d\tau. \end{aligned} \tag{1.2.14}$$

Here $\overset{\circ}{S}(\xi;\tau) = \overset{\circ}{R}(\xi;\tau)\overset{\circ}{P}{}^{\mathrm{c}}(\xi;\tau)$, $\overset{\circ}{T}(\xi;\tau) = \overset{\circ}{Q}(\xi;\tau)\overset{\circ}{P}{}^{\mathrm{c}}(\xi;\tau)$, $\overset{\circ}{T}_-(\xi;\tau)$ is a matrix obtained from $\overset{\circ}{T}(\xi;\tau)$ via a decomposition of the type (1.2.4), while $\overset{\circ}{R}(\xi;\tau)$, $\overset{\circ}{P}(\xi;\tau)$, and $\overset{\circ}{Q}(\xi;\tau)$ are the matrices $R(\xi;\tau)$, $P(\xi;\tau)$ and $Q(\xi;\tau)$ divided by $p(\xi) = [p_0(\xi)]^{1/m}$, respectively; $p_0(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$, and $f = \overset{\circ}{P}(\xi;-\mathrm{i}\,d/dt)I\varphi$ for $t \geqslant 0$ and $f = 0$ for $t < 0$.
2. *Identity (1.2.7) holds for all $\tau, \eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$.*

This assertion obviously follows from Proposition 1.1.26. It will be used in Section 1.3.

We complete this subsection by two propositions about exact representations of the matrix $G(\xi;\tau)$.

Again, let $N = \operatorname{ord} \dot{\mathscr{P}}_+(\xi;\tau)$. In addition to the above-introduced matrices $\dot{S}(\xi;\tau)$, $\dot{T}(\xi;\tau)$, and $\dot{\mathfrak{T}}(\xi)$, we consider the $1 \times mN$ matrix

$$\left. \begin{aligned} &H(\xi;\tau) = \{H_{\varrho\sigma k}(\xi;\tau)\}, \\ &H_{\varrho\sigma k}(\xi;\tau) = \sum_{\beta=0}^{\sigma} \frac{\sigma!}{\beta!} \dot{S}_k^{(\beta)}(\xi;\zeta_\varrho(\xi)) \frac{\dot{\mathscr{P}}_+(\xi;\tau)}{(\tau-\zeta_\varrho(\xi))^{\sigma+1-\beta}} \\ &\qquad (1 \leqslant \varrho \leqslant l(\xi);\ 0 \leqslant \varsigma \leqslant k_\varrho(\xi)-1;\ 1 \leqslant k \leqslant m). \end{aligned} \right\} \tag{1.2.15}$$

Proposition 1.2.9. *Suppose that conditions* 1–4 *of Theorem* 1.2.2 *are fulfilled. Then, for almost all $\xi \in \mathbb{R}^{n-1}$ one can represent the matrix $G(\xi;\tau)$, figuring in condition* 4 *of Theorem* 1.2.2, *in the form*

$$G(\xi;\tau) = \mathscr{M}(\xi;\tau)H(\xi;\tau)\dot{\mathfrak{T}}_R^{-1}(\xi), \tag{1.2.16}$$

where $H(\xi;\tau)$ is the matrix (1.2.15) and $\dot{\mathfrak{T}}_R^{-1}(\xi)$ is an arbitrary matrix right inverse to $\dot{\mathfrak{T}}(\xi)$.

This result is a consequence of Lemma 1.1.17 (see also equation (1.1.61)).

Another representation for $G(\xi;\tau)$ follows from equation (1.2.7).

Proposition 1.2.10. *Suppose that conditions 1–4 of Theorem 1.2.2 be fulfilled. Suppose also that*

$$\mathscr{T}_+(\xi) = \int_{-\infty}^{\infty} \frac{T_+(\xi;\eta)T_+^*(\xi;\eta)}{|\mathscr{P}_+(\xi;\eta)|^2} d\eta, \tag{1.2.17}$$

where $T_+(\xi;\eta)$ is the $N\times m$ matrix defined by (1.2.4), and $T_+^*(\xi;\eta)$ is the $m\times N$ matrix, that is the conjugate transpose of T_+. Then the $1\times N$ matrix $G(\xi;\tau)$, satisfying condition 4 of Theorem 1.2.2, admits for almost all $\xi\in\mathbb{R}^{n-1}$ the representation

$$G(\xi;\tau) = \int_{-\infty}^{\infty} \frac{\mathscr{P}_+(\xi;\eta)S_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)S_+(\xi;\eta)}{(\eta-\tau)|\mathscr{P}_+(\xi;\eta)|2} T_+^*(\xi;\eta)d\eta\mathscr{T}_+^{-1}. \tag{1.2.18}$$

This representation holds also for the matrix G figuring in Proposition 1.2.8.

The proof of Proposition 1.2.10 is based on Proposition 1.1.27 and Remark 1.1.28.

1.2.4 The case of a single boundary operator

The necessary and sufficient conditions for the validity of the estimates (1.2.1) and (1.2.13), established in Theorems 1.2.2 and 1.2.5, respectively, can be formulated more clearly in the case $N = 1$. The formulation of condition 4 of these theorems becomes especially easy. Namely, we have

Corollary 1.2.11. *Let $N = 1$, let $Q(\xi;\tau)$ be a given $1\times m$ matrix, and let $S(\xi;\tau) = \{S_k(\xi;\tau)\}$, $T(\xi;\tau) = \{T_k(\xi;\tau)\}$, and $T_-(\xi;\tau) = \{T_{k-}(\xi;\tau)\}$ be the $1\times m$ matrices defined by (1.2.2) and (1.2.4), respectively. Suppose also that $\dot{\mathscr{P}}_+(\xi;\tau) = \tau - \zeta(\xi)$.*

Then the estimate (1.2.1) holds for all $u\in\mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if the following conditions are satisfied:

1. *For all $\tau\in\mathbb{R}^1$ and almost all $\xi\in\mathbb{R}^{n-1}$ inequality (1.2.5) remains valid.*
2. $T(\xi;\tau)\equiv 0 \pmod{\mathscr{M}(\xi;\tau)}$ *a.e. in* $\mathbb{R}^{n-1}$.
3. $[T(\xi;\tau)/\mathscr{M}(\xi;\tau)]\big|_{\tau=\zeta(\xi)} \neq 0$ *a.e. in* $\mathbb{R}^{n-1}$.
4. *There exists a measurable function $\alpha(\xi)$ in $\mathbb{R}^{n-1}$ such that*

$$\left[\frac{S(\xi;\tau)}{\mathscr{M}(\xi;\tau)}\right]\Bigg|_{\tau=\zeta(\xi)} = \alpha(\xi)\left[\frac{T(\xi;\tau)}{\mathscr{M}(\xi;\tau)}\right]\Bigg|_{\tau=\zeta(\xi)} \quad a.e.\ in\ \mathbb{R}^{n-1}. \tag{1.2.19}$$

5. *The inequality*

$$B(\xi)\frac{|\alpha(\xi)|^2}{\operatorname{Im}\zeta(\xi)}\int_{-\infty}^{\infty}\left|\frac{T_-(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)}\right|^2 d\eta \leqslant \text{const} \tag{1.2.20}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$.

6. *The inequality*

$$B(\xi)|\alpha(\xi)|^2 \leqslant \text{const}\operatorname{Im}\zeta(\xi) \tag{1.2.21}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$.

Moreover, conditions 1–5 *are necessary and sufficient for* (1.2.13) *to hold for all vector functions* $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ *satisfying the equation* $Q(D)u(x;0) = 0$.

Proof. We show that for $N = 1$ conditions 1–6 of Theorem 1.2.2 turn into conditions 1–6 of of this corollary.

Indeed, conditions 1 and 2 are formulated identically in both cases. The condition 3 of Corollary 1.2.11 obviously means nothing else than the "linear independence" of the single-row matrix $T(\xi;\tau)$ modulo $\mathscr{P}_+(\xi;\tau) = (\tau - \zeta(\xi))\mathscr{M}(\xi;\tau)$.

Further, we consider condition 4. Let $\dot{\mathfrak{S}}(\xi)$ and $\dot{\mathfrak{T}}(\xi)$ be the matrices mentioned in Proposition 1.2.6. It is obvious that, in the case $N = 1$,

$$\dot{\mathfrak{S}}(\xi) = \left[\frac{S(\xi;\tau)}{\mathscr{M}(\xi;\tau)}\right]_{\tau=\zeta(\xi)} \quad \text{and} \quad \dot{\mathfrak{T}}(\xi) = \left[\frac{T(\xi;\tau)}{\mathscr{M}(\xi;\tau)}\right]_{\tau=\zeta(\xi)}.$$

Relation (1.2.19) says that for almost all $\xi \in \mathbb{R}^{n-1}$ the single-row matrix $\dot{\mathfrak{S}}(\xi)$ belongs to the subspace generated in $\mathbf{C}^m$ by the single-row matrix $\dot{\mathfrak{T}}(\xi)$. Taking into account Proposition 1.2.6, we observe that conditions 1–4 of Theorem 1.2.2 are overall equivalent to conditions 1–4 of Corollary 1.2.11.

Finally, we turn to conditions 5 and 6. Comparing (1.1.23) and (1.2.15), we note that for $N = 1$ the matrix $H(\xi;\tau)$ does not depend on τ and the equality $H(\xi;\tau) = \dot{\mathfrak{S}}(\xi)$ holds a.e. in $\mathbb{R}^{n-1}$. Then, in accordance with (1.2.19), we have

$$H(\xi;\tau) = \alpha(\xi)\dot{\mathfrak{T}}(\xi).$$

Therefore, for $N = 1$, representation (1.2.16) takes the form

$$G(\xi;\tau) = \alpha(\xi)\mathscr{M}(\xi;\tau), \tag{1.2.22}$$

where $\alpha(\xi)$ is the coefficient on the right-hand side of equation (1.2.19).

Using (1.2.22) and the relation $\dot{\mathscr{P}}_+(\xi;\tau) = \tau - \zeta(\xi)$, it is easy to see that inequalities (1.2.8) and (1.2.9) turn into (1.2.20) and (1.2.21), respectively. $\square$

1.2.5 The case of a polynomial $\mathscr{P}(\xi;\tau)$ with roots in the half-plane $\operatorname{Im}\zeta \leqslant 0$

If all the τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ lie in the half-plane $\operatorname{Im}\zeta \geqslant 0$ ($\zeta = \tau + i\sigma$) for almost all $\xi \in \mathbb{R}^{n-1}$, then $\mathscr{P}_-(\xi;\tau) = 1$ and $T_-(\xi;\tau) = 0$ a.e. in $\mathbb{R}^{n-1}$. Therefore, Theorem 1.2.5 admits the following

Corollary 1.2.12. *Suppose all the τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ lie in the half-plane* $\operatorname{Im}\zeta \geqslant 0$ *for almost all $\xi \in \mathbb{R}^{n-1}$. The estimate* (1.2.13) *holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if conditions* 1–4 *of Theorem* 1.2.2 *are fulfilled.*

We do not dwell on the obvious simplification of Theorem 1.2.2 that is achieved on this class of polynomials. Of course, all the above remarks on the special cases, where condition 4 can be omitted from Theorems 1.2.2 and 1.2.5, remain valid.

1.2.6 Estimates of the types (1.2.1), (1.2.12), (1.2.13) in the norms $\|\cdot\|_{\boldsymbol{\nu}}$ and $⟪\cdot⟫_{\boldsymbol{\mu}}$

In this subsection, we establish several necessary and sufficient conditions for the validity of the estimates (1.2.27), (1.2.31) and (1.2.33) in some more general norms, in comparison with $\|\cdot\|$ and $⟪\cdot⟫$ (Corollaries 1.2.13 and 1.2.14). Different versions of these results will be used in Subsection 1.2.7 (Remark 1.2.16) and in Section 1.4, in the analysis of estimates for quasielliptic generalized-homogeneous matrix operators.

First, we define the norms $\|\cdot\|^2_{\boldsymbol{\nu}}$ and $⟪\cdot⟫_{\boldsymbol{\mu}}$.

Let $\boldsymbol{\nu} = (\nu_1,\dots,\nu_m)$ be a vector with nonnegative integer coordinates, and let $u(x;t) = (u_1(x;t),\dots,u_m(x;t)) \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. We set

$$\|u\|^2_{\boldsymbol{\nu}} = \sum_{j=1}^{m} \|u_j\|^2_{\nu_j},$$

where $\|\cdot\|_{\nu_j}$ is the norm in $\mathscr{H}_{\nu_j}(\mathbb{R}^n_+)$. Further, let $\boldsymbol{\mu} = (\mu_1,\dots,\mu_N)$, and let $\varphi(x) = (\varphi_1(x),\dots,\varphi_N(x)) \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$. We set

$$⟪\varphi⟫^2_{\boldsymbol{\mu}} = \sum_{\beta=1}^{N} ⟪\varphi_\beta⟫^2_{\mu_\beta},$$

where $⟪\cdot⟫_{\mu_\beta}$ is the norm in $\mathscr{H}_{\mu_\beta}(\partial\mathbb{R}^{n-1}_+)$.

Similarly to the beginning of Section 1.2, we consider the $1\times m$, $m\times m$ and $N\times m$ matrices $R(\xi;\tau)$, $P(\xi;\tau)$ and $Q(\xi;\tau)$. The entries of these matrices are polynomials of the variable $\tau \in \mathbb{R}^1$ with measurable locally bounded in $\mathbb{R}^{n-1}$ coefficients that grow no faster than some power of $|\xi|$ as $|\xi| \to +\infty$.

The assumptions about the matrix P are the same as those at the beginning of this section. Namely, $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau) \not\equiv 0$, $J = \operatorname{ord}\mathscr{P}(\xi;\tau) \geqslant 1$, $\operatorname{mes}_{n-1}\mathscr{Z} =$

0, where $\mathscr{Z} = \{\xi : \xi \in \mathbb{R}^{n-1},\ p_0(\xi) = 0\}$ and $p_0(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}(\xi;\tau)$.

Let the matrices $S(\xi;\tau)$ and $T(\xi;\tau)$ be defined by (1.2.2). We assume that on some full-measure set $X \subseteq \mathbb{R}^{n-1} \setminus \mathscr{Z}$ the following conditions hold:

$$\begin{aligned} &\operatorname{ord} S_k(\xi;\tau) \leqslant J + \nu_k, \\ &\operatorname{ord} T_{\alpha k}(\xi;\tau) \leqslant J + \nu_k - 1 \qquad (k = 1,\dots,m;\ \alpha = 1,\dots,N), \\ &\operatorname{ord} \dot{\mathscr{P}}_+(\xi;\tau) = N. \end{aligned}$$

On the set $X \times \mathbb{R}^1$ we define the $1 \times m$ matrices $S_\pm(\xi;\tau) = \{S_{k\pm}(\xi;\tau)\}$ and the $N \times m$ matrices $T_\pm(\xi;\tau) = \{T_{\alpha k\pm}(\xi;\tau)\}$ by means of the following partial fraction decompositions:

$$\begin{aligned} \frac{S_k(\xi;\tau)}{\mathscr{P}(\xi;\tau)(\tau + \mathrm{i}|\xi| + \mathrm{i})^{\nu_k}} = c_k(\xi) + \frac{S_{k+}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \\ + \frac{S_{k-}(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)(\tau + \mathrm{i}|\xi| + \mathrm{i})^{\nu_k}} \end{aligned} \tag{1.2.23}$$

and

$$\left.\begin{aligned} \frac{T_{\alpha k}(\xi;\tau)}{\mathscr{P}(\xi;\tau)(\tau + \mathrm{i}|\xi| + \mathrm{i})^{\nu_k}} = \frac{T_{\alpha k+}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} + \frac{T_{\alpha k-}(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)(\tau + \mathrm{i}|\xi| + \mathrm{i})^{\nu_k}} \\ (k = 1,\dots,m;\ \alpha = 1,\dots,N) \end{aligned}\right\}, \tag{1.2.24}$$

where

$$\begin{aligned} &\operatorname{ord} S_{k+}(\xi;\tau),\ \operatorname{ord} T_{\alpha k+}(\xi;\tau) < \operatorname{ord} \mathscr{P}_+(\xi;\tau), \\ &\operatorname{ord} S_{k-}(\xi;\tau),\ \operatorname{ord} T_{\alpha k-}(\xi;\tau) < \operatorname{ord} \mathscr{P}_-(\xi;\tau) + \nu_k. \end{aligned}$$

Also, we denote by $\mathfrak{R}(\xi;\tau)$ the diagonal $m \times m$ matrix

$$\mathfrak{R}(\xi;\tau) = \{\delta_{jk}(\tau + \mathrm{i}|\xi| + \mathrm{i})^{\nu_k}\}, \tag{1.2.25}$$

and by $\mathfrak{M}(\xi)$ the $N \times N$ matrix

$$\mathfrak{M}(\xi) = \{\delta_{\alpha\beta}(1 + |\xi|^2)^{\mu_\beta/2}\}. \tag{1.2.26}$$

(Here δ_{jk} stands for the Kronecker symbol).

As a generalization of necessary and sufficient conditions for the validity of the estimates (1.2.1) and (1.2.3) we have the following assertion.

Corollary 1.2.13. *Let $N \geqslant 1$. The inequality*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2_{\boldsymbol{\nu}} + ⟪Q(D)u⟫^2_{\boldsymbol{\mu}}\right) \tag{1.2.27}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if the following conditions are fulfilled:

1. *For almost all* $\xi \in \mathbb{R}^{n-1}$ *and all* $\tau \in \mathbb{R}$ *the matrix* $S(\xi;\tau)$*, defined by* (1.2.2)*, satisfies the inequality*

$$B^{1/2}(\xi)|S(\xi;\tau)\mathfrak{R}^{-1}(\xi;\tau)| \leqslant \text{const}\,|\mathscr{P}(\xi;\tau)|, \tag{1.2.28}$$

where $\mathfrak{R}(\xi;\tau)$ *is the matrix* (1.2.25)*.*

2. *The matrix* $T(\xi;\tau)$*, defined by* (1.2.2)*, satisfies conditions* 2 *and* 3 *of Theorem* 1.2.2*.*
3. *The matrices* $S_+(\xi;\tau)$ *and* $T_+(\xi;\tau)$*, defined by decompositions* (1.2.23) *and* (1.2.24)*, respectively, satisfy condition* 4 *of Theorem* 1.2.2*.*
4. *For the* $1 \times N$ *matrix* $G(\xi;\tau)$*, satisfying identity* (1.2.7)*, the inequalities*

$$B(\xi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)T_-(\xi;\eta)\mathfrak{R}^{-1}(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)}\right|^2 d\tau d\eta \leqslant \text{const} \tag{1.2.29}$$

and

$$B(\xi)\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)\mathfrak{M}^{-1}(\xi)}{\mathscr{P}_+(\xi;\tau)}\right|^2 d\tau \leqslant \text{const} \tag{1.2.30}$$

hold for almost all $\xi \in \mathbb{R}^{n-1}$*. Here* $T_-(\xi;\tau)$*,* $\mathfrak{R}$ *and* $\mathfrak{M}$ *are the matrices defined by* (1.2.24)*,* (1.2.25) *and* (1.2.26)*, respectively.*

Moreover, assumptions 1–3 *and inequality* (1.2.29) *are necessary and sufficient for the validity of the estimate*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C_0\|P(D)u\|^2_{\boldsymbol{\nu}} \tag{1.2.31}$$

for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ *satisfying the equation* $Q(D)u(x;0) = 0$*.*

Proof. We introduce the matrices $P_{\boldsymbol{\nu}}(\xi;\tau) = \mathfrak{R}(\xi;\tau)P(\xi;\tau)$ and $Q_{\boldsymbol{\mu}}(\xi;\tau) = \mathfrak{M}(\xi;\tau)Q(\xi;\tau)$ and show that, in accordance with Theorem 1.2.2, conditions 1-4 are necessary and sufficient for the validity of the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P_{\boldsymbol{\nu}}(D)u\|^2 + \langle\!\langle Q_{\boldsymbol{\mu}}(D)u\rangle\!\rangle^2\right) \tag{1.2.32}$$

for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Indeed, if we replace P by $P_{\boldsymbol{\nu}}$ and Q by $Q_{\boldsymbol{\mu}}$, then the matrix S is transformed into the matrix $S\mathfrak{R}^c$, the matrix T into the matrix $\mathfrak{M}T\mathfrak{R}^c$, and the polynomial $\mathscr{P}$ into the polynomial $\mathscr{P}\det\mathfrak{R}$. Therefore, (1.2.3) is replaced by (1.2.23), while (1.2.4) is replaced by the decomposition

$$\frac{(1+|\xi|^2)^{\mu_\alpha/2}T_{\alpha k}(\xi;\tau)}{\mathscr{P}(\xi;\tau)(\tau+\mathrm{i}|\xi|+\mathrm{i})^{\nu_k}} = \frac{(1+|\xi|^2)^{\mu_\alpha/2}T_{\alpha k+}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} + \frac{(1+|\xi|^2)^{\mu_\alpha/2}T_{\alpha k-}(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)(\tau+\mathrm{i}|\xi|+\mathrm{i})^{\nu_k}},$$

which is clearly equivalent to (1.2.24). Inequality (1.2.5) is transferred into (1.2.28), whereas the matrix $G(\xi;\tau)$, appearing in (1.2.7), must be replaced by the matrix $G(\xi;\tau)\mathfrak{M}^{-1}(\xi)$. Therefore, conditions 5–6 of Theorem 1.2.2 take the form of the inequalities (1.2.29) and (1.2.30), respectively. Finally, the matrix $\mathfrak{M}T\mathfrak{N}^c$ satisfies conditions 2 and 3 of Theorem 1.2.2 if and only if the matrix T satisfies these conditions. (The latter follows directly from the definition of the matrices $\mathfrak{M}$ and $\mathfrak{N}$).

Now we show that the estimates (1.2.32) and (1.2.27) are equivalent. Indeed, on the one hand, the norms of $\|P_{\boldsymbol{\nu}}(D)u\|$ and $\|P(D)u\|_{\boldsymbol{\nu}}$ are equivalent, since $\boldsymbol{\nu}$ is an integer vector. On the other hand, from the definition of the norm in $\mathscr{H}_{\mu_\beta}(\partial\mathbb{R}^n_+)$ it follows that $\langle\!\langle Q_{\boldsymbol{\mu}}(D)u\rangle\!\rangle = \langle\!\langle Q(D)u\rangle\!\rangle_{\boldsymbol{\mu}}$ for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Thus, the first part of Corollary 1.2.13 follows from Theorem 1.2.2, and the second part from Theorem 1.2.5, respectively. □

We provide also a generalization of necessary and sufficient conditions for the validity of the estimate (1.2.12).

Corollary 1.2.14. *The condition* 1 *of Theorem* 1.2.3 *and condition* 1 *of Corollary* 1.2.13 *are necessary and sufficient for the validity of the estimate*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\|P(D)u\|^2_{\boldsymbol{\nu}} \tag{1.2.33}$$

for all $u \in \mathbf{C}_0^\infty$. *If, for almost all* $\xi \in \mathbb{R}^{n-1}$, *the* τ*-roots of the polynomial* $\mathscr{P}$ *lie in the half-plane* $\operatorname{Im}\zeta < 0$ ($\zeta = \tau + i\sigma$)*, then condition* 1 *of Corollary* 1.2.13 *is a criterion for the validity of the estimate* (1.2.33) *for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

This corollary follows from Theorem 1.2.3 and Remark 1.2.4.

1.2.7 The case, where the lower-order terms have no influence

In Section 1.0 it has already been noted that the lower order terms of the operators R, P, Q may exert a decisive influence on the validity of the estimate (1.2.1) and similar inequalities. In this subsection we consider a class of estimates that remain true after replacement the operators R, P, and Q by their homogeneous principal parts. We show (Proposition 1.2.15) that under certain natural assumptions on the matrices R, P, and Q, the estimate (1.2.38) is equivalent to the estimate (1.2.39) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Suppose the entries of the matrices R, P, and Q, figuring in the estimate (1.2.38), are polynomials of the variable $(\xi,\tau) \in \mathbb{R}^n$. We assume that the $m \times m$ matrix $P = \{P_{kj}(\xi;\tau)\}$ satisfies the condition

$$\deg\, \mathscr{P}(\xi;\tau) = \max\, \deg\,(P_{1i_1}P_{2i_2}\dots P_{mi_m}), \tag{1.2.34}$$

where $\mathscr{P} = \det\, P$ and the maximum is taken over all permutations

$$\begin{pmatrix} 1 & \dots & m \\ i_1 & \dots & i_m \end{pmatrix}$$

of m natural numbers $1,\dots,m$. A matrix P with this property is called *regular*.

Further, we rely on the following fact stated by L. R. Volevich [Vol60], [Vol63].

For every regular matrix P the Leray–Douglis–Nirenberg condition is fulfilled: there exist nonnegative integers $s_1, \dots, s_m$ ($\min s_k = 0$) and $t_1, \dots, t_m$ such that

$$\deg P_{kj} \leqslant t_j - s_k \qquad \text{and} \qquad \sum_{j=1}^{m} (t_j - s_j) = \deg \mathscr{P}. \tag{1.2.35}$$

For regular polynomial $m \times m$ matrices, one can give the following natural definition of the principal part (see [Vol63]).

Let $P'_{kj}(\xi; \tau)$ be the principal part (the homogeneous part of the maximal degree) of the polynomial $P_{kj}(\xi; \tau)$. We set

$$P' = \{\chi_{kj} P'_{kj}\}, \qquad \chi_{kj} = \begin{cases} 0, & \text{if} \quad \deg P_{kj} < t_j - s_k, \\ 1, & \text{if} \quad \deg P_{kj} = t_j - s_k. \end{cases} \tag{1.2.36}$$

The matrix P' is called the *principal part* of the matrix P. Obviously, $\det P'$ coincides with the principal part $\mathscr{P}'$ of the polynomial $\mathscr{P}(\xi; \tau) = \det P(\xi, \tau)$.

The norms figuring the right-hand side of the estimate (1.2.38) were defined at the beginning of Subsection 1.2.6. Now we define the seminorms that appear on the right-hand side of (1.2.39).

Let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m)$ be a vector with nonnegative integer coordinates, and let $u = (u_1, \dots, u_m) \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. We set

$$|||u|||^2_{\boldsymbol{\nu}} = \sum_{j=1}^{m} \sum_{|\alpha| = \nu_j} \|D^\alpha u_j\|^2.$$

Further, assuming that $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in \mathbb{R}^N$ and $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$, we set

$$\langle\langle\langle \varphi \rangle\rangle\rangle^2_{\boldsymbol{\mu}} = \sum_{\beta=1}^{N} \int_{\mathbb{R}^{n-1}} |\xi|^{2\mu_\beta} |\hat{\varphi}_\beta(\xi)|^2 d\xi.$$

Proposition 1.2.15. *Let $P = \{P_{kj}(\xi; \tau)\}$ be a regular in the sense of definition (1.2.34) $m \times m$ matrix, let the entries of P be polynomials in $(\xi; \tau) \in \mathbb{R}^n$, and let $s_1, \dots, s_m$, $T_1, \dots, t_m$ be the nonnegative integers figuring in (1.2.35).*

Let $R(\xi; \tau) = \{R_j(\xi; \tau)\}$ and $Q(\xi; \tau) = \{Q_{\alpha j}(\xi; \tau)\}$ be $1 \times m$ and $N \times m$ matrices with entries polynomial in $(\xi; \tau) \in \mathbb{R}^n$ such that $\deg R_j = t_j + l$ and $\deg Q_{\alpha j} = t_j - \varkappa_\alpha$, respectively. Here $j = 1, \dots, m$; $\alpha = 1, \dots, N$, while $l, \varkappa_1, \dots, \varkappa_N$ is another tuple of integers.

In addition, let P' be the principal part of the matrix P in the sense of the definition (1.2.36)*, let R'_j and $Q'_{\alpha j}$ be the principal parts of the polynomials R_j and $Q_{\alpha j}$, respectively; and let $R' = \{R'_j\}$ and $Q' = \{Q'_{\alpha j}\}$. We set*

$$\begin{aligned} \mathbf{s}+\mathbf{l} &= (s_1+l,\dots,s_m+l),\\ \mathbf{t}+\mathbf{l}-\mathbf{1} &= (t_1+l-1,\dots,t_m+l-1),\\ \boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2}) &= (\varkappa_1+l-1/2,\dots,\varkappa_N+l-1/2) \end{aligned}$$

and assume that

$$l \geqslant \max(0, 1-t_j, 1-\varkappa_\alpha). \tag{1.2.37}$$

The estimate

$$\|R(D)u\|^2 \leqslant C\left(\|P(D)u\|^2_{\mathbf{s}+\mathbf{l}} + \|u\|^2_{\mathbf{t}+\mathbf{l}-\mathbf{1}} + \langle\!\langle Q(D)u\rangle\!\rangle^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})}\right) \tag{1.2.38}$$

holds for all $u \in \mathbf{C}_0^\infty$ if and only if the inequality

$$\|R'(D)u\|^2 \leqslant C'\left(|||P'(D)u|||^2_{\mathbf{s}+\mathbf{l}} + \langle\!\langle\!\langle Q'(D)u\rangle\!\rangle\!\rangle^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})}\right) \tag{1.2.39}$$

is satisfied for all $u \in \mathbf{C}_0^\infty$.

Proof. Since the right-hand side of (1.2.38) contains the term $\|u\|^2_{\mathbf{t}+\mathbf{l}-\mathbf{1}}$, the *sufficiency* is obvious.

It remains to show the *necessity*. Assume that inequality (1.2.38) holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. Estimating the norms $\|(R-R')(D)u\|^2$, $\|(P-P')(D)u\|^2_{\mathbf{s}+\mathbf{l}}$ and $\langle\!\langle (Q-Q')(D)u\rangle\!\rangle^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})}$ by $\|u\|^2_{\mathbf{t}+\mathbf{l}-(\mathbf{1/2})}$, we see that all operators in (1.2.38) can be replaced by their principal parts.

We substitute in resulting inequality the vector function

$$\begin{aligned} u &= (u_1,\dots,u_m),\\ u_j(x;t) &= h^{(1-n)/2}\varphi\left(\frac{x}{h}\right)e^{\mathrm{i}x\cdot\xi}|\xi|^{-l-t_j}v_j(t) \qquad (j=1,\dots,m), \end{aligned}$$

where $h>0$ is a parameter, $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$, $0 \neq \xi \in \mathbb{R}^{n-1}$, and $v=(v_1,\dots,v_m) \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$. Applying the same arguments as in the proof of the necessity of conditions of Theorem 1.2.2, we obtain

$$\begin{aligned}
&\int_0^\infty \Big| \sum_{j=1}^m |\xi|^{-l-t_j} R'_j\,(\xi; -\mathrm{i}\, d/dt)\, v_j \Big|^2 dt \\
&\leqslant C \Bigg[\sum_{k=1}^m \int_0^\infty \Big| \sum_{\sigma=0}^{s_k+l} |\xi|^{s_k-\sigma} (-\mathrm{i}\, d/dt)^\sigma \sum_{j=1}^m |\xi|^{-t_j} \chi_{kj} P'_{kj}\,(\xi; -\mathrm{i}\, d/dt)\, v_j \Big|^2 dt \\
&\quad + \sum_{j=1}^m \Bigg(\sum_{\sigma=0}^{t_j+l-1} |\xi|^{2(-1-\sigma)} \int_0^\infty \big|(-\mathrm{i}\, d/dt)^\sigma\, v_j\big|^2 dt + |\xi|^{-2(l+t_j)} \int_0^\infty |v_j|^2 dt \Bigg) \\
&\quad + \sum_{\alpha=1}^N |\xi|^{2\varkappa_\alpha - 1} \Big| \sum_{j=1}^m Q'_{\alpha j}\,(\xi; -\mathrm{i}\, d/dt)\, |\xi|^{-t_j} v_j \big|_{t=0} \Big|^2 \Bigg].
\end{aligned} \tag{1.2.40}$$

We put $\xi = |\xi|\theta$ and $\tau = |\xi|t$ here, use the homogeneity of the polynomials R'_j, P'_{kj} and $Q'_{\alpha j}$, multiply both sides of the resulting inequality by $|\xi|$, and pass to the limit $|\xi| \to \infty$. This yields the inequality

$$\begin{aligned}
&\int_0^\infty \Big| \sum_{j=1}^m R'_j\,(\theta; -\mathrm{i}\, d/d\tau)\, v_j \Big|^2 d\tau \\
&\leqslant C' \Bigg[\sum_{k=1}^m \int_0^\infty \Big| \sum_{\sigma=0}^{s_k+l} (-\mathrm{i}\, d/d\tau)^\sigma \sum_{j=1}^m \chi_{kj} P'_{kj}\,(\theta; -\mathrm{i}\, d/d\tau)\, v_j \Big|^2 d\tau \\
&\quad + \sum_{\alpha=1}^N \Big| \sum_{j=1}^m Q'_{\alpha j}\,(\theta; -\mathrm{i}\, d/d\tau)\, v_j \big|_{\tau=0} \Big|^2 \Bigg].
\end{aligned} \tag{1.2.41}$$

We return in (1.2.41) to the variables ξ, t, and set

$$v = v_\xi(t) = (v_{1\xi}(t), \dots, v_{m\xi}(t))$$

with $v_{j\xi}(t) = |\xi|^j u_j(\xi; t)$, where $u = (u_1(x;t), \dots, u_m(x;t))$ is an arbitrary element of the space $\mathbf{C}_0^\infty(\mathbb{R}^n_+)$. Finally, integrating with respect to ξ and applying the inverse Fourier transform we get (1.2.39). □

Remark 1.2.16. Proposition 1.2.15 remains valid if we require additionally in the part concerning the necessity of the assertion that $\operatorname{supp} u \subset \mathcal{D}(0, \varrho)$ for some $\varrho > 0$, where $\mathcal{D}(0, \varrho)$ denotes the n-dimensional ball of radius ϱ centered at the origin. To show this, it suffices to note that the estimate (1.2.38) is valid for all $u \in$

$\mathbf{C}_0^\infty(\mathbb{R}^n_+)$ (possibly with a different constant), if it holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ satisfying $\operatorname{supp} u \subset \mathcal{D}(0, \varrho)$. The last statement can be easily verified by using a partition of unity subordinated to a finite-multiplicity cover of $\mathbb{R}^n_+$ by congruent cubes. Since the commutators of the operators $R_j(D)$, $P_{kj}(D)$ and $Q_{\alpha j}(D)$ with the operator of multiplication by a smooth function have the orders $t_j + l - 1$, $t_j - s_k - 1$ and $t_j - \varkappa_\alpha - 1$, respectively, one can estimate the terms appearing in this case and not figuring on the right-hand side of (1.2.38) by $\|u\|^2_{\mathbf{t}+\mathbf{l}-\mathbf{1}}$.

Remark 1.2.17. Let $\deg \mathscr{P}'(\xi;\tau) = \operatorname{ord} \mathscr{P}'(\xi;\tau) = J \geqslant 1$. We define the polynomials (of τ) $\mathscr{P}'_+(\xi;\tau)$, $\mathscr{P}'_-(\xi;\tau)$, $\mathscr{M}'(\xi;\tau)$ and $\dot{\mathscr{P}}'_+(\xi;\tau)$, which correspond to the polynomial $\mathscr{P}'(\xi;\tau)$ and the matrix $R'(\xi;\tau)$. Suppose that $\operatorname{ord} \dot{\mathscr{P}}'_+(\xi;\tau) = N \geqslant 1$ for all $\xi \in \mathbb{R}^{n-1}$.

Under these assumptions, necessary and sufficient conditions for the validity of the estimate (1.2.39) and, consequently, the estimate (1.2.38) are contained in Corollary 1.2.13. It is necessary only to set $B(\xi) = 1$ in the formulation of this corollary, replace the matrices (1.2.25) and (1.2.26) by

$$\mathfrak{N}'(\xi;\tau) = \{\delta_{jk}(\tau - i|\xi|)^{s_k + l}\} \qquad \text{and} \qquad \mathfrak{M}'(\xi) = \{\delta_{\alpha\beta}|\xi|^{\varkappa_\beta + l - (1/2)}\},$$

and replace the numbers ν_k and μ_α by $s_k + l$ and $\varkappa_\alpha + l - 1/2$, respectively.

Indeed, setting

$$P'_{\mathbf{s}+\mathbf{l}}(\xi;\tau) = \mathfrak{N}'(\xi;\tau)P'(\xi;\tau) \quad \text{and} \quad Q'_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})}(\xi;\tau) = \mathfrak{M}'(\xi)Q'(\xi;\tau),$$

we obviously get the equality $\langle\!\langle\!\langle Q'(D)u\rangle\!\rangle\!\rangle^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})} = \langle\!\langle Q'_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})}(D)u\rangle\!\rangle^2$. In addition, the norms $|||P'(D)u|||_{\mathbf{s}+\mathbf{l}}$ and $\|P'_{\mathbf{s}+\mathbf{l}}(D)u\|$ are equivalent, since $\mathbf{s}+\mathbf{l}$ is an integer vector. Hence, inequality (1.2.39) is an estimate of the type (1.2.27). Finally, we show that in the case under consideration all a priori assumptions necessary for the validity of Corollary 1.2.13 are fulfilled.

The conditions $\det P'(\xi;\tau) \not\equiv 0$ and $\operatorname{mes}_{n-1}\mathscr{Z} = 0$ follow from the relations $\deg \mathscr{P}' = \operatorname{ord} \mathscr{P}' = J \geqslant 1$ and Remark 1.2.1. Consider the matrices $S' = \{S'_k\} = R'P'^{c}$ and $T' = \{T'_{\alpha k}\} = Q'P'^{c}$, where P'^{c} is the adjugate of P'. It can be directly verified that S'_k and $T'_{\alpha k}$ are homogeneous polynomials in $(\xi;\tau) \in \mathbb{R}^n$ with $\deg S'_k = J + s_k + l$ and $\deg T'_{\alpha k} = J + s_k + \varkappa_\alpha$, respectively. Since $\operatorname{ord} \mathscr{P}' = J$ and $l \geqslant 1 - \varkappa_\alpha$, we have $\operatorname{ord} S'_k \leqslant J + s_k + l$ and $\operatorname{ord} T'_{\alpha k} = J + s_k + l - 1$.

1.3 Estimates in a half-space. Sufficient conditions

Let $\mathfrak{M}(\xi)$ be an arbitrary measurable $N \times N$ matrix, which is regular a.e. in $\mathbb{R}^{n-1}$. We generalize definition of the norm $\langle\!\langle Q(D)u\rangle\!\rangle_{\boldsymbol{\mu}}$, which figures in the estimate (1.2.27), by setting for vector functions $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$

$$\langle\!\langle Q(D)u\rangle\!\rangle^2_{\mathfrak{M}} = \int\limits_{\mathbb{R}^{n-1}} |\mathfrak{M}(\xi) Q\,(\xi; -\mathrm{i}\, d/dt)\,\hat{u}(\xi;t)|_{t=0}|^2 \, d\xi.$$

In this section, we consider a modified version of the estimate (1.2.27). Replacing in (1.2.27) $\langle\!\langle Q(D)u \rangle\!\rangle_{\boldsymbol{\mu}}$ by $\langle\!\langle Q(D)u \rangle\!\rangle_{\mathfrak{M}}$ and restricting ourselves, for the sake of simplicity, to the case $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m) = 0$, we formulate some sufficient conditions for the validity of the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2_{\mathfrak{M}} \right) \tag{1.3.1}$$

for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

1.3.1 Sufficient condition for the validity of the estimate (1.3.1)

The main result of this subsection is Theorem 1.3.1, which states that inequality (1.3.3) and conditions 1–4 of Theorem 1.2.2 are sufficient for the validity of estimate (1.3.1) with any measurable $N \times N$ matrix $\mathfrak{M}$, which is regular a.e. in $\mathbb{R}^{n-1}$. Of course, conditions 1–4 of Theorem 1.2.2 are also necessary for the validity of (1.3.1). In addition, all the remarks from Section 1.2, describing the cases where condition 4 of Theorem 1.2.2 can be omitted, remain valid.

We assume that the matrices R, P, and Q satisfy the conditions formulated at the beginning of Section 1.2; the matrices $S(\xi;\tau)$ and $S(\xi;\tau)$ are defined by equations (1.2.2), the matrices $S_\pm(\xi;\tau)$ and $T_\pm(\xi;\tau)$ are defined by the decompositions (1.2.3) and (1.2.4), respectively; and the matrix $\mathscr{T}_+(\xi)$ is defined by equation (1.2.17). We also consider the $N \times N$ matrix

$$\mathscr{T}_-(\xi) = \int_{-\infty}^{\infty} \frac{T_-(\xi;\tau) T_-^*(\xi;\eta)}{|\mathscr{P}_-(\xi;\eta)|^2} d\eta, \tag{1.3.2}$$

where the $m \times N$ matrix T_-^* is the conjugate transpose to T_-.

Theorem 1.3.1. *Let* $N \geqslant 1$. *If conditions* 1–4 *of Theorem* 1.2.2 *are fulfilled, and*

$$B(\xi)\,\mathrm{tr}\left[\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_+ \right)^{-1} \right] \left[1 + \mathrm{tr}\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_- \right) \right] \sup \left| \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \leqslant \text{ const} \tag{1.3.3}$$

for almost all $\xi \in \mathbb{R}^{n-1}$, *then the estimate* (1.3.1) *is valid for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Proof. Let $\mathring{R}(\xi;\tau)$, $\mathring{P}(\xi;\tau)$ and $\mathring{Q}(\xi;\tau)$ be the matrices considered in the proof of Theorem 1.2.2. It suffices to show that the assumptions of Theorem 1.3.1 imply the validity of the estimate

$$\int_0^\infty \left| \mathring{R}\,(\xi; -\mathrm{i}\, d/dt)\, v \right|^2 dt$$

$$\leqslant \frac{C}{B(\xi)} \left[\int_0^\infty \left| \mathring{P}\,(\xi; -\mathrm{i}\, d/dt)\, v \right|^2 dt + \left| \mathfrak{M}(\xi) \mathring{Q}\,(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} \right|^2 \right]$$

for all $v \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ and almost all $\xi \in \mathbb{R}^{n-1}$. For a fixed $\xi \in \mathbb{R}^{n-1}$ this is an estimate of the type (1.1.1). Applying Theorem 1.1.19, we see that it suffices to verify that

$$\int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathfrak{M}^{-1}(\xi)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau \leqslant \text{ const } \operatorname{tr}\left[\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_+ \right)^{-1} \right] \sup \left| \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \quad (1.3.4)$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)T_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)} \right|^2 d\tau d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathfrak{M}^{-1}(\xi)\mathfrak{M}(\xi)T_-(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)} \right|^2 d\tau d\eta \qquad (1.3.5) \\ &\leqslant \text{ const } \operatorname{tr}\left[\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_- \right)^{-1} \right] \operatorname{tr}\left[\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_+ \right)^{-1} \right] \sup \left| \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \end{aligned}$$

for almost all $\xi \in \mathbb{R}^{n-1}$. (Here G is the $1 \times N$ matrix figuring in Condition 4 of Theorem 1.2.2.)

Denote by $(\mathfrak{M}T_-)_j$ the j-th column of the matrix $\mathfrak{M}T_-$. One can directly verify that

$$\sum_{j=1}^{N} \int_{-\infty}^{\infty} \left| \frac{(\mathfrak{M}(\xi)T_-(\xi;\eta))_j}{\mathscr{P}_-(\xi;\eta)} \right|^2 d\eta = \operatorname{tr}\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_- \right),$$

where $\mathscr{T}_-$ is the matrix (1.3.2). It means that (1.3.5) follows from (1.3.4).

Now we prove the estimate (1.3.4). Using representation (1.2.18) for the matrix $G(\xi;\tau)$ and the boundedness of the singular integral in $L^2(\mathbb{R}^1)$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \frac{G(\xi;\tau)\mathfrak{M}^{-1}(\xi)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau \\ &\leqslant \text{ const } \sup \left| \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 \sum_{j=1}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{T_+^*(\xi;\eta)}{\overline{\mathscr{P}}_+(\xi;\eta)} \left(\mathfrak{M}\mathscr{T}_+ \right)^{-1}(\xi) \right)_j \right|^2 d\eta. \end{aligned}$$

Here $\left(T_+^* \left(\mathfrak{M}\mathscr{T}_+ \right)^{-1} \right)_j$ denotes the j-th column of the matrix $T_+^* \left(\mathfrak{M}\mathscr{T}_+ \right)^{-1}$. On the other hand, a direct calculation shows that

$$\operatorname{tr}\left[\left(\mathfrak{M}^* \mathfrak{M} \mathscr{T}_+ \right)^{-1} \right] = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{T_+^*(\xi;\eta)}{\overline{\mathscr{P}}_+(\xi;\eta)} \left(\mathfrak{M}\mathscr{T}_+ \right)^{-1} \right)_j \right|^2 d\eta. \qquad \square$$

Remark 1.3.2. If conditions 1–4 of Theorem 1.2.2 are fulfilled and

$$B(\xi)\,\mathrm{tr}\Big[\big(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_+\big)^{-1}\Big]\big[1+\mathrm{tr}\,\big(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_-\big)\big]\sup\left|\frac{S(\xi;\tau)}{\mathscr{P}(\xi;\tau)}\right|^2 \leqslant \text{const} \tag{1.3.6}$$

for almost all $\xi \in \mathbb{R}^{n-1}$, then the estimate (1.3.1) remains valid for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Indeed, it is well known that there exists a constant $C > 0$, depending only on $\mathrm{ord}\,\mathscr{P}(\xi;\tau) = J$ and on the order m of the matrix $P(\xi;\tau)$, such that for almost all $\xi \in \mathbb{R}^{n-1}$ the inequality

$$\sup\left|\frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right| \leqslant C\sup\left|\frac{S(\xi;\tau)}{\mathscr{P}(\xi;\tau)}\right| \tag{1.3.7}$$

holds[16]. Hence inequality (1.3.6) implies the estimate (1.3.3).

1.3.2 The case $\mathfrak{M}(\xi) = \mathscr{T}_+^{-1/2}(\xi)$

Consider the case $\mathfrak{M} = \mathscr{T}_+^{-1/2}$, where $\mathscr{T}_+$ denotes the matrix (1.2.17). First, we show (Theorem 1.3.3) that assumption (1.3.3) of Theorem 1.3.1 can be replaced by simpler to formulate condition (1.3.8) or by condition (1.3.10).

The sufficient condition of Theorem 1.3.6, related to the estimates with a "large number" of boundary operators, can be formulated in an even more simple way. (In this case, conditions 2 and 4 of Theorem 1.2.2 will be omitted.) This condition, proved by a direct method by M. Schechter [Sch64a], is a simple consequence of some results of Section 1.2 and the arguments used in the proofs of Theorems 1.3.1 and 1.3.3.

Theorem 1.3.3. *Let $N \geqslant 1$, and let the matrices $\mathscr{T}_+(\xi)$ and $\mathscr{T}_-(\xi)$ be defined by* (1.2.17) *and* (1.3.2)*, respectively. If conditions* 1–4 *of Theorem* 1.2.2 *are fulfilled and*

$$B(\xi)\,\mathrm{tr}\,\big(\mathscr{T}_+^{-1}\mathscr{T}_-\big)\sup\left|\frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 \leqslant \text{const} \tag{1.3.8}$$

for almost all $\xi \in \mathbb{R}^{n-1}$, then the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2_{\mathscr{T}_+^{-1/2}}\right) \tag{1.3.9}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Proof. Suppose that $\mathfrak{M}(\xi) = \mathscr{T}_+^{-1/2}(\xi)$. Then

$$\mathrm{tr}\Big[\big(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_+\big)^{-1}\Big] = N \quad\text{and}\quad \mathrm{tr}\,\big(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_-\big) = \mathrm{tr}\,\big(\mathscr{T}_+^{-1}\mathscr{T}_-\big)\,.$$

Taking into account inequality (1.3.7) and condition 1 of Theorem 1.2.2, we conclude that the estimate (1.3.3) follows from (1.3.8). □

[16] This result was proved by V. È. Katsnelson ([Kats67], pp. 58–61).

Remark 1.3.4. The estimate (1.3.9) holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ also in the case when conditions 1-4 of Theorem 1.2.2 are fulfilled and the matrices $\mathscr{T}_+$ and $\mathscr{T}_-$ satisfy for almost all $\xi \in \mathbb{R}^{n-1}$ the relation

$$\mathscr{T}_- \mathscr{T}_+^{-1} \mathscr{T}_- \leqslant \text{const}\, \mathscr{T}_-. \tag{1.3.10}$$

Indeed, condition 1 of Theorem 1.2.2 and the estimate (1.3.7) imply the inequality

$$B(\xi) \sup \left| \frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right| \leqslant \text{const}.$$

On the other hand, (1.3.10) is clearly equivalent to the estimate

$$\operatorname{tr}\left(\mathscr{T}_+^{-1}\mathscr{T}_-\right) \leqslant \text{const}.$$

Remark 1.3.5. Obviously, condition (1.3.10) is equivalent to the following statement: For almost all $\xi \in \mathbb{R}^{n-1}$ we have

$$\int_{-\infty}^{\infty} \left| \frac{\mathscr{L}_{T_-}(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)} \right|^2 d\tau \leqslant \text{const} \int_{-\infty}^{\infty} \left| \frac{\mathscr{L}_{T_+}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right|^2 d\tau, \tag{1.3.11}$$

where the $1 \times m$ matrices $\mathscr{L}_{T_-}$ and $\mathscr{L}_{T_+}$ are determined by the decomposition

$$\frac{\mathscr{L}_T(\xi;\tau)}{\mathscr{P}(\xi;\tau)} = \frac{\mathscr{L}_{T_+}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} + \frac{\mathscr{L}_{T_-}(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)}.$$

Here the $1 \times m$ matrix $\mathscr{L}_T(\xi;\tau)$ is an arbitrary linear combination of the rows of the matrix $T(\xi;\tau)$.

The following result is an example of an assertion that is not related (as the case was up to now) to an individual matrix R, but to the whole class of the matrices R such that the corresponding matrices S satisfy condition 1 of Theorem 1.2.2.

Theorem 1.3.6. *Let $\mathscr{P}_+(\xi;\tau)$ have no real τ-roots, let* ord $\mathscr{P}_+(\xi;\tau) = N \geqslant 1$ *for all $\xi \in \mathbb{R}^{n-1}$, and let the $N \times N$ matrices $\mathscr{T}_+$ and $\mathscr{T}_-$ be defined by* (1.2.17) *and* (3.2), *respectively. If for almost all $\xi \in \mathbb{R}^{n-1}$ the rows of the matrix $T(\xi;\tau)$ are linearly independent modulo $\mathscr{P}_+(\xi;\tau)$ and the matrices $\mathscr{T}_\pm$ satisfy condition* (1.3.10), *then the estimate* (1.3.9) *holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ and for any $1 \times m$ matrix $R(\xi;\tau)$ for which the corresponding matrix $S(\xi;\tau)$ satisfies condition* 1 *of Theorem* 1.2.2.

Proof. According to Proposition 1.2.8, we can construct the $1 \times N$ matrix $G(\xi;\tau)$ that figures in representation (1.2.14) and satisfies identity (1.2.7). By the second part of Proposition 1.2.10, the matrix G admits representation (1.2.18). As noted in the proof of Theorem 1.3.1, inequalities (1.3.4) and (1.3.5) follows from (1.2.18).

Suppose that $\mathfrak{M}(\xi) = \mathscr{T}_+^{-1/2}(\xi)$. We use condition 1 of Theorem 1.2.2, inequality (1.3.7) and condition (1.3.10) which, as we have already mentioned above, is equivalent to the inequality

$$\operatorname{tr}\left(\mathscr{T}_+^{-1}\mathscr{T}_-\right) \leqslant \text{const}.$$

Then (1.3.4) and (1.3.5) imply the inequalities

$$\begin{aligned}
&B(\xi)\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)\mathscr{T}_{+}^{-1/2}(\xi)}{\mathscr{P}_{+}(\xi;\tau)}\right|^{2}d\tau\leqslant\text{const},\\
&B(\xi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(\xi;\tau)\mathscr{T}_{+}^{1/2}(\xi)\mathscr{T}_{+}^{-1/2}(\xi)T_{-}(\xi;\eta)}{\mathscr{P}_{+}(\xi;\tau)\mathscr{P}_{-}(\xi;\tau)}\right|^{2}d\tau\, d\eta\leqslant\text{const}.
\end{aligned}\tag{1.3.12}$$

In representation (1.2.14) we replace $\mathring{T}(\xi;\tau)$ by $\mathscr{T}_{+}^{-1/2}(\xi)\mathring{T}(\xi;\eta)$ and $G(\xi;\tau)$ by $G(\xi;\tau)\mathscr{T}_{+}^{1/2}(\xi)$. Using (1.2.5), (1.3.12) and the result of the substitution, we conclude that the inequality

$$\begin{aligned}
&B(\xi)\int_{0}^{\infty}\left|\mathring{S}\,(\xi;-\mathrm{i}\,d/dt)\,\varphi\right|^{2}dt\\
&\leqslant\text{const}\left[\int_{0}^{\infty}\left|\mathring{\mathscr{P}}\,(\xi;-\mathrm{i}\,d/dt)\,I\varphi\right|^{2}dt+\left|\mathscr{T}_{+}^{-1/2}(\xi)\mathring{T}\,(\xi;-\mathrm{i}\,d/dt)\,\varphi|_{t=0}\right|^{2}\right]
\end{aligned}$$

holds for all $\xi\in\mathbb{R}^{n-1}$ and all $\varphi\in\mathbf{C}_{0}^{\infty}(\mathbb{R}_{+}^{n})$. Then, in view of Lemma 1.1.3, the inequality

$$\begin{aligned}
&B(\xi)\int_{0}^{\infty}\left|\mathring{R}\,(\xi;-\mathrm{i}\,d/dt)\,v\right|^{2}dt\\
&\leqslant\text{const}\left[\int_{0}^{\infty}\left|\mathring{P}\,(\xi;-\mathrm{i}\,d/dt)\,v\right|^{2}dt+\left|\mathscr{T}_{+}^{-1/2}(\xi)\mathring{Q}\,(\xi;-\mathrm{i}\,d/dt)\,v|_{t=0}\right|^{2}\right]
\end{aligned}\tag{1.3.13}$$

is valid for all $v\in\mathbf{C}_{0}^{\infty}(\mathbb{R}_{+}^{1})$.

We consider any $u(x;t)\in\mathbf{C}_{0}^{\infty}(\mathbb{R}_{+}^{n})$ and set in (1.3.13) $v=v_{\xi}(t)=\hat{u}(\xi;t)$. Multiplying both parts of the resulting inequality by $|p(\xi)|^{2}=|p_{0}(\xi)|^{2/m}$, where $p_{0}(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}(\xi;\tau)=\det P(\xi;\tau)$, and integrating over ξ, we conclude that $u(x;t)$ satisfies (1.3.9). □

The scalar version of Theorem 1.3.6 (see Corolary 1.3.7) was proved by a direct method in the work of M. Schechter [Sch64]. Now we show that this result (contrary to Theorem 1.3.6 itself) follows directly from Theorem 1.3.3.

Corollary 1.3.7. *Let $m=1$, let the polynomial $\mathscr{P}_{+}(\xi;\tau)$ has no real τ-roots, and let* ord $\mathscr{P}_{+}(\xi;\tau)=N\geqslant 1$ *for almost all $\xi\in\mathbb{R}^{n-1}$. Define the $N\times N$ matrices $\mathfrak{D}_{\pm}$*

by the equalities

$$\mathfrak{D}_{\pm}(\xi) = \int_{-\infty}^{\infty} \frac{\mathsf{Q}_{\pm}(\xi;\tau)\mathsf{Q}^*_{\pm}(\xi;\tau)}{|\mathscr{P}_{\pm}(\xi;\tau)|^2} d\tau,$$

where

$$\frac{\mathsf{Q}(\xi;\tau)}{\mathscr{P}(\xi;\tau)} = \frac{\mathsf{Q}_{+}(\xi;\tau)}{\mathscr{P}_{+}(\xi;\tau)} + \frac{\mathsf{Q}_{-}(\xi;\tau)}{\mathscr{P}_{-}(\xi;\tau)},$$

and $\mathsf{Q}(\xi;\tau) = \{\mathsf{Q}_j(\xi;\tau)\}$ *denotes a given* $N \times 1$ *matrix satisfying*

$$\operatorname{ord} \mathsf{Q}_j(\xi;\tau) < \operatorname{ord} \mathscr{P}(\xi;\tau) = J.$$

If for almost all $\xi \in \mathbb{R}^{n-1}$ *the polynomials* $\mathsf{Q}_j(\xi;\tau)$ *are linearly independent modulo* $\mathscr{P}_{+}(\xi;\tau)$ *and the matrices* $\mathfrak{D}_{\pm}$ *satisfy the relation*

$$\mathfrak{D}_{-}\mathfrak{D}_{+}^{-1}\mathfrak{D}_{-} \leqslant \operatorname{const} \mathfrak{D}_{-},$$

then the estimate

$$\|\mathscr{R}(D)u\|^2_{B^{1/2}} \leqslant C \left(\|\mathscr{P}(D)u\|^2 + \langle\!\langle \mathsf{Q}(D)u \rangle\!\rangle^2_{\mathfrak{D}^{-1/2}} \right)$$

holds true for all functions $u \in \mathbf{C}_0^{\infty}(\mathbb{R}^n_+)$ *and for any polynomial* $\mathscr{R}(\xi;\tau)$ *satisfying the inequality*

$$B^{1/2}(\xi) \sup \left| \frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right| \leqslant \operatorname{const}$$

a.e. in $\mathbb{R}^{n-1}$.

Proof. We set $\mathscr{R}_1(\xi;\tau) = p_0(\xi)\overline{\mathscr{P}}_{+}(\xi;\tau)\mathscr{P}_{-}(\xi;\tau)$, where $p_0(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}(\xi;\tau)$. Since all the τ-roots of the polynomial $\mathscr{R}_1$ lie in the half-plane $\operatorname{Im}\zeta < 0$ $(\zeta = \tau + i\sigma)$ and

$$B^{1/2}(\xi) \sup \left| \frac{\mathscr{R}(\xi;\tau)}{\mathscr{R}_1(\xi;\tau)} \right| = B^{1/2}(\xi) \sup \left| \frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right| \leqslant \operatorname{const},$$

then, in accordance with Remark 1.2.4 $(m = 1)$, the estimate

$$\|\mathscr{R}(D)u\|^2_{B^{1/2}} \leqslant C \|\mathscr{R}_1(D)u\|^2$$

is valid for all $u \in \mathbf{C}_0^{\infty}(\mathbb{R}^n_+)$.

On the other hand, $\mathscr{R}_1/\mathscr{P} = \overline{\mathscr{P}}_{+}/\mathscr{P}_{+}$, i.e., we have

$$\frac{\mathscr{R}_1(\xi;\tau)}{\mathscr{P}(\xi;\tau)} = c_1(\xi) + \frac{\mathscr{R}_{1+}(\xi;\tau)}{\mathscr{P}_{+}(\xi;\tau)}$$

with $\operatorname{ord} \mathscr{R}_{1+} < \operatorname{ord} \mathscr{P}_{+}$. It follows directly from this decomposition that

$$\sup \left| \frac{\mathscr{R}_{1+}(\xi;\tau)}{\mathscr{P}_{+}(\xi;\tau)} \right| \leqslant \operatorname{const} \sup \left| \frac{\mathscr{R}_1(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right|.$$

We note that the polynomials $\mathscr{R}_1(\xi;\tau)$ and $\mathscr{P}_+(\xi;\tau)$ are relatively prime. Moreover, in the case $m = 1$, condition 4 of Theorem 1.2.2 follows from conditions 1-3 of the same theorem. Therefore, in view of Theorem 1.3.3 ($m = 1$, $B(\xi) = 1$), the estimate

$$\|\mathscr{R}_1(D)u\|^2 \leqslant C\left(\|\mathscr{P}(D)u\|^2 + ⟪\mathsf{Q}(D)u⟫^2_{\mathfrak{D}^{-1/2}}\right)$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. Thus, the statement of Corollary 1.3.7 follows from this theorem. □

We return to the general case of arbitrary m and consider a direct corollary of Theorem 1.3.3 concerning the matrices $P(\xi;\tau)$ with determinants having a unique τ-root with negative imaginary part.

Corollary 1.3.8. *Let $\mathscr{P}_-(\xi;\tau) = \tau - z(\xi)$, and let the $N \times N$ matrix $\mathscr{T}_+(\xi)$ and the $N \times m$ matrix $T_-(\xi) = \{T_{\lambda k-}(\xi)\}$ be defined by (1.2.17) and (1.2.4), respectively. If conditions 1–4 of Theorem 1.2.2 are fulfilled and*

$$\sum_{\lambda=1}^{N}\sum_{k=1}^{m} |T_{\lambda k-}(\xi)|^2 \operatorname{tr}\left[\mathscr{T}_+^{-1}(\xi)\right] \leqslant \operatorname{const}|\operatorname{Im} z(\xi)| \tag{1.3.14}$$

for almost all $\xi \in \mathbb{R}^{n-1}$, then the estimate (1.3.9) holds true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Proof. By hypothesis, the polynomial $\mathscr{P}(\xi;\tau)$ has only one root $\tau = z(\xi)$ in the half-plane $\operatorname{Im}\zeta < 0$ ($\zeta = \tau + i\sigma$). Hence the matrix $T_-(\xi;\tau) = \{T_{\alpha j-}(\xi;\tau)\}$ does not depend on τ, and then it follows from (1.3.2) that the entries of the matrix T_- have the form

$$\mathscr{T}_{\alpha\beta-}(\xi) = \pi\sum_{k=1}^{m} T_{\alpha k-}(\xi)\overline{T}_{\beta k-}(\xi)/|\operatorname{Im} z(\xi)|.$$

Therefore,

$$\operatorname{tr}(\mathscr{T}_-(\xi)) = \pi\sum_{\alpha=1}^{N}\sum_{k=1}^{m}|T_{\alpha k}(\xi)|^2/|\operatorname{Im} z(\xi)|.$$

Since the inequality

$$\operatorname{tr}\left(\mathscr{T}_+^{-1}\mathscr{T}_-\right) \leqslant N\operatorname{tr}\left(\mathscr{T}_+^{-1}\right)\operatorname{tr}(\mathscr{T}_-)$$

is obviously valid, assumption (1.3.14) yields

$$\operatorname{tr}\left(\mathscr{T}_+^{-1}\mathscr{T}_-\right) \leqslant \operatorname{const}.$$

Finally, estimating $\sup|S_+/\mathscr{P}_+|^2$ in accordance with inequality (1.3.7) and using condition 1 of Theorem 1.2.2, we conclude that (1.3.14) implies (1.3.8). □

1.3.3 The case of the diagonal matrix $\mathfrak{M}(\xi)$

In this subsection we formulate sufficient condition for the validity of the estimate (1.3.16) (Theorem 1.3.9). This estimate is a special case of the estimate (1.3.1), which corresponds to the case of the diagonal matrix $\mathfrak{M}(\xi)$ with eigenvalues $(1+|\xi|^2)^{\mu_\beta/2}$. This situation arises often in applications. We will show that the sufficient condition of Theorem 1.3.9 is also necessary in the case $N = 1$. In general, this condition is not necessary for $N > 1$.

Theorem 1.3.9. *Let the matrices S_+ and $\mathscr{T}_\pm = \{\mathscr{T}_{\alpha\beta\pm}\}$ be defined by decomposition (1.2.3) and equalities (1.2.17) and (1.3.2), respectively, and let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in \mathbb{R}^N$. We also consider the $N \times N$ matrices $\mathscr{T}_{-(\boldsymbol{\mu})} = \{(1+|\xi|^2)^{\mu_\alpha} \mathscr{T}_{\alpha\beta-}(\xi)\}$ and $\left(\mathscr{T}_+^{-1}\right)_{(-\boldsymbol{\mu})} = \{(1+|\xi|^2)^{-\mu_\beta} t_{\alpha\beta}(\xi)\}$, where $t_{\alpha\beta}$ are the entries of the matrix $\mathscr{T}_+^{-1}$. If*

$$B(\xi)\,\mathrm{tr}\left(\mathscr{T}_+^{-1}\right)_{(-\boldsymbol{\mu})}\left[1+\mathrm{tr}\,\mathscr{T}_{-(\boldsymbol{\mu})}\right]\sup\left|\frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 \leqslant \mathrm{const} \tag{1.3.15}$$

for almost all $\xi \in \mathbb{R}^{n-1}$ and conditions 1–4 of Theorem 1.2.2 are satisfied, then the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2_{\boldsymbol{\mu}}\right) \tag{1.3.16}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

The proof is based on a direct calculation of the traces $\mathrm{tr}\left[(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_+)^{-1}\right]$ and $\mathrm{tr}\,(\mathfrak{M}^*\mathfrak{M}\mathscr{T}_-)$ figuring in the left-hand side of (1.3.3). Here $\mathfrak{M}(\xi)$ is the matrix (1.2.26).

Remark 1.3.10. In the case $m = 1$, $B(\xi) = 1$, condition (1.3.15) is not necessary for the validity of the estimate (1.3.16) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Suppose, for example, that $N = 2$, $m = 1$, $\mathscr{R}(\xi;\tau) = 1$, $\mathscr{P}(\xi;\tau) = (\tau - i\varkappa_1(\xi))(\tau - i\varkappa_2(\xi))$, $\mathrm{Q}_j(\xi;\tau) = \tau - \varkappa_j(\xi)$ $(j = 1,2)$ and $\boldsymbol{\mu} = (\mu_1,\mu_2) \in \mathbb{R}^2$. In Subsection 1.4.7 we prove that if $\varkappa_j(\xi) > 0$ $(j = 1,2)$ and $\varkappa_1(\xi) \neq \varkappa_2(\xi)$, then assumption (1.3.15) takes for almost all $\xi \in \mathbb{R}^{n-1}$ the form:

$$\left.\begin{aligned}(1+\varkappa_2/\varkappa_1)^2\varkappa_2^{-1}(\varkappa_1-\varkappa_2)^{-2}(1+|\xi|^2)^{-\mu_1} &\leqslant \mathrm{const},\\ (1+\varkappa_1/\varkappa_2)^2\varkappa_1^{-1}(\varkappa_1-\varkappa_2)^{-2}(1+|\xi|^2)^{-\mu_2} &\leqslant \mathrm{const}\end{aligned}\right\}. \tag{1.3.17}$$

It will be also shown that, under the assumptions introduced above, one of necessary conditions for the validity of the estimate (1.3.16) (similarly to condition 6 of Theorem 1.2.2) can be represented a.e. in $\mathbb{R}^{n-1}$ in the form

$$\left.\begin{aligned}\varkappa_2^{-1}(\varkappa_1-\varkappa_2)^{-2}(1+|\xi|^2)^{-\mu_1} &\leqslant \mathrm{const},\\ \varkappa_1^{-1}(\varkappa_1-\varkappa_2)^{-2}(1+|\xi|^2)^{-\mu_2} &\leqslant \mathrm{const}\end{aligned}\right\}. \tag{1.3.18}$$

It is obvious that (1.3.18) follows from (1.3.17), while the converse is, in general, incorrect.

Remark 1.3.11. If $N = 1$, then (1.3.15) is also necessary for the validity of the estimate (1.3.16) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

This claim follows from Corollary 1.2.11. To see this, replace in the formulation of Corollary 1.2.11 the estimate (1.2.1) by the estimate (1.3.16), and inequality (1.2.21) by the inequality

$$B(\xi)|\alpha(\xi)|^2 \leqslant \text{const}\,(1+|\xi|^2)^\mu \operatorname{Im}\zeta(\xi), \tag{1.3.19}$$

where $\mu \in \mathbb{R}^1$ denotes the exponent of the norm $\langle\!\langle \cdot \rangle\!\rangle_\mu$.

Indeed, let $N = 1$, $\dot{\mathscr{P}}_+(\xi;\tau) = \tau - \zeta(\xi)$, let $Q(\xi;\tau) = \{Q_j(\xi;\tau)\}$ be a given $1 \times m$ matrix, let $S = \{S_k(\xi;\tau)\}$ and $T = \{T_k(\xi;\tau)\}$ be the matrices defined by equations (1.2.2), and let $S_\pm = \{S_{k\pm}(\xi;\tau)\}$ and $T_\pm = \{T_{k\pm}(\xi;\tau)\}$ be the $1\times m$ matrices defined by decompositions (1.2.3) and (1.2.4), respectively. Since $N = 1$, the matrices $S_+(\xi;\tau) = S_+(\xi)$ and $T_+(\xi;\tau) = T_+(\xi)$ do not depend on τ. Moreover, it follows from (1.2.19) and the decompositions (1.2.3) and (1.2.4) that

$$S_+(\xi) = \alpha(\xi) T_+(\xi), \tag{1.3.20}$$

where $\alpha(\xi)$ is the measurable function figuring in (1.2.19).

One can verify directly that

$$\operatorname{tr}\left(\mathscr{T}_+^{-1}\right)_{(-\mu)} = \pi^{-1}|T_+(\xi)|^{-2}\operatorname{Im}\zeta(\xi)(1+|\xi|^2)^{-\mu},$$

$$\operatorname{tr}\mathscr{T}_{-(\mu)} = (1+|\xi|^2)^\mu \int_{-\infty}^{\infty} \left|\frac{T_-(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)}\right|^2 d\eta,$$

$$\sup\left|\frac{S_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 = [\operatorname{Im}\zeta(\xi)]^{-2}\,|S_+(\xi)|^2.$$

Using relation (1.3.20), we find that in the considered case condition (1.3.15) is equivalent to inequalities (1.3.19) and (1.2.20). It follows from Corollary 1.2.11 that (1.3.19) and (1.2.20) are necessary conditions for the validity of the estimate (1.3.16).

1.3.4 Sufficient conditions for the validity of the estimate (1.3.21)

In this subsection we consider the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2 + \|u\|^2\right), \tag{1.3.21}$$

which differs from the estimate (1.2.1) by the additional term $\|u\|^2$ on the right-hand side. Of course, the conditions of Theorem 1.2.2 are also sufficient for the validity of (1.3.21) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

In the case when the leading coefficient $p_0(\xi)$ of the polynomial $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$ is uniformly bounded from below in some ball in $\mathbb{R}^{n-1}$, one can formulate the following strengthening of the above assertion:

Proposition 1.3.12. *Let $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ be given constants, let $|p_0(\xi)| \geqslant A_1$ and $B(\xi) \leqslant A_3$ for almost all $\xi \in \{\xi : \xi \in \mathbb{R}^{n-1}, |\xi| \leqslant A_2\}$, and let the conditions of Theorem 1.2.2 be satisfied for almost all $\xi \in \{\xi : \xi \in \mathbb{R}^{n-1}, |\xi| > A_2\}$. Assume also that $\max_k \operatorname{ord} P_{kj} = J_j$ and the polynomials $R_j(\xi;\tau)$ satisfy a.e. in the ball $|\xi| \leqslant A_2$ the condition*

$$\operatorname{ord} R_j(\xi;\tau) \leqslant J_j. \tag{1.3.22}$$

Here J_j are constants that do not depend on ξ. Then the estimate (1.3.21) holds true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$.

Proof. It follows from Theorem 1.1.19 that inequality (1.2.11) is valid for almost all $\xi \in \mathbb{R}^{n-1}$ with $|\xi| > A_2$ and for all $v \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$, in particular, for $v_\xi(t) = \hat{u}(\xi;t)$.

Suppose that $|\xi| \leqslant A_2$. We denote by $p_{kj}(\xi)$ the leading coefficient of the polynomial $P_{kj}(\xi;\tau)$. It is obvious that $p_0(\xi) = \det\{p_{kj}(\xi)\}$. Since $|p_0(\xi)| \geqslant A_1 > 0$ for almost all ξ satisfying $|\xi| \leqslant A_2$, the inequality

$$\sum_{j=1}^{m} \left|(-\mathrm{i}\, d/dt)^{J_j}\, \hat{u}_j(\xi;t)\right| \leqslant \text{const} \sum_{k=1}^{m} \left| \sum_{j=1}^{m} p_{kj}(\xi)\, (-\mathrm{i}\, d/dt)^{J_j}\, \hat{u}(\xi;\tau)\right| \tag{1.3.23}$$

holds a.e. in this ball. Since the coefficients of the polynomials $P_{kj}(\xi;\tau)$ are locally bounded, it follows from (1.3.23) that for almost all ξ satisfying $|\xi| \leqslant A_2$ we have

$$\sum_{j=1}^{m}\sum_{r=0}^{J_j} \left|(-\mathrm{i}\, d/dt)^{r}\, \hat{u}_j(\xi;\tau)\right| \leqslant \text{const} \left\{ \sum_{k=1}^{m} \left| \sum_{j=1}^{m} P_{kj}\,(\xi; -\mathrm{i}\, d/dt)\, \hat{u}_j(\xi;t)\right| + \sum_{j=1}^{m}\sum_{r=0}^{J_j-1} \left|(-\mathrm{i}\, d/dt)^{r}\, \hat{u}_j(\xi;t)\right| \right\}. \tag{1.3.24}$$

Using the well-known inequality

$$\sum_{r=0}^{J_j-1} \int_0^\infty \left|(-\mathrm{i}\, d/dt)^{r}\, \hat{u}_j(\xi;t)\right|^2 dt \leqslant \varepsilon \int_0^\infty \left|(-\mathrm{i}\, d/dt)^{J_j}\, \hat{u}(\varepsilon;t)\right|^2 dt + c(\xi) \int_0^\infty |\hat{u}(\xi;t)|^2 dt$$

and taking into account the local boundedness of the coefficients of polynomials $R_j(\xi;\tau)$, inequalities (1.3.22) and (1.3.24), as well as the validity of the estimate

$B(\xi) \leqslant A_3$ a.e. in the ball $|\xi| \leqslant A_2$, we conclude that the estimate

$$B(\xi) \int_0^\infty |R(\xi; -\mathrm{i}\, d/dt)\, \hat{u}(\xi;t)|^2\, dt \leqslant \text{const} \left[\int_0^\infty |P(\xi; -\mathrm{i}\, d/dt)\, \hat{u}(\xi;t)|^2\, dt + \int_0^\infty |\hat{u}(\xi;t)|^2 dt \right]$$

holds for almost all ξ with $|\xi| \leqslant A_2$. This means that for almost all $\xi \in \mathbb{R}^{n-1}$

$$\int_0^\infty |R(\xi; -\mathrm{i}\, d/dt)\, \hat{u}(\xi;t)|^2\, dt \leqslant \text{const} \left[\int_0^\infty |P(\xi; -\mathrm{i}\, d/dt)\, \hat{u}(\xi;t)|^2\, dt + |Q(\xi; -\mathrm{i}\, d/dt)\, \hat{u}(\xi;t)|_{t=0}|^2 + \int_0^\infty |\hat{u}(\xi;t)|^2 dt \right],$$

which yields (1.3.21). □

Similarly, if $|p_0(\xi)| \geqslant A_1$, $B(\xi) \leqslant A_3$, assumption (1.3.22) is fulfilled a.e. in the ball $|\xi| \leqslant A_2$, and conditions 1–5 of Theorem 1.2.2 are satisfied for almost all $\xi \in \{\xi : \xi \in \mathbb{R}^{n-1}, |\xi| > A_2\}$, then the estimate

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C_0 \left(\|P(D)u\|^2 + \|u\|^2 \right) \tag{1.3.25}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ such that $Q(D)u(x;0) = 0$.

Remark 1.3.13. Condition (1.3.22) is satisfied, for example, if the coefficients of the polynomials $R_j(\xi;\tau)$ and $P_{kj}(\xi;\tau)$ are themselves polynomials of the variable $\xi \in \mathbb{R}^{n-1}$ and inequality (1.2.5) holds true for almost all ξ such that $|\xi| > A_2$ and for all $\tau \in \mathbb{R}^1$.

Indeed, let $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ and $|\xi| > A_2$. From the identity

$$\sum_{l=1}^m R_l \zeta_l = \mathscr{P}^{-1} \sum_{l,k,j=1}^m R_l P^{lk} P_{kj} \zeta_j$$

(here $P^{lk}(\xi;\tau)$ are the entries of the matrix $P^{\mathrm{c}}(\xi;\tau)$) and inequality (1.2.5) it follows that

$$B(\xi) \left| \sum_{l=1}^m R_l \zeta_l \right| \leqslant \text{const} \sum_{k=1}^m \left| \sum_{l=1}^m P_{kl} \zeta_l \right| \tag{1.3.26}$$

for all $\tau \in \mathbb{R}^1$. We set $\zeta_j = 1$ and $\zeta_l = 0$ for $l \neq j$. Then (1.3.26) implies for all $\tau \in \mathbb{R}^1$ and for almost all ξ with $|\xi| > A_2$ the inequality

$$B(\xi) |R_j(\xi;\tau)| \leqslant \text{const} \sum_{k=1}^m |P_{kj}(\xi;\tau)|. \tag{1.3.27}$$

Since $B(\xi) > 0$ a.e. in $\mathbb{R}^{n-1}$ and R_j, P_{kj} are, by assumption, polynomials of $(\xi;\tau) \in \mathbb{R}^n$, relation (1.3.27) yields (1.3.22).

1.4 Examples

In this section, we consider concrete operators and consider some examples of estimates whose validity follows from the general theorems proved above.

1.4.1 Generalized-homogeneous quasielliptic systems

The main result of this subsection is a necessary and sufficient condition for the validity of the estimate (1.4.4) (and, in particular, of the estimate (1.4.9)), where P is a generalized-homogeneous quasielliptic matrix. We shall consider only matrices for which all numbers $s_1, \ldots, s_m; t_1, \ldots, t_m$, occuring in conditions 1 and 2 given below, are integers. However, more general definitions of quasielliptic matrices that allow non-integer (rational) s_j, t_k (see, for example, [Vol60a], [Vol62]) are also possible. It will be seen from the discussion below that the used estimation method requires at least integer numbers s_j.

First, we define the notion of a generalized-homogeneous quasielliptic matrix.

Let $b_\varrho > 0$ ($\varrho = 1, \ldots, n-1$) be given real numbers, and let $\mathbf{b} = (b_1, \ldots, b_{n-1}, 1)$. We say that a function $f(\xi;\tau)$ is a generalized-homogeneous function of degree k with respect to the weight $\mathbf{b}$ ($\deg_{\mathbf{b}} f = k$), if for any $\lambda > 0$ and all $(\xi;\tau) \in \mathbb{R}^n$ the relation

$$f(\lambda^{b_1}\xi_1, \ldots, \lambda^{b_{n-1}}\xi_{n-1}; \lambda\tau) = \lambda^k f(\xi;\tau)$$

is satisfied.

We call the $m \times m$ matrix $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$ generalized-homogeneous with respect to the weight $\mathbf{b}$, if it satisfies the following conditions:

1. There exist nonnegative integers $s_1, \ldots, s_m$ ($\min s_k = 0$) and $t_1, \ldots, t_m$ such that $P_{kj}(\xi;\tau)$ are generalized-homogeneous polynomials in $(\xi;\tau) \in \mathbb{R}^n$ with respect to the weight $\mathbf{b}$ satisfying $\deg P_{kj} = t_j - s_k$, and $P_{kj}(\xi;\tau) \equiv 0$ for $t_j - s_k < 0$.
2. Let $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$. Then $\deg_{\mathbf{b}} \mathscr{P} = J = \sum\limits_{k=1}^{m} (t_j - s_j)$.

Since the last component of the weight vector $\mathbf{b}$ is 1, we have $\operatorname{ord} \mathscr{P}(\xi;\tau) = J$ provided that $p_0(\xi) = 1$.

A generalized-homogeneous matrix $P(\xi;\tau)$ is called quasielliptic, if $\mathscr{P}(\xi;\tau) \neq 0$ for all real $(\xi;\tau) \neq 0$ or, what is the same, if the estimate

$$|\mathscr{P}(\xi;\tau)| \geqslant C \left(|\tau| + \langle\xi\rangle\right)^J \tag{1.4.1}$$

holds for all $(\xi;\tau) \in \mathbb{R}^n$. Here $\langle\xi\rangle = \sum\limits_{\varrho=1}^{n-1} |\xi_\varrho|^{1/b_\varrho}$.

We also say that a generalized-homogeneous quasielliptic matrix $P(\xi;\tau)$ is properly quasielliptic of type h, if for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$ the polynomial $\mathscr{P}(\xi;\tau)$ has the same number h (counting multiplicities) of τ-roots with positive imaginary part.

Finally, we define the seminorms that occur in the estimate (1.4.4). Conditions for the validity of (1.4.4) for quasielliptic systems are established below.

For vector functions $\varphi(x) = (\varphi_1(x), \dots, \varphi_N(x)) \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$ we set

$$[\varphi]^2_{\boldsymbol{\mu}} = \int\limits_{\mathbb{R}^{n-1}} \sum_{\alpha=1}^{N} \langle\xi\rangle^{2\mu_\alpha} |\widehat{\varphi}_\alpha(\xi)|^2 d\xi, \tag{1.4.2}$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in \mathbb{R}^N$ and $\widehat{\varphi}_\alpha(\xi)$ is the Fourier transform of the function $\varphi_\alpha(\xi)$.

Let $u = (u_1, \dots, u_m) \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$, and let $\mathbf{a} = (a_1, \dots, a_m)$ be an integer positive multi-index. We set

$$\|u\|^2_{\mathbf{a},\mathbf{b}} = \sum_{j=1}^{m} \sum_{(\boldsymbol{\alpha},\mathbf{b})=a_j} \|D^{\boldsymbol{\alpha}} u_j\|^2, \tag{1.4.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $(\boldsymbol{\alpha}, \mathbf{b}) = \sum\limits_{\varrho=1}^{n} \alpha_\varrho b_\varrho$ $(b_n = 1)$.

Proposition 1.4.1. *Let $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$, $R(\xi;\tau) = \{R_j(\xi;\tau)\}$ and $Q(\xi;\tau) = \{Q_{\alpha j}(\xi;\tau)\}$ be generalized-homogeneous with respect to the weight $\mathbf{b} = (b_1, b_2, \dots, b_{n-1}, 1)$ polynomial $m \times m$, $1 \times m$, and $N \times m$ matrices, respectively, and let $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau)$ with $p_0(\xi) = 1$. Suppose also that $\deg_{\mathbf{b}} P_{kj} = t_j - s_k$, $\deg_{\mathbf{b}} \mathscr{P} = \sum_{j=1}^m (t_j - s_j) = J \geqslant 1$, $\deg_{\mathbf{b}} R_j = t_j + l$ and $\deg_{\mathbf{b}} Q_{\alpha j} = t_j - \varkappa_\alpha$, where t_j, s_k are the integers defined by conditions 1 and 2, $\varkappa_\alpha$ $(\alpha = 1, \dots, N)$ is another set of integers, and l is an integer satisfying the condition $l \geqslant \max(0, 1 - \varkappa_\alpha)$. Let the matrix $P(\xi;\tau)$ be quasielliptic, and let for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$ the degrees of the polynomials (of τ) $\mathscr{M}(\xi;\tau)$ and $\mathscr{P}_+(\xi;\tau)$ be constant, and let $\operatorname{ord} \dot{\mathscr{P}}_+(\xi;\tau) = N \geqslant 1$. Finally, we set $\mathbf{s} + \mathbf{l} = (s_1 + l, \dots, s_m + l)$ and $\boldsymbol{\varkappa} + \mathbf{l} - \mathbf{1/2} = (\varkappa_1 + l - 1/2, \dots, \varkappa_N + l - 1/2)$.*

The estimate

$$\|R(D)u\|^2 \leqslant C \left(\|P(D)u\|^2_{\mathbf{s}+\mathbf{l},\mathbf{b}} + [Q(D)u]^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})} \right) \tag{1.4.4}$$

holds true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if conditions 2–4 of Theorem 1.2.2 are satisfied for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$.

Proof. Consider the $m \times m$ matrix

$$\mathfrak{R}'(\xi;\tau) = \{\delta_{jk}(\tau + \mathrm{i}\langle\xi\rangle)^{s_k + l}\} \tag{1.4.5}$$

and the $N \times N$ matrix

$$\mathfrak{M}'(\xi) = \{\delta_{\alpha\beta}\langle\xi\rangle^{\varkappa_\beta + l - (1/2)}\}. \tag{1.4.6}$$

We set

$$P_{\mathbf{s}+\mathbf{l}}(\xi;\tau) = \mathfrak{R}'(\xi;\tau)P(\xi;\tau), \tag{1.4.7}$$

$$Q_{\varkappa+\mathbf{l}-(1/2)}(\xi;\tau) = \mathfrak{M}'(\xi;\tau)Q(\xi;\tau). \tag{1.4.8}$$

It is obvious that $\langle\!\langle Q_{\varkappa+\mathbf{l}-(1/2)}(D)u \rangle\!\rangle^2 = [Q(D)u]^2_{\varkappa+\mathbf{l}-(1/2)}$. In addition, since $\mathbf{s}+\mathbf{l}$ is an integer vector, the norms $\|P(D)u\|_{\mathbf{s}+\mathbf{l},\mathbf{b}}$ and $\|P_{\mathbf{s}+\mathbf{l}}(D)u\|$ are equivalent for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. Hence, inequality (1.4.4) is an estimate of the type (1.2.27).

We show that the result of Proposition 1.4.1 follows from Corollary 1.2.13. One only has to put $B(\xi) = 1$ in the formulation of this corollary and replace the matrices (1.2.25) and (1.2.26) by the matrices $\mathfrak{R}'$ and $\mathfrak{M}'$, respectively.

Indeed, the conditions $\det P(\xi;\tau) \not\equiv 0$ and $\mathrm{mes}_{n-1}\mathscr{Z} = 0$ follow from the quasiellipticity of the matrix P. Let $S = \{S_k\} = RP^{\mathrm{c}}$ and $T = \{T_{\alpha k}\} = QP^{\mathrm{c}}$. A direct verification shows that S_k and $T_{\alpha k}$ are generalized-homogeneous polynomials with respect to the weight $\mathbf{b}$ satisfying $\deg_{\mathbf{b}} S_k = J + s_k + l$ and $\deg_{\mathbf{b}} T_{\alpha k} = J + s_k - \varkappa_\alpha$, respectively. Since $\mathrm{ord}\,\mathscr{P}(\xi;\tau) = J$ and $l \geqslant 1 - \varkappa_\alpha$, we have $\mathrm{ord}\, S_k \leqslant J + s_k + l$ and $\mathrm{ord}\, T_{\alpha k} \leqslant J + s_k + l - 1$.

From the quasiellipticity of the matrix $P(\xi;\tau)$ follows the validity of inequality (1.2.28) with $B(\xi) = 1$. To complete the proof, it remains to show that inequalities (1.2.29) and (1.2.30) (with $B(\xi) = 1$ and $\mathfrak{M}'$, $\mathfrak{R}'$ instead of $\mathfrak{M}$ and $\mathfrak{R}$) are satisfied for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$.

By assumption, the matrices R, P, Q are generalized-homogeneous. Hence, the functions $\mathscr{P}_+$, $\mathscr{M}$, $\dot{\mathscr{P}}_+$, S_{k+}, $T_{\alpha k+}$, $T_{\alpha k-}$ and G_α also have this property. Suppose that $\deg_{\mathbf{b}} \mathscr{P}_+ = r$. A direct calculation shows that $\deg_{\mathbf{b}} S_{k+} = r$, $\deg_{\mathbf{b}} \mathscr{P}_- = J - r$, $\deg_{\mathbf{b}} T_{\alpha k+} = r + l - \varkappa_\alpha$, $\deg_{\mathbf{b}} T_{\alpha k-} = J + s_k - \varkappa_\alpha - r$ and $\deg_{\mathbf{b}} G_\alpha = r + l + \varkappa_\alpha - 1$. Therefore, substituting $\xi_\varrho = \langle\xi\rangle^{1/b_\varrho}\xi'_\varrho$ ($\varrho = 1, \dots, n-1$), $\tau = \langle\xi\rangle\tau'$ and $\eta = \langle\xi\rangle\eta'$ in (1.2.29)–(1.2.30), we see that these inequalities hold for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$ if and only if they hold for all $\xi' \in \Sigma^{n-2} = \{\xi : \xi' \in \mathbb{R}^{n-1}, \langle\xi'\rangle = 1\}$.

Notice that $\mathrm{ord}\, G_\alpha(\xi;\tau) < \mathrm{ord}\,\mathscr{P}_+(\xi;\tau)$ and $\mathrm{ord}\, T_{\alpha k-}(\xi;\tau) < \mathrm{ord}\,\mathscr{P}_-(\xi;\tau) + s_k + l$. To prove the validity of inequalities (1.2.29)–(1.2.30) for all $\xi' \in \Sigma^{n-2}$, it suffices to show that the coefficients of the polynomial matrices G, T_- and the polynomials $\mathscr{P}_\pm(\xi;\tau)$ are piecewise continuous on Σ^{n-2}.

As we know (see [Tre59], p. 126), the assumption $p_0(\xi) = 1$ implies the piecewise continuity of the τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ on Σ^{n-2}. Combining this fact with inequalities (1.2.3) and (1.2.4) and the constancy of $\mathrm{ord}\,\mathscr{M}$ and $\mathrm{ord}\,\dot{\mathscr{P}}_+$, we see that the coefficients of the polynomials $\mathscr{P}_+$, $\mathscr{P}_-$, S_k, $T_{\alpha k-}$ and $T_{\alpha k+}$ also have this property. Finally, by the above arguments, we obtain the piecewise continuity of the coefficients of the polynomials G_α on Σ^{n-2} from the representation (1.2.18) and the linear independence of the rows of the matrix T modulo $\mathscr{P}_+$. □

The following assertion is a direct consequence of Proposition 1.4.1.

Corollary 1.4.2. *Let $P(\xi;\tau)$ be a generalized-homogeneous properly quasi-elliptic $m \times n$ matrix of the type $N \geqslant 1$, let $p_0(\xi) = 1$, and let $Q(\xi;\tau)$ be a generalized-homogeneous $N \times m$ matrix of boundary operators. The estimate*

$$\|u\|^2_{\mathbf{t+l,b}} \leqslant C \left(\|P(D)u\|^2_{\mathbf{t+l,b}} + [Q(D)u]^2_{\boldsymbol{\varkappa}+\mathbf{l}-(\mathbf{1/2})} \right), \tag{1.4.9}$$

where the weight $\mathbf{b}$ *and the numbers l, s_k, t_j, $\varkappa_\alpha$ are defined in Proposition 1.4.1, holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if the rows of the matrix $T(\xi;\tau) = Q(\xi;\tau)P^c(\xi;\tau)$ are linearly independent modulo $\mathscr{P}_+(\xi;\tau)$ for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$. (Here $\mathbf{t}+\mathbf{l} = (t_1+l,\dots,t_m+l)$, while $\mathbf{s}+\mathbf{l}$ and $\boldsymbol{\varkappa}+\mathbf{l}-\mathbf{1/2}$ are defined in Proposition 1.4.1).*

Remark 1.4.3. In the case $\mathbf{b} = (1,\dots,1)$, the matrix $P(\xi;\tau)$ is elliptic in the sense of Douglis and Nirenberg [DN55] and, in particular, $P(\xi;\tau)$ with $s_1 = \dots = s_m = 0$ is elliptic in the sense of Petrovsky [Pet39].

If $s_k = s'_k/2b$ and $t_j = t'_j/2b$, where s'_k, t'_k and b are integers, then the weight $\mathbf{b} = (1/2b,\dots,1/2b,1)$ with integers s_k and t_j corresponds to parabolic systems in the sense of Solonnikov [Sol65][17]. In particular, if $s_1 = \dots = s_m = 0$ and $t'_j = 2b - s'_j$, then the matrix $P(\xi;\tau)$ is parabolic in the sense of Petrovsky [Pet38], and if $t'_j = 2b - s'_j$ then $P(\xi;\tau)$ is called parabolic in the sense of T. Shirota [Shi57].

Note also that the weight of the form $(1/2b_1,\dots,1/2b_{n-1},1)$ with integers $b_1,\dots,b_{n-1} > 0$ corresponds to parabolic systems in the sense of Eidelman [Eid60], provided $s_1 = \dots = s_m = 0$ and $t_j = 2b_j n_j$ (here $n_j > 0$ are integers).

1.4.2 The Lamé system of the static elasticity theory

Using the Lamé operator of the isotropic elasticity theory we want to show that for elliptic operators one can also have "nonelliptic" estimates, that is, the estimates that cease to be valid after the operators are replaced by their principal homogeneous parts. Such an estimate is inequality (4.4.12), where $R(\xi;\tau) = \{\tau(\tau - \mathrm{i}|\xi|),0,0)$, $Q(\xi;\tau)$ is the 1×3 matrix $\{(1+\mathrm{i}\xi_1)|\xi|^{1/2}(\tau - \mathrm{i}|\xi|),0,0\}$, and $P(\xi;\tau)$ is the matrix (1.4.10).

It is known that the operator of the Lamé system of static isotropic elasticity theory has the symbol

$$P(\xi;\tau) = \begin{pmatrix} -(c+1)\tau^2 - |\xi|^2 & -c\xi_1\tau & -c\xi_2\tau \\ -c\xi_1\tau & -\tau^2 - |\xi|^2 - c\xi_1^2 & -c\xi_1\xi_2 \\ -c\xi_2\tau & -c\xi_1\xi_2 & -\tau^2 - |\xi|^2 - c\xi_2^2 \end{pmatrix} \tag{1.4.10}$$

[17] In the definition of parabolicity given in [Sol65] it is not required that the numbers t'_j and s'_k be divisible by $2b$. However, this divisibility is presupposed in the derivation of estimates in integral norms (see [Sol65], Theorems 5.4 and 5.5).

(here c is a constant and $|\xi|^2 = \xi_1^2 + \xi_2^2$). Since $\det P(\xi;\tau) = \mathscr{P}(\xi;\tau) = -(1+c)(\tau^2+|\xi|^2)^3$, the matrix (1.4.10) is homogeneous and elliptic in the sense of Petrovsky provided that $c \neq -1$. It is also clear that this matrix is properly elliptic of type 3 with $s_1 = s_2 = s_3 = 0$ and $t_1 = t_2 = t_3 = 2$. (We use the notation introduced in Subsection 1.4.1.)

Suppose that $Q(\xi;\tau) = \{Q_{\alpha j}(\xi;\tau)\}$ is a 3×3 matrix of boundary operators, the entries $Q_{\alpha j}(\xi;\tau)$ of which are homogeneous polynomials of deg $Q_{\alpha j} = 2 - \varkappa_\alpha$ $(\alpha, j = 1,2,3)$. Assume also that l is an integer satisfying the condition $l \geqslant \max(0, 1-\varkappa_\alpha)$. It follows from Corollary 1.4.2 that a necessary and sufficient condition for the validity of an "elliptic estimate" of the type (1.4.9)[18] is the linear independence of the rows of the matrix $Q(\xi;\tau)P^{\mathrm{c}}(\xi;\tau)$ modulo $(\tau - \mathrm{i}|\xi|)^3$, where

$$P^{\mathrm{c}}(\xi;\tau) = (\tau^2+|\xi|^2) \times \begin{pmatrix} \tau^2+(c+1)|\xi|^2 & -c\xi_1\tau & -c\xi_2\tau \\ -c\xi_1\tau & (c+1)\tau^2+|\xi|^2+c\xi_2^2 & -c\xi_1\xi_2 \\ -c\xi_2\tau & -c\xi_1\xi_2 & (c+1)\tau^2+|\xi|^2+c\xi_1^2 \end{pmatrix}. \tag{1.4.11}$$

Now we consider an example of nonelliptic estimate for the Lamé system. Let $P(\xi;\tau)$ be the matrix (1.4.10), and let the constant c satisfy the conditions $c \neq -1$ and $c \neq 0$; the first of this conditions ensures the ellipticity of the matrix P and the second one excludes from our consideration the case $P(D) = \Delta I$, where Δ is the Laplace operator.

Choose $R(\xi;\tau) = \{\tau(\tau - \mathrm{i}|\xi|), 0, 0\}$. Using (1.4.11), we obtain

$$S(\xi;\tau) = (\tau^2+|\xi|^2)\tau(\tau-\mathrm{i}|\xi|)\{\tau^2+(1+c)|\xi|^2, -c\xi_1\tau, -c\xi_2\tau\}.$$

Therefore, $\mathscr{M}(\xi;\tau) = (\tau-\mathrm{i}|\xi|)^2$. Since $\mathscr{P}_+(\xi;\tau) = (\tau-\mathrm{i}|\xi|)^3$, we get $\dot{\mathscr{P}}_+(\xi;\tau) = \tau - \mathrm{i}|\xi|$ and $N = 1$.

We show that for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^3_+)$,

$$\|R(D)u\|^2 \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2\right), \tag{1.4.12}$$

where the 1×3 matrix $Q(\xi;\tau)$ is defined by

$$Q(\xi;\tau) = \{(1+\mathrm{i}\xi_1)|\xi|^{1/2}(\tau-\mathrm{i}|\xi|), 0, 0\}.$$

To do this, we verify that conditions 1–6 of Corollary 1.2.11 with $B(\xi) = 1$ are fulfilled in the considered example.

The validity of condition 1 follows from the ellipticity of the polynomial $\mathscr{P}(\xi;\tau)$.

Calculating $T(\xi;\tau) = Q(\xi;\tau)P^{\mathrm{c}}(\xi;\tau)$, we obtain the equality

$$T(\xi;\tau) = (1+\mathrm{i}\xi_1)|\xi|^{1/2}(\tau-\mathrm{i}|\xi|)^2\{\tau^2+(1+c)|\xi|^2, -c\,\xi_1\tau, -c\,\xi_2\tau\},$$

which establishes condition 2.

[18] In the case under consideration, the weight $\mathbf{b} = (1,1,1)$, $[\cdot]_{\varkappa+l-(1/2)}$ is the norm in $\mathbf{L}^{\varkappa+l-(1/2)}(\mathbb{R}^2)$, while $\|\cdot\|_{l,1}$ and $\|\cdot\|_{2+l,1}$ are the norms in the spaces $\mathbf{L}^l_2(\mathbb{R}^3_+)$ and $\mathbf{L}^{2+l}(\mathbb{R}^3_+)$, respectively.

Since in the considered example $\zeta(\xi) = \mathrm{i}|\xi|$, we have

$$[T(\xi;\tau)/\mathscr{M}(\xi;\tau)]\big|_{\tau=\zeta(\xi)} = 2\mathrm{i}(1+\mathrm{i}\xi_1)|\xi|^{5/2}\{c|\xi|, -c\,\mathrm{i}\xi_1, -c\,\mathrm{i}\xi_2\} \neq 0$$

for $|\xi| \neq 0$. Hence condition 3 is verified.

Calculating

$$[S(\xi;\tau)/\mathscr{M}(\xi;\tau)]\big|_{\tau=\zeta(\xi)} = -2|\xi|^3\{c|\xi|, -c\mathrm{i}\xi_1, -c\mathrm{i}\xi_2\},$$

we see that condition 4 holds. Moreover, the coefficient $\alpha(\xi)$ figuring in (1.2.19) is determined by the equality

$$\alpha(\xi) = \mathrm{i}|\xi|^{1/2}(1+\mathrm{i}\xi_1)^{-1},$$

which immediately yields condition 6.

Finally, condition 5 follows from the representation

$$|T(\xi;\tau)/\mathscr{P}_-(\xi;\tau)|^2 = |\xi|(1+\xi_1^2)(\tau^2+|\xi|^2)^{-2}\left\{\left[\left(1+\frac{c}{4}\right)^2 + \frac{c^2}{16}\right]\tau^2 + \frac{c^2}{16}|\xi|^2\right\},$$

which is established by expanding of $T/\mathscr{P}$ into partial fractions (see (1.2.4)).

However, the estimate (1.4.12) ceases to be true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^3_+)$ if the matrix Q is replaced by its principal part $Q'(\xi;\tau) = \{\mathrm{i}\xi_1|\xi|^{1/2}(\tau - \mathrm{i}|\xi|), 0, 0\}$. Indeed, for the matrix Q' we get

$$[T'(\xi;\tau)/\mathscr{M}(\xi;\tau)]\big|_{\tau=\zeta(\xi)} = -2\xi_1|\xi|^{5/2}\{c|\xi|, -c\,\mathrm{i}\xi_1, -c\,\mathrm{i}\xi_2\}.$$

Therefore, the coefficient $\alpha(\xi)$ in (1.2.19) will be replaced by $\alpha'(\xi) = \xi_1^{-1}|\xi|^{1/2}$. Thus, condition 6 of Corollary 1.2.11 is violated for $B(\xi) = 1$.

1.4.3 The Cauchy–Riemann system

Consider the matrix

$$P(\xi;\tau) = \begin{pmatrix} \mathrm{i}\xi & -\mathrm{i}\tau \\ \mathrm{i}\tau & \mathrm{i}\xi \end{pmatrix} \tag{1.4.13}$$

corresponding to the Cauchy–Riemann system of equations

$$\partial u_1/\partial x - \partial u_2/\partial t = 0,$$
$$\partial u_1/\partial t - \partial u_2/\partial x = 0$$

in the half-space $\mathbb{R}^2_+$. Since $\det P(\xi;\tau) = \mathscr{P}(\xi;\tau) = -(\xi^2+\tau^2)$, the matrix $P(\xi;\tau)$ is homogeneous and elliptic in the sense of Petrovsky ($s_1 = s_2 = 0$, $t_1 = t_2 = 1$). It is obvious that this matrix is also properly elliptic of type 1. Therefore, considering the weight $\mathbf{b} = (1,1)$ and using Corollary 1.4.2, we can formulate necessary and sufficient condition for the validity of an "elliptic" estimate (an estimate of the type (1.4.9)) for the Cauchy–Riemann system. We do not go into the details associated with such "elliptic" estimates. Instead, we consider in more detail some estimates of the type (1.2.1).

Proposition 1.4.4. *Let $P(\xi;\tau)$ be the matrix (1.4.13), and let $Q(\xi) = \{q_1(\xi), q_2(\xi)\}$ be a 1×2 matrix whose entries $q_k(\xi)$ are measurable locally bounded in $\mathbb{R}^1$ functions that grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Set $D = (D_x; D_t)$ where $D_x = -\mathrm{i}\partial/\partial x$ and $D_t = -\mathrm{i}\partial/\partial t$.*

The estimate

$$\int_0^\infty dt \int_{-\infty}^\infty \left(|D_t u_1|^2 + |D_t u_2|^2 + |D_x u_1|^2 + |D_x u_2|^2\right) dx \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle q_1(D_x)u_1 + q_2(D_x)u_2 \rangle\!\rangle^2\right) \tag{1.4.14}$$

holds for all $u = (u_1, u_2) \in \mathbf{C}_0^\infty(\mathbb{R}_+^2)$ if and only if the inequality

$$|\xi|^{1/2} + |\mathrm{i}q_1 - q_2 \operatorname{sgn} \xi| \leqslant \mathrm{const}\, |\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi| \tag{1.4.15}$$

is satisfied for all $\xi \in \mathbb{R}^1$.

Proof. We show that Proposition 1.4.4 follows from Corollary 1.2.11 ($B(\xi) = 1$).

From the definition (1.4.13) of the matrix $P(\xi;\tau)$ it follows that

$$P^{\mathrm{c}}(\xi;\tau) = \begin{pmatrix} \mathrm{i}\xi & \mathrm{i}\tau \\ -\mathrm{i}\tau & \mathrm{i}\xi \end{pmatrix}. \tag{1.4.16}$$

For example, let $R(\xi;\tau) = \{\tau, 0\}$. Then we have $S(\xi;\tau) = \{\mathrm{i}\tau\xi, \mathrm{i}\tau^2\}$. On the other hand, in accordance with definition of the matrix $Q(\xi)$ we get

$$T(\xi;\tau) = \{\mathrm{i}\xi q_1 - \mathrm{i}\tau q_2, \mathrm{i}\tau q_1 + \mathrm{i}\xi q_2\}.$$

Obviously, the ellipticity of the matrix (1.4.13) implies the validity of condition 1 of Corollary 1.2.11.

Since $\mathscr{P}_+(\xi;\tau) = \tau - \mathrm{i}|\xi|$, we have $\zeta(\xi) = \mathrm{i}|\xi|$ and $S(\xi;\zeta(\xi)) \neq 0$ for $\xi \neq 0$. This means that $\mathscr{M}(\xi;\tau) = 1$ for all $\tau \in \mathbb{R}^1$ and all $\xi \neq 0$. Therefore, conditions 2 and 4 of Corollary 1.2.11 are automatically satisfied (for condition 4 see Proposition 1.2.7).

A direct calculation shows that

$$|\alpha(\xi)|^2 = |\xi|^2 |\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi|^{-2},$$

where $\alpha(\xi)$ is the coefficient appearing in (1.2.19). Thus, condition 6 of Corollary 1.2.11 takes the form

$$|\xi|^{1/2} \leqslant \mathrm{const}|\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi|. \tag{1.4.17}$$

Determining the matrix $T_-(\xi)$ from the decomposition

$$\frac{T(\xi;\tau)}{\tau^2 + \xi^2} = \frac{T_+(\xi)}{\tau - \mathrm{i}|\xi|} + \frac{T_-(\xi)}{\tau + \mathrm{i}|\xi|},$$

we get the relation

$$\int_{-\infty}^{\infty} \frac{|T_-(\xi)|^2}{\eta^2+\xi^2}\,d\eta = \frac{\pi}{2|\xi|}|\mathrm{i}q_1 - q_2 \operatorname{sgn} \xi|^2.$$

Therefore, condition 5 of Corollary 1.2.11 takes the form

$$|\mathrm{i}q_1 - q_2 \operatorname{sgn} \xi| \leqslant \text{const}\, |\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi|. \tag{1.4.18}$$

But inequality (1.4.15) is equivalent to inequalities (1.4.17) and (1.4.18).

As to condition 3 of Corollary 1.2.11, a direct check shows that in the example under consideration it takes the form $|\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi| \neq 0$ a.e. in $\mathbb{R}^1$, which here clearly follows from inequality (1.4.15).

The remaining terms on the left-hand side of the estimate (1.4.14) can be treated in a similar way, by taking as matrix $R(\xi;\tau)$ the matrices $\{0,\tau\}$, $\{\xi,0\}$ and $\{0,\xi\}$. ☐

Notice also that if we take as $R(\xi;\tau)$ the matrices $\{1,0\}$ and $\{0,1\}$, and apply Corollary 1.2.11, we obtain

Proposition 1.4.5. *Let $P(\xi;\tau)$ be the matrix (1.4.13), and let $Q(\xi) = \{q_1(\xi), q_2(\xi)\}$ be a 1×2 matrix whose entries $q_k(\xi)$ are measurable locally bounded in $\mathbb{R}^1$ functions that grow no faster than a certain power of $|\xi|$ as $|\xi| \to \infty$. Set $D_x = -\mathrm{i}\partial/\partial x$, $D_t = -\mathrm{i}\partial/\partial t$, and $D = (D_x, D_t)$. The estimate*

$$\|u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle q_1(D_x)u_1 + q_2(D_x)u_2 \rangle\!\rangle^2\right) \tag{1.4.19}$$

holds for all $u = (u_1,u_2) \in \mathbf{C}_0^\infty\left(\mathbb{R}^2_+\right)$ if and only if the inequality

$$\begin{aligned}&|\xi|^{-2} + |\xi|^{-1}|\mathrm{i}q_1 + q_2 \operatorname{sgn} \xi|^{-2}\left(1 + |\xi|^{-1}|\mathrm{i}q_1 - q_2 \operatorname{sgn} \xi|^2\right)\\ &\leqslant \text{const}\,[B(\xi)]^{-1}\end{aligned} \tag{1.4.20}$$

is satisfied for almost all $\xi \in \mathbb{R}^1$.

1.4.4 The stationary linearized Navier–Stokes system

We write the operator of the stationary linearized system of the Navier–Stokes equations in the half-space $\mathbb{R}^3_+$ in the form

$$P(D)u = \begin{cases} -\Delta v + \operatorname{grad} p, \\ \operatorname{div} v, \end{cases}$$

where $u = (v,p)$ and $v = (v_1,v_2,v_3)$.

This operator corresponds to the matrix

$$P(\xi;\tau) = \begin{pmatrix} \tau^2+|\xi|^2 & 0 & 0 & \mathrm{i}\xi_1 \\ 0 & \tau^2+|\xi|^2 & 0 & \mathrm{i}\xi_2 \\ 0 & 0 & \tau^2+|\xi|^2 & \mathrm{i}\tau \\ \mathrm{i}\xi_1 & \mathrm{i}\xi_2 & \mathrm{i}\tau & 0 \end{pmatrix}, \tag{1.4.21}$$

with determinant $\mathscr{P}(\xi;\tau) = (\tau^2 + |\xi|^2)^3$. The matrix (1.4.21) is elliptic in the sense of Douglis–Nirenberg. As the numbers s_k, t_j one can choose $s_1 = s_2 = s_3 = 0$, $s_4 = 1$, $t_1 = t_2 = t_3 = 2$, and $t_4 = 1$. Therefore, the statement of Corollary 1.4.2 with $\mathbf{b} = (1,1,1)$ holds true for the operator $P(D)$. Thus, for example, for all solutions of the system $-\Delta v + \operatorname{grad} p = f$, $\operatorname{div} v = 0$, where $f = (f_1, f_2, f_3)$, one has the estimate

$$\sum_{j=1}^{3} \|v_j\|_{2+l}^2 + \sum_{k=1}^{2} \left\| \frac{\partial p}{\partial x_k} \right\|_l^2 + \left\| \frac{\partial p}{\partial t} \right\|_l^2 \leqslant C \left(\sum_{j=1}^{3} \|f_j\|_l^2 + \sum_{j=1}^{3} \langle\!\langle v_j \rangle\!\rangle_{l+(3/2)}^2 \right). \tag{1.4.22}$$

Here $\|\cdot\|_r$ and $\langle\!\langle \cdot \rangle\!\rangle_\varrho$ are the norms in $L_2^r(\mathbb{R}^3_+)$ and $L_2^\varrho(\partial\mathbb{R}^3_+)$, respectively, and l is an integer, $l \geqslant -1$.

Indeed, in the case under consideration we have

$$P^{\mathrm{c}}(\xi;\tau) = (\tau^2 + |\xi|^2)$$
$$\times \begin{pmatrix} \tau^2 + \xi_2^2 & -\xi_1\xi_2 & -\xi_1\tau & -\mathrm{i}\xi_1(\tau^2+|\xi|^2) \\ -\xi_1\xi_2 & \xi_1^2 + \tau^2 & -\xi_2\tau & -\mathrm{i}\xi_2(\tau^2+|\xi|^2) \\ -\xi_1\tau & -\xi_2\tau & \xi_1^2+\xi_2^2 & -\mathrm{i}\tau(\tau^2+|\xi|^2) \\ -\mathrm{i}\xi_1(\tau^2+|\xi|^2) & -\mathrm{i}\xi_2(\tau^2+|\xi|^2) & -\mathrm{i}\tau(\tau^2+|\xi|^2) & (\tau^2+|\xi|^2)^2 \end{pmatrix}.$$

Therefore, setting

$$Q(\xi;\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we see that the rows of the matrix QP^{c} are linearly independent modulo $\mathscr{P}_+(\xi;\tau) = (\tau - \mathrm{i}|\xi|)^3$.

We note that "nonelliptic" estimates can be also obtained for the Navier–Stokes system. Examples of such estimates can be constructed as we proceeded for the Lamé system (see Subsection 1.4.2): choose a matrix Q in such a way that the rows of the matrix QP^{c} are linearly independent modulo $\mathscr{P}_+(\xi;\tau)$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, while the rows of the matrix $Q'P^{\mathrm{c}}$ (here Q' denotes the principal part of the matrix Q) do not possess this property.

1.4.5 Hyperbolic systems

Consider the $m \times m$ matrix $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$ whose entries are homogeneous polynomials of degree γ of the variable $(\xi;\tau) \in \mathbb{R}^n$. Suppose that the hyperplane $t = 0$ is not characteristic for the polynomial $\mathscr{P}(\xi;\tau) = \det\, P(\xi;\tau)$.

A homogeneous operator $P(D)$ is called *hyperbolic* if the equation $\mathscr{P}(\xi;\tau) = 0$ has only real τ-roots for all $\xi \in \mathbb{R}^{n-1}$. If these τ-roots are, in addition, pairwise distinct, then the operator $P(D)$ is hyperbolic in the sense of Petrovsky [Pet37].

Obviously, if all the τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ are real and condition 1 of Theorem 1.2.2 (or condition 1 of Corollary 1.2.13) is satisfied, then these roots must also be roots of all entries of the matrix S. Therefore, we have $\mathscr{M}(\xi;\tau) = \mathscr{P}_+(\xi;\tau) = \mathscr{P}(\xi;\tau)/p_0(\xi)$, where $p_0(\xi)$ is the leading coefficient of the polynomial $\mathscr{P}$. Hence, $\mathscr{P}_+(\xi;\tau) = 1$ for all $(\xi;\tau) \in \mathbb{R}^n$.

Thus, if $P(D)$ is a homogeneous hyperbolic operator, it is reasonable to consider only estimates corresponding to the case $N = 0$. Below we formulate a result related to such estimates which follows from Corollary 1.2.14.

Let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ be a vector with nonnegative integer coordinates and $u = (u_1, \dots, u_m) \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$. We set

$$|||u|||_{\boldsymbol{\nu}}^2 = \sum_{j=1}^m |||u_j|||_{\nu_j}^2,$$

where $|||\cdot|||_{\nu_j}$ is the norm in $L_2^{\nu_j}(\mathbb{R}^n_+)$.

Proposition 1.4.6. *Let the weight $B(\xi)$ be a homogeneous function of the variable $\xi \in \mathbb{R}^{n-1}$, let $R(\xi;\tau) = \{R_1(\xi;\tau), \dots, R_m(\xi;\tau)\}$ be a homogeneous $1 \times m$ matrix of polynomials of the variables $(\xi;\tau) \in \mathbb{R}^n$, and let $P(\xi;\tau) = \{P_{kj}(\xi;\tau)\}$ be a homogeneous hyperbolic $m \times m$ matrix. The estimate*

$$\|R(D)u\|_{B^{1/2}}^2 \leqslant C\|P(D)u\|_{\boldsymbol{\nu}}^2 \tag{1.4.23}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ if and only if the following conditions are satisfied:

1. *$R(\xi;\tau)$ is a matrix of the form*

$$R(\xi;\tau) = R^k(\xi;\tau) = \{r_k(\xi)P_{k1}(\xi;\tau), \dots, r_k(\xi)P_{km}(\xi;\tau)\}, \tag{1.4.24}$$

where $r_k(\xi)$ $(k = 1, \dots, m)$ are homogeneous polynomials of the variable $\xi \in \mathbb{R}^{n-1}$.

2. *The inequalities*

$$B^{1/2}(\xi)|r_k(\xi)| \leqslant \mathrm{const}\,|\xi|^{\nu_k} \qquad (k = 1, \dots, m) \tag{1.4.25}$$

are fulfilled for all $\xi \in \mathbb{R}^{n-1}$.

Proof. Since $\mathscr{P}_+(\xi;\tau) = \mathscr{P}(\xi;\tau)/p_0(\xi)$, condition 1 of Theorem 1.2.2 is equivalent to condition 1 of this proposition. We replace the norm $\|\cdot\|_{\boldsymbol{\nu}}$ in the formulation of Corollary 1.2.14 by the norm $|||\cdot|||_{\boldsymbol{\nu}}$. This is equivalent to replacing the matrix (1.2.25), which appears in condition 1 of Corollary 1.2.13, by the matrix $\mathfrak{R}'(\xi;\tau) = \{\delta_{jk}(\tau + i|\xi|)^{\nu_k})\}$. Using representation (1.4.24), we find that condition 1 of Corollary 1.2.13 becomes after this replacement condition 2 of Proposition 1.4.6. □

To conclude this subsection, we note that for nonhomogeneous hyperbolic operators it is not necessary to restrict to the the trivial case $N = 0$. Likewise, it is not necessary to assume that the matrix R is proportional to some row of the matrix P (see (1.4.24)).

For example, let $(\xi;\tau) \in \mathbb{R}^2$ and

$$P(\xi;\tau) = \begin{pmatrix} \mathrm{i}\tau & \mathrm{i}\xi - 1 \\ \mathrm{i}\xi - 1 & \mathrm{i}\tau \end{pmatrix}. \tag{1.4.26}$$

Then det $P(\xi;\tau) = \mathscr{P}(\xi;\tau) = -\tau^2+\xi^2+2\mathrm{i}\xi-1$ is a polynomial whose τ-roots are equal to $\pm(\xi+\mathrm{i})$. Hence, the imaginary parts of these roots do not depend on $\xi \in \mathbb{R}^1$. Therefore, the operator $P(D)$ is hyperbolic (see, for example, [H63], pp. 178–180).

We claim that for all $u = (u_1, u_2) \in \mathbf{C}_0^\infty(\mathbb{R}^2_+)$ the estimates

$$\left.\begin{aligned} \|u_1\|^2 + \|u_2\|^2 &\leqslant C\left(\|P(D)u\|^2 + \langle\!\langle u_1\rangle\!\rangle^2\right), \\ \|u_1\|^2 + \|u_2\|^2 &\leqslant C\left(\|P(D)u\|^2 + \langle\!\langle u_2\rangle\!\rangle^2\right) \end{aligned}\right\} \tag{1.4.27}$$

hold.

Indeed, let, for instace, $R(\xi;\tau) = \{1,0\}$. Since $\mathscr{P}_+(\xi;\tau) = \tau - (\xi+\mathrm{i})$ and

$$P^{\mathrm{c}}(\xi;\tau) = \begin{pmatrix} \mathrm{i}\tau & 1-\mathrm{i}\xi \\ 1-\mathrm{i}\xi & \mathrm{i}\tau \end{pmatrix}$$

we get $S(\xi;\tau) = \{\mathrm{i}\tau, 1-\mathrm{i}\xi\}$ and $\mathscr{M}(\xi;\tau) = 1$. One can easily verify that $|S(\xi;\tau)| \leqslant |\mathscr{P}(\xi;\tau)|$ for all $(\xi;\tau) \in \mathbb{R}^2$. Hence, $N = 1$ and we can use Corollary 1.2.11 with $B(\xi) = 1$.

We set $Q(\xi;\tau) = \{1,0\}$. Then $T(\xi;\tau) = S(\xi;\tau)$ and conditions 1–4 of Corollary 1.2.11 are satisfied, and $\alpha(\xi) = 1$. Since $\zeta(\xi) = \xi + \mathrm{i}$, condition 6 of Corollary 1.2.11 is also satisfied.

Calculating the 1×2 matrix $T_-(\xi;\tau)$ and taking into consideration the equality $\mathscr{P}_-(\xi;\tau) = \tau + (\xi+\mathrm{i})$, we get

$$\int_{-\infty}^{\infty} \left|\frac{T_-(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)}\right|^2 d\eta = \frac{\pi}{2}.$$

Therefore, condition 5 of Corollary 1.2.11 is satisfied.

It is clear that we reach similar conclusions by taking $Q(\xi;\tau) = \{0,1\}$. Hence the inequalities

$$\left.\begin{aligned} \|u_1\|^2 &\leqslant C\left(\|P(D)u\|^2 + \langle\!\langle u_1\rangle\!\rangle^2\right), \\ \|u_1\|^2 &\leqslant C\left(\|P(D)u\|^2 + \langle\!\langle u_2\rangle\!\rangle^2\right) \end{aligned}\right\} \tag{1.4.28}$$

hold for all $u = (u_1, u_2) \in \mathbf{C}_0^\infty(\mathbb{R}^2_+)$.

Applying the same arguments to the matrix $R(\xi;\tau) = \{0,1\}$, we find that the right-hand sides of inequalities (1.4.28) are also majorants for $\|u_2\|^2$.

Notice also that for $R(\xi;\tau) = \{\tau, 0\}$ we obtain $S(\xi;\tau) = \tau\{\mathrm{i}\tau, 1-\mathrm{i}\xi\}$. Therefore, $|S(\xi;\xi)| = O(|\xi|^2)$ as $|\xi| \to \infty$. On the other hand, we have $\mathscr{P}(\xi;\xi) = O(|\xi|)$. Hence, if P is the matrix (1.4.26), then for any matrix Q of boundary operators the estimate

$$\|D_t u_1\|^2 \leqslant C\left(\|P(D)u\|^2 + \langle\!\langle Q(D)u \rangle\!\rangle^2\right) \tag{1.4.29}$$

does not hold for all $u = (u_1, u_2) \in \mathbf{C}_0^\infty(\mathbb{R}_+^2)$. Failure of the estimate (1.4.29) is, of course, a consequence of the hyperbolicity of the operator $P(D)$ (cf. Proposition 1.4.4 on conditions for the validity of the estimate (1.4.14) for the elliptic operator of the Cauchy–Riemann system).

1.4.6 Operators of first order in the variable t. Scalar case

In this subsection we consider estimates of the types (1.3.16) and (1.2.12) in the case where $m = 1$ and $P(D)$ is a first-order operator with respect to the variable t. It will be shown that criteria for the validity of such estimates can be formulated explicitly in terms of the coefficients of polynomials $\mathscr{R}$, $\mathscr{P}$, and Q.

Let $r_0(\xi)$, $r_1(\xi)$, $p_0(\xi)$, $p_1(\xi)$, and $q(\xi)$ be measurable functions that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Suppose that $p_0(\xi)$ is not equal to zero a.e. in $\mathbb{R}^{n-1}$. We set $D = (D_x; D_t)$, where $D_t = -\mathrm{i}\partial/\partial t$ and $D_x = (-\mathrm{i}\partial/\partial x_1, \dots, -\mathrm{i}\partial/\partial x_{n-1})$.

Proposition 1.4.7. *Let $\mu \in \mathbb{R}^1$, let $r_0 p_1 - r_1 p_0 \neq 0$, and let $\operatorname{Im}(\bar{p}_0 p_1) < 0$ a.e. in $\mathbb{R}^{n-1}$. The estimate*

$$\|\left(r_0(D_x)D_t + r_1(D_x)\right)u\|_{B^{1/2}}^2 \leqslant C\left[\|\left(p_0(D_x)D_t + p_1(D_x)\right)u\|^2 + \langle\!\langle q(D_x)u \rangle\!\rangle_\mu^2\right] \tag{1.4.30}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ if and only if the inequalities

$$\begin{aligned} &B^{1/2}(\xi)|r_0 p_1 - r_1 p_0| \\ &\leqslant \text{const}\,\min\{|\operatorname{Im}(\bar{p}_0 p_1)|, (1+|\xi|^2)^{\mu/2}|q|\operatorname{Im}(\bar{p}_0 p_1)^{1/2}\}, \\ &B^{1/2}(\xi)|r_0 \bar{p}_1 - r_1 \bar{p}_0| \leqslant \text{const}\,|\operatorname{Im}(\bar{p}_0 p_1)| \end{aligned} \tag{1.4.31}$$

are satisfied for almost all $\xi \in \mathbb{R}^{n-1}$.

Proof. We show that Proposition 1.4.7 follows from Corollary 1.2.11 if in the formulation of the latter we replace the estimate (1.2.1) by the estimate (1.3.16), inequality (1.2.21) by inequality (1.3.19), and the matrices S and T by the polynomials $\mathscr{R}(\xi;\tau) = r_0(\xi)\tau + r_1(\xi)$ and $\mathrm{Q}(\xi;\tau) = q(\xi)$, respectively.

Indeed, a direct calculation shows that

$$\sup\left|\frac{r_0\tau + r_1}{p_0\tau + p_1}\right| = \frac{|r_1 p_0 - r_0 p_1| + |r_1\bar{p}_0 - r_0\bar{p}_1|}{2|\operatorname{Im}(\bar{p}_0 p_1)|}, \tag{1.4.32}$$

provided $r_0 p_1 - r_1 p_0 \neq 0$. It is also obvious that in this case

$$\mathscr{M}(\xi;\tau) = 1, \quad \mathscr{P}_+(\xi;\tau) = \tau + \frac{p_1}{p_0}, \quad \mathscr{P}_-(\xi;\tau) = 1,$$
$$\zeta(\xi) = -\frac{p_1}{p_0}, \quad \mathsf{Q}(\xi;\tau) = 0.$$

Thus, conditions 5, 2, and 3 of Corollary 1.2.11 are evidently fulfilled (the latter one is satisfied since $q(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$). The function $\alpha(\xi)$, defined by (1.2.19), is here equal to

$$\alpha(\xi) = \frac{r_1 p_0 - r_0 p_1}{p_0 q_0}. \tag{1.4.33}$$

Finally, using (1.4.32) and (1.4.33), we conclude that inequality (1.3.19) and condition 1 of Corollary 1.2.11 are equivalent to inequalities (1.4.31). □

Proposition 1.4.8. *Let* $\operatorname{Im}(\bar{p}_0 p_1) = 0$ *a.e. in* $\mathbb{R}^{n-1}$. *The inequality*

$$\| (r_0(D_x)D_t + r_1(D_x))\, u\|^2_{B^{1/2}} \leqslant C \| (p_0(D_x)D_t + p_1(D_x))\, u\|^2 \tag{1.4.34}$$

holds for all $u \in \mathbf{C}_0^\infty(\mathbb{R}^n_+)$ *if and only if the conditions*

$$r_0 p_1 - r_1 p_0 = 0, \tag{1.4.35}$$

$$B^{1/2}(\xi)|r_0| \leqslant \text{const}\,|p_0| \tag{1.4.36}$$

are satisfied a.e. in $\mathbb{R}^{n-1}$.

Proof. The statement of Proposition 1.4.8 follows from Theorem 1.2.3, if in the formulation of this theorem we replace the matrix S by the polynomial $\mathscr{R} = r_0\tau + r_1$. Indeed, the equality $\operatorname{Im}(\bar{p}_0 p_1) = 0$ implies the relation $\mathscr{P}_+(\xi;\tau) = \tau + p_1/p_0$. Thus, conditions (1.4.35) and (1.4.36) are equivalent to conditions 1 and 2 of Theorem 1.2.3, respectively. □

Proposition 1.4.9. *Let* $r_0 p_1 - r_1 p_0 \neq 0$ *and* $\operatorname{Im}(\bar{p}_0 p_1) > 0$ *a.e. in* $\mathbb{R}^{n-1}$. *The estimate* (1.4.34) *holds for all* $u \in C_0^\infty(\mathbb{R}^n_+)$ *if and only if the inequality*

$$B^{1/2}(\xi)\,(|r_0 p_1 - r_1 p_0| + |r_0 \bar{p}_1 - r_1 \bar{p}_0|) \leqslant \text{const}\,\operatorname{Im}(\bar{p}_0 p_1) \tag{1.4.37}$$

is fulfilled a.e. in $\mathbb{R}^{n-1}$.

Proof. We claim that this proposition follows from Remark 1.2.4 (see Subsection 1.2.2). Indeed, since $\operatorname{Im}(\bar{p}_0 p_1) > 0$, the unique τ-root of the polynomial $\mathscr{P}$ lies in the half-plane $\operatorname{Im}\zeta < 0$ ($\zeta = \tau + i\sigma$). On the other hand, it follows from (1.4.32) that condition 2 of Theorem 1.2.3 is equivalent to inequality (1.4.37). □

1.4.7 An example of a second-order operator w.r.t. *t*

In this subsection we consider in detail an example of a second-order operator w.r.t. the variable t, which was already discussed in Remark 1.3.10 (see Subsection 1.3.3). Here we prove all the statements about this operator which were already used in that remark.

Proposition 1.4.10. *Let* $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2$, *let* $\mathscr{R}(\xi;\tau) = 1$, *and let*

$$\mathscr{P}(\xi;\tau) = (\tau - \mathrm{i}\varkappa_1(\xi))(\tau - \mathrm{i}\varkappa_2(\xi)).$$

Suppose also that $\varkappa_j(\xi) > 0$ $(j = 1, 2)$ *and* $\varkappa_1(\xi) \neq \varkappa_2(\xi)$ *for almost all* $\xi \in \mathbb{R}^{n-1}$. *Set* $\mathsf{Q}(\xi;\tau) = \{\mathsf{Q}_1(\xi;\tau), \mathsf{Q}_2(\xi;\tau)\}$, *where* $\mathsf{Q}_j(\xi;\tau) = \tau - \mathrm{i}\varkappa_j(\xi)$ *and* $j = 1, 2$. *The estimate*

$$\|u\|^2 \leqslant C\left(\|\mathscr{P}(D)u\|^2 + \langle\!\langle \mathsf{Q}(D)u\rangle\!\rangle^2_{\boldsymbol{\mu}}\right) \tag{1.4.38}$$

holds true for all $u \in C_0^\infty(\mathbb{R}^n_+)$ *if and only if condition* (1.3.18) *and the inequality*

$$(\varkappa_1\varkappa_2)^{-1} \leqslant \mathrm{const} \tag{1.4.39}$$

are satisfied a.e. in $\mathbb{R}^{n-1}$.

Proof. We show that this proposition follows from Corollary 1.2.13 if in this corollary we put $m = 1$, $B(\xi) = 1$, $R(\xi;\tau) = S(\xi;\tau) = 1$,

$$T(\xi;\tau) = \mathsf{Q}(\xi;\tau) = \left\{\begin{array}{c} \mathsf{Q}_1(\xi;\tau) \\ \mathsf{Q}_2(\xi;\tau) \end{array}\right\},$$

$\boldsymbol{\mu} = (\mu_1, \mu_2)$, and $\boldsymbol{\nu} = \nu_1 = 0$ (and, consequently, $\mathfrak{R} = I$).

Indeed, in the example under consideration we have $\mathscr{P}(\xi;\tau) = \mathscr{P}_+(\xi;\tau) = \dot{\mathscr{P}}_+(\xi;\tau)$. Therefore, condition 2 of Theorem 1.2.2 is automatically satisfied, while condition 3 of Theorem 1.2.2 follows from the assumption $\varkappa_1(\xi) \neq \varkappa_2(\xi)$. Thus, condition 2 of Corollary 1.2.13 is also satisfied. Similarly, condition 3 of this corollary is fulfilled, because $m = 1$. Obviously, the equality

$$\sup\left|\frac{\mathscr{R}(\xi;\tau)}{\mathscr{P}(\xi;\tau)}\right|^2 = (\varkappa_1(\xi)\varkappa_2(\xi))^{-2}$$

holds, and, consequently, condition 1 of Corollary 1.2.13 is equivalent to inequality (1.4.39). From the equation $\mathscr{P} = \mathscr{P}_+$ we obtain $\mathscr{P}_- = 1$, which means that $\mathsf{Q}_- = T_- = 0$. Hence, condition 4 of Corollary 1.2.13 follows from inequality (1.2.30). We show that this inequality is equivalent to condition (1.3.18).

First, we note that in the case $m = 1$ representation (1.2.16) for the $1 \times N$ matrix $G(\xi;\tau)$, related to the estimate

$$\|\mathscr{R}(D)u\|^2_{B^{1/2}} \leqslant C\left(\|\mathscr{P}(D)u\|^2 + \langle\!\langle \mathsf{Q}(D)u\rangle\!\rangle^2\right),$$

takes the form

$$G(\xi;\tau)=\mathscr{M}(\xi;\tau)H(\xi;\tau)\big(\dot{\mathfrak{D}}(\xi)\big)^{-1}. \tag{1.4.40}$$

Here $H(\xi;\tau)$ is the $1\times N$ matrix $\{H_{\varrho\sigma}(\xi;\tau)\}$ with entries

$$H_{\varrho\sigma}(\xi;\tau)=\sigma!\sum_{\gamma=0}^{\sigma}\frac{1}{\gamma!}\dot{\mathscr{R}}^{\gamma}(\xi;\zeta_\varrho(\xi))\dot{\mathscr{P}}_{+}(\xi;\tau)(\tau-\zeta_\varrho(\xi))^{\gamma-\sigma-1}, \tag{1.4.41}$$

and $\big(\dot{\mathfrak{D}}(\xi)\big)^{-1}$ is the inverse of the matrix

$$\dot{\mathfrak{D}}(\xi)=\{\dot{\mathrm{Q}}_\alpha^{(\sigma)}(\xi;\zeta_\varrho(\xi))\}\qquad(\varrho=1,\dots,l;\ \sigma=\sigma(\varrho)=0,\ \dots,k_\varrho(\xi)-1;$$
$$\alpha=1,\dots,N;\ \dot{\mathscr{R}}=\mathscr{R}/\mathscr{M};\ \dot{\mathrm{Q}}_\alpha=\mathrm{Q}_\alpha/\mathscr{M}).$$

In the considered example we have $\mathrm{Q}_\alpha(\xi;\tau)=\tau-\mathrm{i}\varkappa_\alpha(\xi)$. Therefore,

$$\dot{\mathfrak{D}}(\xi)=\begin{pmatrix}0 & \mathrm{i}(\varkappa_2-\varkappa_1)\\ \mathrm{i}(\varkappa_2-\varkappa_1) & 0\end{pmatrix}$$

and

$$\big(\dot{\mathfrak{D}}(\xi)\big)^{-1}=\begin{pmatrix}0 & -\mathrm{i}(\varkappa_1-\varkappa_2)^{-1}\\ \mathrm{i}(\varkappa_1-\varkappa_2)^{-1} & 0\end{pmatrix}.$$

On the other hand, the equalities $\mathscr{R}=1$, $\mathscr{P}_+=\dot{\mathscr{P}}_+=\mathscr{P}=(\tau-\mathrm{i}\varkappa_1(\xi))(\tau-\mathrm{i}\varkappa_2(\xi))$ and equations (1.4.41) imply $\mathscr{M}(\xi;\tau)=1$ and $H(\xi;\tau)=\{\tau-\mathrm{i}\varkappa_1,\tau-\mathrm{i}\varkappa_2\}$. Therefore, it follows from (1.4.40) that

$$G(\xi;\tau)=\{\mathrm{i}(\varkappa_1-\varkappa_2)^{-1}(\tau-\mathrm{i}\varkappa_1),-\mathrm{i}(\varkappa_1-\varkappa_2)^{-1}(\tau-\mathrm{i}\varkappa_2)\}. \tag{1.4.42}$$

Finally, calculating the integral on the left-hand side of (1.2.30), we conclude that inequality (1.2.30) (with $B(\xi)=1$) is equivalent to (1.3.18). □

At the end of this subsection, we show that condition (1.3.15) (with $B(\xi)=1$) is equivalent to (1.3.17) (it was noted in Subsection 1.3.3 that condition (1.3.15) is, in general, not necessary for the validity of (1.3.16)).

Indeed, since $m=1$, it follows from the definition of the polynomials $\mathscr{P}$, Q_1, and Q_2 given in Proposition 1.4.10 that the matrix $\mathscr{T}_+$ has the form

$$\mathscr{T}_+=\begin{pmatrix}\pi\varkappa_2^{-1} & -2\pi(\varkappa_1+\varkappa_2)^{-1}\\ -2\pi(\varkappa_1+\varkappa_2)^{-1} & \pi\varkappa_1^{-1}\end{pmatrix}$$

and $\mathscr{T}_-=0$. Consequently, we have

$$\pi\,\mathrm{tr}\left(\mathscr{T}_+^{-1}\right)_{(-\boldsymbol{\mu})}=\frac{\varkappa_2(\varkappa_1+\varkappa_2)^2}{(\varkappa_1-\varkappa_2)^2(1+|\xi|^2)^{\mu_1}}+\frac{\varkappa_1(\varkappa_1+\varkappa_2)^2}{(\varkappa_1-\varkappa_2)^2(1+|\xi|^2)^{\mu_2}}$$

and $\mathrm{tr}\,\mathscr{T}_{-(\boldsymbol{\mu})}=0$. On the other hand,

$$\sup\left|\frac{\mathscr{R}_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2=\sup\frac{1}{|\mathscr{P}(\xi;\tau)|^2}=(\varkappa_1\varkappa_2)^{-2}.$$

Rewriting condition (1.3.15) with these equalities in mind, we arrive at (1.3.17).

1.5 On well-posed boundary value problems in a half-space

In this section we establish necessary and sufficient conditions for the unique solvability of the boundary value problem (1.5.4)–(1.5.5), where $\exp(\beta t)\cdot f(x;t) \in L^2(\mathbb{R}^n_+)$ for some $\beta \in \mathbb{R}^1$ and $\varphi = (\varphi_1,\dots,\varphi_N) \in \mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)$ ($\boldsymbol{\mu} = (\mu_1,\dots,\mu_N) \in \mathbb{R}^N$). We seek a solution to this problem in the class of functions $u(x;t)$ such that $\exp(\beta t)\cdot u(x;t) \in W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)$.

We denote by $W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)$ the closure of the space $C^\infty_0(\mathbb{R}^n_+)$ w.r.t. the norm $\|v : W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)\|$ defined by

$$\|v : W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)\|^2 = \sum_{|\alpha|\leqslant J} \|D^\alpha_{x,t} v : \mathscr{H}_{(s,0)}(\mathbb{R}^n_+)\|^2 \tag{1.5.1}$$

with $s \in \mathbb{R}^1$ and an integer $J \geqslant 1$. It is well known that for the elements $v \in W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)$ there exist the successive traces

$$D^j_t v\big|_{t=0} \in \mathscr{H}_{J+s-j-(1/2)}(\partial\mathbb{R}^n_+) \quad (j = 0,1,\dots,J-1).$$

Let $\mathscr{P}(\xi;\tau)$ and $\mathsf{Q}_\alpha(\xi;\tau)$ ($\alpha = 1,\dots,N$) be polynomials of the variable $\tau \in \mathbb{R}^1$ with measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Suppose that the leading coefficient $p_0(\xi)$ of the polynomial $\mathscr{P}$ is not equal to zero a.e. in $\mathbb{R}^{n-1}$. Suppose also that $J = \operatorname{ord}\mathscr{P}(\xi;\tau) \geqslant 1$ and $J_\alpha = \operatorname{ord}\mathsf{Q}_\alpha(\xi;\tau) = J-1$ ($\alpha = 1,\dots,N$) a.e. in $\mathbb{R}^{n-1}$.

For any $\beta \in \mathbb{R}^1$ we denote by $[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+$ the polynomial of τ with the leading coefficient 1, the roots of which (counting multiplicities) coincide with the τ-roots of the polynomial $\mathscr{P}(\xi;\tau+\beta\mathrm{i})$ in the half-plane $\operatorname{Im}\zeta \geqslant 0$ ($\zeta = \tau + \mathrm{i}\sigma$). We set

$$[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_- = \mathscr{P}(\xi;\tau+\beta\mathrm{i})/p_0(\xi)\,[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+$$

and define the polynomials $[\mathsf{Q}_\alpha(\xi;\tau+\beta i)]_\pm$ by the partial fraction decompositions (w.r.t. τ):

$$\left.\begin{array}{c}\dfrac{\mathsf{Q}_\alpha(\xi;\tau+\beta\mathrm{i})}{\mathscr{P}(\xi;\tau+\beta\mathrm{i})} = \dfrac{[\mathsf{Q}_\alpha(\xi;\tau+\beta\mathrm{i})]_+}{[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+} + \dfrac{[\mathsf{Q}_\alpha(\xi;\tau+\beta\mathrm{i})]_-}{[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_-} \\ (\alpha = 1,\dots,N)\end{array}\right\}. \tag{1.5.2}$$

We consider also the polynomials $\left[\mathscr{R}_j(\xi;\tau+\beta\mathrm{i})\right]_\pm$ which are defined by the following partial fraction decompositions (w.r.t. τ):

$$\left.\begin{array}{c}\dfrac{(\tau+\beta\mathrm{i})^j}{\mathscr{P}(\xi;\tau+\beta\mathrm{i})} = c_j(\xi) + \dfrac{\left[\mathscr{R}_j(\xi;\tau+\beta\mathrm{i})\right]_+}{[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+} + \dfrac{\left[\mathscr{R}_j(\xi;\tau+\beta\mathrm{i})\right]_-}{[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_-} \\ (J = 0,1,\dots,N)\end{array}\right\}. \tag{1.5.3}$$

It is obvious that $c_j(\xi) = 0$ ($j = 0,1,\dots,J-1$) and $c_J = [p_0(\xi)]^{-1}$. It is also clear that in the case $\beta = 0$ we have $[\mathscr{P}(\xi;\tau)]_\pm = \mathscr{P}_\pm(\xi;\tau)$, $[\mathsf{Q}_\alpha(\xi;\tau)]_\pm = \mathsf{Q}_{\alpha\pm}(\xi;\tau)$, and $\left[\mathscr{R}_j(\xi;\tau)\right]_{\pm} = \mathscr{R}_{j\pm}(\xi;\tau)$. Here $\mathscr{P}_{\pm}(\xi;\tau)$, $\mathsf{Q}_{\alpha\pm}(\xi;\tau)$ and $\mathscr{R}_{j\pm}(\xi;\tau)$ are the polynomials defined in Subsections 1.0.1, 1.0.2 by the polynomials $\mathscr{P}(\xi;\tau)$, $\mathsf{Q}_{\alpha}(\xi;\tau)$ and $\mathscr{R}_j(\xi;\tau) = \tau^j$, respectively.

Consider the Hilbert spaces $L^2(\exp(\beta t);\mathbb{R}^n_+)$ and $W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+)$ with the norms

$$\|f : L^2(\exp(\beta t);\mathbb{R}^n_+)\| = \|\exp(\beta t)\cdot f : L^2(\mathbb{R}^n_+)\|$$

and

$$\|u : W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+)\| = \|\exp(\beta t)\cdot u : W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)\|,$$

respectively.

Let $\varphi = (\varphi_1,\dots,\varphi_N) \in \mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)$ ($\boldsymbol{\mu} = (\mu_1,\dots,\mu_N) \in \mathbb{R}^n$) and $f \in L^2(\exp(\beta t);\mathbb{R}^n_+)$ with some $\beta \in \mathbb{R}^1$. We say that a generalized function u is a *solution* of the boundary value problem

$$\mathscr{P}(D)u = f(x;t) \qquad (x \in \mathbb{R}^{n-1},\ t > 0), \tag{1.5.4}$$

$$\mathsf{Q}_{\alpha}(D)u\big|_{t=0} = \varphi_{\alpha}(x) \qquad (x \in \mathbb{R}^{n-1};\ \alpha = 1,\dots,N), \tag{1.5.5}$$

if there exists a number $s \in \mathbb{R}^1$ such that $u \in W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+)$ and u satisfies equation (1.5.4) and boundary conditions (1.5.5) in the sense of generalized functions[19].

The boundary value problem is called *well-posed* in the pair of spaces

$$\left[W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+),\, L^2(\exp(\beta t);\mathbb{R}^n_+)\times\mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)\right]$$

if for any $f \in L^2(\exp(\beta t);\mathbb{R}^n_+)$ and $\varphi \in \mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)$ there exists a unique solution of the problem (1.5.4)–(1.5.5) satisfying the estimate

$$\begin{aligned}&\|u : W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+)\|\\ &\leqslant \mathrm{const}\left(\|f : L^2(\exp(\beta t);\mathbb{R}^n_+)\| + \langle\!\langle\varphi\rangle\!\rangle_{\boldsymbol{\mu}}\right).\end{aligned} \tag{1.5.6}$$

Theorem 1.5.1. *Assume that the equality*

$$\operatorname{ord}\,[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+ = N \quad \textit{a.e. in} \quad \mathbb{R}^{n-1}, \tag{1.5.7}$$

holds for some $\beta \in \mathbb{R}^1$, and for some $s \in \mathbb{R}^1$ the polynomials $\mathscr{P}(\xi;\tau+\beta\mathrm{i})$ and $\mathsf{Q}_{\alpha}(\xi;\tau+\beta\mathrm{i})$ satisfy for all $\tau \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$ the following conditions:

$$|\tau|^j \leqslant \mathrm{const}(1+|\xi|^2)^{(j-s-J)/2}|\mathscr{P}(\xi;\tau+\beta\mathrm{i})| \qquad (j = 0,1,\dots,J); \tag{1.5.8}$$

the polynomials $\mathsf{Q}_{\alpha}(\xi;\tau+\beta\mathrm{i})$ $(\alpha = 1,\dots,N)$, are linearly independent modulo $[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+$; (1.5.9)

[19] The boundary values are understood in the sense of traces.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(1+|\xi|^2)^J \left|\sum_{\alpha=1}^{N} G_{\alpha 0}(\xi;\tau+\beta\mathrm{i})\,[\mathsf{Q}_\alpha(\xi;\eta+\beta\mathrm{i})]_-\right|^2}{\left|[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+\,[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_-\right|^2} d\tau d\eta$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left|\sum_{\alpha=1}^{N} G_{\alpha J}(\xi;\tau+\beta\mathrm{i})\,[\mathsf{Q}_\alpha(\xi;\eta+\beta\mathrm{i})]_-\right|^2}{\left|[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+\,[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_-\right|^2} d\tau d\eta \tag{1.5.10}$$
$$\leqslant \mathrm{const}(1+|\xi|^2)^{-s};$$

$$\int_{-\infty}^{\infty}\sum_{\alpha=1}^{N} \frac{(1+|\xi|^2)^J |G_{\alpha 0}(\xi;\tau+\beta\mathrm{i})|^2 + |G_{\alpha J}(\xi;\tau+\beta\mathrm{i})|^2}{(1+|\xi|^2)^{\mu_\alpha}|\,[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+\,|^2} d\tau \tag{1.5.11}$$
$$\leqslant \mathrm{const}(1+|\xi|^2)^{-s},$$

where $G_{\alpha 0}(\xi;\tau+\beta\mathrm{i})$ *are polynomials* (*in* τ) *of degree at most* $N-1$*, satisfying the identity*

$$\sum_{\alpha=1}^{N} G_{\alpha 0}(\xi;\tau+\beta\mathrm{i})\,[\mathsf{Q}_\alpha(\xi;\eta+\beta\mathrm{i})]_+ = (\eta-\tau)^{-1}$$
$$\times\left\{[\mathscr{P}(\xi;\eta+\beta\mathrm{i})]_+\,[\mathscr{R}_0(\xi;\tau+\beta\mathrm{i})]_+ - [\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+\,[\mathscr{R}_0(\xi;\eta+\beta\mathrm{i})]_+\right\}; \tag{1.5.12}$$

here $G_{\alpha J}(\xi;\tau+\beta\mathrm{i})$ *denote the reminders of the division of the polynomials* $(\tau+\beta\mathrm{i})^J G_{\alpha 0}(\xi;\tau+\beta\mathrm{i})$ *by* $[\mathscr{P}(\xi;\tau+\beta\mathrm{i})]_+$*. The polynomials* $[\mathsf{Q}_\alpha(\xi;\tau+\beta\mathrm{i})]_\pm$ *and* $[\mathscr{R}_0(\xi;\tau+\beta\mathrm{i})]_+$ *are defined by decompositions* (1.5.2) *and* (1.5.3) *with* $j=0$*, respectively.*

Then the boundary value problem (1.5.4)–(1.5.5) *is well-posed in the pair of spaces*

$$\left[W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+),\, L^2(\exp(\beta t);\mathbb{R}^n_+)\times\mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)\right].$$

Conversely, if condition (1.5.7) *is satisfied for some* $\beta\in\mathbb{R}^1$ *and inequality* (1.5.6) *holds for some* $s\in\mathbb{R}^1$ *and all* $u\in W^{J+s,J}_{2,x,t}(\exp(\beta t);\mathbb{R}^n_+)$*, where* $f(x;t)$ *and* $\varphi(x)=(\varphi_1(x),\ldots,\varphi_N(x))$ *are the right-hand sides of* (1.5.4) *and* (1.5.5)*, respectively, then assumptions* (1.5.8)–(1.5.11) *are fulfilled for all* $\tau\in\mathbb{R}^1$ *and almost all* $\xi\in\mathbb{R}^{n-1}$*.*

Proof. Without loss of generality, we may assume $\beta=0$. Indeed, the case of arbitrary $\beta\in\mathbb{R}^1$ is reduced to the case $\beta=0$ by the substitution $u(x;t)=\exp(-\beta t)\cdot v(x;t)$. After this substitution the argument $-\mathrm{i}\partial/\partial t$ is shifted by $\beta\mathrm{i}$ in all operators.

Suppose that conditions (1.5.7)–(1.5.11) are satisfied for $\beta=0$. We prove the well-posedness of the problem (1.5.4)–(1.5.5) in the pair of spaces

$$\left[W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+),\, L^2(\mathbb{R}^n_+)\times\mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)\right].$$

We represent the polynomial $\mathscr{P}_+(\xi;\tau)$ as

$$\mathscr{P}_+(\xi;\tau) = \prod_{\varrho=1}^{l(\xi)} (\tau - \zeta_\varrho(\xi))^{k_\varrho(\xi)}. \tag{1.5.13}$$

From condition (1.5.7) it follows that $k_1(\xi) + \cdots + k_{l(\xi)}(\xi) = N$, while inequality (1.5.8) guarantees that $\operatorname{Im}\zeta_\varrho(\xi) > 0$ ($\varrho = 1, \dots, l(\xi)$) a.e. in $\mathbb{R}^{n-1}$.

Consider the $N \times N$ matrix

$$\mathfrak{D}(\xi) = \{\mathrm{Q}_\alpha^{(\sigma)}(\xi;\zeta_\varrho(\xi))\}, \tag{1.5.14}$$

where the rows are labeled by the index α, and the columns are labeled by the indices ϱ, $\sigma = \sigma(\varrho)$. These indices take the values $\alpha = 1, \dots, N$; $\varrho = 1, \dots, l(\xi)$, and $\sigma = \sigma(\varrho) = 0, 1, \dots, k_\varrho(\xi) - 1$. The assumption (1.5.9) is equivalent to the nondegeneracy of the matrix (1.5.14) for almost all $\xi \in \mathbb{R}^{n-1}$.

Let $f \in L^2(\mathbb{R}^n_+)$ and $\varphi \in \mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}^n_+)$. The solution (in the sense of distributions) of the boundary value problem (1.5.4)–(1.5.5) will be constructed as follows. For all $t \geqslant 0$ and almost all $\xi \in \mathbb{R}^{n-1}$ we set

$$\hat{u}(\xi;t) = F_{\tau\to t}\left[\frac{F_{t\to\tau}\, g(\xi;t)}{\mathscr{P}(\xi;t)}\right] + \sum_{\varrho=1}^{l(\xi)} \sum_{\sigma=0}^{k_\varrho(\xi)-1} c_{\varrho\sigma}(\xi)(\mathrm{i}t)^\sigma \exp(\mathrm{i}\zeta_\varrho(\xi)t), \tag{1.5.15}$$

where $g(\xi;t) = 0$ for $t < 0$, and $g(\xi;t) = \hat{f}(\xi;t)$, and $\{c_{\varrho\sigma}(\xi)\}$ is the uniquely determined solution of the system

$$\begin{aligned} &\sum_{\varrho=1}^{l(\xi)} \sum_{\sigma=0}^{k_\varrho(\xi)-1} \mathrm{Q}_\alpha^{(\sigma)}(\xi;\zeta_\varrho(\xi))c_{\varrho\sigma}(\xi) \\ &= \hat{\varphi}_\alpha(\xi) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{Q}_{\alpha-}(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)} F_{t\to\eta}\, g(\xi;t)\,d\eta \\ &(\alpha = 1, \dots, N). \end{aligned} \tag{1.5.16}$$

Using representation (1.5.13) and equality (1.5.16), we obtain by direct differentiation of (1.5.15) w.r.t. t that

$$\mathscr{P}(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t) = \hat{f}(\xi;t) \tag{1.5.17}$$

for all $t > 0$ and almost all $\xi \in \mathbb{R}^{n-1}$; and

$$\mathrm{Q}_\alpha\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0} = \hat{\varphi}_\alpha(\xi) \tag{1.5.18}$$

a.e. in $\mathbb{R}^{n-1}$. Hence, the inverse ($\xi \to x$) Fourier transform u of the function $\hat{u}$ is a generalized solution of the boundary value problem (1.5.4)–(1.5.5).

Further we show that $u \in W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)$ and the norm $\|u : W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)\|$ admits the estimate (1.5.6) (with $\beta = 0$) for any $s \in \mathbb{R}^1$ satisfying conditions (1.5.8), (1.5.10) and (1.5.11) ($\beta = 0$).

Let $G_{\alpha j}(\xi;\tau)$ be the polynomials (in τ) of degree at most $N-1$ that satisfy identity (1.0.24)[20] with $\mathscr{R}(\xi;\tau) = \tau^j$ ($\alpha = 1,\dots,N$; $j = 0,1,\dots,J$)) for all $\tau \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$. It follows from Remark 1.1.16, Lemma 1.1.15 and Propositions 1.1.26 and 1.2.8 that these polynomials admit the following representation of the type (1.2.16):

$$G_j(\xi;\tau) = \{G_{\alpha j}(\xi;\tau)\} = H_j(\xi;\tau)\mathfrak{D}^{-1}(\xi) \qquad (j = 0,1,\dots,J). \tag{1.5.19}$$

This representation is obtained, if we put in (1.2.16) $m = 1$, $\mathscr{M}(\xi;\tau) = 1$, $\dot{\mathfrak{T}}_R^{-1}(\xi) = \mathfrak{D}^{-1}(\xi)$, where $\mathfrak{D}(\xi)$ is the matrix (1.5.14), and define the matrix $H_j(\xi;\tau)$ by setting in (1.2.15) $m = 1$, $\dot{S}(\xi;\tau) = \tau^j$, and $\dot{\mathscr{P}}_+(\xi;\tau) = \mathscr{P}_+(\xi;\tau)$. Since ord $G_{\alpha j}(\xi;\tau) \leqslant N-1$, it follows directly from (1.5.19) that the polynomial $G_{\alpha j}(\xi;\tau)$ is equal to the reminder of the division $\tau^j G_{\alpha 0}(\xi;\tau)$ by $\mathscr{P}_+(\xi;\tau)$ ($\alpha = 1,\dots,N$; $j = 0,1,\dots,J$).

Differentiating (1.5.15) w.r.t. t, calculating the Fourier transform of the derivatives of the second term of this equation and taking into account (1.5.16), we obtain for all $t > 0$ and almost all $\xi \in \mathbb{R}^{n-1}$ the representation

$$\begin{aligned} \frac{\partial^j \hat{u}(\xi;t)}{(\mathrm{i}\partial t)^j} &= F^{-1}_{\tau\to t}\Bigg\{\tau^j\,[\mathscr{P}(\xi;\tau)]^{-1}\,(F_{t\to\tau}g(\xi;\tau)) + \frac{\mathrm{i}}{\sqrt{2\pi}} \\ &\times \sum_{\alpha=1}^{N}\frac{G_{\alpha j}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{Q}_\alpha(\xi;\eta)}{\mathscr{P}_-(\xi;\eta)}(F_{t\to\eta}g(\xi;t))d\eta - \hat{\varphi}_\alpha(\xi)\right]\Bigg\}, \end{aligned} \tag{1.5.20}$$

where $G_{\alpha j}(\xi;\tau)$ are the polynomials defined by (1.5.19)[21].

Using the Parseval identity and the definition of $g(\xi;t)$, we conclude that representation (1.5.20) implies the inequalities

$$\begin{aligned} \int_{-\infty}^{\infty}\left|\frac{\partial^j \hat{u}(\xi;t)}{\partial t^j}\right|^2 dt \leqslant \mathrm{const}\Bigg[\Bigg(&\sup\left|\frac{\tau^j}{\mathscr{P}(\xi;\tau)}\right|^2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\sum_{\alpha=1}^N G_{\alpha j}(\xi;\tau)\mathrm{Q}_{\alpha-}(\xi;\eta)}{\mathscr{P}_+(\xi;\tau)\mathscr{P}_-(\xi;\eta)}\right|^2 d\tau\, d\eta\Bigg)\int_0^{\infty}|\hat{f}(\xi;t)|^2 dt \\ &+ \int_{-\infty}^{\infty}\sum_{\alpha=1}^{N}(1+|\xi|^2)^{-\mu_\alpha}\left|\frac{G_{\alpha j}(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)}\right|^2 d\tau \sum_{\alpha=1}^{N}(1+|\xi|^2)^{\mu_\alpha}|\hat{\varphi}_\alpha(\xi)|^2\Bigg] \end{aligned} \tag{1.5.21}$$

[20] In particular, for $\boldsymbol{\beta = 0}$ the polynomials $\boldsymbol{G_{\alpha 0}(\xi;\tau)}$ satisfy the identity (1.5.12).

[21] The attentive reader will certainly note that the calculation referring to the second summand of equation (1.5.15) has already been carried out in the proof of Lemma 1.1.12 (see (1.1.37)–(1.1.39) as well as (1.1.31)).

for almost all $\xi \in \mathbb{R}^{n-1}$. On the other hand, it is obvious that the norm defined by (1.5.1) is equivalent to the norm whose square is equal to

$$\int_0^\infty dt \int_{\mathbb{R}^{n-1}} (1+|\xi|^2)^s \left[(1+|\xi|^2)^J |\hat{u}(\xi;\tau)|^2 + \left| \frac{\partial^J \hat{u}(\xi;t)}{\partial t^J} \right|^2 \right] d\xi.$$

Hence if $s \in \mathbb{R}^1$ satisfies conditions (1.5.8), (1.5.10), and (1.5.11) with $\beta = 0$, then the estimate (1.5.6) (with $\beta = 0$) of the solution of boundary value problem (1.5.4)–(1.5.5) follows directly from inequalities (1.5.21) for $\hat{u}(\xi;t)$ and $\partial^J u(\xi;t)/\partial t^J$, and the problem (1.5.4)–(1.5.5) is well-posed in the pair of spaces

$$\left[W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+) \times \mathscr{H}_{\boldsymbol{\mu}}(\partial \mathbb{R}^n_+) \right].$$

We proceed to the proof of the second part of Theorem 1.5.1. Suppose that for $\beta = 0$, some $s \in \mathbb{R}^1$, and all $u \in W^{J+s,J}_{2,x,t}(\mathbb{R}^n_+)$ inequality (1.5.6) is satisfied, where $f(x;t)$ and $\varphi(x)$ denote the right-hand sides of equations (1.5.4) and (1.5.5), respectively. Then, the estimates

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} (1+|\xi|^2)^{s+J-j} \left| \frac{\partial^j \hat{u}(\xi;t)}{\partial t^j} \right|^2 dt \leqslant \text{const} \left(\|\mathscr{P}(D)u\|^2 + \langle\!\langle \mathsf{Q}(D)u \rangle\!\rangle^2_{\boldsymbol{\mu}} \right) \tag{1.5.22}$$

($j = 0, 1, \ldots, J$) hold true for all $u \in C_0^\infty(\mathbb{R}^n_+)$. Here $\mathsf{Q}(D) = \left\{ \begin{matrix} \mathsf{Q}_1(D) \\ \vdots \\ \mathsf{Q}_N(D) \end{matrix} \right\}$.

Each of these estimates is a special case of the estimate (1.2.27), specified by $m = 1$, $\boldsymbol{\nu} = 0$, $\mathscr{R}(\xi;\tau) = \tau^j$, and $B(\xi) = (1+|\xi|^2)^{s+J-j}$ ($j = 0, 1, \ldots, J$). Thus, taking in conditions (1.5.8)–(1.5.11) $\beta = 0$, we see that these conditions follow from conditions 1, 2, and 3 of Corollary 1.2.13. □

1.6 Notes

The questions discussed in Chapter 1 were studied in the case $m = 1$ in the authors' paper [MG75], where it was also mentioned that the established theorems can be generalized to matrix operators in spaces of vector functions (see [MG75], p. 242). Such a generalization was carried out by the authors in [GM85]. Some results of [MG75] were announced in [GM72].

Section 1.0 A priori estimates for differential and pseudodifferential operators (under various assumptions on the type of the operator, its coefficients and the domain in $\mathbb{R}^n$, where the functions u_j are defined) were treated by many authors. Without touching on results, obtained in this direction for concrete types of operators, we mention here several papers on estimates for general differential operators with constant coefficients in the spaces of vector-functions that are directly related to our topic.

In the papers of B. Fuglede [Fug61] and B. P. Paneyakh [Pan61], the well-known result of L. Hörmander [H55] on L^2-estimates for minimal operators in a bounded domain or in $\mathbb{R}^n$ is generalized to certain systems of differential operators (in [Fug61] deals also with overdetermined systems). Further development of these results – necessary and sufficient conditions for the validity of estimates in interior of a domain for certain systems of operators acting in vector spaces $\mathbf{H}^{\boldsymbol{\mu}}$ – can be found in the article of B.P. Paneyakh [Pan66].

Necessary and sufficient conditions for the coercivity of integro-differential forms in spaces of vector functions satisfying homogeneous boundary conditions were obtained by D. G. de Figueiredo [Fig63].

In [Sch64a] M. Schechter established sufficient conditions for the validity of inequality (1.0.1) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^n)$ under the assumptions that $R(D)$, $P(D)$ and $Q(D)$ are matrices of differential operators with constant coefficients, $\mathscr{P}(\xi;\tau) = \det P(\xi;\tau) \not\equiv 0$, and the τ-roots of the polynomial $\mathscr{P}(\xi;\tau)$ are not real. The proof of these results in the scalar case ($m = 1$) is given in [Sch64]. Judjing by remark at the end of the article [Sch64a], Schechter intended to publish some generalizations of these results; but we have no references to such publications.

The role of the matrix P^c n the study of a priori estimates for matrix differential operators has already been noted by A. A. Dezin in [Dez59]. The matrix P^c figures also in the formulation of results in the papers [Fug61], [Pan61], [Pan66], [Sch64a].

Section 1.1 The main results of Section 1.1 were established by the authors in [MG75] for $m = 1$ and in [GM85] for arbitrary m.

An assertion of the type of Lemma 1.1.1 was proved for $m = 1$ in the authors' paper [GM74] (see [GM74, Lemma 7]), where one can find also an assertion of the type of Lemma 1.1.7 for $m = 1$ (see [GM74, Lemma 5]). Lemma 1.1.5 with $m = 1$ was proved by the authors in [GM74] (see [GM74], item 3 in the proof of necessity of the conditions of Theorem 1.2.2). Lemma 1.1.18 is also proved therein (see [GM74, Lemma 2.1]).

The integral representation (1.1.31) was verified for $m = 1$ in [GM74] (see [GM74, Lemma 2.2]). The identity (1.1.40) for the matrix $G(\tau)$ was proved by another method in [GM74] (see [GM74], Lemmas 1.3 and 1.2).

Finally, Theorems 1.1.19, 1.1.29, and 1.1.30 were proved in [GM74] for $m = 1$ (see [GM74], Theorems 2.1, 2.2 and 2.1′, respectively).

Section 1.2 The main results of Section 1.2 were obtained by the authors in [GM74] for $m = 1$ and in [GM85] for arbitrary m. Also in [GM74] a version of the estimate (1.2.1), corresponding to the case $B(\xi) = 1$, is considered. Instead of the norm $\langle\!\langle \cdot \rangle\!\rangle$, [GM74] uses the norm $\langle\!\langle \cdot \rangle\!\rangle_{\boldsymbol{\mu}}$, so that the results of [GM74] (see Theorems 3.1, 3.2 and 3.1′ therein) are particular cases of Corollaries 1.2.13 and 1.2.14 of this section.

The result of Proposition 1.2.15, concerning the case $m = 1$, is also established in [GM74] (see [GM74, Proposition 3.1]). The results of Subsection 1.2.7, concerning the case of arbitrary m, are published here for the first time.

Section 1.3 The results of Section 1.3 were established in [GM74] and [GM85]. The proofs presented in this book are almost identical to the corresponding proofs in [GM74], concerned with the case $m = 1$. The only exception is the proof of Theorem 1.3.6 which follows directly from Theorem 1.3.3 just for $m = 1$ (see Corollary 1.3.7).

Condition (1.3.11) (see Remark 1.3.5) was apparently considered for the first time by J. Peetre [Pee61]. Under the assumption that $\mathscr{P}(\xi;\tau)$ is a hypoelliptic polynomial, Peetre showed that condition (1.3.11) is necessary and sufficient for the validity of certain estimates for solutions of the problem $\mathscr{P}(D)u = f$, $D_t^j u\big|_{t=0} = 0$ $(j = 0, 1, \dots, N-1)$ in the half-space $\mathbb{R}^n_+$

The proof of Proposition 1.3.12 uses some arguments found in the work of M. Schechter [Sch64a] (see [Sch64a], p. 424 and p. 433). The result of Remark 1.3.13 belongs to M. Schechter (see [Sch64a], p. 426 and p. 433).

Section 1.4 The results of Subsections 1.4.1–1.4.5 were established in the authors' paper [GM85], and the results of Subsections 1.4.6–1.4.7 were obtained in [GM74].

An estimate of the type (1.4.9) for quasielliptic systems in a class of the Sobolev spaces with fractional exponent (Slobodeckij spaces) was studied by K. K. Golovkin and V. A. Solonnikov [GolSol68]. The class of spaces considered in [GolSol68] covers the Hölder spaces, but not the L^p-spaces. From this point of view, Corollary 1.4.2 can be considered as a supplement to Theorem 19 from [GolSol68].

Many authors have studied estimates of the type (1.4.9) for solutions of elliptic systems. Among the works devoted to this topic we mention those by of S. Agmon, A. Douglis and L. Nirenberg [AGN64], L. P. Volevich [Vol65], and V. A. Solonnikov [Sol64]. Works dealing with general parabolic and quasielliptic systems have already been mentioned in Subsection 1.4.1. For the scalar quasielliptic $P(D)$, results similar to Corollary 1.4.2 were obtained by V. T. Purmonen ([Pur77] and [Pur79]), who used Schechter's method ([Sch63] and [Sch64]) and results by T. Matsuzawa [Mat68].

Section 1.5 The result of this section is published here for the first time. It is close to the papers [Dik62], [DikSi60a], [DikSi60b], [Pal60] (see also [Sil65, Chapter IV]) of G. V. Dikopolov, V. P. Palamodov and G. E. Shilov, where the general question of describing well-posed problems in a half-space for equations and systems solved with respect to the highest derivative w.r.t. the variable t is discussed. Well-posed problems for equations not solved with respect to the highest derivative w.r.t. t were studied by A. L. Pavlov [Pav77].

In the papers [DikSi60a], [Pal60] and [DikSi60b], solutions are sought in the classes of distributions that depend on a parameter $t \geqslant 0$ and belong to the space $\mathscr{H}$ for each $t \geqslant 0$. The space $\mathscr{H}$ consists of ordinary functions that are square integrable in the whole space $\mathbb{R}^{n-1}$ and together with their generalized derivatives w.r.t. t up to the order $J-1$ grow in $\mathscr{H}$ as $t \to +\infty$ no faster than a certain power of t.

Well-posed problems in spaces that contain growing functions (for example, in the space S') were studied in [Dik62] and [Pav77].

We also note that the problem considered in this section is regular (the definition of regular problem can be found, for instance, in [Sil65], p. 253), since we assume that condition (1.5.7) is satisfied. This enables us to prove (under additional conditions (1.5.8)–(1.5.12)) its well-posedness in the pair of spaces

$$\left[W_{2,x,t}^{J+s,J}(\exp(\beta t);\mathbb{R}_+^n),\, L^2(\exp(\beta t);\mathbb{R}_+^n) \times \mathscr{H}_{\boldsymbol{\mu}}(\partial\mathbb{R}_+^n)\right].$$

Chapter 2
Boundary estimates for differential operators

2.0 Introduction

2.0.1 Description of results

In this chapter we formulate necessary and sufficient conditions for the validity of the estimate

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \left(\sum_{j=1}^{m} \|P_j(D)u\|^2 + \sum_{\alpha=1}^{N} \langle\!\langle Q_\alpha(D)u \rangle\!\rangle^2 \right) \tag{2.0.1}$$
$$u = u(x;t) \in \mathbf{C}_0^\infty(\mathbb{R}^n_+),$$

and give the exact description of the "trace space" $R(D)u\big|_{t=0}$ for elements u belonging to the completion of the space $\mathbf{C}_0^\infty(\mathbb{R}^n_+)$ in the metric $\sum_{j=1}^m \|P_j(D)u\|^2$.

We assume that $R(\xi;\tau)$, $P_j(\xi;\tau)$ and $Q_\alpha(\xi;\tau)$ are polynomials of the variable $\tau \in \mathbb{R}^1$ with complex measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than a power of $|\xi|$ as $|\xi| \to \infty$. It is also supposed that the inequalities

$$0 \leqslant \operatorname{ord} R(\xi;\tau), \qquad \operatorname{ord} Q_\alpha(\xi;\tau) \leqslant \max_{1\leqslant j\leqslant m} \operatorname{ord} P_j(\xi;\tau) - 1$$

hold a.e. in $\mathbb{R}^{n-1}$.

A criterion for the validity of the estimate (2.0.1) is established in Section 2.2. To formulate this criterion we need the polynomials $H_\pm(\xi;\tau)$ and $T_1(\xi;\tau), \ldots, T_m(\xi;\tau)$ (in τ), which are defined as follows:

Polynomials $H_\pm(\xi;\tau)$. We set

$$\sum_{j=1}^{m} |P_j(\xi;\tau)|^2 = H_+(\xi;\tau)H_-(\xi;\tau). \tag{2.0.2}$$

Here, $H_+(\xi;\tau) = \sum_{s=0}^{J} h_s(\xi)\tau^{J-s}$ is a polynomial with roots lying in the half-plane $\operatorname{Im}\zeta \geqslant 0$ ($\zeta = \tau + i\sigma$), and $H_-(\xi;\tau) = \overline{H_+(\xi;\tau)}$. We assume that $h_0(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$.

Polynomials $T_j(\xi;\tau)$. For a point $\xi \in \mathbb{R}^{n-1}$ with $h_0(\xi) \neq 0$ we denote by $\Pi_+(\xi;\tau)$ the greatest common divisor of the polynomials $H_+(\xi;\tau)$, $P_1(\xi;\tau), \ldots, P_m(\xi;\tau)$ with leading coefficients equal to 1. Suppose that there exist functions $\beta_\alpha(\xi)$ such that

$$D(\xi;\tau) \stackrel{\text{def}}{=} R(\xi;\tau) - \sum_{\alpha=1}^{N} \beta_\alpha(\xi) Q_\alpha(\xi;\tau) \equiv 0 \ (\text{mod}\, \Pi_+(\xi;\tau)) \tag{2.0.3}$$

a.e. in $\mathbb{R}^{n-1}$.

Denote by $T_j(\xi;\tau)$ $(j = 1, \dots, m)$ the polynomials in τ (ord $T_j(\xi;\tau) \leqslant J-1$) that satisfy for all $\tau \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$ the following conditions:

$$\overline{T}_j(\xi;\tau) \equiv 0 \ (\text{mod}\, \Pi_+(\xi;\tau)) \qquad (j = 1, \dots, m); \tag{2.0.4}$$

$$D(\xi;\tau) H_-(\xi;\tau) = \sum_{j=1}^{m} P_j(\xi;\tau) T_j(\xi;\tau); \tag{2.0.5}$$

$$P_i(\xi;\tau)\overline{T}_j(\xi;\tau) \equiv P_j(\xi;\tau)\overline{T}_i(\xi;\tau) \ (\text{mod}\, \Pi_+(\xi;\tau) H_+(\xi;\tau)) \tag{2.0.6}$$

$(i \neq j; i, j = 1, \dots, m$; condition (2.0.6) is omitted if $m = 1$).

From the results of Subsection 2.2.1 (Lemma 2.1.1) it follows that for every N-tuple of functions $\beta_\alpha(\xi)$ satisfying condition (2.0.3), the polynomials $T_j(\xi;\tau)$ exist and are uniquely determined by conditions (2.0.4)–(2.0.6).

In Subsection 2.2.1 (Theorem 2.2.2) it is stated that the estimate (2.0.1) holds true if and only if the following conditions are satisfied:

1. There exist functions $\beta_\alpha(\xi)$ such that relation (2.0.3) is valid a.e. in $\mathbb{R}^{n-1}$.
2. The inequality

$$\sup_{\xi \in \mathbb{R}^{n-1}} \left\{ B(\xi) \inf_{\{\beta_\alpha(\xi)\}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau + \sum_{\alpha=1}^{N} |\beta_\alpha(\xi)|^2 \right] \right\} < \infty \tag{2.0.7}$$

   holds true. Here the infimum is taken over all N-tuples $\{\beta_\alpha(\xi)\}$ satisfying (2.0.3), and $T_j(\xi;\tau)$ denote the polynomials determined by conditions (2.0.4)–(2.0.6).

The left-hand side of (2.0.7) is the sharp constant in (2.0.1).

Some corollaries of this result are derived in Subsection 2.2.2. In Subsection 2.2.3 we consider an inequality of the type (2.0.1) with an additional term on the right-hand side. It turns out that this inequality (cf. (2.2.28)) remains valid if we replace all operators by their principal homogeneous parts. Finally, an example of estimate for operators $P_j(D)$ of first order w.r.t. t is discussed in Subsection 2.2.4.

The main tool for obtaining the above-mentioned results is the theorem on the sharp constant in an inequality of the type (2.0.1) for ordinary differential operators on the semi-axis $t \geqslant 0$ which is proved in Subsection 2.2.1.

In Subsection 2.2.3, a special case of inequality (2.0.1), namely

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+) \tag{2.0.8}$$

is studied. From the main result of Subsection 2.2.2 it follows that inequality (2.0.8) holds true if and only if

$$R(\xi;\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)) \quad \text{a.e. in } \mathbb{R}^{n-1}, \tag{2.0.9}$$

and

$$\sup_{\xi\in\mathbb{R}^{n-1}} B(\xi)\Lambda(\xi) < \infty, \tag{2.0.10}$$

where

$$\Lambda(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau \tag{2.0.11}$$

and $T_j(\xi;\tau)$ are the polynomials (of τ) of degree at most $J-1$, satisfying conditions (2.0.4)–(2.0.6) (with $D(\xi;\tau)$ replaced by $R(\xi;\tau)$ in condition (2.0.5)).

The aim of Section 2.3 is to prove the converse of this result, namely the continuation theorem (Theorem 2.3.8). Here it turns out that if $R(D)$, $P_j(D)$ ($j = 1,\dots,m$) are differential operators with constant coefficients, then the "trace space" $R(D)u\big|_{t=0}$ of the elements u belonging to the completion of the space $C_0^\infty(\mathbb{R}^n_+)$ in the metric $\sum_{j=1}^m \|P_j(D)u\|^2$ coincides with closure of the linear space of functions $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ that satisfy the inequality

$$\langle\!\langle \varphi\rangle\!\rangle^2_{\Lambda^{-1/2}} = \int_{\mathbb{R}^{n-1}} \frac{|\hat\varphi(\xi)|^2}{\Lambda(\xi)} d\xi < \infty$$

w.r.t. the norm $\langle\!\langle \cdot \rangle\!\rangle_{\Lambda^{-1/2}}$.

Finally, in Section 2.3 we establish a corollary of these theorems (Proposition 2.3.11) related to the problem of extension with "preservation of the class" for functions with finite norm

$$\left(\sum_{j=1}^m \|P_j(D)u : L^2(\mathbb{R}^n_+)\|^2\right)^{1/2}$$

in the whole space $\mathbb{R}^n$.

Other applications of results of this chapter will be provided in Chapter 4.

2.0.2 Outline of the proof of the main result

To shorten the explanations, we consider in this section only a special case of the fundamental inequality, namely the estimate (2.0.8). For the sake of simplicity, we assume that the leading coefficient of the polynomial $H_+(\xi;\tau)$ is equal to 1, and the τ-roots of this polynomial $\zeta_1(\xi),\dots,\zeta_J(\xi)$ are pairwise distinct a.e. in $\mathbb{R}^{n-1}$.

As in Subsection 1.0.2, we begin by observing that estimate (2.0.8) holds if and only if the inequality

$$|R\,(\xi; -\mathrm{i}\,d/dt)\,v|_{t=0}|^2 \leqslant \Lambda(\xi) \int_0^\infty \sum_{j=1}^m \left|P_j\,(\xi; -\mathrm{i}\,d/dt)\,v\right|^2 dt, \qquad v \in C_0^\infty(\mathbb{R}_+^1), \tag{2.0.12}$$

is satisfied for almost all $\xi \in \mathbb{R}^{n-1}$, and the sharp constant $\Lambda(\xi)$ in (2.0.12) satisfies condition (2.0.10).

Necessity of condition (2.0.9). Let $\zeta(\xi)$ be a τ-root of the polynomial $\Pi_+(\xi;\tau)$. We substitute the function $v_\xi(t) = \exp(\mathrm{i}\zeta(\xi)t)$ in inequality (2.0.12) (see Lemma 2.1.8). Since $P_j\,(\xi; -\mathrm{i}\,d/dt)\,v_\xi(t) = 0$ $(j = 1,\dots,m)$, it follows from (2.0.12) that $R(\xi;\zeta(\xi)) = 0$. Since $\zeta(\xi)$ is an arbitrary root of the polynomial $\Pi_+(\xi;\tau)$, the last equation is equivalent to (2.0.9).

Without loss of generality, we may assume that $\Pi_+(\xi;\tau) = 1$. (Otherwise, we should replace all the polynomials in (2.0.12) by the corresponding quotients arising after dividing on $\Pi_+(\xi;\tau)$, and use the resulting estimate instead of (2.0.12).)

Estimate in a finite-dimensional space. For each fixed $\xi \in \mathbb{R}^{n-1}$ we consider the vector $\mathbf{a}(\xi) \in \mathbb{C}^J$ and the $J \times J$ matrix $\mathfrak{B}(\xi)$, defined as follows:

$$\mathbf{a}(\xi) = \{R(\xi;\zeta_1(\xi)),\dots,R(\xi;\zeta_J(\xi))\}, \tag{2.0.13}$$

$$\mathfrak{B}(\xi) = \{\mathscr{P}_{\varrho\nu}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\},$$

$$\mathscr{P}_{\varrho\nu}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi)) = i\sum_{j=1}^m \frac{P_j(\xi;\zeta_\varrho(\xi))\overline{P}_j(\xi;\overline{\zeta_\nu(\xi)})}{\zeta_\varrho(\xi) - \overline{\zeta}_\nu(\xi)}. \tag{2.0.14}$$

We show that the estimate (2.0.12) is valid if and only if the inequality

$$|(\mathbf{a}(\xi),\mathbf{x})|^2 \leqslant \Lambda(\xi)\,(\mathfrak{B}(\xi)\mathbf{x},\mathbf{x}), \qquad \mathbf{x} \in \mathbb{C}^J, \tag{2.0.15}$$

holds a.e. in $\mathbb{R}^{n-1}$.

Indeed, let $v \in C_0^\infty(\mathbb{R}_+^1)$. We represent $v(t)$ in the form

$$v(t) = z_\xi(t) + \sum_{k=1}^J x_k(\xi)\exp(\mathrm{i}\zeta_k(\xi)t), \tag{2.0.16}$$

where $z_\xi^{(\nu)}(t)\big|_{t=0} = 0$ $(\nu = 0,1,\dots,J-1)$, and $x_k(\xi)$ is determined by the Cauchy data of the function v. Since

$$\sum_{j=1}^m \int_0^\infty P_j\,(\xi; -\mathrm{i}\,d/dt)\,z_\xi(t)\overline{P_j\,(\xi; -\mathrm{i}\,d/dt)\,[\exp(\mathrm{i}\zeta_k(\xi)t)]}dt$$

$$= \int_0^\infty z_\xi(t)\sum_{j=1}^m |P_j|^2\,(\xi; -\mathrm{i}\,d/dt)\,[\exp(\mathrm{i}\zeta_k(\xi)t)]dt = 0, \quad (k = 1,\dots,J),$$

one can recast (2.0.12) as

$$|(\mathbf{a}(\xi),\mathbf{x}(\xi))| \leqslant \Lambda(\xi)\left[\int_0^\infty \sum_{j=1}^m \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\,z_\xi(t)\right|^2 dt + (\mathfrak{B}(\xi)\mathbf{x}(\xi),\mathbf{x}(\xi))\right] \tag{2.0.17}$$

with $\mathbf{x}(\xi) = (\bar{x}_1(\xi),\ldots,\bar{x}_J(\xi))$.

It is obvious that (2.0.15) implies the estimate (2.0.17). Conversely, if (2.0.12) holds true, then approximation of the function

$$x_\xi(t) = \sum_{k=1}^J x_k(\xi)\exp(\mathrm{i}\zeta_k(\xi)t)$$

by a sequence of compactly supported functions (see Lemma 2.1.8 for details), yields inequality (2.0.15).

Proof of inequality (2.0.15). From the definition (2.0.14) of the matrix $\mathfrak{B}(\xi)$ it follows that the equality

$$(\mathfrak{B}(\xi)\mathbf{x},\mathbf{x}) = \int_0^\infty \sum_{j=1}^m \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\left[\sum_{k=1}^J x_k\exp(\mathrm{i}\zeta_k(\xi)t)\right]\right|^2 dt \tag{2.0.18}$$

remains valid for any vector $\mathbf{x} = (x_1,\ldots,x_J)$.

Due to the assumption $\Pi_+(\xi;\tau) = 1$, (2.0.18) shows that the matrix $\mathfrak{B}(\xi)$ is positive definite. Therefore, the equation

$$\mathfrak{B}(\xi)\mathbf{x}(\xi) = \mathbf{a}(\xi) \tag{2.0.19}$$

has a unique solution $\mathbf{x}_0(\xi)$, and, consequently, estimate (2.0.15) with the sharp constant

$$\Lambda(\xi) = (\mathfrak{B}(\xi)\mathbf{x}_0(\xi),\mathbf{x}_0(\xi)) \tag{2.0.20}$$

holds true for almost all $\xi \in \mathbb{R}^{n-1}$.

Computation of $\Lambda(\xi)$. Now we explain how the formula (2.0.11) for the sharp constant $\Lambda(\xi)$ in the estimate (2.0.15) can be derived from equation (2.0.20). The complete derivation of this formula is given in Section 2.1. Here, for simplicity, we restrict ourselves to the case $m = 2$ and $P_2(\xi;\tau) = 1$. It is shown in Subsection 2.1.1 (see Remark 2.1.4) that under these additional assumptions the polynomials $T_1(\xi;\tau)$ and $T_2(\xi;\tau)$ are completely determined by the equation

$$RH_- = P_1T_1 + T_2. \tag{2.0.21}$$

From (2.0.21) it follows that

$$T_2(\xi;\bar{\zeta}_\nu(\xi)) = -P_1(\xi;\bar{\zeta}_\nu(\xi))T_1(\xi;\bar{\zeta}_\nu(\xi)) \qquad (\nu = 1,\ldots,J). \tag{2.0.22}$$

We also note that in the case under consideration the identities

$$P_1(\xi;\bar{\zeta}_\nu(\xi))\overline{P}_1(\xi;\bar{\zeta}_\nu(\xi)) = -1 \qquad (\nu = 1,\dots,J) \tag{2.0.23}$$

follow from (2.0.2).

Let $\mathbf{x}_0(\xi) = (\bar{x}_{10}(\xi),\dots,\bar{x}_{J0}(\xi))$ be the unique solution of equation (2.0.19). Taking into account (2.0.13) and (2.0.14), we can rewrite (2.0.19) in the form

$$R(\xi;\zeta_\varrho(\xi)) = \sum_{\nu=1}^{J} \mathrm{i}\frac{P_1(\xi;\zeta_\varrho(\xi))\overline{P}_1(\xi;\bar{\zeta}_\nu(\xi))) + 1}{\zeta_\varrho(\xi) - \bar{\zeta}_\nu(\xi)}\bar{x}_{\nu 0}(\xi), \quad (\varrho = 1,\dots,J). \tag{2.0.24}$$

On the other hand, equality (2.0.21) implies the relation

$$R(\xi;\zeta_\varrho(\xi)) = P(\xi;\zeta_\varrho(\xi))\frac{T_1(\xi;\zeta_\varrho(\xi))}{H_-(\xi;\zeta_\varrho(\xi))} + \frac{T_2(\xi;\zeta_\varrho(\xi))}{H_-(\xi;\zeta_\varrho(\xi))}. \tag{2.0.25}$$

Applying the Lagrange interpolation formula to $T_1(\xi;\zeta_\varrho(\xi))/H_-(\xi;\zeta_\varrho(\xi))$ and $T_2(\xi;\zeta_\varrho(\xi))/H_-(\xi;\zeta_\varrho(\xi))$ we transform (2.0.25) into the equality

$$R(\xi;\zeta_\varrho(\xi)) = \sum_{\nu=1}^{J} \frac{P_1(\xi;\zeta_\varrho(\xi))T_1(\xi;\bar{\zeta}_\nu(\xi)) + T_2(\xi;\bar{\zeta}_\nu(\xi))}{H'_-(\xi;\bar{\zeta}_\nu(\xi))\left(\zeta_\varrho(\xi) - \bar{\zeta}_\nu(\xi)\right)}. \tag{2.0.26}$$

Next, using (1.0.22) and (1.0.23), we convert (1.0.26) as follows:

$$\begin{aligned} R(\xi;\zeta_\varrho(\xi)) &= \sum_{\nu=1}^{J} \frac{P_1(\xi;\zeta_\varrho(\xi)) - P_1(\xi;\bar{\zeta}_\nu(\xi))}{H'_-(\xi;\bar{\zeta}_\nu(\xi))\left(\zeta_\varrho(\xi) - \bar{\zeta}_\nu(\xi)\right)} T_1(\xi;\bar{\zeta}_\nu(\xi)) \\ &= \sum_{\nu=1}^{J} \mathrm{i}\,\frac{P_1(\xi;\zeta_\varrho(\xi))\overline{P}_1(\xi;\bar{\zeta}_\nu(\xi)) + 1}{\left(\zeta_\varrho(\xi) - \bar{\zeta}_\nu(\xi)\right)}\,\mathrm{i}\,\frac{T_1(\xi;\bar{\zeta}_\nu(\xi))P_1(\xi;\bar{\zeta}_\nu(\xi))}{H'_-(\xi;\bar{\zeta}_\nu(\xi))}, \\ &\qquad (\varrho = 1,\dots,J). \end{aligned} \tag{2.0.27}$$

Since system (2.0.19) has the unique solution, we deduce from (1.0.24) and (1.0.27) the relations

$$x_{\nu 0}(\xi) = -\mathrm{i}\frac{\overline{T}_1(\xi;\zeta_\nu(\xi))\overline{P}_1(\xi;\zeta_\nu(\xi))}{H'_+(\xi;\zeta_\nu(\xi))} \qquad (\nu = 1,\dots,J). \tag{2.0.28}$$

In view of (2.0.22), these relations can also be recast as

$$x_{\nu 0}(\xi) = \mathrm{i}\,\frac{\overline{T}_2(\xi;\zeta_\nu(\xi))}{H'_+(\xi;\zeta_\nu(\xi))} \qquad (\nu = 1,\dots,J). \tag{2.0.29}$$

Finally, we compute $(\mathfrak{B}(\xi)\mathbf{x}_0(\xi), \mathbf{x}_0(\xi))$ with the help of (2.0.19), (2.0.24), and (2.0.29), and also the equation $P_1(\xi;\zeta_\varrho(\xi))\overline{P}_1(\xi;\zeta_\varrho(\xi)) = 1$, which follows from (2.0.21). This yields

$$
\begin{aligned}
&(\mathfrak{B}(\xi)\mathbf{x}_0(\xi), \mathbf{x}_0(\xi)) = (\mathbf{a}(\xi), \mathbf{x}_0(\xi)) = \sum_{\varrho=1}^{J} R(\xi;\zeta_\varrho(\xi)) x_{\varrho 0}(\xi) \\
&= \sum_{\varrho=1}^{J} \frac{P_1(\xi;\zeta_\varrho(\xi))T_1(\xi;\zeta_\varrho(\xi)) + T_2(\xi;\zeta_\varrho(\xi))}{H_-(\xi;\zeta_\varrho(\xi))} x_{\varrho 0}(\xi) \\
&= \sum_{\varrho=1}^{J} \Bigg\{ -\mathrm{i}\, \frac{P_1(\xi;\zeta_\varrho(\xi))\overline{P}_1(\xi;\zeta_\varrho(\xi))T_1(\xi;\zeta_\varrho(\xi))\overline{T}_1(\xi;\zeta_\varrho(\xi))}{H'_+(\xi;\zeta_\varrho(\xi))H_-(\xi;\zeta_\varrho(\xi))} \\
&\qquad + \mathrm{i}\, \frac{T_2(\xi;\zeta_\varrho(\xi))\overline{T}_2(\xi;\zeta_\varrho(\xi))}{H'_+(\xi;\zeta_\varrho(\xi))H_-(\xi;\zeta_\varrho(\xi))} \Bigg\} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|T_1(\xi;\tau)|^2 + |T_2(\xi;\tau)|^2}{H_+(\xi;\tau)H_-(\xi;\tau)} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|T_1(\xi;\tau)|^2 + |T_2(\xi;\tau)|^2}{|P_1(\xi;\tau)|^2 + 1} d\tau.
\end{aligned}
$$

Taking into account relation (1.0.20), we arrive at (1.0.11).

2.1 Estimates for ordinary differential operators on the semi-axis

Let $R(\tau)$, $P_1(\tau), \ldots, P_m(\tau)$, $Q_1(\tau), \ldots, Q_N(\tau)$ be polynomials of the variable $\tau \in \mathbb{R}^1$ with constant coefficients, let $\max_i \operatorname{ord} P_j(\tau) = J \geqslant 1$, and let $\operatorname{ord} R(\tau)$, $\operatorname{ord} Q_\alpha \leqslant J - 1$ $(\alpha = 1, \ldots, N)$. In this section, necessary and sufficient conditions for the validity of the estimates

$$
\begin{aligned}
&|R(-\mathrm{i}\, d/dt)\, u|_{t=0}|^2 \\
&\leqslant \Lambda \left[\int_0^\infty \sum_{j=1}^{m} \left| P_j(-\mathrm{i}\, d/dt)\, u \right|^2 dt + \sum_{\alpha=1}^{N} |Q_\alpha(-\mathrm{i}\, d/dt)\, u|_{t=0}|^2 \right], \qquad (2.1.1) \\
&u \in C_0^\infty(\mathbb{R}^1_+),
\end{aligned}
$$

$$
\left.
\begin{aligned}
&|R(-\mathrm{i}\, d/dt)\, u|_{t=0}|^2 \leqslant \Lambda_0 \int_0^\infty \sum_{j=1}^{m} \left| P_j(-\mathrm{i}\, d/dt)\, u \right|^2 dt, \\
&u \in C_0^\infty(\mathbb{R}^1_+), \\
&Q_\alpha(-\mathrm{i}\, d/dt)\, u\, |_{t=0} = 0 \qquad (\alpha = 1, \ldots, N)
\end{aligned}
\right\} \qquad (1.1')
$$

are established, and the sharp constants Λ, Λ_0 in inequalities (2.1.1), (1.1′) are calculated.

In what follows, we denote by $H_\pm(\tau)$ the polynomials of order J defined by the relations

$$\sum_{j=1}^{m} |P_j(\tau)|^2 = H_+(\tau)H_-(\tau) \tag{2.1.2}$$

and $H_-(\tau) = \overline{H_+(\tau)}$, as well as the property that all roots of the polynomial $H_+(\tau)$ lie in the half-plane $\operatorname{Im}\zeta \geqslant 0$ $(\zeta = \tau + i\sigma)$. By $\Pi_+(\xi)$ we denote the greatest common divisor of the polynomials $P_1(\tau), \dots, P_m(\tau)$ and $H_+(\tau)$ with leading coefficients 1.

2.1.1 A lemma on polynomials

The main result of this subsection is the following lemma on the unique solvability of the system of congruences and equalities (2.1.3)–(2.1.5). The solutions of this system (the polynomials $T_j(\tau)$) are repeatedly used in this and the following chapters. In particular, they appear in formula (2.1.58) for the sharp constant Λ from inequality (2.1.1).

Lemma 2.1.1. *Suppose the polynomial $D(\tau)$ is such that $D(\tau) \equiv 0 \pmod{\Pi_+(\tau)}$ and* ord $D(\tau) \leqslant J - 1$. *Then there exist uniquely determined polynomials $T_j(\tau)$ satisfying the relations $T_j(\tau) \leqslant J - 1$ $(j = 1, \dots, m)$ and the following conditions:*

$$\overline{T}_j(\tau) \equiv 0 \pmod{\Pi_+(\tau)} \qquad (j = 1, \dots, m), \tag{2.1.3}$$

$$D(\tau)H_-(\tau) = \sum_{j=1}^{m} P_j(\tau)T_j(\tau), \tag{2.1.4}$$

$$P_i(\tau)\overline{T}_J(\tau) \equiv P_j(\tau)\overline{T}_i(\tau) \pmod{H_+(\tau)\Pi_+(\tau)} \tag{2.1.5}$$

($i \neq j$, $i, j = 1, \dots, m$; condition (2.1.5) *is omitted for $m = 1$).*

Proof. Consider the polynomials $p_j(\tau), h_+(\tau), h_-(\tau), d(\tau)$, defined by the formulas

$$P_j(\tau) = \Pi_+(\tau)p_j(\tau), \qquad H_+(\tau) = \Pi_+(\tau)h_+(\tau),$$
$$h_-(\tau) = \overline{h_+(\tau)}, \qquad D(\tau) = \Pi_+(\tau)d(\tau) \qquad (j = 1, \dots, m).$$

Let $k = \operatorname{ord} h_+(\tau)$, and let ζ_ν be the roots of $h_+(\tau)$ with multiplicities k_ν $(\nu = 1, \dots, l;\ k_1 + \cdots + k_l = k)$, so that $h_+(\tau) = \prod_{\nu=1}^{l} (\tau - \zeta_\nu)^{k_\nu}$.

Due to (2.1.3)–(2.1.5), the polynomials $T_j(\tau)$ can be written in the form $T_j(\tau) = \overline{\Pi_+(\tau)}t_j(\tau)$, where $t_j(\tau)$ are polynomials $(\operatorname{ord} t_j(\tau) \leqslant k - 1;\ j = 1, \dots, m)$ that satisfy the conditions

$$d(\tau)h_-(\tau) = \sum_{j=1}^{m} p_j(\tau)t_j(\tau), \tag{2.1.6}$$

$$p_i(\tau)\overline{t_j(\tau)} \equiv p_j(\tau)\overline{t_i(\tau)} \pmod{h_+(\tau)}. \tag{2.1.7}$$

Let us show that the polynomials $t_j(\tau)$ are uniquely determined by (2.1.6) and (2.1.7).

We set $h_\nu(\tau) = (\tau - \zeta_\nu)^{-k_\nu} h_+(\tau)$ $(\nu = 1, \dots, l)$. From the definition of the polynomials $h_+(\tau)$ and $p_1(\tau), \dots, p_m(\tau)$ it follows that they are relatively prime. Hence for each value of $\nu = 1, \dots, l$ we can select an index $a = a(\nu)$, $1 \leqslant a(\nu) \leqslant m$, such that $p_a(\zeta_\nu) \neq 0$. Then (2.1.7) yields

$$\left[p_a\left(\frac{\bar{t}_j}{h_\nu} - \frac{\bar{t}_a p_i}{p_a h_\nu}\right)\right]^{(s)}\Bigg|_{\tau=\zeta_\nu} = 0,$$

$$\left(\frac{t_j}{h_\nu}\right)^{(s)}\Bigg|_{\tau=\zeta_\nu} = \sum_{\gamma=0}^{s} C_s^\gamma \left(\frac{\bar{t}_a}{p_a h_\nu}\right)^{(\gamma)}\Bigg|_{\tau=\zeta_\nu} p_j^{(s-\gamma)}(\zeta_\nu)$$

$$(s = 0, \dots, k_\nu - 1;\ \nu = 1, \dots, l;\ j = 1, \dots, m).$$

According to the Lagrange–Sylvester interpolation theorem we have

$$\bar{t}_j(\tau) = h_+(\tau) \sum_{\nu=1}^{l} \sum_{s=0}^{k_\nu - 1} \frac{1}{s!} \left(\frac{\bar{t}_j}{h_\nu}\right)^{(s)}\Bigg|_{\tau=\zeta_\nu} \frac{1}{(\tau - \zeta_\nu)^{k_\nu - s}},$$

and, consequently, condition (2.1.6) takes on the form

$$d(\tau) = \sum_{\nu=1}^{l} \sum_{s=0}^{k_\nu - 1} \sum_{\gamma=0}^{s} \sum_{j=1}^{m} \frac{\bar{c}_{\nu\gamma}}{(s-\gamma)!} \bar{p}_j^{(s-\gamma)}(\bar{\zeta}_\nu) \frac{p_j(\tau)}{(\tau - \zeta_\nu)^{k_\nu \nu - s}},$$

where $c_{\nu\gamma} = \left(\dfrac{1}{\gamma!} \dfrac{t_a}{p_a h_\nu}\right)^{(\gamma)}\Bigg|_{t=\zeta_\nu}$ $(\nu = 1, \dots, l;\ \gamma = 0, \dots, k_\nu - 1)$. Setting $\mu = s - \gamma$ and

$$l_{\nu\gamma}(\tau) = \sum_{j=1}^{m} \frac{p_j(\tau)}{(\tau - \bar{\zeta}_\nu)^{k_\nu - \gamma}} \sum_{\mu=0}^{k_\nu - 1 - \gamma} \frac{\bar{p}_j^{(\mu)}(\bar{\zeta}_\nu)(\tau - \bar{\zeta}_\nu)^\mu}{\mu!}, \tag{2.1.8}$$

we get the equality

$$d(\tau) = \sum_{\nu=1}^{m} \sum_{\gamma=0}^{k_\nu - 1} \bar{c}_{\nu\gamma} l_{\nu\gamma}(\tau). \tag{2.1.9}$$

We claim that

$$\sum_{j=1}^{m} p_j(\tau) \sum_{\mu=0}^{k_\nu-1-\gamma} \frac{1}{\mu!} \bar{p}_j^{(\mu)}(\bar{\zeta}_\nu)(\tau-\bar{\zeta}_\nu)^\mu \equiv 0 \; (\operatorname{mod}(\tau-\zeta_\nu)^{k_\nu-\gamma}). \tag{2.1.10}$$

Indeed, it is obvious that

$$\bar{p}_j(\tau) = \sum_{\mu=0}^{k_\nu-1-\gamma} \frac{1}{\mu!} \bar{p}_j^{(\mu)}(\zeta_\nu)(\tau-\zeta_\nu)^\mu + (\tau-\zeta_\nu)^{k_\nu-\gamma} q_j(\tau), \tag{2.1.11}$$

where $\nu = 1, \dots, l$, $\gamma = 0, \dots, k_\nu - 1$, $j = 1, \dots, m$, and $q_j(\tau)$ is a polynomial of τ. On the other hand, we have

$$\sum_{j=1}^{m} |p_j(\tau)|^2 = h_+(\tau)h_-(\tau) \equiv 0 \; (\operatorname{mod}(\tau-\zeta_\nu)^{k_\nu}) \qquad (\nu = 1, \dots, l). \tag{2.1.12}$$

Thus, (2.1.11) and (2.1.12) immediately imply (2.1.10).

In view of the congruence (2.1.10) and the equality (2.1.8), we conclude that $l_{\nu\gamma}(\tau)$ are polynomials of τ of degree less than or equal to $k-1$. The same is true for the polynomial $d(\tau)$. Therefore, if we prove that the $k \times k$ matrix $\{l_{\nu\gamma}^{(\sigma)}(\zeta_\nu)\}$ (whose rows and colums are labeled by the indices ν, γ and ϱ, σ, respectively; $\nu, \varrho = 0, \dots, l$; $\gamma = \gamma(\nu) = 0, \dots, k_\nu - 1$; $\sigma = \sigma(\varrho) = 0, \dots, k_\varrho - 1$) is regular, then the constants $c_{\nu\gamma}$, as well as the polynomials $t_j(\tau)$, are uniquely determined by the relation (2.1.9).

To prove the regularity of the matrix $\{l_{\nu\gamma}^{(\sigma)}(\zeta_\nu)\}$, we consider the Gram matrix of the system of k vector functions

$$\left\{p_1(-\mathrm{i}\,d/dt)\left[(\mathrm{i}\,t)^\sigma \exp(\mathrm{i}\,\zeta_\varrho t)\right], \dots, p_m(-\mathrm{i}\,d/dt)\left[(\mathrm{i}\,t)^\sigma \exp(\mathrm{i}\,\zeta_\varrho t)\right]\right\}, \tag{2.1.13}$$

where $\varrho = 1, \dots, l$ and $\sigma = 0, \dots, k_\varrho - 1$. Based on (2.1.8) one can easily verify that the entry

$$\begin{aligned}
&\sum_{j=1}^{m} \int_0^\infty p_j(-\mathrm{i}\,d/dt)\left[(\mathrm{i}\,t)^\sigma \exp(\mathrm{i}\,\zeta_\varrho t)\right] \overline{p_j(-\mathrm{i}\,d/dt)\left[(\mathrm{i}\,t)^{k_\nu-1-\gamma} \exp(\mathrm{i}\,\zeta_\varrho t)\right]} \\
&= \mathrm{i} \sum_{g=0}^{k_\nu-1-\gamma} \sum_{h=0}^{\sigma} \frac{(-1)^{\sigma-h}(k_\nu-\gamma-1+\sigma-g-h)!\,C_{k_\nu-1-\gamma}^{g} C_\sigma^h}{(\zeta_\varrho-\bar{\zeta}_\nu)^{k_\nu-\gamma+\sigma-g-h}} \\
&\quad \times \sum_{j=1}^{m} p_j^{(h)}(\zeta_\varrho)\bar{p}_j^{(g)}(\bar{\zeta}_\nu)
\end{aligned}$$

of this matrix is equal to $\mathrm{i}\,(k_\nu-\gamma-1)!\{l_{\nu\gamma}^{(\sigma)}(\zeta_\varrho)\}$. Hence the nondegeneracy of the matrix $\{l_{\nu\gamma}^{(\sigma)}(\zeta_\varrho)\}$ is equivalent to the linear independence of vector functions (2.1.13).

Let us established their linear independence. By contradiction, suppose that there exist constants $\varphi_{\varrho\sigma}$ such that not all $\varphi_{\varrho\sigma}$ equal zero and

$$\sum_{\varrho=1}^{l}\sum_{\sigma=0}^{k_\varrho-1}\varphi_{\varrho\sigma}\,p_j\,(-\mathrm{i}\,d/dt)\big[(\mathrm{i}\,t)^{\sigma}\exp(\mathrm{i}\,\zeta_\varrho t)\big]=0 \qquad (j=1,\dots,m;\ t>0),$$

or, equivalently,

$$\sum_{\varrho=1}^{l}\sum_{\sigma=0}^{k_\varrho-1}\sum_{h=0}^{\sigma}C_\sigma^h\varphi_{\varrho\sigma}\,p_j^{(h)}(\zeta_\varrho)(\mathrm{i}\,t)^{\sigma-h}\exp(\mathrm{i}\,\zeta_\varrho t)=0.$$

This last condition is equivalent to the system of equations

$$\sum_{\sigma=0}^{k_\varrho-1}\sum_{h_1=0}^{\sigma}C_\sigma^{h_1}\varphi_{\varrho\sigma}\,p_j^{(\sigma-h_1)}(\zeta_\varrho)(\mathrm{i}\,t)^{h_1}=0 \qquad (j=1,\dots,m;\ t>0),$$

which can also be written as

$$\sum_{h_1=0}^{k_\varrho-1}\sum_{\sigma=h_1}^{k_\varrho-1}C_\sigma^{h_1}\varphi_{\varrho\sigma}\,p_j^{(\sigma-h_1)}(\zeta_\varrho)(\mathrm{i}\,t)^{h_1}=0 \qquad (j=1,\dots,m;\ t>0). \tag{2.1.14}$$

It is clear that equalities (2.1.14) holds true if and only if

$$\sum_{\sigma=h_1}^{k_\varrho-1}C_\sigma^{h_1}\varphi_{\varrho\sigma}\,p_j^{(\sigma-h_1)}(\zeta_\varrho)=0$$
$$(j=1,\dots,m;\quad \varrho=1,\dots,l;\quad h_1=0,\dots,k_\varrho-1). \tag{2.1.15}$$

Because of the triangular structure of system (2.1.15) for each fixed j, we can see that if for some ϱ $(1\leqslant\varrho\leqslant l)$ and for all $j=1,\dots,m$ we have $p_j(\zeta_\varrho)\neq 0$, then $\varphi_{\varrho 0}=\cdots=\varphi_{\varrho,k_\varrho-1}=0$. Since not all of $\varphi_{\varrho\sigma}$ equal zero, we see that for some ϱ_0 $(1\leqslant\varrho_0\leqslant l)$, we have $p_j(\zeta_{\varrho_0})=0$ $(j=1,\dots,N)$, which contradicts the definition of the polynomials $p_j(\tau)$. □

Remark 2.1.2. We consider a factorization $\Pi_+(\tau)=\Pi_0(\tau)\Pi_1(\tau)$, where $\Pi_0(\tau)$ is a polynomial with real roots and $\Pi_1(\tau)$ a polynomial with non-real roots, and the leading coefficients of these polynomials equal 1. We set

$$\mathscr{D}(\tau)=D(\tau)/\Pi_1(\tau),\quad \mathscr{P}_j(\tau)=P_j(\tau)/\Pi_1(\tau),\quad \mathscr{H}_+(\tau)=H_+(\tau)/\Pi_1(\tau),$$
$$\mathscr{H}_-(\tau)=\overline{\mathscr{H}}_+(\tau),\quad \mathscr{H}_\nu(\tau)=\mathscr{H}_+(\tau)(\tau-\zeta_\nu)^{-k_\nu},\quad \mathscr{T}_j(\tau)=T_j(\tau)/\Pi_1(\tau),$$

and assume that

$$\mathcal{L}_{\nu\gamma}(\tau) = \sum_{j=1}^{m} \frac{\mathscr{P}_j(\tau)}{(\tau - \bar{\zeta}_\nu)^{k_\nu - \gamma}} \sum_{\mu=0}^{k_\nu - 1 - \gamma} \frac{\bar{\mathscr{P}}_j^{(\mu)}(\bar{\zeta}_\nu)(\tau - \bar{\zeta}_\nu)^\mu}{\mu!} \tag{2.1.16}$$
$$(\nu = 1, \dots, l; \quad \gamma = 1, \dots, k_\nu - 1).$$

We will prove the following assertions:

1. *For each* $\nu = 1, \dots, l$ *there exists an index* $a = a(\nu)$, $1 \leqslant a(\nu) \leqslant m$, *such that* $\mathscr{P}_a(\zeta_\nu) \neq 0$.
2. *The system of equations*

$$\mathscr{D}^{(\sigma)}(\zeta_\varrho) = \sum_{\nu=1}^{l} \sum_{\gamma=0}^{k_\nu - 1} \mathcal{L}_{\nu\gamma}^{(\sigma)}(\zeta_\varrho) \bar{d}_{\nu\gamma} \qquad (\varrho = 1, \dots, l;\ \sigma = 0, \dots, k_{\varrho - 1}) \tag{2.1.17}$$

has the uniquely determined solution

$$d_{\nu\gamma}^0 = \frac{1}{\gamma!} \left(\frac{\overline{\mathscr{T}}_a}{\mathscr{P}_a \mathscr{H}_\nu} \right)^{(\gamma)} \Bigg|_{\tau = \zeta_\nu}. \tag{2.1.18}$$

First, we note that the roots of $h_+(\tau)$ are not real. Indeed, assuming that τ_0 is a real root of $H_+(\tau)$ with multiplicity k_0, (2.1.2) yields the congruence

$$P_j(\tau) \equiv 0 \ (\mathrm{mod}\, (\tau - \tau_0)^{k_0}) \qquad (j = 1, \dots, m).$$

Hence $(\tau - \tau_0)^{k_0}$ is a divisor of Π_+ (and also of Π_0), but is not a divisor of h_+.

On the other hand, the polynomials p_j and h_+ are relatively prime, $\mathscr{P}_j = p_j \Pi_0$, and the roots of Π_0 are real. Hence the polynomials $\mathscr{P}_j$ and h_+ are also relatively prime. This implies assertion 1.

Using this statement and differentiating the right-hand side of (2.1.16), we find that the determinant of (2.1.17) is not zero.

Now let us prove (2.1.18). From (2.1.4) and the definition of the polynomials $\mathscr{D}$, $\mathscr{T}_j$, $\mathscr{P}_j$ and $\mathscr{H}_-$ it follows that

$$\mathscr{D}(\tau)\mathscr{H}_-(\tau) = \sum_{j=1}^{m} \mathscr{P}_j(\tau)\mathscr{T}_j(\tau). \tag{2.1.19}$$

In addition, we have $\mathscr{T}_j / \mathscr{H}_- = t_j / h_-$ and $\mathscr{T}_j / \overline{\mathscr{H}}_\nu = t_j / \bar{h}_\nu$, and hence

$$\mathscr{T}_j(\tau) = \mathscr{H}_-(\tau) \sum_{\nu=1}^{l} \sum_{s=0}^{k_\nu - 1} \frac{1}{s!} \left(\frac{\overline{\mathscr{T}}_j}{\mathscr{H}_\nu} \right)^{(s)} \Bigg|_{\tau = \zeta_\nu} (\tau - \zeta_\nu)^{s - k_\nu}.$$

Since $\mathscr{P}_i\overline{\mathscr{T}}_j \equiv \mathscr{P}_j\overline{\mathscr{T}}_i \pmod{h_+}$, we have also the relation

$$\left(\frac{\mathscr{T}_j}{\mathscr{H}_\nu}\right)^{(s)}\Big|_{\tau=\zeta_\nu} = \left(\frac{\mathscr{P}_j\overline{\mathscr{T}}_a}{\mathscr{P}_a\mathscr{H}_\nu}\right)^{(s)}\Big|_{\tau=\zeta_\nu} = \sum_{\gamma=0}^{s} C_s^\gamma \left(\frac{\overline{\mathscr{T}}_a}{\mathscr{P}_a\mathscr{H}_\nu}\right)^{(\gamma)}\Big|_{\tau=\zeta_\nu} \mathscr{P}_j^{(s-\gamma)}(\zeta_\nu).$$

Substituting these relations into the right-hand side of (2.1.19), we obtain

$$\mathscr{D}(\tau) = \sum_{\nu=1}^{l}\sum_{\gamma=0}^{k_\nu-1}\frac{1}{\gamma!}\overline{\left(\frac{\overline{\mathscr{T}}_a}{\mathscr{P}_a\mathscr{H}_\nu}\right)^{(\gamma)}\Big|_{\tau=\zeta_\nu}}\mathcal{L}_{\nu\gamma}(\tau),$$

where $\mathcal{L}_{\nu\gamma}(\tau)$ denote the polynomials (1.1.16). Thus, (2.1.18) follows from the uniqueness of the solution of (2.1.17).

Remark 2.1.3. Suppose that $P_j(\tau) = P(\tau)$ $(j = 1,\dots,m)$. Consider a factorization $P(\tau) = P_+(\tau)P_-(\tau)$, where the roots of P_+ coincide (including multiplicity) with the roots of P in the half-plane $\operatorname{Im}\zeta \geqslant 0$ $(\zeta = \tau + \mathrm{i}\sigma)$. In this case we have

$$T_j = m^{-1/2} D\overline{P}_+/P_+ \qquad (j = 1,\dots,m). \tag{2.1.20}$$

Indeed, under these assumptions $H_+ = m^{1/2}P_+\overline{P}_-$ and $H_- = m^{1/2}\overline{P}_+P_-$. Let p_0 be the leading coefficient of P_-. Then we obviously have the relations $\Pi_+ = m^{1/2}\bar{p}_0P_+$, $h_+ = \overline{P}_-/\bar{p}_0$, $h_- = P_-/p_0$, $p_j = m^{-1/2}P_-/\bar{p}_0$. Therefore, equation (1.1.6) can be written as

$$d(\tau) = m^{-1/2}p_0\sum_{j=1}^{m} t_j(\tau)/\bar{p}_0,$$

and congruence (1.1.7) as

$$P_-(\bar{t}_j - \bar{t}_i) \equiv 0 \pmod{\overline{P}_-}.$$

Since P_- and $\overline{P}_-$ are relatively prime, the last formula implies the congruence

$$\bar{t}_j - \bar{t}_i \equiv 0 \pmod{\overline{P}_-}.$$

However, we have $\operatorname{ord} P_-(\tau) = k$ and $\operatorname{ord} t_j(\tau) \leqslant k-1$. Hence, $t_i = t_j$ $(i,j = 1,\dots,m)$, which yields $t_j = m^{-1/2}\bar{p}_0 d/p_0$.

Finally, since $T_j = t_j\overline{\Pi}_+$ and $d = D/\Pi_+$, we get (2.1.20) for the polynomials T_j.

Remark 2.1.4. Let $m = 2$, and let the polynomials P_1 and P_2 be relatively prime. Then, T_1 and T_2 are completely determined by the equation

$$D(\tau)H_-(\tau) = P_1(\tau)T_1(\tau) + P_2(\tau)T_2(\tau). \tag{2.1.21}$$

In particular, if $P_1(\tau) = P(\tau)$ and $P_2(\tau) = 1$, then T_1 and T_2 are the quotient and the remainder of the division of DH_- by P.

Indeed, in the case under consideration, condition (2.1.4) has the form (2.1.21). Therefore, it suffices to show that (2.1.5) follows from (2.1.21).

By virtue of (2.1.21),

$$\overline{P}_1\overline{T}_1 + \overline{P}_2\overline{T}_2 \equiv 0 \ (\mathrm{mod}\, H_+),$$

and consequently

$$|P_1|^2\overline{T}_1 + \overline{P}_2 P_1\overline{T}_2 \equiv 0 \ (\mathrm{mod}\, H_+).$$

Using the equality $|P_1|^2 + |P_2|^2 = H_+H_-$ one can verify that

$$\overline{P}_2\left(P_1\overline{T}_2 - P_2\overline{T}_1\right) \equiv 0 \ (\mathrm{mod}\, H_+).$$

If $\overline{P}_2$ and H_+ have no common roots, then our assertion is proved.

Now let ζ be a common root of $\overline{P}_2$ and H_+. From the definition of H_+ and the fact that P_1 and P_2 are relatively prime, it follows that $\overline{P}_1(\zeta) \neq 0$ and $P_1(\zeta) \neq 0$. Since $\overline{P}_1\overline{T}_1 + \overline{P}_2\overline{T}_2 \equiv 0 \ (\mathrm{mod}\, H_+)$, we get $\overline{T}_1(\zeta) = 0$. But then $P_1(\zeta)\overline{T}_2(\zeta) - P_2(\zeta)\overline{T}_1(\zeta) = 0$. The proof is complete.

Remark 2.1.5. For polynomials D, T_j satisfying the hypothesis of Lemma 2.1.1, we have

$$|D(\tau)|^2 \leqslant \sum_{j=1}^{m} |T_j(\tau)|^2. \tag{2.1.22}$$

Indeed, we have $|DH_-|^2 = \left|\sum_{j=1}^{m} P_jT_j\right|^2 \leqslant \sum_{j=1}^{m} |P_j|^2 \sum_{i=1}^{m} |T_i|^2$. It remains to note that $|H_-|^2 = \sum_{j=1}^{m} |P_j|^2$.

2.1.2 A variational problem in finite-dimensional space

As already shown in Subsection 2.0.2, the estimate (2.0.12) is equivalent to inequality (2.0.15). A similar statement for the estimate (2.1.1) will be proved in Subsection 2.1.3: we will see that (2.1.1) is equivalent to (2.1.43). In this subsection we consider a variational problem equivalent to (2.1.43), give necessary and sufficient conditions for the boundedness of the function (1.1.23), and calculate its supremum.

Let $(\cdot,\cdot)_\mu$ and $(\cdot,\cdot)_\lambda$ be the scalar products in the spaces $\mathbb{C}^\mu$ and $\mathbb{C}^\lambda$, respectively, and let $\{\cdot,\cdot\}$ denote the scalar product in $\mathbb{C}^\mu \times \mathbb{C}^\lambda$. Elements $\mathbf{z} \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$ will be written in the form $\mathbf{z} = (\mathbf{x};\mathbf{y})$, $\mathbf{x} \in \mathbb{C}^\mu$, $\mathbf{y} \in \mathbb{C}^\lambda$.

We consider a nonnegative $\mu \times \mu$ matrix $\mathfrak{B}$ in the space $\mathbb{C}^\mu$, and denote by $\mathbb{X}$ the orthogonal complement in $\mathbb{C}^\mu$ of the subspace $\ker \mathfrak{B}$. Let $\mathfrak{B}^{1/2}$ be the nonnegative square root of $\mathfrak{B}$.

Let $\mathbf{a}, \mathbf{c}_\alpha \in \mathbb{C}^\mu$, $\mathbf{b}, \mathbf{d}_\alpha \in \mathbb{C}^\lambda$ ($\alpha = 1,\dots,N$) and $(\mathbf{a};\mathbf{b}) \neq 0$. We consider the function

$$\Phi(z) = \frac{|(\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda|^2}{(\mathfrak{B}\mathbf{x},\mathbf{x})_\mu + \sum_{\alpha=1}^{N} |(\mathbf{c}_\alpha,\mathbf{x})_\mu + (\mathbf{d}_\alpha,\mathbf{y})_\lambda|^2}, \tag{2.1.23}$$

and set

$$\Lambda = \sup_{\mathbf{z}\in\mathbb{C}^\mu\times\mathbb{C}^\lambda} \Phi(\mathbf{z}). \tag{2.1.24}$$

It is evident that $\Lambda > 0$.

The proof of Lemma 2.1.6, formulated below, will be essentially based on the following result from the theory of nonnegative quadratic forms.

Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be nonnegative $(\mu+\lambda)\times(\mu+\lambda)$ matrices. The ratio $\{\mathfrak{K}_1\mathbf{z},\mathbf{z}\}/\{\mathfrak{K}_2\mathbf{z},\mathbf{z}\}$ of their quadratic forms is bounded in $\mathbb{C}^\mu\times\mathbb{C}^\lambda$ if and only if $\ker\mathfrak{K}_2\subset\ker\mathfrak{K}_1$. *If this condition is satisfied, there exists an extremal element $\mathbf{z}^0\in\mathbb{C}^\mu\times\mathbb{C}^\lambda$ such that*

$$\frac{\{\mathfrak{K}_1\mathbf{z}^0,\mathbf{z}^0\}}{\{\mathfrak{K}_2\mathbf{z}^0,\mathbf{z}^0\}} = \sup_{\mathbf{z}\in\mathbb{C}^\mu\times\mathbb{C}^\lambda}\frac{\{\mathfrak{K}_1\mathbf{z},\mathbf{z}\}}{\{\mathfrak{K}_2\mathbf{z},\mathbf{z}\}}.$$

Lemma 2.1.6. *The function $\Phi(\mathbf{z})$ defined by (2.1.23) is bounded in $\mathbb{C}^\mu\times\mathbb{C}^\lambda$ if and only if there exist constants β_α $(\alpha=1,\dots,N)$ such that*

$$\sum_{\alpha=1}^{N}\beta_\alpha\mathbf{d}_\alpha = \mathbf{b} \tag{2.1.25}$$

and the equation

$$\mathfrak{B}\mathbf{x} = \mathbf{a} - \sum_{\alpha=1}^{N}\beta_\alpha\mathbf{c}_\alpha \tag{2.1.26}$$

is solvable.

If these conditions are satisfied and $\mathbf{x}_0$ is an arbitrary solution of (2.1.26), then

$$(\mathfrak{B}\mathbf{x}_0,\mathbf{x}_0)_\mu = \sup_{\mathbf{x}\in\mathbb{C}^\mu}\frac{\left|\left(\mathbf{a}-\sum_{\alpha=1}^{N}\beta_\alpha\mathbf{c}_\alpha,\mathbf{x}\right)_\mu\right|^2}{(\mathfrak{B}\mathbf{x},\mathbf{x})_\mu}, \tag{2.1.27}$$

and for the constant Λ defined by (2.1.24) it holds that

$$\Lambda = \inf_{\{\beta_\alpha\}}\left[(\mathfrak{B}\mathbf{x}_0,\mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N}|\beta_\alpha|^2\right]. \tag{2.1.28}$$

Here the infimum is taken over all β_α satisfying the conditions of the lemma.

Proof. Denote by $\{\mathfrak{K}_1\mathbf{z},\mathbf{z}\}$ and $\{\mathfrak{K}_2\mathbf{z},\mathbf{z}\}$ the numerator and the denominator of the right-hand side of (2.1.23). We set $\mathbf{a}=\mathbf{a}^{(1)}+\mathbf{a}^{(2)}$, $\mathbf{c}_\alpha=\mathbf{c}_\alpha^{(1)}+\mathbf{c}_\alpha^{(2)}$, where $\mathbf{a}^{(1)},\mathbf{c}_\alpha^{(1)}\in\ker\mathfrak{B}$ and $\mathbf{a}^{(2)},\mathbf{c}_\alpha^{(2)}\in\mathbb{X}$ $(\alpha=1,\dots,N)$.

Necessity. Let $\Lambda<\infty$. If $\mathbf{z}=(\mathbf{x},\mathbf{y})\in\ker\mathfrak{K}_2$, i.e., if $\mathbf{x}\in\ker\mathfrak{B}$ and $(\mathbf{c}_\alpha^{(1)},\mathbf{x})_\mu+(\mathbf{d}_\alpha,\mathbf{y})_\lambda=0$ $(\alpha=1,\dots,N)$, then $\mathbf{z}\in\ker\mathfrak{K}_1$ and $(\mathbf{a}_\alpha^{(1)},\mathbf{x})_\mu+(\mathbf{b},\mathbf{y})_\lambda=0$. Thus, the vector $(\mathbf{a}^{(1)}; \mathbf{b}) \in \ker \mathfrak{B} \times \mathbb{C}^{\lambda}$ belongs to the linear span of the vectors $(\mathbf{c}_\alpha^{(1)}; \mathbf{d}_\alpha)$. Hence there exist constants β_α such that $\mathbf{b} = \sum\limits_{\alpha=1}^{N} \beta_\alpha \mathbf{d}_\alpha$ and $\mathbf{a}^{(1)} = \sum\limits_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha^{(1)}$.

Since $\mathbf{a}^{(2)} - \sum\limits_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha^{(2)} \in \mathbb{X}$, the equation

$$\mathfrak{B}\mathbf{x} = \mathbf{a}^{(2)} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha^{(2)}$$

is solvable. Then equation (2.1.26) is also solvable.

Sufficiency. Suppose that there exist constants β_α such that (1.1.25) is satisfied, and $\mathbf{x}_0$ is a solution of (2.1.26). Consider an arbitrary element $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \ker \mathfrak{K}_2$. We claim that $\mathbf{z} \in \ker \mathfrak{K}_1$. Indeed, if $\mathbf{z} \in \ker \mathfrak{K}_2$, then

$$\left(\sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha^{(1)}, \mathbf{x} \right)_\mu + \sum_{\alpha=1}^{N} (\beta_\alpha \mathbf{d}_\alpha, \mathbf{y})_\lambda = 0,$$

and consequently

$$\left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha^{(2)} - \mathfrak{B}\mathbf{x}_0, \mathbf{x} \right)_\mu + (\mathbf{b}, \mathbf{y})_\lambda = 0.$$

But $\mathbf{x} \in \ker \mathfrak{B}$ and $\mathbf{c}_\alpha^{(2)} \in \mathbb{X}$. Hence $(\mathbf{a}, \mathbf{x})_\mu + (\mathbf{b}, \mathbf{y})_\lambda = 0$; that is, $\mathbf{z} \in \ker \mathfrak{K}_1$.

Computation of $(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)$. Let β_α be the constants satisfying (2.1.25), and let $\mathbf{x}_0$ be a solution of (2.1.26). Then

$$\left| \left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha, \mathbf{x} \right)_\mu \right|^2 = \left| (\mathfrak{B}\mathbf{x}_0, \mathbf{x})_\mu \right|^2$$
$$\leqslant \left(\mathfrak{B}^{1/2}\mathbf{x}_0, \mathfrak{B}^{1/2}\mathbf{x}_0 \right)_\mu \left(\mathfrak{B}^{1/2}\mathbf{x}, \mathfrak{B}^{1/2}\mathbf{x} \right)_\mu = \left(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0 \right)_\mu (\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu .$$

Therefore,

$$\frac{\left| \left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha, \mathbf{x} \right)_\mu \right|^2}{(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu} \leqslant (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu$$

for all $x \in \mathbb{C}^\mu$.

On the other hand, we have

$$\frac{\left| \left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha, \mathbf{x}_0 \right)_\mu \right|^2}{(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu} = (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu .$$

Combining all these relations we arrive at (2.1.27).

Computation of Λ. Let $\mathbf{z}^0 = (\mathbf{x}^0, \mathbf{y}^0)$ be an extremal element of the function $\Phi(\mathbf{z})$. In other words, let

$$\Phi(\mathbf{z}^0) = \frac{\{\mathfrak{K}_1 \mathbf{z}^0, \mathbf{z}^0\}}{\{\mathfrak{K}_2 \mathbf{z}^0, \mathbf{z}^0\}} = \Lambda.$$

Varying the right-hand side of (2.1.23) over all possible vectors $\mathbf{y}$ and $\mathbf{x}$, we obtain at the extremal point $\mathbf{z}^0$ the equalities

$$\overline{\big((\mathbf{a}, \mathbf{x}^0)_\mu + (\mathbf{b}, \mathbf{y}^0)_\lambda\big)}\mathbf{b} = \Lambda \sum_{\alpha=1}^{N} \overline{\big((\mathbf{c}_\alpha, \mathbf{x}^0)_\mu + (\mathbf{d}_\alpha, \mathbf{y}^0)_\lambda\big)}\mathbf{d}_\alpha, \tag{2.1.29}$$

$$\overline{\big((\mathbf{a}, \mathbf{x}^0)_\mu + (\mathbf{b}, \mathbf{y}^0)_\lambda\big)}\mathbf{a} = \Lambda \left[\mathfrak{B}\mathbf{x}^0 + \sum_{\alpha=1}^{N} \overline{\big((\mathbf{c}_\alpha, \mathbf{x}^0)_\mu + (\mathbf{d}_\alpha, \mathbf{y}^0)_\lambda\big)}\mathbf{c}_\alpha \right]. \tag{2.1.30}$$

Since $\Lambda > 0$, it follows that $(\mathbf{a}, \mathbf{x}^0)_\mu + (\mathbf{b}, \mathbf{y}^0)_\lambda \neq 0$. We set

$$\mathbf{x}^0 = \Lambda\overline{\big((\mathbf{a}, \mathbf{x}^0)_\mu + (\mathbf{b}, \mathbf{y}^0)_\lambda\big)}^{-1}\mathbf{x}^0, \quad \mathbf{y}^0 = \Lambda\overline{\big((\mathbf{a}, \mathbf{x}^0)_\mu + (\mathbf{b}, \mathbf{y}^0)_\lambda\big)}^{-1}\mathbf{y}^0 \tag{2.1.31}$$

and

$$\beta_\alpha^0 = \overline{\big((\mathbf{c}_\alpha, \mathbf{x}^0)_\mu + (\mathbf{d}_\alpha, \mathbf{y}^0)_\lambda\big)} \qquad (\alpha = 1, \dots, N). \tag{2.1.32}$$

Equation (2.1.29) can then be written in the form $\mathbf{b} = \sum_{\alpha=1}^{N} \beta_\alpha^0 \mathbf{d}_\alpha$, while equation (2.1.30) can be recast as $\mathbf{a} = \mathfrak{B}\mathbf{x}_0 + \sum_{\alpha=1}^{N} \beta_\alpha^0 \mathbf{c}_\alpha$. Thus, the constants β_α^0 defined by (2.1.32) satisfy (2.1.25), and the element $\mathbf{x}_0$ given by (2.1.31) is a solution of (2.1.26).

Let $\mathbf{z}_0 = (\mathbf{x}_0; \mathbf{y}_0)$. Since $\Phi(\mathbf{z})$ is a homogeneous function of degree zero and $\mathbf{z}^0$ is an extremal element of this function, it follows from (2.1.31) that $\mathbf{z}_0$ is also an extremal element, i.e.,

$$\Lambda = \Phi(\mathbf{z}_0) = \frac{\{\mathfrak{K}_1 \mathbf{z}_0, \mathbf{z}_0\}}{\{\mathfrak{K}_2 \mathbf{z}_0, \mathbf{z}_0\}}. \tag{2.1.33}$$

From (2.1.25), (2.1.26), and (2.1.32) we obtain

$$\begin{aligned} \{\mathfrak{K}_1 \mathbf{z}_0, \mathbf{z}_0\} &= \left| (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} \beta_\alpha^0 \big((\mathbf{c}_\alpha, \mathbf{x}_0)_\mu + (\mathbf{d}_\alpha, \mathbf{y}_0)_\lambda\big) \right|^2 \\ &= \left[(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha^0|^2 \right]^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \{\mathfrak{K}_2 \mathbf{z}_0, \mathbf{z}_0\} &= (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} \big|(\mathbf{c}_\alpha, \mathbf{x}_0)_\mu + (\mathbf{d}_\alpha, \mathbf{y}_0)_\lambda\big|^2 \\ &= (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha^0|^2. \end{aligned}$$

Substituting these expressions in (2.1.33), we get

$$\Lambda = (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha^0|^2,$$

and, consequently,

$$\Lambda \geqslant \inf_{\{\beta_\alpha\}} \left[(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha^0|^2 \right].$$

To prove the opposite inequality, we consider an arbitrary element $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$, a set of constants β_α satisfying (2.1.25), and an arbitrary solution $\mathbf{x}_0$ of (2.1.26). It is obvious that

$$\begin{aligned} \left|(\mathbf{a}, \mathbf{x})_\mu + (\mathbf{b}, \mathbf{y})_\lambda\right| &\leqslant \left| \left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha, \mathbf{x} \right)_\mu \right| + \sum_{\alpha=1}^{N} \left| \beta_\alpha \left[(\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\lambda \right]^2 \right|^{1/2} \\ &\leqslant \left[(\mathfrak{B}^{1/2}\mathbf{x}_0, \mathfrak{B}^{1/2}\mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha|^2 \right]^{1/2} \\ &\quad \times \left[(\mathfrak{B}^{1/2}\mathbf{x}, \mathfrak{B}^{1/2}\mathbf{x})_\mu + \sum_{\alpha=1}^{N} |(\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\lambda|^2 \right]^{1/2}. \end{aligned}$$

The latter implies $\Phi(\mathbf{z}) \leqslant (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha|^2$ and

$$\Lambda \leqslant \inf_{\{\beta_\alpha\}} \left[(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha|^2 \right].$$

The proof is complete. □

2.1.3 Reduction of the estimate for ordinary differential operators on the semi-axis to a variational problem in a finite-dimensional space

The main result of this section is Lemma 2.1.9, which establishes the equivalence of inequalities (2.1.1) and (2.1.43). The proof is based on a special decompositions of elements $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ (Lemma 2.1.7) and on the approximation of solutions of the equation $H_+(-\mathrm{i}\, d/dt)\, z = 0$ by elements $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ (Lemma 2.1.8).

We define the dimensions μ, λ, the matrix $\mathfrak{B}$, and the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_\alpha$, $\mathbf{d}_\alpha$ that appear in (2.1.43). Let the polynomials P_j, p_j, H_+, H_-, Π_+, Π_1, Π_0 and h_+ be defined as in Subsection 2.1.1. We set $H^+(\tau) = h_+(\tau)\Pi_1(\tau)$, and we define

$$\mu = \operatorname{ord} H^+(\tau), \qquad \lambda = J - \mu = \operatorname{ord} \Pi_0(\tau). \tag{2.1.34}$$

The roots of the polynomial $H^+(\tau)$ will be denoted (as will the roots of $h_+(\tau)$) by ζ_ϱ ($\varrho = 1, \dots, l_1 \geqslant l$), and their multiplicities will be denoted by $\varkappa_\varrho$ ($\varkappa_\varrho \geqslant k_\varrho$, $\varrho = 1, \dots, l$; $\varkappa_1 + \varkappa_2 + \cdots + \varkappa_l + \cdots + \varkappa_{l_1} = \mu$). The roots of the polynomial $\Pi_0(\tau)$ and their multiplicities will be denoted by η_δ and g_δ, respectively. (Here $\delta = 1, \dots, l_2$; $g_1 + \cdots + g_{l_2} := \lambda$).

Consider the $\mu \times \mu$ matrix $\mathfrak{B} = \{P_{\varrho\sigma\nu\gamma}(\zeta_\varrho, \zeta_\nu)\}$. Its rows and columns are labeled by the indices ϱ, $\sigma = \sigma(\varrho)$ and ν, $\gamma = \gamma(\nu)$, respectively, where $\varrho, \nu = 1, \dots, l_1$; $\sigma(\varrho) = 0, \dots, \varkappa_\varrho - 1$, and $\gamma(\nu) = 0, \dots, \varkappa_\nu - 1$. The entries of this matrix are defined by the formula

$$\begin{aligned} P_{\varrho\sigma\nu\gamma}(\zeta_\varrho, \zeta_\nu) = \mathrm{i} \sum_{g=0}^{\gamma} \sum_{h=0}^{\sigma} & \frac{(-1)^{\sigma-h} C_\gamma^g C_\sigma^h (\gamma - g + \sigma - h)!}{(\zeta_\varrho - \bar{\zeta}_\nu)^{\gamma-g+\sigma-h+1}} \\ & \times \sum_{j=1}^{m} P_j^{(h)}(\zeta_\varrho) \overline{P}_j^{(g)}(\bar{\zeta}_\nu). \end{aligned} \tag{2.1.35}$$

The matrix $\mathfrak{B}$ is nonnegative, since for any $\mathbf{x} = (\bar{x}_{\varrho\sigma}) \in \mathbb{C}^\mu$ we have the easily verified identity

$$(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu = \sum_{j=1}^{m} \int_0^\infty \left| \sum_{\varrho=1}^{l_1} \sum_{\sigma=0}^{\varkappa_\varrho - 1} x_{\varrho\sigma} \sum_{h=0}^{\sigma} C_\sigma^h P_j^{(h)}(\zeta_\varrho)(\mathrm{i}\, t)^{\sigma-h} \exp(\mathrm{i}\, \zeta_\varrho t) \right|^2 dt. \tag{2.1.36}$$

Finally, we associate to the polynomials $R(\tau)$ and $Q_\alpha(\tau)$ the vectors

$$\left.\begin{gathered} \mathbf{a} = (R^{(\sigma)}(\zeta_\varrho)), \quad \mathbf{c}_\alpha = (Q_\alpha^{(\sigma)}(\zeta_\varrho)) \in \mathbb{C}^\mu, \quad \mathbf{b} = (R^{(\beta)}(\eta_\delta)), \\ \mathbf{d}_\alpha = (Q_\alpha^{(\beta)}(\eta_\delta)) \in \mathbb{C}^\lambda \\ \big(\varrho = 1, \dots, l_1; \quad \sigma = 0, \dots, \varkappa_\varrho - 1; \quad \delta = 1, \dots, l_2; \\ \beta = 0, \dots, g_\delta - 1; \quad \alpha = 1, \dots, N) \end{gathered}\right\}. \tag{2.1.37}$$

We will assume in the following that the polynomial $R(\tau)$ is not identically zero, so that $(\mathbf{a}; \mathbf{b}) \neq 0$.

Lemma 2.1.7. *Any function $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ has a unique representation*

$$u(t) = x(t) + y(t) + v(t), \tag{2.1.38}$$

where $x(t)$ and $y(t)$ are solutions of the equations $H^+(-\mathrm{i}\, d/dt)\, x = 0$ and $\Pi_0(-\mathrm{i}\, d/dt)\, y = 0$, respectively, and $v(t)$ is an infinitely differentiable function such that $v^{(p)}(0) = 0$, $p = 0, \dots, J - 1$.

Proof. Define the functions $x(t)$, $y(t)$ by

$$x(t) = \sum_{\varrho=1}^{l_1} \sum_{\sigma=0}^{\varkappa_\varrho - 1} x_{\varrho\sigma} f_{\varrho\sigma}(t), \qquad f_{\varrho\sigma}(t) = (\mathrm{i}\,t)^\sigma \exp(\mathrm{i}\,\zeta_\varrho t), \tag{2.1.39}$$

$$y(t) = \sum_{\delta=1}^{l_2} \sum_{\beta=0}^{g_\delta - 1} y_{\delta\beta} h_{\delta\beta}(t), \qquad h_{\delta\beta}(t) = (\mathrm{i}\,t)^\beta \exp(\mathrm{i}\,\eta_\delta t), \tag{2.1.40}$$

where $\mathbf{x} = (\bar{x}_{\varrho\sigma}) \in \mathbb{C}^\mu$ and $\mathbf{y} = (\bar{y}_{\delta\beta}) \in \mathbb{C}^\lambda$.

The representation (2.1.38) is obtained as follows. First, we find the constants $x_{\varrho\sigma}$, $y_{\delta\beta}$ by solving the system

$$\sum_{\varrho=1}^{l_1} \sum_{\sigma=0}^{\varkappa_\varrho - 1} f_{\varrho\sigma}^{(p)}(0) x_{\varrho\sigma} + \sum_{\delta=1}^{l_2} \sum_{\beta=0}^{g_\delta - 1} h_{\delta\beta}^{(p)}(0) y_{\delta\beta} = u^{(p)}(0) \quad (p = 0, \ldots, J-1).$$

It is obvious that the determinant of this system (the value of the Wronskian of the linearly independent functions $[f_{\varrho\sigma}(t), h_{\delta\beta}(t)]$ at $t = 0$) is not zero.

Setting $v(t) = u(t) - x(t) - y(t)$, we obtain an infinitely differentiable function which satisfies the conditions $v^{(p)}(0) = 0$, $p = 0, \ldots, J-1$. □

Lemma 2.1.8.[1] *For an arbitrary solution $z(t)$ of the equation*

$$H_+(-\mathrm{i}\,d/dt)\, z = 0$$

there exists a sequence $z_s \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ such that

$$\begin{aligned} R\,(-\mathrm{i}\,d/dt)\,(z_s - z)|_{t=0} = 0, \qquad & Q_\alpha\,(-\mathrm{i}\,d/dt)\,(z_s - z)|_{t=0} = 0 \\ (\alpha = 1, \ldots, N; \qquad & s = 1, 2, 3, \ldots) \end{aligned} \tag{2.1.41}$$

and

$$\lim_{s\to\infty} \sum_{j=1}^{m} \int_0^\infty \left| P_j\,(-\mathrm{i}\,d/dt)\,(z_s - z)\right|^2 dt = 0. \tag{2.1.42}$$

Proof. We represent $z(t)$ as $z(t) = x(t) + y(t)$, where $x(t)$ and $y(t)$ are given by (2.1.39) and (2.1.40), respectively.

Consider a cut-off function $\eta(t) \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ such that $\eta(t) = 1$ if $0 \leqslant t \leqslant 1$ and $\eta(t) = 0$ if $2 \leqslant t < \infty$. Set $z_s(t) = z(t)\eta(t/s)$ $(s = 1, 2, \ldots)$. Clearly, $z_s \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$.

The definition of η immediately yields (2.1.41). To prove (2.1.42), we fix the indices j, δ, β. In view of (2.1.40), we have $P_j^{(\omega)}\,(-\mathrm{i}\,d/dt)\,h_{\delta\beta} = 0$, if $\omega < g_\delta$. In the case $\omega \geqslant g_\delta$, we use the obvious estimate

$$\left| P_j^{(\omega)}\,(-\mathrm{i}\,d/dt)\,(h_{\delta\beta}(t))\,(-\mathrm{i}\,d/dt)^\omega\,(\eta(t/s))\right| \leqslant \frac{c s^{\beta - J + \omega}}{s^\omega} = c s^{\beta - J},$$

[1] cf. Lemma 1.1.7, Chapter 1.

where $C > 0$ is a constant, and obtain

$$\int_0^\infty \left|P_j\,(-\mathrm{i}\,d/dt)\,(h_{\delta\beta}(t) - h_{\delta\beta}(t)\eta(t/s))\right|^2 dt \leqslant cs^{1+2(\beta-J)}.$$

The last inequality evidently implies (2.1.42). □

Lemma 2.1.9. *The estimate* (2.1.1) *holds with some* $\Lambda < \infty$ *if and only if*

$$\left|(\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda\right|^2 \leqslant \Lambda\left[(\mathfrak{B}\mathbf{x},\mathbf{x})_\mu + \sum_{\alpha=1}^{N}\left|(\mathbf{c}_\alpha,\mathbf{x})_\mu + (\mathbf{d}_\alpha,\mathbf{y})_\lambda\right|^2\right], \tag{2.1.43}$$

for all vectors $(\mathbf{x},\mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$. *Here the dimensions* μ *and* λ, *the vectors* $(\mathbf{a};\mathbf{b})$ *and* $(\mathbf{c}_\alpha;\mathbf{d}_\alpha)$, *and the matrix* $\mathfrak{B}$ *are defined by* (2.1.34), (2.1.37) *and* (2.1.35), *respectively.*

Proof. Suppose that (2.1.43) holds for all $(\mathbf{x},\mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$. Consider an arbitrary function $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$, and write it in the form (2.1.38), where $x(t)$, $y(t)$ are given by (2.1.39) and (2.1.40), respectively.

From (2.1.37)–(2.1.40) it follows that

$$R\,(-\mathrm{i}\,d/dt)\,u|_{t=0} = (\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda,$$

$$Q_\alpha\,(-\mathrm{i}\,d/dt)\,u|_{t=0} = (\mathbf{c}_\alpha,\mathbf{x})_\mu + (\mathbf{d}_\alpha,\mathbf{y})_\lambda,$$

$$P_j\,(-\mathrm{i}\,d/dt)\,u = P_j\,(-\mathrm{i}\,d/dt)\,v + \sum_{\varrho=1}^{l_1}\sum_{\sigma=0}^{\varkappa_\varrho-1} x_{\varrho\sigma}\sum_{h=0}^{\sigma} C_\sigma^h P_j^{(h)}(\zeta_\varrho)(\mathrm{i}\,t)^{\sigma-h}\exp(\mathrm{i}\,\zeta_\varrho t).$$

Since

$$\sum_{j=1}^{m}\int_0^\infty P_j\,(-\mathrm{i}\,d/dt)\,v\,\overline{P_j\,(-\mathrm{i}\,d/dt)\,[(it)^\sigma\exp(i\zeta_\varrho t)]}dt$$
$$= \int_0^\infty v(t)\sum_{j=1}^{m}|P_j|^2\,(-\mathrm{i}\,d/dt)\,[(\mathrm{i}\,t)^\sigma\exp(\mathrm{i}\,\zeta_\varrho t)]dt = 0$$
$$(\varrho = 1,\dots,l_1;\quad \sigma = 0,\dots,\varkappa_\varrho - 1),$$

the inequality (2.1.1) can be written in the form

$$\begin{aligned}\left|(\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda\right|^2 \leqslant \Lambda\Bigg[&\int_0^\infty\sum_{j=1}^{m}\left|P_j\,(-\mathrm{i}\,d/dt)\,v\right|^2 dt \\ &+ (\mathfrak{B}\mathbf{x},\mathbf{x})_\mu + \sum_{\alpha=1}^{N}\left|(\mathbf{c}_\alpha,\mathbf{x})_\mu + (\mathbf{d}_\alpha,\mathbf{y})_\lambda\right|^2\Bigg].\end{aligned} \tag{2.1.44}$$

It is evident that (2.1.43) implies (2.1.44).

Conversely, suppose that for some $\Lambda < \infty$ the inequality (2.1.1) holds true for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Consider an arbitrary vector $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$. Following (2.1.39) and (2.1.40), construct a solution $z(t) = x(t) + y(t)$ of the equation $H_+ (-\mathrm{i}\, d/dt)\, z = 0$. Using Lemma 2.1.8, approximate this solution by a sequence $z_s \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. Substituting z_s in (2.1.1) and passing to the limits as $s \to \infty$ we find that for the given Λ the vector $(\mathbf{x}, \mathbf{y})$ satisfies (2.1.43). □

2.1.4 Two properties of the matrix $\mathfrak{B}$

In this subsection we study the properties of the operator $\mathfrak{B} : \mathbb{C}^\mu \to \mathbb{C}^\mu$, where $\mathfrak{B}$ is the matrix defined by (2.1.35). Lemma 2.1.10 provides a description of the kernel of $\mathfrak{B}$. A criterion for a vector (2.1.45) to lie in the range of the operator $\mathfrak{B}$ is given in Lemma 2.1.11. The proof of Lemma 2.1.11 is based on Lemma 2.1.10. Lemma 2.1.11 will be used in Subsection 2.1.5.

Lemma 2.1.10. *Let $\Pi_1(\tau)$ be the polynomial defined in Remark 2.1.2, and let $\mathfrak{B}$ be the matrix (2.1.35). The function $x(t)$ given by (2.1.39) is a solution of the equation $\Pi_1 (-\mathrm{i}\, d/dt)\, x = 0$ if and only if $\overline{\mathbf{x}} = (x_{\varrho\sigma}) \in \ker \mathfrak{B}$.*

Proof. Suppose that $\Pi_1 (-\mathrm{i}\, d/dt)\, x = 0$, where $x(t)$ is the function (2.1.39). Then the equations $P_j (-\mathrm{i}\, d/dt)\, x = 0\ (j = 1, \dots, m)$ follow from the definition of Π_1. Taking into account (2.1.36), we obtain $(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu = 0$, $\mathfrak{B}^{1/2}\mathbf{x} = 0$ and $\mathbf{x} \in \ker \mathfrak{B}$. Conversely, if $\mathbf{x} \in \ker \mathfrak{B}$, then $(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu = 0$, and, in accordance with (2.1.36), we have $P_j (-\mathrm{i}\, d/dt)\, x = 0\ (j = 1, \dots, m)$, where $x(t)$ is the function defined by (2.1.39). Therefore,

$$\Pi_0 (-\mathrm{i}\, d/dt)\, P_j (-\mathrm{i}\, d/dt)\, \Pi_1 (-\mathrm{i}\, d/dt)\, x(t) = 0 \qquad (j = 1, \dots, m),$$

where Π_0, P_j, Π_1 are the polynomials introduced in Subsection 2.1.1. We set $\varphi = \Pi_1 (-\mathrm{i}\, d/dt)\, x(t)$. In view of

$$H^+ (-\mathrm{i}\, d/dt)\, x(t) = h_+ (-\mathrm{i}\, d/dt)\, \Pi_1 (-\mathrm{i}\, d/dt)\, x(t) = 0,$$

we get $\varphi(t) = \sum_{\varrho=1}^{l} \sum_{\sigma=0}^{k_\varrho - 1} \varphi_{\varrho\sigma} (\mathrm{i}\, t)^\sigma \exp(\mathrm{i}\, \zeta_\varrho t)$ with $\varphi_{\varrho\sigma} = \mathrm{const}$. Since $\operatorname{Im} \zeta_\varrho > 0\ (\varrho = 1, \dots, l)$ and the roots of the polynomial $\Pi_0(\tau)$ are real, we have $P_j (-\mathrm{i}\, d/dt)\, \varphi(t) = 0\ (j = 1, \dots, m)$. Thanks to the linear independence of the system (2.1.13), $\varphi_{\varrho\sigma} = 0$ for all the coefficients (see the proof of Lemma 2.1.1). Hence $\Pi_1 (-\mathrm{i}\, d/dt)\, x(t) = 0$. □

Lemma 2.1.11. *Let $D(\tau)$ be a not identically vanishing polynomial such that* $\operatorname{ord} D(\tau) \leqslant J - 1$, *let*

$$\mathbf{g} = (D^{(\sigma)}(\zeta_\varrho)) \in \mathbb{C}^\mu \qquad (\varrho = 1, \dots, l_1;\ \sigma = 0, \dots, \varkappa_\varrho - 1), \tag{2.1.45}$$

and let $\mathfrak{B}$ be the matrix (2.1.35). Then the equation

$$\mathfrak{B}\mathbf{x} = \mathbf{g} \tag{2.1.46}$$

is solvable if and only if

$$D(\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_1(\tau)). \tag{2.1.47}$$

Proof. Equation (2.1.46) is solvable if and only if $(\mathbf{g}, \mathbf{x})_\mu = 0$ for all $\mathbf{x} \in \ker \mathfrak{B}$. Let $x(t)$ be the function (2.1.39), and let $\mathbf{x} = (\bar{x}_{\varrho\sigma})$. Then

$$(\mathbf{g}, \mathbf{x})_\mu = \sum_{\varrho=1}^{l_1} \sum_{\sigma=0}^{\varkappa_\varrho - 1} D^{(\sigma)}(\zeta_\varrho) x_{\varrho\sigma} = D\,(-\mathrm{i}\, d/dt)\, x|_{t=0}\,.$$

Applying Lemma 2.1.10, we see that Eq. (2.1.46) is solvable if and only if each solution $x(t)$ of the equation $\Pi_1(-\mathrm{i}\, d/dt)\, x(t) = 0$ satisfies the condition $D\,(-\mathrm{i}\, d/dt)\, x|_{t=0} = 0$. This last condition is equivalent to the statement that $D^{(\varkappa)}(\zeta) = 0$ ($\varkappa = 0, \ldots, \gamma - 1$) for every root of multiplicity γ of the polynomial $\Pi_1(\tau)$. Hence $D(\tau)$ satisfies (2.1.47). □

2.1.5 An estimate without boundary operators in the right-hand side

In this subsection we study a special case of inequality (2.1.1), that is, the estimate

$$|D\,(-\mathrm{i}\, d/dt)\, u|_{t=0}|^2 \leqslant \Lambda \int_0^\infty \sum_{j=1}^m \left|P_j\,(-\mathrm{i}\, d/dt)\, u\right|^2 dt, \tag{2.1.48}$$

where $D(\tau)$ is a not identically vanishing polynomial such that $\mathrm{ord}\, D(\tau) \leqslant J - 1$. We also study the equivalent estimate (2.1.52). The results obtained here will be used in Subsection 2.1.6 to prove a criterion for the validity of the estimate (2.1.1).

Lemma 2.1.12. *Let $D(\tau)$ be a not identically vanishing polynomial such that* $\mathrm{ord}\, D(\tau) \leqslant J - 1$*, let $\mathbf{g} \in \mathbb{C}^\mu$ be the vector* (2.1.45)*, let*

$$\mathbf{e} = (D^{(\beta)}(\eta_\delta)) \in \mathbb{C}^\mu \qquad (\delta = 1, \ldots, l_2; \quad \beta = 0, \ldots, g_\delta - 1), \tag{2.1.49}$$

and let $\mathfrak{B}$ be the matrix (2.1.35)*. Inequality* (2.1.48) *holds true with some $\Lambda < \infty$ for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ if and only if $\mathbf{e} = 0$ and equation* (2.1.46) *is solvable. The sharp constant Λ in* (2.1.48) *is given by*

$$\Lambda = (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu, \tag{2.1.50}$$

where $\mathbf{x}_0$ is an arbitrary solution of (2.1.46).

This lemma is a direct consequence of Lemmas 2.1.6 and 2.1.9.

Remark 2.1.13. Lemma 2.1.11 implies that (2.1.47) is a necessary condition for the validity of the estimate (2.1.48) for all $u \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$. In what follows we assume that (2.1.47) is satisfied.

Lemma 2.1.14.[2] *For any function* $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ *there exists a solution* $u \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ *of the equation* $\Pi_1\,(-\mathrm{i}\,d/dt)\,u = \varphi$.

Proof. Let $\varkappa = \mu - k = \operatorname{ord}\Pi_1(\tau)$, let $u_1(\tau), \dots, u_\varkappa(\tau)$ be a system of linearly independent solutions of the equation $\Pi_1\,(-\mathrm{i}\,d/dt)\,u = 0$, and let $W(u_1, \dots, u_\varkappa)$ be the Wronskian of this system. Let $W_\varrho(u_1, \dots, u_\varkappa)$ be the determinant obtained from $W(u_1, \dots, u_\varkappa)$ when the ϱ-th column $(1 \leqslant \varrho \leqslant \varkappa)$ is replaced by the vector $(0, \dots, 0, 1)$. Then the function

$$u(t) = -\sum_{\varrho=1}^{\varkappa} u_\varrho(t) \int_t^{+\infty} \frac{W_\varrho(u_1, \dots, u_\varkappa)(\tau)}{W(u_1, \dots, u_\varkappa)(\tau)} \varphi(\tau)\,d\tau$$

is a solution of the equation $\Pi_1\,(-\mathrm{i}\,d/dt)\,u = \varphi$ in the space $\mathbf{C}_0^\infty(\mathbb{R}^1_+)$. □

We set

$$\mathscr{D}(\tau) = \frac{D(\tau)}{\Pi_1(\tau)}, \qquad \mathscr{P}_j(\tau) = \frac{P_j(\tau)}{\Pi_1(\tau)} \qquad (j = 1, \dots, m). \tag{2.1.51}$$

By Remark 2.1.13, $\mathscr{D}(\tau)$ is a polynomial in τ. Observe that Lemma 2.1.14 and equalities (2.1.51) imply the following statement:

Lemma 2.1.15. *Inequality* (2.1.48) *holds with some* $\Lambda < \infty$ *for all* $u \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$ *if and only if the inequality*

$$\left|\mathscr{D}\,(-\mathrm{i}\,d/dt)\,\varphi|_{t=0}\right|^2 \leqslant \Lambda \int_0^\infty \sum_{j=1}^m \left|\mathscr{P}_j\,(-\mathrm{i}\,d/dt)\,\varphi\right|^2 dt \tag{2.1.52}$$

holds for all $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^1_+)$.

We associate with the polynomials $\mathscr{D}$ and $\mathscr{P}_j$ the vectors

$$\mathbf{d} = (\mathscr{D}^{(\sigma)}(\zeta_\varrho)) \in \mathbb{C}^k, \qquad \mathbf{s} = (\mathscr{D}^{(\beta)}(\eta_\delta)) \in \mathbb{C}^\lambda \tag{2.1.53}$$

and the positive definite $k \times k$ matrix $\mathscr{P} = \{\mathscr{P}_{\varrho\sigma\nu\gamma}(\zeta_\varrho, \zeta_\nu)\}$, where

$$\begin{aligned} \mathscr{P}_{\varrho\sigma\nu\gamma}(\zeta_\varrho, \zeta_\nu) = \mathrm{i} \sum_{g=0}^{\gamma} \sum_{h=0}^{\sigma} & \frac{(-1)^{\sigma-h} C_\gamma^g C_\sigma^h (\gamma - g + \sigma - h)!}{(\zeta_\varrho - \bar{\zeta}_\nu)^{\gamma-g+\sigma-h+1}} \\ & \times \sum_{j=1}^m \mathscr{P}_j^{(h)}(\zeta_\varrho) \overline{\mathscr{P}}_j^{(g)}(\bar{\zeta}_\nu), \end{aligned} \tag{2.1.54}$$

and ζ_ϱ are the roots of the polynomial $h_+(\tau)$ $(\varrho, \nu = 1, \dots, l;\ \sigma = 0, \dots, k_\varrho - 1,$ $\gamma = 0, \dots, k_\nu - 1,\ \delta = 1, \dots, l_2;\ \beta = 0, \dots, g_\delta - 1)$.

Replacing in the formulation of Lemma 2.1.12 μ, D, P_j, $\mathbf{g}$, $\mathbf{e}$, $\mathfrak{B}$, and $\mathbf{x}$ by k, $\mathscr{D}$, $\mathscr{P}_j$, $\mathbf{d}$, $\mathbf{s}$, $\mathscr{P}$, and φ, respectively, we obtain the following assertion:

[2] cf. Lemma 1.1.1, Chapter 1.

Lemma 2.1.16. *Inequality* (2.1.52) *holds with some* $\Lambda < \infty$ *for all* $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}_+^1)$ *if and only if* $\mathbf{s} = \mathbf{0}$, *where* $\mathbf{s}$ *is the vector in* (2.1.53).

The sharp constant Λ *in* (2.1.52) *is given by*

$$\Lambda = (\mathscr{P}\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)_k, \tag{2.1.55}$$

where $\boldsymbol{\varphi}_0 \in \mathbb{C}^k$ *is the* (*unique*) *solution of the equation*

$$\mathscr{P}\boldsymbol{\varphi} = \mathbf{d}. \tag{2.1.56}$$

Here the vector $\mathbf{d}$ *and the matrix* $\mathscr{P}$ *are given by* (2.1.53) *and* (2.1.54), *respectively.*

2.1.6 Necessary and sufficient conditions for the validity of inequality (2.1.1)

Now we turn to the proof of the fundamental result of this section.

Theorem 2.1.17. *The estimate* (2.1.1) *holds with some* $\Lambda < \infty$ *if and only if there exist constants* β_α *such that*

$$D(\tau) \stackrel{\text{def}}{=} R(\tau) - \sum_{\alpha=1}^{N} \beta_\alpha Q_\alpha(\tau) \equiv 0 \ (\operatorname{mod} \Pi_+(\tau)). \tag{2.1.57}$$

The sharp constant Λ *in* (2.1.1) *is given by*

$$\Lambda = \inf_{\{\beta_\alpha\}} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2} d\tau + \sum_{\alpha=1}^{N} |\beta_\alpha|^2 \right\}, \tag{2.1.58}$$

where the polynomials $T_j(\tau)$ *with* $\operatorname{ord} T_j(\tau) \leqslant J-1$, $j = 1, \ldots, m$, *satisfy conditions* (2.1.3)–(2.1.5), *and the infimum is taken over all* $\{\beta_\alpha\}$ *entering in* (1.1.57).

Proof. Necessity. Suppose that (2.1.1) holds with some $\Lambda < \infty$ for all $u \in C_0^\infty(\mathbb{R}_+^1)$. We consider the matrix (2.1.35) and the vectors (2.1.37). By Lemma 2.1.9, the function $\Phi(\mathbf{z})$ defined by (2.1.23) is bounded in $\mathbb{C}^\mu \times \mathbb{C}^\lambda$. Then by Lemma 2.1.6 there exist constants β_α satisfying (2.1.25) such that (2.1.26) is solvable. Consider the polynomial $D(\tau) = R(\tau) - \sum_{\alpha=1}^N \beta_\alpha Q_\alpha(\tau)$. Let $\mathbf{g}$ and $\mathbf{e}$ be the vectors defined by (2.1.45) and (2.1.49), respectively. Then, (2.1.26) coincides with (2.1.46). This means that identity (2.1.47) holds true in accordance with Lemma 2.1.11. On the other hand, condition (2.1.25) implies that $\mathbf{e} = \mathbf{0}$, and so

$$D(\tau) \equiv 0 \ (\operatorname{mod} \Pi_0(\tau)). \tag{2.1.59}$$

The identities (1.1.47), (1.1.50) are equivalent to (2.1.57).

Sufficiency. Let β_α be a collection of constants satisfying (2.1.57). Then we have (2.1.47) and (2.1.59). From (2.1.59) it follows that $\mathbf{e} = \mathbf{0}$, that is, the constants β_α satisfy (2.1.25). According to Lemma 2.1.11, identity (2.1.47) ensures the solvability of (2.1.45), which in turn is equivalent to the solvability of (2.1.26).

Using Lemma 2.1.6, we deduce from these conditions the boundedness of the function $\Phi(\mathbf{z})$ defined by (2.1.23). Hence, by Lemma 2.1.9, the estimate (2.1.1) is true with some $\Lambda < \infty$ for all $u \in C_0^\infty(\mathbb{R}_+^1)$.

Computation of Λ. By Lemmas 2.1.9 and 2.1.6, the sharp constant in (2.1.1) can be calculated by the recipe (2.1.28). The first term $(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu$ on the right-hand side of (2.1.28) equals the sharp constant in (2.1.48) according to Lemma 2.1.12. By Lemma 2.1.15, this constant is equal to the sharp constant in (2.1.52). Using Lemma 2.1.16, we obtain

$$(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu = (\mathscr{P}\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)_k,$$

where $\boldsymbol{\varphi}_0 = (\overline{\varphi^0_{\nu\gamma}})$ is the solution of (2.1.56). Let us calculate $(\mathscr{P}\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)_k$. We rewrite equation (2.1.56) in the form

$$\begin{aligned}\mathscr{D}^\sigma(\zeta_\varrho) &= \sum_{\nu=1}^{l}\sum_{\gamma=0}^{k_\nu-1} \mathscr{P}_{\varrho\sigma\nu\gamma}(\zeta_\varrho, \zeta_\nu)\overline{\varphi^0_{\nu\gamma}} \\ &= \sum_{\nu=1}^{l}\sum_{\gamma=0}^{k_\nu-1} \mathscr{P}_{\varrho\sigma\nu k_\nu-1-\gamma}(\zeta_\varrho, \zeta_{k_\nu-1-\gamma})\overline{\varphi^0_{\nu k_\nu-1-\gamma}}.\end{aligned}$$

Differentiating the right-hand side of (2.1.16), we get

$$\mathrm{i}\,(k_\nu - 1 - \gamma)!\mathscr{B}^{(\sigma)}_{\nu\gamma}(\zeta_\varrho) = \mathscr{P}_{\varrho\sigma\nu k_\nu-1-\gamma}(\zeta_\varrho, \zeta_{k_\nu-1-\gamma}).$$

Since each of the equations (2.1.17) and (2.1.56) has a unique solution, the application of (1.1.18) yields the relation

$$\varphi^0_{\nu\gamma} = \frac{\mathrm{i}d^0_{\nu k_\nu-1-\gamma}}{\gamma!} = \frac{\mathrm{i}}{\gamma!(k_\nu-1-\gamma)!}\left(\frac{\overline{\mathscr{T}}_a}{\mathscr{P}_a\mathscr{H}_\nu}\right)^{(k_\nu-1-\gamma)}\Bigg|_{\tau=\zeta_\nu},$$

which in conjunction with the equations $\mathscr{D}^{(\gamma)}(\zeta_\nu) = \sum_{j=1}^{m}\left(\frac{\mathscr{P}_j\mathscr{T}_j}{\mathscr{H}_-}\right)^{(\gamma)}\Big|_{\tau=\zeta_\nu}$, which follow from (2.1.19), yields

$$
\begin{aligned}
(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu &= (\mathscr{P}\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0)_k = (\mathbf{d}, \boldsymbol{\varphi}_0)_k \\
&= \mathrm{i} \sum_{\nu=1}^{l} \sum_{\gamma=0}^{k_\nu-1} \left(\sum_{j=1}^{m} \frac{\mathscr{P}_j \mathscr{T}_j}{\mathscr{H}_-} \right)^{(\gamma)} \Bigg|_{\tau=\zeta_\nu} \frac{1}{\gamma!(k_\nu-1-\gamma)!} \left(\frac{\overline{\mathscr{T}}_a}{\mathscr{P}_a \mathscr{H}_\nu} \right)^{(k_\nu-1-\gamma)} \Bigg|_{\tau=\zeta_\nu} \\
&= \mathrm{i} \sum_{\nu=1}^{l} \frac{1}{(k_\nu-1)!} \sum_{j=1}^{m} \left(\frac{\mathscr{T}_j}{\mathscr{H}_-} \frac{\mathscr{P}_j \overline{\mathscr{T}}_a}{\mathscr{P}_a \mathscr{H}_\nu} \right)^{(k_\nu-1)} \Bigg|_{\tau=\zeta_\nu} \\
&= \mathrm{i} \sum_{j=1}^{m} \sum_{\nu=1}^{l} \frac{1}{(k_\nu-1)!} \sum_{\gamma=0}^{k_\nu-1} C_{k_\nu-1}^{\gamma} \left(\frac{\mathscr{T}_j}{\mathscr{H}_-} \right)^{(k_\nu-1-\gamma)} \Bigg|_{\tau=\zeta_\nu} \left(\frac{\mathscr{P}_j \overline{\mathscr{T}}_a}{\mathscr{P}_a \mathscr{H}_\nu} \right)^{(\gamma)} \Bigg|_{\tau=\zeta_\nu}.
\end{aligned}
$$

As was shown in Remark 2.1.2, we have

$$
\left(\frac{\mathscr{P}_j \overline{\mathscr{T}}_a}{\mathscr{P}_a \mathscr{H}_\nu} \right)^{(\gamma)} \Bigg|_{\tau=\zeta_\nu} = \left(\frac{\overline{\mathscr{T}_j}}{\mathscr{H}_-} \right)^{(\gamma)} \Bigg|_{\tau=\zeta_\nu}, \quad \nu = 1, \dots, l;\ \gamma = 0, \dots, k_\nu - 1.
$$

Therefore

$$
(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu = \mathrm{i} \sum_{j=1}^{m} \sum_{\nu=1}^{l} \frac{1}{(k_\nu-1)!} \left(\frac{\mathscr{T}_j \overline{\mathscr{T}}_j}{\mathscr{H}_- \mathscr{H}_\nu} \right)^{(k_\nu-1)} \Bigg|_{\tau=\zeta_\nu}.
$$

Therefore, by the residue theorem,

$$
(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |\mathscr{T}_j(\tau)|^2}{\sum_{j=1}^{m} |\mathscr{P}_j(\tau)|^2} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |T_j(\tau)|^2}{\sum_{j=1}^{m} |P_j(\tau)|^2} d\tau. \quad (2.1.60)
$$

□

2.1.7 Estimates for functions satisfying homogeneous boundary conditions

In this subsection we formulate necessary and sufficient conditions for the validity of (2.1.1) and find the sharp constant Λ_0 for this inequality. We begin the study of (2.1.1) with a consideration of the equivalent variational problem in the finite-dimensional space $\mathbb{C}^\mu \times \mathbb{C}^\lambda$.

Lemma 2.1.18. *Let the $\mu \times \mu$ matrix $\mathfrak{B}$ and the vectors $\mathbf{a}, \mathbf{c}_\alpha \in \mathbb{C}^\mu$, $\mathbf{b}, \mathbf{d}_\alpha \in \mathbb{C}^\lambda$ $(\alpha = 1, \dots, N)$ be the same as in Lemma 2.1.1, and $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\lambda$. Let $\mathbb{B}$ be the subspace of $\mathbb{C}^\mu \times \mathbb{C}^\lambda$ defined by*

$$
\mathbb{B} = \{\mathbf{z} : (\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}, \mathbf{y})_\lambda = 0,\ \alpha = 1, \dots, N\}. \quad (2.1.61)
$$

Further, let

$$\Phi_0(\mathbf{z}) = \frac{|(\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda|^2}{(\mathfrak{B}\mathbf{x},\mathbf{x})_\mu}, \tag{2.1.62}$$

and

$$\Lambda_0 = \sup_{\mathbf{z}\in\mathbb{B}} \Phi_0(\mathbf{z}). \tag{2.1.63}$$

The function $\Phi_0(\mathbf{z})$ is bounded on the subspace $\mathbb{B}$ if and only if there exist constants β_α $(\alpha = 1,\dots,N)$ that satisfy (2.1.25) and (2.1.26). If these conditions are fulfilled and $\mathbf{x}_0$ is an arbitrary solution of (2.1.26), then the constant Λ_0 defined by (2.1.63) satisfies

$$\Lambda_0 = \inf_{\beta_\alpha} (\mathfrak{B}\mathbf{x}_0,\mathbf{x}_0)_\mu, \tag{2.1.64}$$

where the infimum is taken over all systems $\{\beta_\alpha\}$ satisfying the assumptions of the lemma.

Proof. The boundedness of the function (2.1.62) on the subspace (2.1.61) is equivalent to the following assertion: if $\mathbf{z} = (\mathbf{x};\mathbf{y}) \in \mathbb{B}$ and $(\mathfrak{B}\mathbf{x},\mathbf{x})_\mu = 0$, then $(\mathbf{a},\mathbf{x})_\mu + (\mathbf{b},\mathbf{y})_\lambda = 0$. This in turn is equivalent to the boundedness of the function $\Phi(\mathbf{z})$ defined by (2.1.23) on $\mathbb{C}^\mu \times \mathbb{C}^\lambda$. Hence, the first part of the lemma follows from Lemma 2.1.6.

We now prove (2.1.64). Let $\mathbf{z}^0 = (\mathbf{x}^0;\mathbf{y}^0) \in \mathbb{B}$ be an extremal element of $\Phi_0(\mathbf{z})$, that is, $\Lambda_0 = \Phi_0(\mathbf{z}^0)$. Applying the method of Lagrange multipliers to the problem of finding the extremum of (2.1.62) on the subspace (2.1.61) under additional constraints, we conclude that for some constants β_α^0 the following equations are satisfied:

$$\overline{\big((\mathbf{a},\mathbf{x}^0)_\mu + (\mathbf{b},\mathbf{y}^0)_\lambda\big)}\,\mathbf{a} - \Lambda_0\mathfrak{B}\mathbf{x}^0 - \sum_{\alpha=1}^N \beta_\alpha^0\mathbf{c}_\alpha = 0, \tag{2.1.65}$$

$$\overline{\big((\mathbf{a},\mathbf{x}^0)_\mu + (\mathbf{b},\mathbf{y}^0)_\lambda\big)}\mathbf{b} - \sum_{\alpha=1}^N \beta_\alpha^0\mathbf{d}_\alpha = 0. \tag{2.1.66}$$

We set

$$\left.\begin{aligned} \mathbf{x}_0 &= \Lambda_0\overline{\big((\mathbf{a},\mathbf{x}^0)_\mu + (\mathbf{b},\mathbf{y}^0)_\lambda\big)}^{-1}\,\mathbf{x}^0,\\ \mathbf{y}_0 &= \Lambda_0\overline{\big((\mathbf{a},\mathbf{x}^0)_\mu + (\mathbf{b},\mathbf{y}^0)_\lambda\big)}^{-1}\,\mathbf{y}^0,\\ \beta_{0\alpha} &= \beta_\alpha^0\overline{\big((\mathbf{a},\mathbf{x}^0)_\mu + (\mathbf{b},\mathbf{y}^0)_\lambda\big)}^{-1} \qquad (\alpha = 1,\dots,N) \end{aligned}\right\}. \tag{2.1.67}$$

Then, inequality (2.1.65) can be rewritten as

$$\mathbf{a} = \mathfrak{B}\mathbf{x}_0 + \sum_{\alpha=1}^N \beta_{0\alpha}\mathbf{c}_\alpha, \tag{2.1.68}$$

and equation (2.1.66) as

$$\mathbf{b} = \sum_{\alpha=1}^{N} \beta_{0\alpha} \mathbf{d}_\alpha. \tag{2.1.69}$$

Thus, the constants $\beta_{0\alpha}$ and the element $\mathbf{x}_0$, defined by (2.1.67), satisfy conditions (2.1.25) and (2.1.26). Let $\mathbf{z}_0 = (\mathbf{x}_0; \mathbf{y}_0)$. Since $\mathbf{z}^0 = (\mathbf{x}^0; \mathbf{y}^0)$, equations (2.1.67) imply $\mathbf{z}_0 \in \mathbb{B}$. Since $\Phi_0(\mathbf{z})$ is a homogeneous function of degree zero and $\mathbf{z}^0$ is an extremum of this function on the subspace $\mathbb{B}$, it follows from (2.1.67) that $\mathbf{z}_0$ is also an extremum. From (2.1.69) and (2.1.61) we obtain

$$\sum_{\alpha=1}^{N} (\beta_{0\alpha} \mathbf{c}_\alpha, \mathbf{x}_0)_\mu = -(\mathbf{b}, \mathbf{y}_0)_\lambda.$$

Therefore

$$\Lambda_0 = \Phi_0(\mathbf{z}) = \frac{|(\mathbf{a}, \mathbf{x}_0)_\mu + (\mathbf{b}, \mathbf{y}_0)_\lambda|^2}{(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu} = (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu,$$

and, consequently, $\Lambda_0 \geqslant \inf_{\{\beta_\alpha\}} (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu$.

To prove the opposite inequality, consider an arbitrary solution $\mathbf{x}_0$ of (2.1.26), where the constants β_α satisfy (1.1.25). Let $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \mathbb{B}$. By (2.1.25), (2.1.26) and (1.1.61),

$$\begin{aligned} |(\mathbf{a}, \mathbf{x})_\mu + (\mathbf{b}, \mathbf{y})_\lambda| &= \left| \left(\mathbf{a} - \sum_{\alpha=1}^{N} \beta_\alpha \mathbf{c}_\alpha, \mathbf{x} \right)_\mu \right| = |(\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu| \\ &\leqslant (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu^{1/2} (\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu^{1/2}. \end{aligned}$$

Consequently $\Lambda_0 = \inf_{\{\beta_\alpha\}} (\mathfrak{B}\mathbf{x}_0, \mathbf{x}_0)_\mu$. □

Now we can formulate a result about equivalence of the estimate (1.1′) for ordinary differential operators in a half-space to inequality (2.1.70).

Lemma 2.1.19. *The estimate* (1.1′) *is true with some* $\Lambda_0 < \infty$ *if and only if the inequality*

$$|(\mathbf{a}, \mathbf{x})_\mu + (\mathbf{b}, \mathbf{y})_\lambda|^2 \leqslant \Lambda_0 (\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu, \qquad (\mathbf{x}; \mathbf{y}) \in \mathbb{B} \tag{2.1.70}$$

holds, where the subspace $\mathbb{B} \subset \mathbb{C}^\mu \times \mathbb{C}^\lambda$*, the dimensions* μ *and* λ*, the vectors* $(\mathbf{a}, \mathbf{b})$ *and* $(\mathbf{c}_\alpha, \mathbf{d}_\alpha)$*, and the matrix* $\mathfrak{B}$ *are defined by* (2.1.61)*,* (2.1.34)*,* (2.1.37) *and* (2.1.35)*, respectively.*

This statement is deduced from Lemmas 2.1.7 and 2.1.8 in the same way as Lemma 2.1.9 was derived from these lemmas. It is only necessary to consider solutions $z(t)$ of the equation $H_+(-\mathrm{i}\, d/dt)\, z = 0$ satisfying the boundary conditions $Q_\alpha(-\mathrm{i}\, d/dt)\, z|_{t=0} = 0$ $(\alpha = 1, \dots, N)$ instead of an arbitrary solution of this equation.

We now formulate the main result of this subsection.

Theorem 2.1.20. *The estimate* (1.1′) *holds with some* $\Lambda_0 < \infty$ *if and only if the conditions of Theorem* 2.1.17 *are satisfied. The sharp constant* Λ_0 *in* (1.1′) *satisfies the equation*

$$\Lambda_0 = \inf_{\{\beta_\alpha\}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2} d\tau, \tag{2.1.71}$$

where the polynomials $T_j(\tau)$ *and the constants* β_α *are the same as in Theorem* 2.1.17.

For the proof it suffices to make the following changes in the proof of Theorem 2.1.17: Lemma 2.1.9 should be replaced by Lemma 2.1.19. Inequality (2.1.70) is equivalent to the boundedness of the function $\Phi_0(\mathbf{z})$, defined by (2.1.62), on the subspace (2.1.61). Therefore, the references to Lemma 2.1.6 in the proof of Theorem 2.1.17 should be replaced by the references to Lemma 2.1.18. Then, formula (2.1.71) follows from (2.1.64) and (2.1.60).

2.2 Estimates in a half-space. Necessary and sufficient conditions

Let $R(\xi;\tau)$, $P_j(\xi;\tau)$, $Q(\xi;\tau)$ ($j = 1,\dots,m$; $\alpha = 1,\dots,N$) be polynomials in the variable $\tau \in \mathbb{R}^1$ with measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Write

$$\sum_{j=1}^m |P_j(\xi;\tau)|^2 = H_+(\xi;\tau) H_-(\xi;\tau), \tag{2.2.1}$$

where $H_+(\xi;\tau) = \sum_{s=0}^J h_s(\xi)\tau^{J-s}$ is a polynomial with τ-roots lying in the half-plane $\operatorname{Im}\zeta \geqslant 0$, $\zeta = \tau + i\sigma$, and $H_-(\xi;\tau) = \overline{H_+(\xi;\tau)}$. We will assume that $h_0(\xi) \neq 0$, and $\operatorname{ord} R(\xi;\tau)$, $\operatorname{ord} Q_\alpha(\xi;\tau) \leqslant J-1$ a.e. in $\mathbb{R}^{n-1}$ ($\alpha = 1,\dots,N$).[3]

In this section we establish necessary and sufficient conditions for the validity of the estimates:

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \left(\sum_{j=1}^m \|P_j(D)u\|^2 + \sum_{\alpha=1}^N \langle\!\langle Q_\alpha(D)u \rangle\!\rangle^2 \right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{2.2.2}$$

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \sum_{j=1}^m \|P_j(D)u\|^2, \quad u \in C_0^\infty(\mathbb{R}^n_+), \quad Q_\alpha(D)u(x;0) = 0, \quad \alpha = 1,\dots,N. \tag{2.2.2′}$$

[3] This condition is fulfilled, for example, if $h_0(\xi)$ is a polynomial in the variable $\xi \in \mathbb{R}^{n-1}$ (cf. Remark 1.2.1, Chapter 1).

We will also derive some corollaries of these results. All these assertions follow directly from analogous results of Section 2.1 on estimates for ordinary differential operators on the semi-axis $t \geqslant 0$.

2.2.1 Theorems on necessary and sufficient conditions for the validity of the estimates in a half-space

The main result of this subsection is a criterion for the validity of the estimate (2.2.2) (Theorem 2.2.2). Before deriving this result, we formulate the following lemma, which follows directly from Lemma 2.1.1. For every point $\xi \in \mathbb{R}^{n-1}$ such that $h_0(\xi) \neq 0$, we denote by $\Pi_+(\xi;\tau)$ the greatest common divisor of the polynomials $H_+(\xi;\tau)$ and $P_1(\xi;\tau), \ldots, P_m(\xi;\tau)$ with leading coefficients equal 1.

Lemma 2.2.1. *Let $D(\xi;\tau)$ be a polynomial of the variable $\tau \in \mathbb{R}^1$ with measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$, let* $\operatorname{ord} D(\xi;\tau) \leqslant J-1$*, and let the congruence*

$$D(\xi;\tau) \equiv 0 \ (\operatorname{mod} \Pi_+(\xi;\tau))$$

holds a.e. in $\mathbb{R}^{n-1}$. Then there exist uniquely determined polynomials $T_j(\xi;\tau)$ (in τ) with $\operatorname{ord} T_j(\xi;\tau) \leqslant J-1$ $(j=1,\ldots,m)$ *which satisfy a.e. in $\mathbb{R}^{n-1}$ the following conditions:*

$$\overline{T}_j(\xi;\tau) \equiv 0 \ (\operatorname{mod} \Pi_+(\xi;\tau)) \quad (j=1,\ldots,m), \tag{2.2.3}$$

$$D(\xi;\tau)H_-(\xi;\tau) = \sum_{j=1}^{m} P_j(\xi;\tau)T_j(\xi;\tau); \tag{2.2.4}$$

$$\begin{gathered} P_i(\xi;\tau)\overline{T}_j(\xi;\tau) \equiv P_j(\xi;\tau)\overline{T}_i(\xi;\tau) \ (\operatorname{mod} \Pi_+(\xi;\tau)H_+(\xi;\tau)), \\ (i \neq j,\ i,j=1,\ldots,m). \end{gathered} \tag{2.2.5}$$

(*Condition* (2.2.5) *is omitted for* $m=1$).

Theorem 2.2.2. *The estimate* (2.2.2) *is valid if and only if the following conditions are satisfied:*

1. *There exist functions $\beta_\alpha(\xi)$ $(\alpha = 1,\ldots,N)$ such that the congruence*

$$D(\xi;\tau) \stackrel{\text{def}}{=} R(\xi;\tau) - \sum_{\alpha=1}^{N} \beta_\alpha(\xi) Q_\alpha(\xi;\tau) \equiv 0 \ (\operatorname{mod} \Pi_+(\xi;\tau)) \tag{2.2.6}$$

   holds for almost all $\xi \in \mathbb{R}^{n-1}$.

2. *The inequality*

$$\sup_{\xi \in \mathbb{R}^{n-1}} \left\{ B(\xi) \inf_{\{\beta_\alpha(\xi)\}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau + \sum_{\alpha=1}^{N} |\beta_\alpha(\xi)|^2 \right] \right\} < \infty \tag{2.2.7}$$

holds true. Here, the infimum is taken over all systems $\{\beta_{\alpha(\xi)}\}$ *that satisfy* (2.2.6)*, and* $T_j(\xi;\tau)(j=1,\dots,m)$ *are polynomials satisfying* (2.2.3)–(2.2.5)*. The left-hand side of* (2.2.7) *is the sharp constant in* (2.2.2)*.*

Proof. Necessity. Suppose that (2.2.2) is valid for all $u \in C_0^\infty(\mathbb{R}^n_+)$ and C is the exact constant in this inequality. Localizing (2.2.2) in ξ (cf. the proof of the necessity of conditions of Theorem 1.2.2, Chapter 1), we find that for almost all $\xi \in \mathbb{R}^{n-1}$

$$R\,|(\xi;-\mathrm{i}\,d/dt)\,v(t)|_{t=0}|^2 = \frac{C}{B(\xi)}\Bigg[\int_0^\infty \sum_{j=1}^m \big|P_j\,(\xi;-\mathrm{i}\,d/dt)\,v(t)\big|^2\,dt + \sum_{\alpha=1}^N |Q_\alpha\,(\xi;-\mathrm{i}\,d/dt)\,v(t)|_{t=0}|^2\Bigg], \tag{2.2.8}$$

for all $v \in C_0^\infty(\mathbb{R}^1_+)$. For each fixed $\xi \in \mathbb{R}^{n-1}$, inequality (2.2.8) is an estimate of the type (2.1.1). Applying Theorem 2.1.17, we find that (2.2.6) must hold a.e. in $\mathbb{R}^{n-1}$ and the sharp constant in (2.2.8) satisfies

$$\Lambda(\xi) = \inf_{\{\beta_\alpha(\xi)\}}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2}d\tau + \sum_{\alpha=1}^N |\beta_\alpha(\xi)|^2\right], \tag{2.2.9}$$

where the infimum is taken over all systems $\{\beta_\alpha(\xi)\}$ entering in (2.2.6), and $T_j(\xi;\tau)$ are the polynomials defined in Lemma 2.1.1. Hence, $\Lambda(\xi) = C/B(\xi)$. This implies (2.2.7) and the inequality

$$C \geqslant \sup_{\xi\in\mathbb{R}^{n-1}} \Lambda(\xi)B(\xi). \tag{2.2.10}$$

Sufficiency. Suppose that (2.2.6) holds a.e. in $\mathbb{R}^{n-1}$. Then, in accordance with Theorem 2.1.17, for almost all $\xi \in \mathbb{R}^{n-1}$ we have the inequality

$$R\,|(\xi;-\mathrm{i}\,d/dt)\,v(t)|_{t=0}|^2 \leqslant \Lambda(\xi)\Bigg[\int_0^\infty \sum_{j=1}^m \big|P_j\,(\xi;-\mathrm{i}\,d/dt)\,v(t)\big|^2\,dt + \sum_{\alpha=1}^N |Q_\alpha\,(\xi;-\mathrm{i}\,d/dt)\,v(t)|_{t=0}|^2\Bigg], \tag{2.2.11}$$

where $\Lambda(\xi)$ is defined by (2.2.9), and v is an arbitrary function in $C_0^\infty(\mathbb{R}^1_+)$. Assume that condition (2.2.7) holds. Denote by C the left-hand side of this condition. It follows from (2.2.11) that for all $v \in C_0^\infty(\mathbb{R}^1_+)$ and almost all $\xi \in \mathbb{R}^{n-1}$ inequality (2.2.8) is valid. Substituting $v = v_\xi(t) = \hat{u}(\xi;t)$ with $u \in C_0^\infty(\mathbb{R}^n_+)$ in (2.2.8), and multiplying both parts of the resulting inequality by $B(\xi)$, we obtain after integration over $\mathbb{R}^{n-1}$ that inequality (2.2.2) with the constant $C = \sup_{\xi\in\mathbb{R}^{n-1}} \Lambda(\xi)B(\xi)$ holds for all $u \in C_0^\infty(\mathbb{R}^n_+)$. As it was shown above, the sharp constant in (2.2.2) satisfies (2.2.10). This means that it is equal to the left-hand side of (2.2.7). □

Remark 2.2.3. If R, P_j, and Q_α are differential operators, then it is more appropriate to consider instead of (2.2.2) the inequality

$$\langle\!\langle R(D)u\rangle\!\rangle^2_{B^{1/2}} \leqslant C\left(\sum_{j=1}^{m}\|P_j(D)u\|^2 + \sum_{\alpha=1}^{N}\langle\!\langle Q_\alpha(D)u\rangle\!\rangle^2_{\mu_\alpha}\right), \qquad (2.2.2'')$$
$$u \in C_0^\infty(\mathbb{R}^n_+),$$

where $\langle\!\langle\cdot\rangle\!\rangle_{\mu_\alpha}$ is the norm in the space $\mathscr{H}_{\mu_\alpha}(\partial\mathbb{R}^n_+)$. A criterion for the validity of estimate (2.2.2) is contained in Theorem 2.2.2. It is only necessary to replace in the formulation of this theorem $Q_\alpha(\xi;\tau)$ by $(1+|\xi|^2)^{\mu_\alpha/2}Q_\alpha(\xi;\tau)$.

We end of this subsection with a result cincerning functions that satisfy homogeneous boundary conditions.

Theorem 2.2.4. *The estimate* (2.2.2′) *holds if and only if the first condition of Theorem 2.2.2 and the inequality*

$$\sup_{\xi\in\mathbb{R}^{n-1}}\left[B(\xi)\inf_{\{\beta_\alpha(\xi)\}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sum_{j=1}^m|T_j(\xi;\tau)|^2}{\sum_{j=1}^m|P_j(\xi;\tau)|^2}d\tau\right] < \infty \qquad (2.2.7')$$

is satisfied, where the polynomials T_j and the functions β_α are the same as in Theorem 2.2.2. The left-hand side of (2.2.7) *is the sharp constant in* (2.2.2).

This theorem is deduced from Theorem 2.1.20 in the same way as Theorem 2.2.2 was from Theorem 2.1.17.

2.2.2 Corollaries

In this subsection we study some special cases of the estimate (2.2.2), namely, inequalities (2.2.12), (2.2.15) and (2.2.19). The criteria for the validity of these inequalities follow from Theorem 2.2.2, Remarks 2.1.3 and 2.1.4, and from results of Subsection 2.1.6. First, we consider the inequality

$$\langle\!\langle R(D)u\rangle\!\rangle^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \sum_{\alpha=1}^{N}\langle\!\langle Q_\alpha(D)u\rangle\!\rangle^2\right), \qquad (2.2.12)$$

which is a particular case of estimate (2.2.2) corresponding to the polynomials $P_j(D) = P(D)$.

Corollary 2.2.5. *Let $P(\xi;\tau) = P_+(\xi;\tau)P_-(\xi;\tau)$, where the τ-roots of the polynomial $P_+(\xi;\tau)$ coincide (counting multiplicities) with the τ-roots of the polynomial $P(\xi;\tau)$ in the half-plane* $\operatorname{Im}\zeta \geqslant 0$, $\zeta = \tau + i\sigma$. *The estimate* (2.2.12) *is valid for all $u \in C_0^\infty(\mathbb{R}^n_+)$ if and only if the following conditions are satisfied:*

1. *There exist functions* $\{\beta_\alpha(\xi)\}$ *such that the polynomial* $D(\xi;\tau)$ *defined by* (2.2.6) *satisfies the congruence*
$$D(\xi;\tau) \equiv 0 \ (\mathrm{mod}\ P_+(\xi;\tau)) \tag{2.2.13}$$
for almost all $\xi \in \mathbb{R}^{n-1}$*; and*
2. *the inequality*
$$\sup_{\xi\in\mathbb{R}^{n-1}} \left\{ B(\xi) \inf_{\{\beta_\alpha(\xi)\}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{D(\xi;\tau)}{P(\xi;\tau)} \right|^2 d\tau + \sum_{\alpha=1}^{N} |\beta_\alpha(\xi)|^2 \right] \right\} < \infty \tag{2.2.14}$$
holds true. Here, the infimum is taken over all $\{\beta_\alpha(\xi)\}$ *entering in* (2.2.13). *The left-hand side of* (2.2.14) *is the sharp constant in* (2.2.12).

This result follows directly from Theorem 2.2.2 and Remark 2.1.3. Indeed, it follows from (2.1.20) that
$$\sum_{j=1}^{m} |T_j(\xi;\tau)|^2 = |D(\xi;\tau)|^2.$$

In particular, if polynomials $Q_\alpha(\xi;\tau)$ are all identically equal to zero, we obtain the following assertion.

Corollary 2.2.6. *The inequality*
$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C\|P(D)u\|^2 \tag{2.2.15}$$
is valid if and only if the following conditions are satisfied:
1. *The congruence*
$$R(\xi;\tau) \equiv 0 \ (\mathrm{mod}\ P_+(\xi;\tau)) \tag{2.2.16}$$
holds a.e. in $\mathbb{R}^{n-1}$.
2. *The inequality*
$$\sup_{\xi\in\mathbb{R}^{n-1}} \left\{ B(\xi) \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{R(\xi;\tau)}{P(\xi;\tau)} \right|^2 d\tau \right\} < \infty \tag{2.2.17}$$
holds true. The left-hand side of (2.2.17) *is the sharp constant in* (2.2.15).

Remark 2.2.7. It is well-known that condition (2.2.17) is necessary and sufficient for the validity of the estimate
$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \int_{\mathbb{R}^n} |P(D)u|^2 dx\,dt \tag{2.2.18}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Thus, the validity of (2.2.15) for all $u \in C_0^\infty(\mathbb{R}_+^n)$ implies the validity of (2.2.18) for all $u \in C_0^\infty(\mathbb{R}_+^n)$. The converse statement is in general not true.

We now consider the estimate

$$\langle\!\langle R(D)u \rangle\!\rangle_{B^{1/2}}^2 \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \tag{2.2.19}$$

which is a particular case of the estimate (2.2.2) where the polynomials $Q_\alpha(\xi;\tau)$ ($\alpha = 1, \dots, N$) are identically equal to zero. Next assertion follows directly from Theorem 2.2.2.

Corollary 2.2.8. *The estimate (2.2.19) is valid for all $u \in C_0^\infty(\mathbb{R}_+^n)$ if and only if the following conditions are satisfied:*

1. *The congruence*
$$R(\xi;\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)) \tag{2.2.20}$$
holds a.e. in $\mathbb{R}^{n-1}$.
2. *The inequality*
$$\sup_{\xi\in\mathbb{R}^{n-1}} \left\{ B(\xi) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau \right\} < \infty \tag{2.2.21}$$
holds true. Here $T_j(\xi;\tau)$ are the polynomials that satisfy conditions (2.2.3)–(2.2.5) of Lemma 2.2.1 (where $D(\xi;\tau)$ is replaced by $R(\xi;\tau)$). The left-hand side of (2.2.21) is the sharp constant in (2.2.19).

Remark 2.2.9. Set

$$\Lambda(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T_j(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau, \tag{2.2.22}$$

where $T_j(\xi;\tau)$ are the polynomials defined in Corollary 2.2.8. Let

$$L(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau. \tag{2.2.23}$$

Replacing in (2.1.22) $D(\tau)$ by $R(\xi;\tau)$ and $T_j(\tau)$ by $T_j(\xi;\tau)$, we obtain, in accordance with Remark 2.1.5, that

$$L(\xi) \leqslant \Lambda(\xi) \tag{2.2.24}$$

a.e. in $\mathbb{R}^{n-1}$.

It is well-known that the condition

$$\sup_{\xi \in \mathbb{R}^{n-1}} B(\xi)L(\xi) < \infty \tag{2.2.25}$$

is equivalent to the validity of the estimate

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{m} |P_j(D)u|^2 dx dt \tag{2.2.26}$$

for all $u \in C_0^\infty(\mathbb{R}^n_+)$. By (2.2.24) and Corollary 2.2.8 the validity of (2.2.19) for all $u \in C_0^\infty(\mathbb{R}^n_+)$ follows from the validity of (2.2.26) for all $u \in C_0^\infty(\mathbb{R}^n)$.

We show that the opposite statement is generally not true. Let $R(\xi;\tau) = 1$, let $N = 2$, let $P_1(\xi;\tau) = \mathrm{i}\tau + |\xi|^2$ be the symbol of the heat operator, and let $P_2(\xi;\tau) = 1$. Then, according to (2.2.23),

$$L(\xi) = ((|\xi|^4 + 1)^{-1/2})2^{-1}.$$

On the other hand, we have

$$H_+(\xi;\tau) = -\mathrm{i}\tau - (|\xi|^4 + 1)^{1/2}, \qquad H_-(\xi;\tau) = \mathrm{i}\tau - (|\xi|^4 + 1)^{1/2},$$
$$T_1(\xi;\tau) = 1, \qquad T_2(\xi;\tau) = -(|\xi|^2 + (|\xi|^4 + 1)^{1/2}).$$

Thus, according to (2.2.22), we find that $\Lambda(\xi) = |\xi|^2 + (|\xi|^4 + 1)^{1/2}$ and, consequently, the opposite of inequality (2.2.24) does not hold.

2.2.3 The case when the lower-order terms play no role

Let $R(D)$, $P_j(D)$ and $Q_\alpha(D)$ be differential operators with constant coefficients of orders μ_0, J_j and μ_α ($j = 1,\dots,m$ $\alpha = 1,\dots,N$), respectively, and let $J = \max_{1 \leqslant j \leqslant m} J_j$. We assume that the orders of these operators w.r.t. t are also equal to μ_0, J_j and μ_α, respectively. We denote by $R'(\xi;\tau)$, $P'_j(\xi;\tau)$ and $Q'_\alpha(\xi;\tau)$ the respective homogeneous principal parts of the orders μ_0, J_j and μ_α w.r.t. t of the polynomials $R(\xi;\tau)$, $P_j(\xi;\tau)$ and $Q_\alpha(\xi;\tau)$.

Next, we introduce the polynomials $H'_+(\xi;\tau)$, $H'_-(\xi;\tau)$, $\Pi'_+(\xi;\tau)$ corresponding to the polynomials $P'_j(\xi;\tau)$. The validity of (2.2.2) and similar estimates depends essentially on the lower-order terms of the operators R, P_j, and Q_α. For example, consider the estimate

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \|u\|^2 \right), \tag{2.2.27}$$

where $R(\xi;\tau) = R'(\xi;\tau)$, $P(\xi;\tau) = P'(\xi;\tau)$, and ord $P \geqslant 1$. Then, we have $P'_2(\xi;\tau) = 0$. If we replace the operators by their principal parts, (2.2.27) takes the form

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \|P(D)u\|^2. \tag{2.2.27'}$$

We already know (see Corollary 2.2.6) that the congruence $R \equiv 0 \pmod{P_+}$ is necessary for the validity of (1.2.27) for all $u \in C_0^\infty(\mathbb{R}^n_+)$. On the other hand, this condition has no relation to the estimate (1.2.27).

In this subsection we consider a class of estimates that remain valid after one replaces the operators R, P_j, and Q_α by their homogeneous principal parts. The result formulated in Proposition 2.2.10 is analogous to Proposition 1.2.15 from Chapter 1.

Proposition 2.2.10. *The estimate*

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{J-\mu_0-1/2} \leqslant C\Big(\sum_{j=1}^m \|P_j(D)u\|^2 + \|u\|^2_{\mathscr{H}_{J-1}(\mathbb{R}^n_+)} + \sum_{\alpha=1}^N \langle\!\langle Q_\alpha(D)u \rangle\!\rangle^2_{J-\mu_\alpha-1/2} \Big) \tag{2.2.28}$$

is valid for all $u \in C_0^\infty(\mathbb{R}^n_+)$ if and only if the inequality

$$\langle\!\langle R'(D)u \rangle\!\rangle^2_{J-\mu_0-1/2} \leqslant C'\Big(\sum_{j=1}^m \|P'_j(D)u\|^2 + \sum_{\alpha=1}^N \langle\!\langle Q'_\alpha(D)u \rangle\!\rangle^2_{J-\mu_\alpha-1/2} \Big) \tag{2.2.29}$$

holds true, or, what is the same, if and only if

1. *There exist functions $\{\beta_\alpha(\xi)\}$ such that the congruence*

$$\begin{aligned} D'(\xi;\tau) &\overset{\text{def}}{=} R'(\xi;\tau) - \sum_{\alpha=1}^m \beta_\alpha(\xi)(1+|\xi|^2)^{1/2(J-\mu_\alpha-(1/2))} Q'_\alpha(\xi;\tau) \\ &\equiv \pmod{\Pi'_+(\xi;\tau)} \end{aligned} \tag{2.2.30}$$

   holds a.e. in $\mathbb{R}^{n-1}$;

2. *The inequality*

$$\sup_{\xi\in\mathbb{R}^{n-1}} \Big\{ (1+|\xi|^2)^{J-\mu_0-(1/2)} \inf_{\{\beta_\alpha(\xi)\}} \Big[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |T'_j(\xi;\tau)|^2}{\sum_{j=1}^m |P'_j(\xi;\tau)|^2} d\tau + \sum_{\alpha=1}^N |\beta_\alpha(\xi)|^2 \Big] \Big\} < \infty \tag{2.2.31}$$

   holds. Here $T'_j(\xi;\tau)$ are the polynomials constructed in accordance with Lemma 2.2.1 for the polynomials $P'_j(\xi;\tau)$ and $D'(\xi;\tau)$, and the infimum is taken over all systems $\{\beta_\alpha(\xi)\}$ satisfying (2.2.30).

Proof. The *sufficiency* is obvious. Let us prove the *necessity*. Suppose (2.2.28) holds for all $u \in C_0^\infty(\mathbb{R}^n_+)$. Estimating the norms

$$\langle\!\langle (R - R')(D)u \rangle\!\rangle_{J-\mu_0-(1/2)}, \ \|(P_j - P'_j)(D)u\|, \text{ and } \langle\!\langle (Q_\alpha - Q'_\alpha)(D)u \rangle\!\rangle_{J-\mu_\alpha-(1/2)}$$

by $\|u\|_{\mathscr{H}_{J-1}}(\mathbb{R}^n_+)$, we see that all the operators in (2.2.28) can be replaced by their homogeneous principal parts of orders μ_0, J, and μ_α, respectively. Localizing the obtained inequality in ξ, we find that the estimate

$$\begin{aligned} &|\xi|^{2J-2\mu_0-1} \left| R'(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} \right|^2 \\ &\leqslant C \Bigg(\sum_{j=1}^m \int_0^\infty \left| P'_j(\xi; -\mathrm{i}\, d/dt)\, v \right|^2 dt \\ &\qquad + \int_0^\infty |v|^2 dt + \sum_{s=0}^{J-1} |\xi|^{2(J-1-s)} \int_0^\infty \left| (-\mathrm{i}\, d/dt)^s\, v \right|^2 dt \\ &\qquad + \sum_{\alpha=1}^N |\xi|^{2J-2\mu_\alpha-1} \left| Q'_\alpha(\xi; -\mathrm{i}\, d/dt)\, v|_{t=0} \right|^2 \Bigg) \end{aligned}$$

holds for all $v \in C_0^\infty(\mathbb{R}^1_+)$. We set here $\xi = |\xi|\theta$ and $\tau = |\xi|t$, divide both sides of this inequality by $|\xi|^{2J-1}$, and take the limit as $|\xi| \to \infty$, obtaining

$$\begin{aligned} \left| R'(\theta; -\mathrm{i}\, d/dt)\, v|_{\tau=0} \right|^2 \leqslant C \Bigg(&\sum_{j=1}^m \int_0^\infty \left| P'_j(\theta; -\mathrm{i}\, d/d\tau)\, v \right|^2 d\tau \\ &+ \sum_{\alpha=1}^N \left| Q'_\alpha(\theta; -\mathrm{i}\, d/d\tau)\, v|_{\tau=0} \right|^2 \Bigg). \end{aligned} \tag{2.2.32}$$

If we revert in (2.2.32) to the variables ξ and t, set $v = v_\xi(t) = \hat{u}(\xi; \tau)$, where $u(x; t)$ is an arbitrary function in $C_0^\infty(\mathbb{R}^n_+)$, integrate w.r.t. ξ and apply the inverse Fourier transform, then we arrive at (2.2.29). □

Remark 2.2.11. The statement of Proposition 2.2.10 remains valid if we require additionally in the necessity part that $\operatorname{supp} u \subset \mathscr{D}(0, \varrho)$ for some $\varrho > 0$, where $\mathscr{D}(0, \varrho)$ denotes the n-dimensional ball of radius $\varrho > 0$ centered at the origin (cf. Remark 1.2.16, Chapter 1).

2.2.4 An example of estimate for operators of first order with respect to *t*

In this subsection we consider an estimate of the type (2.2.19) in the case when $P_j(D)$ are the first-order operators in t, and $R(D) = 1$. It will be shown that a criterion for the validity of such estimate can be formulated explicitly in the form of necessary and sufficient conditions on the coefficients of $P_j(\xi; \tau)$.

Proposition 2.2.12. *Let* $P_j(\xi;\tau) = \mathrm{i}\tau - p_j(\xi)$ $(j = 1,\dots,m,\ m > 1)$*, where* $p_j(\xi)$ *are the measurable functions that are locally bounded in* $\mathbb{R}^{n-1}$ *and grow no faster than some power of* $|\xi|$ *as* $|\xi| \to \infty$*. Suppose that* $\sum_{j=1}^{m} |p_j(\xi)| \neq 0$ *a.e. in* $\mathbb{R}^{n-1}$*. The estimate*

$$\langle\!\langle u \rangle\!\rangle^2_{B^{1/2}} \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2 \tag{2.2.33}$$

is true for all $u \in C_0^\infty(\mathbb{R}^n_+)$ *if and only if the following conditions are satisfied:*

1. *The inequality*
$$\sum_{j,h=1}^{m} |p_j - p_h| \neq 0 \tag{2.2.34}$$
holds for almost all $\xi \in \bigcap_{j=1}^{m} \{\xi : \mathrm{Re}\, p_j(\xi) \leqslant 0\}$.
2. *The inequality*
$$B(\xi) \leqslant \mathrm{const} \left(\sum_{j=1}^{m} |\mathrm{Re}\, p_j| + \sum_{j,h=1}^{m} |\mathrm{Im}\,(p_j - p_h)| \right) \tag{2.2.35}$$
holds for almost all $\xi \in \left\{\xi : \sum_{j=1}^{m} \mathrm{Re}\, p_j(\xi) \geqslant 0\right\}$.
3. *The inequality*
$$B(\xi) \left(\sum_{j=1}^{m} |\mathrm{Re}\, p_j| + \sum_{j,h=1}^{m} |\mathrm{Im}\,(p_j - p_h)| \right) \leqslant \mathrm{const} \sum_{j,h=1}^{m} |p_j - p_h|^2 \tag{2.2.36}$$
holds for almost all $\xi \in \left\{\xi : \sum_{j=1}^{m} \mathrm{Re}\, p_j(\xi) < 0\right\}$.

Proof. We show that Proposition 2.2.12 follows from Corollary 2.2.8. Since (2.2.33) is a special case of the estimate (2.2.19) related to the polynomial $R(\xi;\tau) = 1$, we see that condition 1 of Corollary 2.2.8 is fulfilled in the considered example if and only if $\Pi_+(\xi;\tau) = 1$ a.e. in $\mathbb{R}^{n-1}$. The last condition is equivalent to condition 1 of the proposition to be proved, since the τ-roots of the polynomial $\mathrm{i}\tau - p_j$ lie in the half-plane $\mathrm{Im}\,\zeta \geqslant 0$, $\zeta = \tau + \mathrm{i}\sigma$, if and only if $\mathrm{Re}\, p_j \leqslant 0$. We show that condition 2 of Corollary 2.2.8 is equivalent to conditions 2 and 3 of the proposition to be proved. One can verify directly that $P_j = \mathrm{i}\tau - p_j(\xi)$ satisfies the equality

$$H_\pm(\xi;\tau) = m^{1/2}(\tau - \tau_\pm(\xi)), \tag{2.2.37}$$

where

$$\tau_\pm(\xi) = m^{-1} \left(\sum_{j=1}^{m} \mathrm{Im}\, p_j(\xi) \pm \mathrm{i}\alpha(\xi) \right) \tag{2.2.38}$$

and

$$\alpha(\xi) = \left(m \sum_{j=1}^{m} |p_j(\xi)|^2 - \left(\sum_{j=1}^{m} \operatorname{Im} p_j(\xi) \right)^2 \right)^{1/2}. \tag{2.2.39}$$

The polynomials $T_j(\xi;\tau)$ (of degree zero w.r.t. τ) are calculated for the polynomials $D(\xi;\tau) = R(\xi;\tau) = 1$ and $P_j(\xi;\tau) = \mathrm{i}\tau - p_j(\xi)$ in accordance with Lemma 2.2.1, yielding

$$\overline{T}_j = \frac{\mathrm{i}m^{1/2}\big(\mathrm{i}\tau_+(\xi) - p_j(\xi)\big)}{\sum\limits_{k=1}^{m} (\mathrm{i}\tau_+(\xi) - p_k(\xi))} \qquad (j = 1, \dots, m), \tag{2.2.40}$$

where $\tau_+(\xi)$ is defined by (2.2.38). Since

$$\int_{-\infty}^{\infty} \frac{d\tau}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} = \frac{\pi}{\alpha(\xi)},$$

where $\alpha(\xi)$ is the function defined by (2.2.39), we conclude that condition 2 of Corollary 2.2.8 is equivalent a.e. in $\mathbb{R}^{n-1}$ to the inequality

$$\frac{B}{\alpha} \frac{\sum_{j=1}^{m} |i\tau_+ - p_j|^2}{\left|\sum_{j=1}^{m} (i\tau_+ - p_j)\right|^2} \leqslant \text{const.} \tag{2.2.41}$$

In accordance with (2.2.38), we have

$$\sum_{j=1}^{m} |\mathrm{i}\tau_+ - p_j|^2 = 2m^{-1}(\alpha^2 + \alpha\beta) \tag{2.2.42}$$

and

$$\left| \sum_{j=1}^{m} (\mathrm{i}\tau_+ - p_j) \right|^2 = (\alpha + \beta)^2, \tag{2.2.43}$$

where

$$\beta(\xi) = \sum_{j=1}^{m} \operatorname{Re} p_j(\xi) \tag{2.2.44}$$

and $\alpha(\xi)$ is defined by (2.2.39). Therefore, condition (2.2.41) can be written in the form

$$B(\xi) \leqslant \text{const}\,(\alpha(\xi) + \beta(\xi)) \quad \text{a.e. in} \quad \mathbb{R}^{n-1}. \tag{2.2.45}$$

Suppose that $\beta(\xi) \geqslant 0$. It follows from (2.2.39) and (2.2.44) that (2.2.45) is equivalent to the inequality $B \leqslant \text{const}\,\alpha$, which in turn is equivalent to (2.2.35).

Suppose now that $\beta(\xi) < 0$. Representing $\alpha + \beta$ in the form

$$\alpha \left[1 - \left(1 - 2^{-1}\alpha^{-2} \sum_{j,h=1}^{m} |p_j - p_h|^2 \right)^{1/2} \right],$$

we conclude that (2.2.45) can be written as

$$B \left[1 - \left(1 - 2^{-1}\alpha^{-2} \sum_{j,h=1}^{m} |p_j - p_h|^2 \right)^{1/2} \right] \leqslant \text{const}\, \alpha^{-1} \sum_{j,h=1}^{m} |p_j - p_h|^2. \tag{2.2.46}$$

Inequality (2.2.46) is equivalent to the inequality

$$B\alpha \leqslant \text{const} \sum_{j,h=1}^{m} |p_j - p_h|^2. \tag{2.2.47}$$

The equivalence of (2.2.47) and (2.2.36) is obvious. □

2.3 Description of the trace space

In this section we will assume that $R(\xi;\tau)$ and $P_j(\xi;\tau)$ $(j = 1,\dots,m)$ are polynomials in the variables $(\xi;\tau) \in \mathbb{R}^n$. Hence, $R(D)$, $P_j(D)$ are differential operators with constant coefficients. Our goal is to study the "trace space" $R(D)u\big|_{t=0}$ of elements u belonging to the completion of $C_0^\infty(\mathbb{R}^n_+)$ in the metric $\sum_{j=1}^m \|P_j(D)u\|^2$. The main result of this section (Theorem 2.3.8) will be established in Subsection 2.3.2. We will show that the considered "trace space" coincides with the closed linear span of the functions $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^{n-1})$, which satisfy

$$\langle\!\langle \varphi \rangle\!\rangle^2_{\Lambda^{-1/2}} = \int\limits_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda(\xi)} d\xi < \infty$$

with $\Lambda(\xi)$ defined by (2.2.22). Some preliminary results, needed for the proof of Theorem 2.3.8, are presented in Subsection 2.3.1.

2.3.1 Preliminary results

In this subsection, we show (Proposition 2.3.6) that the function $\Lambda(\xi)$ defined by (2.2.22) is infinitely differentiable in each component Ξ_α of some open set $\Xi \subset \mathbb{R}^{n-1}$ of full measure. This result is essentially used in the proof of the main theorem (Theorem 2.3.8) of this section. A description of Ξ is given in Proposition 2.3.4.

Before we state this proposition, let us recall several well-known statements about polynomials of the variables $(\xi;\tau)\in\mathbb{R}^n$ with complex coefficients that will be used in its proof.

Lemma 2.3.1 (Hörmander [H63], p. 275). *Let $h(\xi;\tau)$ be a polynomial of n variables $\xi=(\xi_1,\dots,\xi_{n-1})$ and τ. If $h(\xi;\tau)=0$ and $\dfrac{\partial h(\xi;\tau)}{\partial\tau}\neq 0$ for $\xi=0$, $\tau=0$, then there exists exactly one function $\tau(\xi)$ which is analytic in a neighborhood of zero, equals zero for $\xi=0$, and satisfies the equation $P(\xi;\tau(\xi))=0$.*

Following Hörmander ([H63], p. 277), we say that some assertion, depending on $\xi\in\mathbb{C}^{n-1}$, holds for a generic ξ, if there exists a non-identically vanishing polynomial of ξ such that the assertion holds for all ξ, for which the polynomial does not vanish.

Lemma 2.3.2 (Hörmander [H63], p. 277). *If the polynomials $h_j(\xi;\tau)$ $(j=1,\dots,m)$ have no common divisors other than constants, then they, as polynomials of τ, have no common zeros for generic ξ.*

Lemma 2.3.3 (Hörmander [H63], p. 277). *If the polynomial $h(\xi;\tau)$ does not have multiple zeros, then the zeros of $h(\xi;\tau)$, regarded as a polynomial of τ, are distinct for generic ξ.*

We define some polynomials (of τ) which will be used later. Denote by $H(\xi;\tau)$ the polynomial $\sum_{j=1}^m |P_j(\xi;\tau)|^2$ and consider the polynomials $H_+(\xi;\tau)$, $H_-(\xi;\tau)$, and $\Pi_+(\xi;\tau)$ defined at the beginning of Section 2.2. Let $g_0(\xi)$ and $h_0(\xi)$ be the leading coefficients of $H(\xi;\tau)$ and $H_+(\xi;\tau)$, respectively. Since $H(\xi;\tau)$ is a polynomial of $(\xi;\tau)\in\mathbb{R}^n$ and $g_0(\xi)=|h_0(\xi)|^2$, we obtain, according to Remark 1.2.1, Chapter 1, that $h_0(\xi)\neq 0$ a.e. in $\mathbb{R}^{n-1}$. As in Section 2.2, we will assume that $\operatorname{ord} H_+=J\geqslant 1$ and $\operatorname{ord} R=J-1$ a.e. in $\mathbb{R}^{n-1}$. We consider the decomposition

$$\Pi_+(\xi;\tau)=\Pi_0(\xi;\tau)\Pi_1(\xi;\tau), \tag{2.3.1}$$

where Π_0 and Π_1 are polynomials of τ with leading coefficients equal to 1 and with real and non-real τ-roots, respectively. Let $\lambda(\xi)=\operatorname{ord}\Pi_0(\xi;\tau)$, $\varkappa(\xi)=\operatorname{ord}\Pi_1(\xi;\tau)$. We set

$$\mathscr{P}_j(\xi;\tau)=\frac{P_j(\xi;\tau)}{\Pi_1(\xi;\tau)}\quad (j=1,\dots,m),\qquad \mathscr{R}(\xi;\tau)=\frac{R(\xi;\tau)}{\Pi_1(\xi;\tau)},$$

$$h_+(\xi;\tau)=\frac{H_+(\xi;\tau)}{\Pi_1(\xi;\tau)}=\prod_{\nu=1}^{l(\xi)}(\tau-\tau_\nu(\xi))^{k_\nu(\xi)},\quad (k_1(\xi)+\dots+k_{l(\xi)}(\xi)=k(\xi)).$$

Proposition 2.3.4. *There exists an open full-measure set $\Xi\subset\mathbb{R}^{n-1}$, (i.e., $\mathrm{mes}_{n-1}(\mathbb{R}^{n-1}\setminus\Xi)=0$), with the following properties:*

1. *The orders of the polynomials $R(\xi;\tau)$ and $H(\xi;\tau)$ are constant for all $\xi \in \Xi$.*
2. *The τ-roots of the polynomials Π_0, Π_1, and h_+ are analytic in each component Ξ_α of the set Ξ.*
3. *The orders $\lambda(\xi)$, $\varkappa(\xi)$, and $k(\xi)$ of the polynomials Π_0, Π_1, and h_+ and the multiplicities of their τ-roots are constant in each component Ξ_α.*

Proof. Since the leading coefficients of $H(\xi;\tau)$ and $R(\xi;\tau)$ are polynomials of ξ, the orders of these polynomials (in the variable ξ) for all ξ are independent of ξ. Let $H(\xi;\tau) = h_1(\xi;\tau)\cdots h_s(\xi;\tau)$ be a decomposition of $H(\xi;\tau)$ into irreducible polynomial (in ξ and τ) factors. Since each of the polynomials $h_k(\xi;\tau)$, $1 \leqslant k \leqslant s$, does not have multiple factors, we conclude on the basis of Lemmas 2.3.3 and 2.3.1 that its roots $\tau = \zeta(\xi)$ are distinct. Moreover, for all ξ these roots are analytic functions. Since the coefficients of $H(\xi;\tau) = \sum_{j=1}^{m} |P_j(\xi;\tau)|^2$ are real, it follows that each factor $h_k(\xi;\tau)$ enters into the decomposition of $H(\xi;\tau)$ together with its complex-conjugate $\bar{h}_k(\xi;\tau)$. Take two arbitrary complex-conjugate τ-roots of the polynomial $H(\xi;\tau)$, namely $\zeta(\xi) = \varkappa_1(\xi) + i\varkappa_2(\xi)$ and $\bar{\zeta}(\xi) = \varkappa_1(\xi) - i\varkappa_2(\xi)$. If both $\zeta(\xi)$ and $\bar{\zeta}(\xi)$ are τ-roots of the irreducible factors of $h_k(\xi;\tau)$, then, as it was already noted, they must be different for all ξ. However, if $\zeta(\xi)$ is a root of h_k and $\bar{\zeta}(\xi)$ is a root of $\bar{h}_k$, and not all coefficients of $h_k(\xi;\tau)$ and $h_k(\xi;\tau)$ are real, then $h_k(\xi;\tau)$ and $\bar{h}_k(\xi;\tau)$ do not have common divisors different from 1. According to Lemma 2.3.2, in this case $\zeta(\xi) \neq \bar{\zeta}(\xi)$ for any ξ. We denote by H the closed subset of $\mathbb{R}^{n-1}$, where at each point ξ at least one of the following conditions holds:

a) The leading coefficient of the polynomial $R(\xi;\tau)H(\xi;\tau)$ is equal to zero.
b) At least one of the polynomials of ξ listed in the definition of "generic ξ", for which the statement on the complex-conjugate τ-roots of irreducible factors of $H(\xi;\tau)$ holds true, equals zero.

It is obvious that $\mathrm{mes}_{n-1} H = 0$. We set $\Xi = \mathbb{R}^{n-1} \setminus H$. In each component Ξ_α of the open set Ξ, the multiplicities of the roots $\tau = \zeta(\xi)$ of polynomials $H(\xi;\tau)$ are constant, the functions $\zeta(\xi)$ are analytic, and their imaginary parts $\operatorname{Im}\zeta(\xi)$ either are identicaly zero or preserve the sign. The last statement follows from the definition of Ξ. In accordance with this definition, it follows that the τ-roots $\zeta(\xi)$ and $\bar{\zeta}(\xi)$ for each fixed α must either coincide or differ for all $\xi \in \Xi_\alpha$. Thus, for each fixed α, decompositions (2.2.1) and (2.3.1) can be realized by setting

$$H_+(\xi;\tau) \equiv 0 \ (\mathrm{mod}\,(\tau - \zeta(\xi))^k) \quad \text{for} \quad \operatorname{Im}\zeta(\xi) > 0,$$

and

$$\Pi_0^2(\xi;\tau) \equiv 0 \ (\mathrm{mod}\,(\tau - \zeta(\xi))^k) \quad \text{for} \quad \operatorname{Im}\zeta(\xi) = 0 \quad \text{for all} \quad \xi \in \Xi_\alpha,$$

where k is the multiplicity of the root $\tau = \zeta(\xi)$ of $H(\xi;\tau)$ in the component Ξ_α. □

Now we derive a formula for the function $\Lambda(\xi)$ defined by (2.2.22). Using this formula we will establish the main result of this subsection, i.e., the infinite differentiability of $\Lambda(\xi)$ is each component Ξ_α of the set Ξ.

Proposition 2.3.5. *Let $\zeta_\varrho(\xi)$ be the τ-roots of the polynomial $h_+(\xi;\tau)$, and let*

$$\mathscr{P} = \{\mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\}$$

be the positive definite $k(\xi)\times k(\xi)$ matrix with the entries

$$\begin{aligned}\mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi)) = \mathrm{i}\sum_{g=0}^{\gamma}\sum_{h=0}^{\sigma}&\frac{(-1)^{\sigma-h}C_\gamma^g C_\sigma^h(\gamma-g+\sigma-h)!}{\big(\zeta_\varrho(\xi)-\bar{\zeta}_\nu(\xi)\big)^{\gamma-g+\sigma-h+1}}\\ &\times\sum_{j=1}^{m}\mathscr{P}_j^{(h)}(\xi;\zeta_\varrho(\xi))\overline{\mathscr{P}_j^{(g)}}(\xi;\bar{\zeta}_\nu(\xi))\end{aligned}\tag{2.3.2}$$

($\varrho,\nu = 1,\dots,l(\xi)$, $\sigma=\sigma(\varrho)=0,\dots,k_\varrho(\xi)-1$, $\gamma=\gamma(\nu)=0,\dots,k_\nu(\xi)-1$). *Here $\Lambda(\xi)$ is the function defined by (2.2.22), while $T_j(\xi;\tau)$ are the polynomials satisfying conditions (2.2.3)–(2.2.5) of Lemma 2.2.1, where $D(\xi;\tau)$ is replaced by $R(\xi;\tau)$. Then, for almost all $\xi\in\mathbb{R}^{n-1}$ we have the equality*

$$\Lambda(\xi) = \sum_{\nu=1}^{l(\xi)}\sum_{\gamma=0}^{k_\nu(\xi)-1}\mathscr{R}^{(\gamma)}(\xi;\zeta_\nu(\xi))\varphi_{\nu\gamma}^0(\xi),\tag{2.3.3}$$

where $\{(\varphi_{\nu\gamma}^0(\xi))\}$ is the (unique) solution of the system

$$\mathscr{R}^{(\gamma)}(\xi;\zeta_\nu(\xi)) = \sum_{\nu=1}^{l(\xi)}\sum_{\gamma=0}^{k_\nu(\xi)-1}\mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\bar{\varphi}_{\nu\gamma}^0(\xi),\tag{2.3.4}$$

($\varrho=1,\dots,l(\xi)$, $\sigma=\sigma(\varrho)=0,\dots,k_\nu(\xi)-1$).

The proof follows immediately if in Lemma 2.1.16 we replace the polynomials $\mathscr{D}(\tau)$ and $\mathscr{P}_j(\tau)$ by the polynomials $\mathscr{R}(\xi;\tau)$ and $\mathscr{P}_j(\xi;\tau)$, respectively.

We now turn to the main result of this subsection.

Proposition 2.3.6. *Let* $\operatorname{ord} H_+(\xi;\tau) = J \geqslant 1$ *and* $\operatorname{ord} R(\xi;\tau)\leqslant J-1$ *for all $\xi\in\Xi$. Then the function $\Lambda(\xi)$ defined by (2.2.22) is infinitely differentiable in each component Ξ_α of the set Ξ. Here $T_j(\xi;\tau)$ are the polynomials satisfying conditions (2.2.3)–(2.2.5) of Lemma 2.2.1, where $D(\xi;\tau)$ is replaced by $R(\xi;\tau)$.*

Proof. Let Ξ_α be a fixed component of the set Ξ. By Proposition 2.3.4, the functions $\zeta_\nu(\xi)$ are analytic, while $l(\xi)$ and $k_\nu(\xi)$ are constant in Ξ_α. Since Lemma 2.2.1 was proved under the assumption that

$$D(\xi;\tau) = R(\xi;\tau)\equiv 0 \ (\operatorname{mod}\Pi_+(\xi;\tau))\quad\text{a.e. in}\quad\mathbb{R}^{n-1},$$

one can assume that the functions $\mathscr{R}^{(\gamma)}(\xi;\zeta_\nu(\xi))$ and $\mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))$ are infinitely differentiable in Ξ_α. Since the matrix (2.3.2) is nondegenerate, the functions $\varphi_{\nu\gamma}^0(\xi)$ are also infinitely differentiable in Ξ_α. Then, from (2.3.3) we obtain this property for the function $\Lambda(\xi)$ as well. □

2.3.2 Embedding and extensions theorems

It was shown in Proposition 2.3.6 that the function $\Lambda(\xi) > 0$ is infinitely differentiable a.e. in $\mathbb{R}^{n-1}$. This means that $B(\xi) = 1/\Lambda(\xi)$ is a measurable function satisfying condition 2 of Corollary 2.2.8. Thus, Corollary 2.2.8 yields the following embedding theorem.

Theorem 2.3.7. *Let (2.2.20) be fulfilled a.e. in $\mathbb{R}^{n-1}$. Suppose also that $\Lambda(\xi)$ is the function defined by (2.2.22), where $T_j(\xi;\tau)$ are the polynomials that satisfy conditions (2.2.3)–(2.2.5) of Lemma 2.2.1 with $D(\xi;\tau)$ replaced by $R(\xi;\tau)$. Then, for all $u \in C_0^\infty(\mathbb{R}_+^n)$ we have*

$$\int_{\mathbb{R}^{n-1}} |R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0}|^2\,\frac{d\xi}{\Lambda(\xi)} \leqslant \sum_{j=1}^{m} \|P_j(D)u\|^2. \tag{2.3.5}$$

In this subsection we formulate an extension theorem (Theorem 2.3.8), which in a certain sense is a converse to Theorem 2.3.7. These two theorems give a complete characterisation of the “trace space” $R(D)u\big|_{t=0}$ of the elements u belonging to completion of $C_0^\infty(\mathbb{R}_+^n)$ in the metric $\sum_{j=1}^m \|P_j(D)u\|^2$.

Theorem 2.3.8. *Let the polynomial $R(\xi;\tau)$ satisfy (2.2.20) for almost all $\xi \in \mathbb{R}^{n-1}$, and let the function $\Lambda(\xi)$ be defined by (2.2.22). Then, for any function $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\int_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda(\xi)} d\xi < \infty$ there exists a sequence $u_p \in C_0^\infty(\mathbb{R}_+^n)$ satisfying the following conditions:*

$$\lim_{p\to\infty} \int_{\mathbb{R}^{n-1}} \left|R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}_p(\xi;t)\big|_{t=0} - \hat{\varphi}(\xi)\right|^2 \frac{d\xi}{\Lambda(\xi)} = 0,$$

$$\lim_{p,k\to\infty} \sum_{j=1}^{m} \|P_j(D)(u_p - u_k)\|^2 = 0, \qquad \lim_{p\to\infty} \sum_{j=1}^{m} \|P_j(D)u_p\|^2 = \int_{\mathbb{R}^{n-1}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{\Lambda(\xi)}.$$

Proof. Consider the functions $\varphi_{\nu\gamma}^0(\xi)$ and $\Lambda(\xi)$ defined by (2.3.4) and (2.3.3), respectively. Set

$$\varphi_{\nu\gamma}(\xi) = \frac{\hat{\varphi}(\xi)\varphi_{\nu\gamma}^0(\xi)}{\Lambda(\xi)} \qquad (\nu = 1, \ldots, l(\xi),\ \gamma = 0, \ldots, k_\nu(\xi) - 1).$$

Clearly, $\varphi_{\nu\gamma}(\xi)$ are infinitely differentiable in each component $\Xi_\alpha \subset \Xi$. Let $\xi \in \Xi$, let $u_1(\xi;t), \ldots, u_{\varkappa(\xi)}(\xi;t)$ be the fundamental system of the operator $\Pi_1(\xi; -\mathrm{i}\,d/dt)$, and let $W[u_1(\xi;t), \ldots, u_{\varkappa(\xi)}(\xi;t)]$ be the Wronskian of this system. Assume further that $W_\varrho[u_1, \dots, u_{\varkappa(\xi)}]$ is the determinant obtained from W by replacing the ϱ-th column ($1 \leqslant \varrho \leqslant \varkappa(\xi)$) by $(0, \dots, 0, 1)$. Set

$$g(\xi;t) = \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} \varphi_{\nu\gamma}(\xi)(\mathrm{i}t)^\gamma \exp(\mathrm{i}\zeta_\nu(\xi)t),$$

$$v(\xi;t) = -\sum_{\varrho=1}^{\varkappa(\xi)} u_\varrho(\xi;t) \int_t^{+\infty} g(\xi;\tau) \frac{W_\varrho[u_1, \dots, u_{\varkappa(\xi)}]}{W[u_1, \dots, u_{\varkappa(\xi)}]}(\xi;\tau)d\tau.$$

It is obvious that $\Pi_1\,(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t) = g(\xi;t)$ for all $\xi \in \Xi$. Therefore, we have

1. $$R\,(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t)|_{t=0} = \mathscr{R}\,(\xi;-\mathrm{i}\,d/dt)\,g(\xi;t)|_{t=0}$$
$$= \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} \mathscr{R}^{(\gamma)}(\xi;\zeta_\nu(\xi))\varphi_{\nu\gamma}(\xi)$$
$$= \hat{\varphi}(\xi) \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} \frac{\mathscr{R}^{(\gamma)}(\xi;\zeta_\nu(\xi))\varphi^0_{\nu\gamma}(\xi)}{\Lambda(\xi)} = \hat{\varphi}(\xi)$$

and

2. $$\sum_{j=1}^{m} \int_{\mathbb{R}^{n-1}} d\xi \int_0^\infty \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t)\right|^2 dt$$
$$= \sum_{j=1}^{m} \int_{\mathbb{R}^{n-1}} d\xi \int_0^\infty \left|\mathscr{P}_j\,(\xi;-\mathrm{i}\,d/dt)\,g(\xi;t)\right|^2 dt$$
$$= \int_{\mathbb{R}^{n-1}} \sum_{\varrho=1}^{l(\xi)} \sum_{\gamma=0}^{k_\varrho(\xi)-1} \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} \mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\varphi^0_{\varrho\sigma}(\xi)\bar{\varphi}^0_{\nu\gamma}(\xi)d\xi$$
$$= \int_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda^2(\xi)} \sum_{\varrho\sigma\nu\gamma} \mathscr{P}_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\varphi^0_{\varrho\sigma}(\xi)\bar{\varphi}^0_{\nu\gamma}(\xi)d\xi$$
$$= \int_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda^2(\xi)} \sum_{\varrho=1}^{l(\xi)} \sum_{\gamma=0}^{k_\varrho(\xi)-1} \mathscr{R}^{(\sigma)}(\xi;\zeta_\nu(\xi))\varphi^0_{\varrho\sigma}(\xi)d\xi = \int_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda^2(\xi)}d\xi.$$

Consider the closed set $H = \mathbb{R}^{n-1} \setminus \Xi$; then $\mathrm{mes}_{n-1}H = 0$. Let H_k be a neighborhood (in $\mathbb{R}^{n-1}$) of the set H such that $\mathrm{mes}_{n-1}H_k < 1/k$ $(k = 1, 2, \dots)$. Define a sequence of infinitely differentiable "cut-off" functions $\chi_k(\xi)$ by

$$\chi_k(\xi) = \begin{cases} 0, & \text{if} \quad |\xi| \geqslant 2k, \\ 0, & \text{if} \quad \xi \in H_k, \\ 1, & \text{if} \quad |\xi| \leqslant k \quad \text{and} \quad \xi \notin H_k \quad (k = 1, 2, \dots). \end{cases}$$

Set $g_k(\xi;t) = g(\xi;t)\chi_k(\xi)$. The functions

$$v_k(\xi;t) = -\sum_{\varrho=1}^{\varkappa(\xi)} u_\varrho(\xi;t) \int_t^\infty g_k(\xi;\tau)\frac{W_\varrho[u_1,\dots,u_{\varkappa(\xi)}]}{W[u_1,\dots,u_{\varkappa(\xi)}]}(\xi;\tau)d\tau$$

satisfy the equation $\Pi_1(\xi;-\mathrm{i}\,d/dt)\,v_k(\xi;t) = g_k(\xi;t)$ $(k = 1,2,\dots)$. Taking into account the definition of $\chi_k(\xi)$ as well as the fact that the imaginary parts of the τ-roots of $\Pi_1(\xi;\tau)$ are bounded from below on the set $\Xi_\alpha \setminus H_k$ by a positive constant (depending on α and k), we conclude that the functions

$$w_k(x;t) = (2\pi)^{(1-n)/2} \int_{\mathbb{R}^{n-1}} v_k(\xi;t)e^{\mathrm{i}x\xi}d\xi$$

are infinitely differentiable and decay at infinity (in $\mathbb{R}^n_+$) faster than any power of $(|x|^2 + t^2)^{-1}$ together with all their derivatives. We now consider a sequence of infinitely differentiable "cut-off" functions

$$\eta_r(x;t) = \begin{cases} 0, & \text{if} \quad (|x|^2+t^2)^{1/2} \geqslant 2r, \\ 1, & \text{if} \quad (|x|^2+t^2)^{1/2} \leqslant r \end{cases} \qquad (r = 1,2,\dots),$$

and set $w_{kr}(x;t) = w_k(x;t)\eta_r(x;t)$ $(k, r = 1,2,\dots)$.

We show that the sequence $w_{kr}(x;t)$ satisfies all conditions of the theorem to be proved. It is evident that $w_{kr} \in C_0^\infty(\mathbb{R}^n_+)$ $(k, r = 1,2,\dots)$.

Since the equalities

$$P_j(\xi;-\mathrm{i}\,d/dt)v_k(\xi;t) = \mathscr{P}_j(\xi;-\mathrm{i}\,d/dt)g_k(\xi;t) = \chi_k(\xi)\mathscr{P}_j(\xi;-\mathrm{i}\,d/dt)g(\xi;t)$$

hold, and the function $g(\xi;t)$ and all their derivatives w.r.t. t tend to zero as $t \to +\infty$ faster than any power of t^{-1} uniformly w.r.t. ξ on the sets $\Xi_\alpha \setminus H_k$, we have

$$\lim_{k,s\to\infty} \sum_{j=1}^m \int_{\mathbb{R}^{n-1}} d\xi \int_0^\infty \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\,[v_k(\xi;t) - v_s(\xi;t)]\right|^2 dt = 0,$$

and, consequently, $\lim\limits_{k,s\to\infty} \sum_{j=1}^m \|P_j(D)(w_k - w_s)\|^2 = 0$. From the definition of $\eta_r(x;t)$ and the properties of $w_k(x;t)$ it follows that

$$\lim_{r,p\to\infty} \sum_{j=1}^m \|P_j(D)(w_{kr} - w_{kp})\|^2 = 0 \qquad (k = 1,2,\dots).$$

Then we also have $\lim\limits_{k,r,s,p\to\infty} \sum_{j=1}^m \|P_j(D)(w_{kr} - w_{sp})\|^2 = 0$. Analogous arguments show that

$$\lim_{r\to\infty} \sum_{j=1}^m \|P_j(D)(w_{kr} - w_k)\|^2 = 0 \qquad (k = 1,2,\dots).$$

Applying Theorem 2.3.7 to the difference $w_{kr} - w_k$ we get

$$\lim_{r\to\infty} \int\limits_{\mathbb{R}^{n-1}} |R\,(\xi;-\mathrm{i}\,d/dt)\,(\hat{w}_{kr}(\xi;t) - \hat{w}_k(\xi;t))|_{t=0}|^2 \,\frac{d\xi}{\Lambda(\xi)} = 0.$$

However, the equalities

$$R\,(\xi;-\mathrm{i}\,d/dt)\,\hat{w}_k(\xi;t)|_{t=0} = \chi_k(\xi) R\,(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t)|_{t=0} = \chi_k(\xi)\hat{\varphi}(\xi)$$

and

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^{n-1}} (\chi_k(\xi) - 1)^2 |\hat{\varphi}(\xi)|^2 \frac{d\xi}{\Lambda(\xi)} = 0$$

hold true. Hence,

$$\lim_{k,r\to\infty} \int\limits_{\mathbb{R}^{n-1}} |R\,(\xi;-\mathrm{i}\,d/dt)\,\hat{w}_{kr}(\xi;t)|_{t=0} - \hat{\varphi}(\xi)|^2 \,\frac{d\xi}{\Lambda(\xi)} = 0.$$

From Parseval's identity and the properties of $\eta_k(x;t)$ and $\chi_k(\xi)$ it follows that

$$\begin{aligned}
\lim_{k,r\to\infty} \sum_{j=1}^{m} \|P_j(D) w_{kr}\|^2 &= \lim_{k\to\infty} \sum_{j=1}^{m} \|P_j(D) w_k\|^2 \\
&= \lim_{k\to\infty} \sum_{j=1}^{m} \int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\,v_k(\xi;t)\right|^2 dt \\
&= \sum_{j=1}^{m} \int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty \left|P_j\,(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t)\right|^2 dt \\
&= \int\limits_{\mathbb{R}^{n-1}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{\Lambda(\xi)}.
\end{aligned}$$

□

Remark 2.3.9. In general, the condition

$$\langle\!\langle \varphi \rangle\!\rangle^2_{\lambda-1/2} = \int\limits_{\mathbb{R}^{n-1}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{\Lambda(\xi)} < \infty$$

is not satisfied for all elements of the space $C_0^\infty(\mathbb{R}^{n-1})$.

Let, for example, $n = 2$, $m = 2$, $P_1(D) = P\,(-\mathrm{i}\,\partial/\partial x; -\mathrm{i}\,\partial/\partial t) = \partial/\partial t$, $P_2(D) = 1$, $R(D) = R\,(-\mathrm{i}\,\partial/\partial x; -\mathrm{i}\,\partial/\partial t) = \partial/\partial x$. Then, we have $P(\xi;\tau) = \mathrm{i}\tau$, $P_2(\xi;\tau) = 1$, $R(\xi;\tau) = \mathrm{i}\xi$, $H_+(\xi,\tau) = -\mathrm{i}\tau - 1$, $H_-(\xi;\tau) = \mathrm{i}\tau - 1$ $T_1(\xi;\tau) = \mathrm{i}\xi$, $T_2(\xi;\tau) = -\mathrm{i}\xi$, $\Lambda(\xi) = \xi^2$. Hence, for every function $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\hat{\varphi}(0) \neq 0$, we have $\langle\!\langle \varphi \rangle\!\rangle_{\Lambda-1/2} = \infty$.

Remark 2.3.10. Convergence to zero in the topology defined by the norm $\langle\langle\cdot\rangle\rangle_{\Lambda^{-1/2}}$ does not always imply convergence to zero in the space of Schwartz distributions $\mathscr{D}'$.

Let, for example, $n = 2$, $m = 2$, $P_1(D) = P(D) = \partial^2/\partial x\partial t$, $P_2(D) = 1$, $R(D) = 1$. Then $P(\xi;\tau) = -\xi\tau$, $R(\xi;\tau) = 1$, and for all $\xi \neq 0$ we have

$$H_+(\xi;\tau) = -\xi\tau + \mathrm{i}\,\mathrm{sgn}\,\xi, \qquad H_-(\xi;\tau) = -\xi\tau - \mathrm{i}\,\mathrm{sgn}\,\xi,$$
$$T_1(\xi;\tau) = 1, \qquad T_2(\xi;\tau) = -\mathrm{i}\,\mathrm{sgn}\,\xi, \qquad \Lambda(\xi) = |\xi|^{-1}.$$

Let $r = (x^2 + t^2)^{1/2}$. Consider the sequence of functions $u_k(x;t) \in \mathscr{H}_1(\mathbb{R}^2)$ given by

$$u_k(x;t) = \begin{cases} 1, & \text{if} \quad r \leqslant 1, \\ \left(\log \dfrac{1}{k}\right)^{-1} \log \dfrac{r}{k}, & \text{if} \quad 1 \leqslant r \leqslant k, \quad (k = 1, 2, \dots), \\ 0, & \text{if} \quad r > k \end{cases}$$

and set $\varphi_k(x) = u_k(x;0)$. It follows from $\|\nabla u_k\|^2_{L^2(\mathbb{R}^2)} = O((\log k)^{-1})$ and

$$\int\limits_{-\infty}^{\infty} |\xi| \cdot |\hat{u}_k(\xi;0)|^2 d\xi \leqslant \|\nabla u_k\|^2_{L^2(\mathbb{R}^2)}$$

that $\lim\limits_{k\to\infty} \langle\langle\varphi\rangle\rangle_{\Lambda^{-1/2}} = 0$.

On the other hand, for any nonnegative function $\eta \in C_0^\infty(\mathbb{R}^1)$ it holds that

$$\int\limits_{\mathbb{R}^{n-1}} \varphi_k(x)\eta(x)dx \geqslant \int\limits_{|x|\leqslant 1} \eta(x)dx.$$

Hence, φ_k does not tend to zero in $\mathscr{D}'(\mathbb{R}^1)$.

2.3.3 On the extension of functions from $\mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_+)$

In this subsection we establish a corollary of Theorems 2.3.7 and 2.3.8 and of Proposition 2.2.12, which is possibly interesting in its own right. It is concerned with the extension of functions having finite norm $\left(\sum_{j=1}^m \|P_j(D)u : L^2(\mathbb{R}^n_+)\|^2\right)^{1/2}$ to the whole space $\mathbb{R}^n$ with class preservation.

Let $P_j(D) = \partial/\partial t - p_j(D_x)$ ($j = 1, \dots, m$; $m > 1$) be differential operators with constant coefficients, with the polynomials $p_j(\xi)$ pairwise distinct. We denote by $\mathscr{H}(\mathbb{R}^n_+)$ and $\mathscr{H}(\mathbb{R}^n)$ the completions of the spaces $C_0^\infty(\mathbb{R}^n_+)$ and $C_0^\infty(\mathbb{R}^n)$ in the metrics

$$\|u : \mathscr{H}(\mathbb{R}^n_+)\|^2 = \sum_{j=1}^m \|P_j(D)u : L^2(\mathbb{R}^n_+)\|^2$$

and

$$\|u : \mathscr{H}(\mathbb{R}^n)\|^2 = \sum_{j=1}^{m} \|P_j(D)u : L^2(\mathbb{R}^n)\|^2,$$

respectively. The restriction of the elements of $\mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_+)$ is defined in a natural way.

Proposition 2.3.11. *An element $u \in \mathscr{H}(\mathbb{R}^n_+)$ is the restriction of some $v \in \mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_+)$ such that*

$$\|v : \mathscr{H}(\mathbb{R}^n)\| \leqslant \text{const}\|u : \mathscr{H}(\mathbb{R}^n_+)\|, \tag{2.3.6}$$

if and only if the inequality

$$\sum_{j=1}^{m} |\operatorname{Re} p_j| \leqslant \text{const} \sum_{j,h=1}^{m} |p_j - p_h| \tag{2.3.7}$$

is fulfilled a.e. on the set $\left\{\xi : \sum_{j=1}^{m} \operatorname{Re} p_j < 0\right\}$.

Proof. Let $\mathbb{R}^n_- = \{(x;t) : x \in \mathbb{R}^{n-1}, t \leqslant 0\}$, and let $C_0^\infty(\mathbb{R}^n_-)$ be the space of functions from $C_0^\infty(\mathbb{R}^n)$ restricted to $\mathbb{R}^n_-$. We denote by $\mathscr{H}(\mathbb{R}^n_-)$ the completion of $C_0^\infty(\mathbb{R}^n_-)$ in the metric

$$\|u : \mathscr{H}(\mathbb{R}^n_-)\|^2 = \sum_{j=1}^{m} \|P_j(D)u : L^2(\mathbb{R}^n_-)\|^2.$$

The restriction of the elements from $\mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathscr{R}^n_-)$ is defined in a natural way. By v^- we denote the restriction of $v \in \mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_-)$. Further, notice that (2.3.6) is equivalent to the inequality

$$\|v^- : \mathscr{H}(\mathbb{R}^n_-)\| \leqslant \text{const}\,\|u : \mathscr{H}(\mathbb{R}^n_+)\|. \tag{2.3.8}$$

Since the polynomials $p_j(\xi)$ are pairwise distinct, we have $\sum_{j=1}^{m} |p_j| \neq 0$ and $\sum_{j,h=1}^{m} |p_j - p_h| \neq 0$ a.e. in $\mathbb{R}^{n-1}$. Therefore, condition 1 of Proposition 2.2.12 is fulfilled.

Necessity. Set

$$\Lambda_+(\xi) = 2m^{-1}(\alpha + \beta)^{-1}, \qquad \Lambda_-(\xi) = 2m^{-1}(\alpha - \beta)^{-1}, \tag{2.3.9}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are defined by (2.2.22) and (2.2.44), respectively.

Theorem 2.3.7 implies the inequality

$$\langle\!\langle u \rangle\!\rangle_{\Lambda_+^{-1/2}} \leqslant \|u : \mathscr{H}(\mathbb{R}^n_+)\|. \tag{2.3.10}$$

Replacing $H_+(\xi;\tau)$ by $H_-(\xi;\tau)$ in all arguments used in proof of Theorem 2.3.7 and taking into account (2.3.8), we obtain

$$\begin{aligned}\langle\!\langle v^- \rangle\!\rangle_{\Lambda_-^{-1/2}} = \langle\!\langle u \rangle\!\rangle_{\Lambda_-^{-1/2}} &\leqslant \text{const}\, \| v^- : \mathscr{H}(\mathscr{R}_-^{n-1}) \| \\ &\leqslant \text{const}\, \| u : \mathscr{H}(\mathscr{R}_+^{n-1}) \|. \end{aligned} \tag{2.3.11}$$

Combining (2.3.10) and (2.3.11) and taking into account the results of Corollary 2.2.8 (see (2.2.21)), we find that the inequality $\Lambda_+\Lambda_-^{-1} \leqslant$ const or, what is the same,

$$\alpha - \beta \leqslant \text{const}\,(\alpha + \beta) \tag{2.3.12}$$

holds a.e. in $\mathbb{R}^n$.

Let $\beta(\xi) < 0$. From the definition (2.2.39) of the function $\alpha(\xi)$ it follows that one can replace α in (2.3.12) by

$$\sum_{j=1}^{m} |\text{Re}\, p_j| + \sum_{j,h=1}^{m} |\text{Im}\,(p_j - p_h)|.$$

Representing $\alpha + \beta$ in the form

$$\alpha + \beta = \alpha \left[1 - \left(1 - 2^{-1}\alpha^{-2} \sum_{j,h=1}^{m} |p_j - p_h|^2 \right)^{1/2} \right],$$

we see that on the set $\left\{\xi : \sum_{j=1}^{m} \text{Re}\, p_j < 0\right\}$ inequality (2.3.12) is equivalent to (2.3.7).

Sufficiency. Suppose that (2.3.7) holds a.e. on $\left\{\xi : \sum_{j=1}^{m} \text{Re}\, p_j < 0\right\}$. Then (2.3.12) is satisfied a.e. on this set (see the proof of necessity). On the other hand, (2.3.12) is automatically fulfilled on the set $\left\{\xi : \sum_{j=1}^{m} \text{Re}\, p_j = \beta \geqslant 0\right\}$. Hence this inequality holds for almost all $\xi \in \mathbb{R}^{n-1}$. Therefore, the functions $\Lambda_+(\xi)$ and $\Lambda_-(\xi)$ defined by (2.3.9) satisfy the condition $\Lambda_+\Lambda_-^{-1} \leqslant$ const a.e. in $\mathbb{R}^{n-1}$.

Let $u \in \mathscr{H}(\mathbb{R}_+^n)$. Using Theorem 2.3.7 we get

$$\langle\!\langle u(x;0) \rangle\!\rangle_{\Lambda_+^{-1/2}} \leqslant \| u : \mathscr{H}(\mathbb{R}_+^n) \|. \tag{2.3.13}$$

Since $\Lambda_+\Lambda_-^{-1} \leqslant$ const a.e. in $\mathbb{R}^{n-1}$, it follows that

$$\langle\!\langle u(x;0) \rangle\!\rangle_{\Lambda_-^{-1/2}} \leqslant \text{const}\, \langle\!\langle u(x;0) \rangle\!\rangle_{\Lambda_+^{-1/2}}. \tag{2.3.14}$$

Replacing $H_+(\xi;\tau)$ by $H_-(\xi;\tau)$ in all arguments used in the proof of Theorem 2.3.8, we construct an element $v^- \in \mathscr{H}(\mathbb{R}_-^n)$ such that $v^-(x;0) = u(x;0)$ and

$$\langle\!\langle u(x;0) \rangle\!\rangle_{\Lambda_-^{-1/2}} = \langle\!\langle v^-(x;0) \rangle\!\rangle_{\Lambda_-^{-1/2}} = \| v^- : \mathscr{H}(\mathbb{R}_-^n) \|. \tag{2.3.15}$$

From (2.3.13), (2.3.14), and (2.3.15) it follows that

$$\|v^- : \mathscr{H}(\mathbb{R}^n_-)\| \leqslant \text{const}\, \|u : \mathscr{H}(\mathbb{R}^n_+)\|.$$

We show (this would be sufficient to complete the proof) that the elements u and v^- are restrictions of some element $v \in \mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_+)$ and $\mathscr{H}(\mathbb{R}^n_-)$, respectively. Following the recipe described in the proof of Theorem 2.3.8, we construct the fundamental sequence $v^-_{kr} \in \mathbf{C}^\infty_0(\mathbb{R}^n_-)$, which determines the element $v^- \in \mathscr{H}(\mathbb{R}^n_-)$. Let $\chi_k(\xi)$ and $\eta_r(\xi)$ be the "cut-off" functions used in this procedure.

Consider the sequence $\hat{u}_k(\xi;t) = \hat{u}(\xi;t)\chi_k(\xi)$, where $\hat{u}(\xi;t)$ is the Fourier transform of the element $u \in \mathscr{H}(\mathbb{R}^n_+)$ w.r.t. x. Further, let

$$w_k(x;t) = (2\pi)^{(1-n)/2} \int\limits_{\mathbb{R}^{n-1}} \hat{u}_k(\xi;t) e^{ix\xi} d\xi,$$

and let $w_{kr}(x;t) = w_k(x;t)\chi_k(\xi)$. It is obvious that $w_{kr} \in \mathbf{C}^\infty_0(\mathbb{R}^n_+)$, and this sequence converges in $\mathscr{H}(\mathbb{R}^n_+)$ to the element u as $k, r \to \infty$.

Since $u(x;0) = v^-(x;0)$ and the sequences $v^-_{kr} \in C^\infty_0(\mathbb{R}^n_-)$ and $w_{kr} \in C^\infty_0(\mathbb{R}^n_+)$ have been constructed by using the same system of "cut-off" functions $\chi_k(\xi)$ and $\eta_r(x;t)$, we have $v^-_{kr}(x;0) = w_{kr}(x;0)$. Therefore, for each fixed pair k, r, the functions $v^-_{kr}(x;t)$ and $w_{kr}(x;t)$ are restrictions of some function $v_{kr} \in C^\infty_0(\mathbb{R}^n)$ to $\mathbb{R}^n_-$ and $\mathbb{R}^n_+$, respectively. Since $v^-_{kr}(x;t)$ converges in $\mathscr{H}(\mathbb{R}^n_-)$ to v^- and $w_{kr}(x;t)$ converges in $\mathscr{H}(\mathbb{R}^n_+)$ to u, we conclude that the sequence $v_{kr}(x;t)$ converges in $\mathscr{H}(\mathbb{R}^n)$ to some limit function v. The functions v^- and u are restrictions of v to $\mathscr{H}(\mathbb{R}^n)$ and $\mathscr{H}(\mathbb{R}^n_+)$, respectively. □

2.4 Notes

The main results of this chapter were established by the authors (sometimes in a less general form) in the paper [GM74]; some of the results were announced in the note [H58].

Chapter 3
Dominance of differential operators

3.0 Introduction

3.0.1 Description of results

In this chapter we formulate necessary and sufficient conditions for the validity of the estimates

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\sum_{j=1}^{m}\|P_j(D)u\|^2 + \sum_{\alpha=1}^{N}\langle\!\langle Q_\alpha(D)u\rangle\!\rangle^2\right), \qquad u \in C_0^\infty(\mathbb{R}^n_+) \tag{3.0.1}$$

and

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\sum_{j=1}^{m}\|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \qquad Q_\alpha u(x;0) = 0 \quad (\alpha = 1, \dots, N). \tag{3.0.2}$$

It is assumed that $R(\xi;\tau)$, $P_j(\xi;\tau)$, and $Q_\alpha(\xi;\tau)$ are polynomials in the variable $\tau \in \mathbb{R}^1$ with complex measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. We suppose also that ord $R(\xi;\tau) \leqslant J = \max_{1\leqslant j\leqslant m}$ ord $P_j(\xi;\tau)$ and ord $Q_\alpha(\xi;\tau) \leqslant J-1$.

Criteria for the validity of (3.0.1) and (3.0.2) are given in Section 3.2. To formulate these criteria, we consider the polynomials (of τ) $H_+(\xi;\tau)$ and $\Pi_+(\xi;\tau)$ defined in Chapter 2 (see Subsection 2.0.1), and the polynomials (of $\eta, \tau \in \mathbb{R}^1$)

$$\Omega(\xi;\eta,\tau) = (\eta-\tau)^{-1}[H_+(\xi;\tau)R(\xi;\tau) - R(\xi;\tau)H_+(\xi;\tau)]. \tag{3.0.3}$$

There exist functions $\beta_\alpha(\xi;\eta)$ $(1 \leqslant \alpha \leqslant N)$ such that the relations

$$\int_{-\infty}^{\infty}\sum_{\alpha=1}^{N}|\beta_\alpha(\xi;\eta)|^2 d\eta < \infty \tag{3.0.4}$$

and

$$D(\xi;\eta,\tau) \overset{\text{def}}{=} [H_+(\xi;\tau)]^{-1}\Omega(\xi;\eta,\tau) - \sum_{\alpha=1}^{N}\beta_\alpha(\xi;\eta)Q_\alpha(\xi;\tau) \equiv 0 \ (\operatorname{mod} \Pi_+(\xi;\tau)) \tag{3.0.5}$$

hold for all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$.

Denote by $D_j(\xi;\eta,\tau)$ the polynomials of τ (ord $D_j(\xi;\eta,\tau) \leqslant J-1$) satisfying for all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$ the conditions

$$\overline{D_j(\xi;\eta,\tau)} \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)) \qquad (j=1,\dots,m); \tag{3.0.6}$$

$$D(\xi;\eta,\tau)H_-(\xi;\tau) = \sum_{j=1}^m P_j(\xi;\tau)D_j(\xi;\eta,\tau); \tag{3.0.7}$$

$$\begin{aligned} &P_i(\xi;\tau)\overline{D_j(\xi;\eta,\tau)} \equiv P_j(\xi;\tau)\overline{D_i(\xi;\eta,\tau)} \ (\mathrm{mod}\, \Pi_+(\xi;\tau)H_+(\xi;\tau)) \\ &(i \neq j;\ i,j=1,\dots,m). \end{aligned} \tag{3.0.8}$$

(Condition (3.0.8) is omitted in the case $m=1$).

From Lemma 2.2.1, Chapter 2 it follows that for each system $\{\beta_\alpha(\xi;\eta)\}$ satisfying (3.0.5) there exist polynomials $D_j(\xi;\eta,\tau)$, which are uniquely determined by conditions (3.0.6)–(3.0.8). In Section 3.2 (Theorem 3.2.2) it is shown that inequality (3.0.1) holds if and only if

1. there exist functions $\beta_\alpha(\xi;\eta)$ ($\alpha = 1,\dots,N$) that satisfy conditions (3.0.4) and (3.0.6);
2. the inequality

$$\begin{aligned} &\int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\xi;\eta)\}} \left\{ \sum_{\alpha=1}^N |\beta_\alpha(\xi;\eta)|^2 + \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau \right\} d\eta \\ &+ \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} \leqslant \frac{\mathrm{const}}{B(\xi)} \end{aligned} \tag{3.0.9}$$

   holds for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\tau \in \mathbb{R}^1$. Here $D_j(\xi;\eta,\tau)$ are the polynomials defined by conditions (3.0.6)–(3.0.8), and the infimum is taken over all systems $\{\beta_\alpha(\xi;\eta)\}$ satisfying (3.0.5).

If all polynomials $Q_\alpha(\xi;\tau)$ vanish identically, then (3.0.1) takes the form

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \sum_{j=1}^m \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+). \tag{3.0.1$'$}$$

It is established in Theorem 3.2.4 that the estimate (3.0.1′) holds if and only if

1. for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\eta \in \mathbb{R}^1$ the polynomial (3.0.3) satisfies the congruence
$$\Omega(\xi;\eta,\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)); \tag{3.0.10}$$
2. the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau\, d\eta + \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} \leqslant \frac{\mathrm{const}}{B(\xi)} \tag{3.0.11}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\eta \in \mathbb{R}^1$. Here $D_j(\xi;\eta,\tau)$ are the polynomials of τ satisfying conditions (3.0.6)–(3.0.8) with $D(\xi;\eta,\tau) = [H_+(\xi;\eta)]^{-1}\Omega(\xi;\eta,\tau)$.

Finally, the criterion for the validity of the estimate (3.0.2) (Theorem 3.2.3) is as follows: condition 1 of Theorem 3.2.2 is satisfied and the inequality

$$\int\limits_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\xi;\eta)\}} \left\{ \int\limits_{-\infty}^{\infty} \frac{\sum_{j=1}^m |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} d\tau \right\} d\eta + \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2} \leqslant \frac{\text{const}}{B(\xi)} \tag{3.0.11$'$}$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$ and for all $\tau \in \mathbb{R}^1$, where the polynomials D_j and the functions β_α are the same as in Theorem 3.2.2.

All these results are deduced from necessary and sufficient conditions for the validity of inequalities of the types (3.0.1) and (3.0.2) for ordinary differential operators on the semi-axis $t \geqslant 0$, as well as from the estimates of the sharp constants in these inequalities.

In Section 3.3 we consider several examples. In Subsection 3.3.1 it is shown that the well-known theorem by N. Aronszajn [Aro54] on necessary and sufficient coercivity conditions for a system of operators $P_j(D)$ in the half-space $\mathbb{R}^n_+$ is a direct consequence of the results of Section 3.2. In Subsection 3.3.2 we consider the case when $m = 1$ and the number N of the boundary operators $Q_\alpha(D)$ is (in a certain sense) minimal. Necessary and sufficient conditions for this case were established by the authors in [MG75], where we used other methods and a different terminology. In Subsection 3.3.2 it is shown that the main result of [MG75] follows from the general criterion obtained in Section 3.2. The estimates of the type (3.0.1) for operators P_j of the first order in t are studied in Subsection 3.3.3.

Other applications of the results of this chapter are given in Chapter 4.

3.0.2 Remarks on the method of proving the main result

The study of the validity conditions for the estimates (3.0.1), (3.0.2), and (3.0.1$'$) is based to a large extent on the results of Chapter 2. We explain this relationship using as example the estimate (3.0.1$'$). Applying the method of localization in ξ and using the Fourier transform w.r.t. x, we obtain similarly to Subsection 1.0.2 that (3.0.1$'$) is valid if and only if the estimate

$$\int\limits_0^\infty |R(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt \leqslant \Lambda(\xi) \int\limits_0^\infty \sum_{j=1}^m \left|P_j(\xi;-\mathrm{i}\,d/dt)\,v\right|^2 dt \tag{3.0.12}$$

with $v \in C_0^\infty(\mathbb{R}^1_+)$ holds true for almost all $\xi \in \mathbb{R}^{n-1}$, and the sharp constant $\Lambda(\xi)$ in (3.0.12) satisfies the condition

$$B(\xi)\Lambda(\xi) \leqslant C. \tag{3.0.13}$$

Necessity of (3.0.10) is established in the same manner as necessity of (2.0.9) in Subsection 2.0.2. Notice only that, in accordance with (3.0.3), the congruences $\Omega(\xi;\eta,\tau) \equiv 0 \pmod{\Pi_+(\xi;\tau)}$ and $R(\xi;\tau) \equiv 0 \pmod{\Pi_+(\xi;\tau)}$ are equivalent. Without loss of generality we can assume that $\Pi_+(\xi;\tau) = 1$. For the sake of simplicity, we assume also that the leading coefficient of the polynomial $H_+(\xi;\tau)$ is equal to 1, and its τ-roots $\zeta_1(\xi),\dots,\zeta_J(\xi)$ are pairwise distinct a.e. in $\mathbb{R}^{n-1}$.

Variational problem in a finite-dimensional space and estimates of the sharp constants. We consider the positive definite $J \times J$ matrix $\mathfrak{B}(\xi)$ defined by (2.0.14) from Chapter 2. Let $\mathbf{a}(\xi;\eta) = (a_\varrho(\xi;\eta)) \in \mathbb{C}^J$ be the vector-function with the components

$$a_\varrho(\xi;\eta) = [H_+(\xi;\eta)]^{-1}\Omega(\xi;\eta,\zeta_\varrho(\xi)) \qquad (\varrho = 1,\dots,J). \tag{3.0.14}$$

By (3.0.3), these components are continuous and belong to $L^2(\mathbb{R}^1)$ as functions of the variable η. Let $\mathbf{x} = (\bar{x}_1,\dots,\bar{x}_J) \in \mathbb{C}^J$, and let $x(t) = \sum_{\varrho=1}^{J} x_\varrho \exp(i\zeta_\varrho(\xi)t)$. Using the easily verifiable identity

$$\int_{-\infty}^{\infty} |\,(\mathbf{a}(\xi;\eta),\mathbf{x})_J\,|^2 d\eta = 2\pi \int_0^{\infty} |R\,(\xi;-i\,d/dt)\,x(t)|^2\,dt$$

and arguing in the same way as in Subsection 2.0.2, we can show that (3.0.12) is true if and only if the estimate

$$\int_{-\infty}^{\infty} |\,(\mathbf{a}(\xi;\eta),\mathbf{x})_J\,|^2 d\eta \leqslant 2\pi\Lambda_1(\xi)\,(\mathfrak{B}(\xi)\mathbf{x},\mathbf{x})_J\,, \qquad \mathbf{x} \in \mathbb{C}^J \tag{3.0.15}$$

holds a.e. in $\mathbb{R}^{n-1}$ and the inequality

$$\begin{gathered}\int_0^{\infty} |R\,(\xi;-i\,d/dt)\,v|^2\,dt \leqslant \Lambda_2(\xi)\int_0^{\infty}\sum_{j=1}^{m}\left|P_j\,(\xi;-i\,d/dt)\,v\right|^2 dt,\\ v \in C_0^\infty(\mathbb{R}^1_+), \quad v^{(p)}(0) = 0 \quad (p = 0,\dots,J-1)\end{gathered} \tag{3.0.16}$$

is satisfied. Here, the sharp constants $\Lambda(\xi)$, $\Lambda_1(\xi)$, and $\Lambda_2(\xi)$ from inequalities (3.0.12), (3.0.15), and (3.0.16), respectively, satisfy the estimate[1]

$$2^{-1}\Lambda(\xi) \leqslant \max\,(\Lambda_1(\xi),\Lambda_2(\xi)) \leqslant \Lambda(\xi).$$

Since (3.0.16) must be fulfilled for functions $v(t)$ satisfying the homogeneous Cauchy data, we have[2]

$$\Lambda_2(\xi) = \sup \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2}. \tag{3.0.17}$$

[1] A complete proof of an analogous statement corresponding to the estimate (3.0.1) is given in Lemma 3.1.6.
[2] For more details, see the proof of Theorem 3.1.9.

Along with (3.0.15) we consider the inequality

$$|\,(\mathbf{a}(\xi;\eta),\mathbf{x})_J\,|^2 \leqslant \lambda_1(\xi;\eta)\,(\mathfrak{B}(\xi)\mathbf{x},\mathbf{x})_J\,, \qquad \mathbf{x}\in\mathbb{C}^J. \tag{3.0.18}$$

It is obvious that each of inequalities (3.0.15) and (3.0.18) holds true if and only if for every $\eta \in \mathbb{R}^1$ all elements $\mathbf{x} \in \ker\mathfrak{B}$ satisfy the condition $(\mathbf{a}(\xi;\eta),\mathbf{x})_J = 0$. Since the left-hand side of (3.0.15) is the result of integration of the left-hand side of (3.0.18) over $\mathbb{R}^1$, the sharp constants $\Lambda_1(\xi)$ and $\lambda_1(\xi;\eta)$ in (3.0.15) and (3.0.18), respectively, satisfy the estimate

$$c_1\Lambda_1(\xi) \leqslant \int_{-\infty}^{\infty} \lambda_1(\xi;\eta)d\eta \leqslant c_2\Lambda_1(\xi).[3] \tag{3.0.19}$$

From the results of Chapter 2 and (3.0.14) it follows that the sharp constant $\lambda_1(\xi;\eta)$ coincides with the sharp constant appearing in the boundary estimate

$$\left|[H_+(\xi;\eta)]^{-1}\Omega\;(\xi;\eta,-\mathrm{i}\,d/dt)\,v|_{t=0}\right|^2 \leqslant \lambda_1(\xi;\eta)\sum_{j=1}^{m}\int_0^{\infty}\left|P_j\;(\xi;-\mathrm{i}\,d/dt)\,v\right|^2 dt,$$
$$v\in C_0^\infty(\mathbb{R}^1_+) \tag{3.0.20}$$

for ordinary differential operators. According to Theorem 2.1.17, Chapter 2, we have

$$\lambda_1(\xi;\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sum_{j=1}^m |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\xi;\tau)|^2}d\tau, \tag{3.0.21}$$

where $D_j(\xi;\eta,\tau)$ are the polynomials satisfying conditions (3.0.6)-(3.0.8) with $D(\xi;\eta,\tau) = [H_+(\xi;\eta)]^{-1}\Omega(\xi;\eta,\tau)$. Using (3.0.17), (3.0.19), (3.0.21) and (3.0.13), we arrive at (3.0.11).

3.1 Estimates for ordinary differential operators on the semi-axis

Let R, P_j, and Q_α be polynomials of the variable $\tau\in\mathbb{R}^1$ with complex coefficients, and let $J = \max\operatorname{ord} P_j \geqslant 1$, $\operatorname{ord} R \leqslant J$, and $\max\operatorname{ord} Q_\alpha \leqslant J-1$. In this section

[3] A complete proof of this assertion is given in Lemma 3.1.1.

we establish necessary and sufficient conditions for the validity of the inequalities

$$\begin{aligned}\int_0^\infty |R(-\mathrm{i}\,d/dt)\,u|^2\,dt \leqslant \Lambda\Bigg[\int_0^\infty \sum_{j=1}^m \big|P_j(-\mathrm{i}\,d/dt)\,u\big|^2\,dt \\ + \sum_{\alpha=1}^N |Q_\alpha(-\mathrm{i}\,d/dt)\,u|_{t=0}|^2\Bigg], \\ u \in C_0^\infty(\mathbb{R}^1_+),\end{aligned} \tag{3.1.1}$$

$$\begin{aligned}\int_0^\infty |R(-\mathrm{i}\,d/dt)\,u|^2\,dt \leqslant \Lambda_0 \int_0^\infty \sum_{j=1}^m \big|P_j(-\mathrm{i}\,d/dt)\,u\big|^2\,dt, \\ Q_\alpha(-\mathrm{i}\,d/dt)\,u|_{t=0} = 0, \\ u \in C_0^\infty(\mathbb{R}^1_+) \qquad (\alpha = 1,\dots,N)\end{aligned} \tag{3.1.1$'$}$$

and give upper and lower bounds for the sharp constants Λ, Λ_0 figuring in (3.1.1) and (3.1.1$'$), respectively.

3.1.1 A variational problem in a finite-dimensional space

As already noted, the estimates for ordinary differential operators on the semi-axis are equivalent to certain inequalities in a finite-dimensional space. In Subsection 3.1.3 it will be shown that (3.1.1) is equivalent to (3.1.26) and (3.1.27), while (3.1.1$'$) is equivalent to (3.1.36) and (3.1.27). In this subsection, we consider the variational problems which are equivalent to (3.1.26) and (3.1.36): we find necessary and sufficient conditions ensuring the boundedness of the function $\Psi_1(\mathbf{z})$, defined by (3.1.4), on $\mathbb{C}^J$ and the functions (3.1.14) on the subspace (3.1.10), respectively; and give estimates for the suprema of these functions (Lemmas 3.1.2-3.1.4). To prove these results, we begin with a statement related to a variational problem with a parameter.

Suppose that quadratic forms $\mathscr{A}(\eta,\mathbf{z};\mathbf{z}) \geqslant 0$, $\mathscr{B}(\mathbf{z};\mathbf{z}) \geqslant 0$, $\mathbf{z} \in \mathbb{B}$, $\eta \in \mathbb{R}^1$ are given on the subspace $\mathbb{B}$ of the complex space $\mathbb{C}^J$. We assume that $\mathscr{A}(\eta,\mathbf{z};\mathbf{z})$ is continuous w.r.t. η and $\mathscr{A}(\eta,\mathbf{z};\mathbf{z}) \in L^1(\mathbb{R}^1)$ for all $\mathbf{z} \in \mathbb{B}$. We set

$$\Phi(\eta,\mathbf{z}) = \frac{\mathscr{A}(\eta,\mathbf{z};\mathbf{z})}{\mathscr{B}(\mathbf{z};\mathbf{z})}, \qquad \Psi(\mathbf{z}) = \int_{-\infty}^{\infty} \Phi(\eta,\mathbf{z})\,d\eta. \tag{3.1.2}$$

It follows from (3.1.2) that $\Psi(\mathbf{z})$ is bounded on $\mathbb{B}$ if and only if we have $\ker \mathscr{B}(\mathbf{z};\mathbf{z}) \subseteq \ker \mathscr{A}(\eta,\mathbf{z};\mathbf{z})$ for all $\eta \in \mathbb{R}^1$, or, what is the same, if and only if the function $\Phi(\eta,\mathbf{z})$ is bounded for all $\eta \in \mathbb{R}^1$.

Let $\mathbb{X}$ be the orthogonal complement of the subspace $\ker\mathscr{B}(\mathbf{z};\mathbf{z})$ in $\mathbb{B}$, and let $r = \dim\mathbb{X}$.

Lemma 3.1.1. *Let the functions* $\Phi(\eta, \mathbf{z})$ *and* $\Psi(\mathbf{z})$ *be defined by* (3.1.2)*, let the function* $\Psi(\mathbf{z})$ *be bounded on* $\mathbb{B}$*, and let* $\lambda(\eta) = \sup_{\mathbf{z}\in\mathbb{B}} \Phi(\eta, \mathbf{z})$ *and the constant* $\Lambda = \sup_{\mathbf{z}\in\mathbb{B}} \Psi(\mathbf{z})$*. Then* $\lambda(\eta) \in L^1(\mathbb{R}^1)$ *and* Λ *satisfies the estimates*

$$r^{-1}\Lambda \leqslant \int_{-\infty}^{\infty} \lambda(\eta)d\eta \leqslant r\Lambda. \tag{3.1.3}$$

Proof. It is obvious that $\mathscr{B}(\mathbf{z}; \mathbf{z})$ is positive definite on $\mathbb{X}$, and $\lambda(\eta) = \sup_{\mathbf{z}\in\mathbb{X}} \Phi(\eta, \mathbf{z})$, $\Lambda = \sup_{\mathbf{z}\in\mathbb{X}} \Psi(\mathbf{z})$. Let $\mathscr{B}(\mathbf{z}_1; \mathbf{s}_2)$ be the bilinear form corresponding to $\mathscr{B}(\mathbf{z}; \mathbf{z})$. We define in $\mathbb{X}$ the scalar product $\{\mathbf{z}_1, \mathbf{z}_2\} = \mathscr{B}(\mathbf{z}_1; \mathbf{z}_2)$ and consider a nonnegative $r \times r$ matrix $\mathcal{U}(\eta)$ satisfying $\{\mathcal{U}(\eta)\mathbf{z}, \mathbf{z}\} = \mathscr{A}(\eta, \mathbf{z}; \mathbf{z})$ for $\mathbf{z} \in \mathbb{X}$. It is evident that its entries are continuous functions from $L^1(\mathbb{R}^1)$, and $\lambda(\eta)$ is the largest eigenvalue of $\int_{-\infty}^{\infty} \mathcal{U}(\eta)d\eta$. Therefore, $r^{-1}\operatorname{tr}\mathcal{U}(\eta) \leqslant \lambda(\eta) \leqslant \operatorname{tr}\mathcal{U}(\eta)$ and

$$r^{-1}\operatorname{tr}\left(\int_{-\infty}^{\infty} \mathcal{U}(\eta)d\eta\right) \leqslant \Lambda \leqslant \operatorname{tr}\left(\int_{-\infty}^{\infty} \mathcal{U}(\eta)d\eta\right).$$

From these inequalities it follows that $\lambda(\eta) \in L^1(\mathbb{R}^1)$ and Λ satisfies (3.1.3). □

Now we turn to the consideration of variational problems discussed at the beginning of this subsection.

We denote by $(\cdot,\cdot)_\mu$ and $(\cdot,\cdot)_\nu$ the scalar products in the complex spaces $\mathbb{C}^\mu$ and $\mathbb{C}^\nu$, respectively, and set $\mathbb{C}^J = \mathbb{C}^\mu \times \mathbb{C}^\nu$. The elements $\mathbf{z} \in \mathbb{C}^J$ will be written as $\mathbf{z} = (\mathbf{x}; \mathbf{y})$, where $\mathbf{x} \in \mathbb{C}^\mu$ and $\mathbf{y} \in \mathbb{C}^\nu$. Suppose that $\mathbf{c}_\alpha \in \mathbb{C}^\mu$ and $\mathbf{d}_\alpha \in \mathbb{C}^\nu$ ($\alpha = 1, \dots, N$). Let $\mathbf{a}(\eta) \neq 0$ be a μ-dimensional vector-function with continuous components belonging to $L^2(\mathbb{R}^1)$, and let $\mathscr{B}$ be a non-negative $\mu \times \mu$ matrix. For $\mathbf{z} \in \mathbb{C}^J$, $\eta \in \mathbb{R}^1$ set

$$\begin{aligned}\Phi_1(\eta, \mathbf{z}) &= \frac{|\,(\mathbf{a}(\eta), \mathbf{x})_\mu\,|^2}{(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu + \sum_{\alpha=1}^{N} |\,(\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\nu\,|^2},\\ \Psi_1(\mathbf{z}) &= \int_{-\infty}^{\infty} \Phi_1(\eta, \mathbf{z})d\eta,\end{aligned} \tag{3.1.4}$$

$$\Lambda_1 = \sup_{\mathbf{z}\in\mathbb{C}^J} \Psi_1(\mathbf{z}). \tag{3.1.5}$$

Lemma 3.1.2. *The function* $\Psi_1(\mathbf{z})$ *defined by* (3.1.4) *is bounded on* $\mathbb{C}^J$ *if and only if there exist functions* $\beta_\alpha(\eta) \in L^2(\mathbb{R}^1)$ *that satisfy the following conditions:*

1. *The equality*

$$\sum_{\alpha=1}^{N} \beta_\alpha(\eta)\mathbf{d}_\alpha = 0 \tag{3.1.6}$$

holds for all $\eta \in \mathbb{R}^1$.

2. *The equation*

$$\mathfrak{B}\mathbf{x}(\eta) = \mathbf{a}(\eta) - \sum_{\alpha=1}^{N} \beta_\alpha(\eta)\mathbf{c}_\alpha \tag{3.1.7}$$

is solvable for every $\eta \in \mathbb{R}^1$.

If these conditions are satisfied and $\mathbf{x}_0(\eta)$ *is an arbitrary solution of* (3.1.7)*, then the constant* Λ_1 *defined by* (3.1.5) *satisfies the estimate*

$$r^{-1}\Lambda_1 \leqslant \int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\eta)\}} \left[(\mathfrak{B}\mathbf{x}_0(\eta), \mathbf{x}_0(\eta))_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha(\eta)|^2 \right] d\eta \leqslant r\Lambda_1, \tag{3.1.8}$$

where the infimum is taken over all $\{\beta_\alpha(\eta)\}$ *satisfying conditions* 1 *and* 2*, and* r *is an integer such that* $0 < r \leqslant J$.

Proof. The boundedness of the function $\Psi_1(\mathbf{z})$ on $\mathbb{C}^J$ is equivalent to the boundedness of the function $\Phi_1(\eta, \mathbf{z})$ for all $\eta \in \mathbb{R}^1$. Substituting $\mathbf{a} = \mathbf{a}(\eta)$ and $\mathbf{b} = \mathbf{0}$ in Lemma 2.1.6, Chapter 2, we see that this boundedness, in turn, is equivalent to conditions 1 and 2 of the present lemma. In addition, we have

$$\lambda_1(\eta) = \sup_{\mathbf{z}\in\mathbb{C}^J} \Phi_1(\eta, \mathbf{z}) = \inf_{\{\beta_\alpha(\eta)\}} \left[(\mathfrak{B}\mathbf{x}_0(\eta), \mathbf{x}_0(\eta))_\mu + \sum_{\alpha=1}^{N} |\beta_\alpha(\eta)|^2 \right]. \tag{3.1.9}$$

Thus, (3.1.8) follows from Lemma 3.1.1 (with $\mathbb{B} = \mathbb{C}^J$) and (3.1.9). □

Suppose that the matrix $\mathfrak{B}$, the vectors $\mathbf{c}_\alpha$, $\mathbf{d}_\alpha$, and the vector-function $\mathbf{a}(\eta)$ are the same as in Lemma 3.1.2. Let $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \mathbb{C}^J$, and let

$$\mathbb{B} = \{\mathbf{z} : (\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\nu = 0, \quad \alpha = 1, \dots, N\}. \tag{3.1.10}$$

For $\mathbf{z} \in \mathbb{B}$ we set

$$\Phi_{10}(\eta, \mathbf{z}) = |(\mathbf{a}(\eta), \mathbf{x})_\mu|^2 [(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu]^{-1}, \tag{3.1.11}$$

$$\lambda_{10}(\eta) = \sup_{\mathbf{z}\in\mathbb{B}} \Phi_{10}(\eta, \mathbf{z}). \tag{3.1.12}$$

Lemma 3.1.3. *The function* $\Phi_{10}(\eta, \mathbf{z})$ *is bounded on the subspace* $\mathbb{B}$ *for every* $\eta \in \mathbb{R}^1$ *if and only if there exist functions* $\beta_\alpha(\eta) \in L^2(\mathbb{R}^1)$ *satisfying conditions* 1 *and* 2 *of Lemma* 3.1.2*. If these conditions are satisfied and* $\mathbf{x}_0(\eta)$ *is an arbitrary solution of* (3.1.7)*, then*

$$\lambda_{10}(\eta) = \inf_{\{\beta_\alpha(\eta)\}} (\mathfrak{B}\mathbf{x}_0(\eta), \mathbf{x}_0(\eta))_\mu, \tag{3.1.13}$$

where the infimum is taken over all $\{\beta_\alpha(\eta)\}$ *satisfying conditions* 1 *and* 2 *of Lemma* 3.1.2.

Proof. All statements of this lemma are essentially contained in Lemma 2.1.18, Chapter 2. It is only necessary to replace in that lemma $\mathbf{a}$ by $\mathbf{a}(\eta)$, set $\mathbf{b} = \mathbf{0}$, and notice that the boundedness of the function (3.1.11) on the subspace (3.1.10) is equivalent to the boundedness of the function $\Phi_1(\eta, \mathbf{z})$, defined by (3.1.4), on the space $\mathbb{C}^\mu \times \mathbb{C}^\nu$. □

Lemmas 3.1.1 and 3.1.3 imply immediately the following assertion.

Lemma 3.1.4. *Let the function $\Phi_{10}(\eta, \mathbf{z})$ and the subspace $\mathbb{B}$ be defined by (3.1.11) and (3.1.10), respectively, and let*

$$\Psi_{10}(\mathbf{z}) = \int_{-\infty}^{\infty} \Phi_{10}(\eta, \mathbf{z}) d\eta, \qquad \mathbf{z} \in \mathbb{B}. \tag{3.1.14}$$

The function (3.1.14) is bounded on the subspace $\mathbb{B}$ if and only if there exist functions $\beta_\alpha(\eta) \in L^2(\mathbb{R}^1)$ $(1 \leqslant \alpha \leqslant N)$ satisfying conditions 1 and 2 of Lemma 3.1.2. If these conditions are satisfied, $\mathbf{x}_0(\eta)$ is an arbitrary solution of (3.1.7), and

$$\Lambda_{10} = \sup_{\mathbf{z} \in \mathbb{B}} \Psi_{10}(\mathbf{z}), \tag{3.1.15}$$

then

$$r^{-1}\Lambda_{10} \leqslant \int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\eta)\}} (\mathfrak{B}\mathbf{x}_0(\eta), \mathbf{x}_0(\eta))_\mu \, d\eta \leqslant r\Lambda_{10}. \tag{3.1.16}$$

(*Here, r is an integer such that $0 < r \leqslant J$, and the infimum is taken over all systems $\{\beta_\alpha\}$ satisfying conditions 1 and 2 of Lemma 3.1.2*).

3.1.2 The simplest lower bound for the constant Λ

In this subsection we obtain the lower bound (3.1.17) for the constant Λ from inequality (3.1.1). It is a direct consequence of inequality (3.1.1). It can be also regarded as the first natural restriction on the class of operators R for which (3.1.1) holds. In particular, the polynomial $R(\tau)$ must satisfy condition (3.1.23) below.

Lemma 3.1.5.[4] *If for some $\Lambda < \infty$ the estimate (3.1.1) holds for all $u \in C_0^\infty(0, +\infty)$, then*

$$\Lambda \geqslant \sup \frac{|R(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2}. \tag{3.1.17}$$

Proof. We substitute in (3.1.1) $u(t) = v(t+a)$, where $v \in C_0^\infty(\mathbb{R}^1)$ and the constant $a \in \mathbb{R}^1$ satisfies the condition $\operatorname{supp} v \cap (-\infty, a) = \emptyset$. Since $Q_\alpha\,(-\mathrm{i}\, d/dt)\, u|_{t=0} = 0$

[4]Cf. Lemma 1.1.5, Chapter 1.

($\alpha = 1, \dots, N$), we have

$$\int\limits_{-\infty}^{\infty} |R\,(-\mathrm{i}\,d/dt)\,v|^2\,dt \leqslant \Lambda \int\limits_{-\infty}^{\infty} \sum_{j=1}^{m} \left|P_j\,(-\mathrm{i}\,d/dt)\,v\right|^2 dt, \quad v \in C_0^\infty(\mathbb{R}^1). \tag{3.1.18}$$

Inequality (3.1.17) is a trivial consequence of (3.1.18). □

3.1.3 Reduction of the estimates for ordinary differential operators on the semi-axis to variational problems in a finite-dimensional space

In this subsection we show that (3.1.1) is equivalent to inequalities (3.1.26) and (3.1.27), whereas (3.1.1′) is equivalent to inequalities (3.1.36) and (3.1.27).

Let us define the dimensions μ, ν, the vectors $\mathbf{c}_\alpha$, $\mathbf{d}_\alpha$, the matrix $\mathfrak{B}$ and the vector-function $\mathbf{a}(\eta)$ that appear in (3.1.26) and (3.1.36). Let R, P_j, and Q_α be the polynomials considered at the beginning of Section 3.1. Set

$$\sum_{j=1}^{m} |P_j(\tau)|^2 = H_+(\tau)H_-(\tau), \tag{3.1.19}$$

where $H_+(\tau)$ is a polynomial of degree J with roots lying in the half-plane $\operatorname{Im}\zeta \geqslant 0$, $\zeta = \tau + \mathrm{i}\sigma$, and $H_-(\tau) = \overline{H_+(\tau)}$. Let $\Pi_+(\tau)$ denote the greatest common divisor of H_+ and $P_1, \dots, P_m$, and let the leading coefficient of $\Pi_+(\tau)$ be equal to 1. Consider the factorization $\Pi_+ = \Pi_0\Pi_1$, where $\Pi_0(\tau)$ is a polynomial with real roots, while the roots of the polynomial $\Pi_1(\tau)$ are non-real. We set $H^+ = H_+/\Pi_0$.

We define the dimensions μ and ν by

$$\mu = \operatorname{ord} H^+(\tau), \qquad \nu = \operatorname{ord} \Pi_0(\tau). \tag{3.1.20}$$

Clearly, $\mu + \nu = J$.

Let ζ_ϱ ($\varrho = 1, \dots, l_1$) be the roots of the polynomial H^+, and let h_ϱ ($h_1 + \dots + h_{l_1} = \mu$) be their multiplicities. Similarly, let τ_δ ($\delta = 1, \dots, l_2$) be the roots of the polynomial Π_0, and let g_δ ($g_1 + \dots + g_{l_2} = \nu$) be their multiplicities. We define the vectors $\mathbf{c}_\alpha \in \mathbb{C}^\mu$ and $\mathbf{d}_\alpha \in \mathbb{C}^\nu$ as follows:

$$\begin{aligned} &\mathbf{c}_\alpha = \left(Q_\alpha^{(\varkappa)}(\zeta_\varrho)\right), \qquad \mathbf{d}_\alpha = \left(Q_\alpha^{(\beta)}(\tau_\delta)\right) \\ &(\varrho = 1, \dots, l_1, \quad \varkappa = 0, \dots, h_\varrho - 1, \quad \delta = 1, \dots, l_2, \\ &\ \beta = 0, \dots, g_\delta - 1, \quad \alpha = 1, \dots, N). \end{aligned} \tag{3.1.21}$$

Let $\mathfrak{B} = \left(P_{\varrho\varkappa\sigma\beta}(\zeta_\varrho, \zeta_\sigma)\right)$ be a $\mu \times \mu$ matrix, with its rows labeled by the indices ϱ, $\varkappa = \varkappa(\varrho)$ and its columns labeled by the indices σ, $\beta(\sigma)$. These indices take the values $\varrho, \sigma = 1, \dots, l_1$; $\varkappa(\varrho) = 0, \dots, h_\varrho - 1$; $\beta(\sigma) = 0, \dots, h_\sigma - 1$. The entries of this matrix are defined by

$$P_{\varrho\varkappa\sigma\beta}(\zeta_\varrho, \zeta_\sigma) = \mathrm{i} \sum_{b=0}^{\beta} \sum_{k=0}^{\varkappa} \frac{(-1)^{\varkappa-k} C_\beta^b C_\varkappa^k (\beta - b + \varkappa - k)!}{(\zeta_\varrho - \bar{\zeta}_\sigma)^{\beta-b+\varkappa-k+1}} \times \sum_{j=1}^{m} P_j^{(k)}(\zeta_\varrho) \overline{P}_j^{(b)}(\bar{\zeta}_\sigma). \tag{3.1.22}$$

Since $\mathfrak{B}$ is the matrix defined by relations (2.1.35) from Chapter 2, we have $\mathfrak{B} \geqslant 0$. Therefore, inequality (2.1.36) from Chapter 2 holds true.

Further, to define the vector-function $\mathbf{a}(\eta)$, we will assume that

$$R \equiv 0 \ (\mathrm{mod}\ \Pi_0). \tag{3.1.23}$$

This congruence follows directly from (3.1.17), which is a necessary condition for the validity of (3.1.1) for functions $u \in C_0^\infty(0, +\infty)$.

We introduce the polynomial

$$\Omega(\eta, \tau) = (\eta - \tau)^{-1} [H_+(\eta) R(\tau) - H_+(\tau) R(\eta)] \tag{3.1.24}$$

and put

$$\mathbf{a}(\eta) = \left(a_{\varrho\varkappa}(\eta)\right), \qquad a_{\varrho\varkappa}(\eta) = [H_+(\eta)]^{-1} \left.\frac{\partial^\varkappa \Omega}{\partial \tau^\varkappa}\right|_{\tau=\zeta_\varrho}, \tag{3.1.25}$$

where $\varrho = 1, \dots, l_1$ and $\varkappa = 0, \dots, h_\varrho - 1$. By (3.1.24) and (3.1.23), the components of $\mathbf{a}(\eta)$ are continuous and belong to $L^2(\mathbb{R}^1)$.

We now turn to the main results of this subsection.

Lemma 3.1.6. *The estimate (3.1.1) is valid for some $\Lambda < \infty$ if and only if the following conditions are satisfied:*

1. *The inequality*

$$\int_{-\infty}^{\infty} |(\mathbf{a}(\eta), \mathbf{x})_\mu|^2 d\eta \leqslant 2\pi\Lambda_1 \left[(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu + \sum_{\alpha=1}^{N} |(\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\nu|^2 \right] \tag{3.1.26}$$

holds true for all $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{C}^\mu \times \mathbb{C}^\nu$. Here μ, ν are the dimensions defined by (3.1.20), $\mathbf{c}_\alpha$, $\mathbf{d}_\alpha$ are the vectors defined by (3.1.21), $\mathfrak{B}$ is the matrix (3.1.22), and $\mathbf{a}(\eta)$ is the vector-function (3.1.25).

2. *The estimate*

$$\int_0^\infty |R(-\mathrm{i}\, d/dt)\, v|^2\, dt \leqslant \Lambda_2 \int_0^\infty \sum_{j=1}^{m} \left|P_j(-\mathrm{i}\, d/dt)\, v\right|^2 dt, \qquad v \in C_0^\infty(\mathbb{R}_+^1), \qquad v(p)(0) = 0 \quad (p = 0, \dots, J-1) \tag{3.1.27}$$

holds true.

If Λ, Λ_1, *and* Λ_2 *are the sharp constants in* (3.1.1), (3.1.26), *and* (3.1.27), *then* $2^{-1}\Lambda \leqslant \max(\Lambda_1, \Lambda_2) \leqslant \Lambda$.

Proof. The necessity of condition 2 and the validity of the estimate $\Lambda \geqslant \Lambda_2$ are trivial. We show necessity of condition 1.

Let $(\mathbf{x}; \mathbf{y}) = (\bar{x}_{\varrho\varkappa}; \bar{y}_{\delta\alpha}) \in \mathbb{C}^\mu \times \mathbb{C}^\nu$, and let $z(t) = x(t) + y(t)$ be a solution of the equation $H_+(-\mathrm{i}\, d/dt)\, z = 0$, where

$$x(t) = \sum_{\varrho=1}^{l_1} \sum_{\varkappa=0}^{h_\varrho - 1} x_{\varrho\varkappa} (\mathrm{i}t)^\varkappa \exp(\mathrm{i}\zeta_\varrho t), \quad t \geqslant 0; \quad x(t) = 0, \quad t < 0; \tag{3.1.28}$$

$$y(t) = \sum_{\delta=1}^{l_2} \sum_{\alpha=0}^{g_\delta - 1} y_{\delta\alpha} (\mathrm{i}t)^\alpha \exp(\mathrm{i}\tau_\delta t), \quad t \geqslant 0; \quad y(t) = 0, \quad t < 0. \tag{3.1.29}$$

Taking into account (3.1.23) and applying Lemma 1.1.7 of Chapter 1, one can construct a sequence $z_s \in C_0^\infty(\mathbb{R}^1_+)$ such that $(z_s - z)^{(p)}\big|_{t=0} = 0$ ($s = 1, 2, \dots;$ $p = 0, \dots, J-1$) and

$$\lim_{s\to\infty} \left[\int_0^\infty |R(-\mathrm{i}\, d/dt)(z_s - z)|^2\, dt + \sum_{j=1}^m \int_0^\infty \left| P_j(-\mathrm{i}\, d/dt)(z_s - z) \right|^2 dt \right] = 0.$$

Therefore, $z(t)$ satisfies inequality (3.1.1). From (3.1.29) it follows that $P_j(-\mathrm{i}\, d/dt)\, y(t) = 0$ ($j = 1, \dots, m$). On the other hand, in view of (3.1.28) one can recast equation (2.1.36) from Chapter 2 as

$$(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu = \sum_{j=1}^m \int_0^\infty \left| P_j(-\mathrm{i}\, d/dt)\, x(t) \right|^2 dt. \tag{3.1.30}$$

Consequently, we have

$$\sum_{j=1}^m \int_0^\infty \left| P_j(-\mathrm{i}\, d/dt)\, z \right|^2 dt = (\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu\,. \tag{3.1.31}$$

Using (3.1.28), (3.1.29), and (3.1.21), we get

$$Q_\alpha(-\mathrm{i}\, d/dt)\, z|_{t=0} = (\mathbf{c}_\alpha, \mathbf{x})_\mu + (\mathbf{d}_\alpha, \mathbf{y})_\nu \qquad (\alpha = 1, \dots, N). \tag{3.1.32}$$

Let $\Omega(\eta, \tau)$ be the polynomial (3.1.24). Since $H_+ \equiv 0 \pmod{H_+}$,

$$[H_+(\eta)]^{-1} \frac{\partial^\varkappa \Omega}{\partial \tau^\varkappa}\bigg|_{\tau=\zeta_\varrho} = \frac{\partial^\varkappa}{\partial \tau^\varkappa} \left[\frac{R(\tau)}{\eta - \tau} \right] \bigg|_{\tau=\zeta_\varrho} \tag{3.1.33}$$
$$(\varrho = 1, \dots, l_1; \quad \varkappa = 0, \dots, h_\varrho - 1).$$

The integral $\int\limits_{-\infty}^{\infty} |\,(\mathbf{a}(\eta),\mathbf{x})_\mu\,|^2 d\eta$ is calculated by means of the residue theorem. In doing so, we have to use (3.1.25) and (3.1.33). Then, we arrive at

$$\int\limits_{-\infty}^{\infty} |\,(\mathbf{a}(\eta),\mathbf{x})_\mu\,|^2 d\eta = 2\pi \int\limits_{0}^{\infty} |R\,(-\mathrm{i}\,d/dt)\,x(t)|^2\,dt. \tag{3.1.34}$$

Now let us substitute the function $z(t)$ in (3.1.1). It follows from (3.1.23) that $R\,(-\mathrm{i}\,d/dt)\,z(t) = R\,(-\mathrm{i}\,d/dt)\,x(t)$. Hence, inequality (3.1.1) for $z(t)$ is transformed into (3.1.26) with the constant $2\pi\Lambda$ (see (3.1.34), (3.1.31), and 3.1.32)). Therefore, $\Lambda \geqslant \Lambda_1$.

To prove the sufficiency of conditions 1 and 2, one has to use the representation (2.1.38) from Chapter 2 for an arbitrary function $u \in C_0^\infty(\mathbb{R}^1_+)$, where $x(t)$ and $y(t)$ are the functions (3.1.28) and (3.1.29), respectively. Further, condition 2 implies (see the proof of Lemma 3.1.5) the validity of (3.1.18) with $\Lambda = \Lambda_2$. This yields (3.1.17) (with Λ replaced by Λ_2) and (3.1.23). Setting $z(t) = x(t) + y(t)$, we get $R\,(-\mathrm{i}\,d/dt)\,z(t) = R\,(-\mathrm{i}\,d/dt)\,x(t)$. Therefore, taking into account (3.1.31) and (3.1.32), we can rewrite (3.1.26) in the form

$$\begin{aligned}\int\limits_0^\infty |R\,(-\mathrm{i}\,d/dt)\,z|^2\,dt \leqslant \Lambda_1 \Bigg[\sum_{j=1}^m \int\limits_0^\infty \left|P_j\,(-\mathrm{i}\,d/dt)\,z\right|^2 dt \\ + \sum_{\alpha=1}^N |Q_\alpha\,(-\mathrm{i}\,d/dt)\,z|_{t=0}|^2 \Bigg].\end{aligned} \tag{3.1.35}$$

Arguing in the same way as in the proof of necessity of condition 1, we approximate $z(t)$ by functions $z_s \in C_0^\infty(\mathbb{R}^1_+)$. Then the functions $v_s = u - z_s$ satisfy (3.1.27), and

$$\lim_{s\to\infty} \left[\int\limits_0^\infty |R\,(-\mathrm{i}\,d/dt)\,(v_s - v)|^2\,dt + \sum_{j=1}^m \int\limits_0^\infty \left|P_j\,(-\mathrm{i}\,d/dt)\,(v_s - v)\right|^2 dt \right] = 0.$$

Hence, the function v in representation (2.1.38), Chapter 2, satisfies also inequality (3.1.27). Since

$$\begin{aligned}&\sum_{j=1}^m \int\limits_0^\infty P_j\,(-\mathrm{i}\,d/dt)\,v\,\overline{P_j\,(-\mathrm{i}\,d/dt)\,[(\mathrm{i}t)^\varkappa \exp(\mathrm{i}\zeta_\varrho t)]}dt \\ &= \int\limits_0^\infty v(t) \sum_{j=1}^m |P_j|^2\,(-\mathrm{i}\,d/dt)\,[(\mathrm{i}t)^\varkappa \exp(\mathrm{i}\zeta_\varrho t)]dt = 0\end{aligned}$$

($\varrho = 1, \ldots, l_1;\ \varkappa = 0, \ldots, h_\varrho - 1$), we have

$$\int_0^\infty \sum_{j=1}^m \left|P_j\,(-\mathrm{i}\,d/dt)\,u\right|^2 dt = \int_0^\infty \sum_{j=1}^m \left|P_j\,(-\mathrm{i}\,d/dt)\,v\right|^2 dt + \int_0^\infty \sum_{j=1}^m \left|P_j\,(-\mathrm{i}\,d/dt)\,z\right|^2 dt.$$

It is also evident that $Q_\alpha\,(-\mathrm{i}\,d/dt)\,u|_{t=0} = Q_\alpha\,(-\mathrm{i}\,d/dt)\,z|_{t=0}$, ($\alpha = 1, \ldots, N$). Combining (3.1.27) and (3.1.35), we obtain (3.1.1) with $\Lambda \leqslant 2\max(\Lambda_1, \Lambda_2)$. □

An analogous statement is valid for functions satisfying homogeneous boundary conditions.

Lemma 3.1.7. *The estimate* (3.1.1′) *is true for some* $\Lambda < \infty$ *if and only if condition 2 of Lemma* 3.1.6 *is satisfied and the inequality*

$$\int_{-\infty}^{\infty} |\,(\mathbf{a}(\eta), \mathbf{x})_\mu\,|^2 d\eta \leqslant 2\pi\Lambda_{10}\,(\mathfrak{B}\mathbf{x}, \mathbf{x})_\mu \tag{3.1.36}$$

holds for all $\mathbf{z} = (\mathbf{x}; \mathbf{y}) \in \mathbb{B}$, *where* $\mathbb{B}$ *is the subspace* (3.1.10). *Here,* $\mathbf{c}_\alpha$ *and* $\mathbf{d}_\alpha$ *are the vectors defined by* (3.1.21), $\mathfrak{B}$ *is the matrix* (3.1.22), *and* $\mathbf{a}(\eta)$ *is the vector-function definded by* (3.1.25). *If* Λ_0, Λ_{10}, *and* Λ_2 *are the sharp constants in the estimates* (3.1.1′), (3.1.36), *and* (3.1.27), *respectively, then* $2^{-1}\Lambda_0 \leqslant \max(\Lambda_{10}, \Lambda_2) \leqslant \Lambda_0$.

The proof of this lemma is a verbatim repetition of the proof of Lemma 3.1.6, where Λ and Λ_1 are replaced by Λ_0 and Λ_{10}, respectively.

3.1.4 Necessary and sufficient conditions for the validity of inequalities (3.1.1) and (3.1.1′)

We now turn to the main results of this section.

Lemma 3.1.8. *For any polynomial* $D(\eta, \tau)$ *of the variable* $\tau \in \mathbb{R}^1$ (*which depends on a parameter* $\eta \in \mathbb{R}^1$) *such that* $D(\eta, \tau) \equiv 0\ (\mathrm{mod}\,\Pi_+(\tau))$ *and* $\operatorname{ord} D \leqslant J - 1$ *for all* $\eta \in \mathbb{R}^1$ *there exist uniquely determined polynomials* (*of* τ) $D_j(\eta, \tau)$ *with* $\operatorname{ord} D_j \leqslant J - 1$ ($j = 1, \ldots, m$) *that satisfy for all* $\eta \in \mathbb{R}^1$ *the following conditions:*

$$\overline{D_j(\eta, \tau)} \equiv 0\ (\mathrm{mod}\,\Pi_+(\tau)) \qquad (j = 1, \ldots, m); \tag{3.1.37}$$

$$\begin{gathered} P_l(\tau)\overline{D_j(\eta, \tau)} \equiv P_j(\tau)\overline{D_l(\eta, \tau)}\ (\mathrm{mod}\,\Pi_+(\tau)H_+(\tau)) \\ (l \neq j; \quad l, j = 1, \ldots, m); \end{gathered} \tag{3.1.38}$$

$$D(\eta,\tau)H_-(\tau) = \sum_{j=1}^{m} P_j(\tau)D_j(\eta,\tau). \tag{3.1.39}$$

(*In the case* $m = 1$ *condition* (3.1.38) *is omitted.*)

This lemma is a natural generalization of Lemma 2.1.1, Chapter 2 to polynomials depending on a parameter.

Theorem 3.1.9. *The estimate* (3.1.1) *is true for some* $\Lambda < \infty$ *if and only if* $\sup \frac{|R(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2} < \infty$ *and there exist functions* $\beta_\alpha(\eta) \in L^2(\mathbb{R}^1)$ *such that*

$$D(\eta,\tau) \stackrel{\text{def}}{=} [H_+(\eta)]^{-1}\Omega(\eta,\tau) - \sum_{\alpha=1}^{N} \beta_\alpha(\eta)Q_\alpha(\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\tau)) \tag{3.1.40}$$

for all $\eta \in \mathbb{R}^1$. *Here* $\Omega(\eta,\tau)$ *is the polynomial defined by* (3.1.24). *The sharp constant* Λ *in inequality* (3.1.1) *satisfies the estimates*

$$\begin{aligned} c_1\Lambda \leqslant \int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\eta)\}} \left\{ \sum_{\alpha=1}^{N} |\beta_\alpha(\eta)|^2 + \int_{-\infty}^{\infty} \frac{\sum_{j=1}^m |D_j(\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2} d\tau \right\} d\eta \\ + \sup \left(\frac{|R(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2} \right) \leqslant c_2\Lambda, \end{aligned} \tag{3.1.41}$$

where the polynomials $D_j(\eta,\tau)$ *of the variable* τ *with* $\operatorname{ord} D_j \leqslant J-1$ $(j = 1,\dots,m)$ *satisfy conditions* (3.1.37)–(3.1.39)*, and the infimum is taken over all systems* $\{\beta_\alpha(\eta)\}$ *figuring in* (3.1.40).

Proof. Necessity. The necessity of the first condition of this theorem follows from Lemma 3.1.5. We show that the necessity of the second condition follows from Lemmas 3.1.2 and 3.1.6. Let Ψ_1 be the function defined in (3.1.4) w.r.t. the vectors (3.1.21), the matrix (3.1.22), and the vector-function (3.1.25). By Lemma 3.1.6, Ψ_1 is bounded on $\mathbb{C}^J$. Therefore, there exist functions $\beta_\alpha(\eta) \in L^2(\mathbb{R}^1)$ that satisfy conditions 1 and 2 of Lemma 3.1.2. On the other hand, for all $\eta \in \mathbb{R}^1$ these conditions are equivalent to (3.1.40). To establish this, it suffices to replace the polynomial $R(\tau)$ by the polynomials (3.1.24) and the numbers β_α by the functions $\beta_\alpha(\eta)$ in the proof of the necessity of the conditions of Theorem 2.1.17, Chapter 2.

Sufficiency. Condition 1 of Lemma 3.1.6 follows from Lemma 3.1.2 and the second condition of the theorem. We show that, under the assumptions of the theorem, condition 2 of Lemma 3.1.6 is also satisfied.

Let $v \in C_0^\infty(\mathbb{R}^1_+)$ and $v^{(p)}(0) = 0$ $(p = 0,\dots,J-1)$. We extend v to the whole $\mathbb{R}^1$ by setting $v(t) = 0$ for $t < 0$. Since $\operatorname{ord} R$, $\operatorname{ord} P_j \leqslant J$ and $v^{(p)}(0) = 0$ for $p \leqslant J-1$, we have

$$F_{t\to\tau}[R(-\mathrm{i}\, d/dt)\, v] = R(\tau)F_{t\to\tau}v, \quad F_{t\to\tau}\left[P_j(-\mathrm{i}\, d/dt)\, v\right] = P_j(\tau)F_{t\to\tau}v.$$

Therefore, the inequality

$$\int_{-\infty}^{\infty} |F_{t\to\tau}[R(-\mathrm{i}\,d/dt)\,v]|^2\,d\tau \leqslant \sup \frac{|R(\tau)|^2}{\sum\limits_{j=1}^{m} |P_j(\tau)|^2} \int_{-\infty}^{\infty} \left|F_{t\to\tau}\left[P_j(-\mathrm{i}\,d/dt)\right]v\right|^2 d\tau$$

is valid. From here and Parseval's identity we obtain (3.1.27) with $\Lambda_2 \leqslant \sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^{m}|P_j(\tau)|^2}\right)$.

Estimates of the sharp constant Λ. Let Λ_2 be the sharp constant in (3.1.27). It is not difficult to show that $\Lambda_2 \geqslant \sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^{m}|P_j(\tau)|^2}\right)$. To do this, it suffices to apply to (3.1.27) the arguments from the proof of Lemma 3.1.5. Taking into account the above-established estimate for Λ_2, we find that

$$\Lambda_2 = \sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^{m}|P_j(\tau)|^2}\right). \tag{3.1.42}$$

Let $\beta_\alpha(\eta)$ be the functions figuring in (3.1.40), and let $\mathbf{x}_0(\eta)$ be a solution of (3.1.7), which is written for the vectors (3.1.21), the vector-function (3.1.24) and the matrix (3.1.22). Then, in accordance with equation (2.1.60) from Chapter 2, we obtain:

$$(\mathfrak{B}\mathbf{x}_0(\eta), \mathbf{x}_0(\eta))_\mu = (2\pi)^{-1} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{m} |P_j(\tau)|^2\right)^{-1} \sum_{j=1}^{m} |D_j(\eta,\tau)|^2 d\tau, \tag{3.1.43}$$

where $D_j(\eta,\tau)$ are the polynomials of τ discussed in Lemma 3.1.8.

Therefore, (3.1.41) follows from Lemma 3.1.6, equations (3.1.42)–(3.1.43), and the estimates (3.1.8). □

Finally, we present a result on inequalities for functions satisfying homogeneous boundary conditions.

Theorem 3.1.10. *The estimate* (3.1.1′) *holds true for some* $\Lambda_0 < \infty$ *if and only if conditions of Theorem* 3.1.9 *are satisfied. The sharp constant* Λ_0 *in inequality* (3.1.1′) *satisfies the estimates*

$$\begin{aligned} c_1\Lambda_0 \leqslant \int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\eta)\}} \left\{ \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m}|D_j(\eta,\tau)|^2}{\sum_{j=1}^{m}|P_j(\tau)|^2} d\tau \right\} d\eta \\ + \sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^{m}|P_j(\tau)|^2}\right) \leqslant c_2\Lambda_0, \end{aligned} \tag{3.1.44}$$

where the polynomials D_j *and the functions* β_α *are the same as in Theorem* 3.1.9.

Proof. To prove this assertion, it suffices to make several changes in the proof of Theorem 3.1.9. The role of Lemma 3.1.6 is now played by Lemma 3.1.7. Inequality (3.1.36) is equivalent to the boundedness of the function Ψ_{10}, defined by (3.1.14), on the subspace (3.1.10). Therefore, the references to Lemma 3.1.2 in the proof of Theorem 3.1.9 should be replaced by references to Lemma 3.1.4. □

3.1.5 Inequalities for functions without boundary conditions

In this subsection, we consider two estimates for functions $u \in C_0^\infty(\mathbb{R}_+^1)$ without boundary conditions: inequalities (3.1.45) and (3.1.48). A criterion for the validity of the first of these inequalities follows directly from Theorem 3.1.9, if we set $Q_\alpha(\tau) = 0$ $(\alpha = 1, \dots, N)$ there. The second inequality is a special case of the first one for $m = 1$, which is related to polynomials with the roots lying in the lower complex half-plane.

Theorem 3.1.11. *The estimate*

$$\int_0^\infty |R(-\mathrm{i}\,d/dt)\,u|^2\,dt \leqslant \Lambda \int_0^\infty \sum_{j=1}^m \left|P_j(-\mathrm{i}\,d/dt)\,u\right|^2 dt, \quad u \in C_0^\infty(\mathbb{R}_+^1) \tag{3.1.45}$$

holds for some $\Lambda < \infty$ *if and only if*

$$\sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2}\right) < \infty$$

and the congruence

$$\Omega(\eta, \tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\tau)) \tag{3.1.46}$$

holds for all $\eta \in \mathbb{R}^1$*. Here* Ω *is the polynomial defined by* (3.1.24)*. The sharp constant* Λ *in* (3.1.45) *satisfies the estimate*

$$c_1\Lambda \leqslant \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\sum_{j=1}^m |D_j(\eta,\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2}\,d\tau\,d\eta + \sup\left(\frac{|R(\tau)|^2}{\sum_{j=1}^m |P_j(\tau)|^2}\right) \leqslant c_2\Lambda, \tag{3.1.47}$$

where the polynomials D_j *are constructed for the polynomial* $D(\eta, \tau) = [H_+(\eta)]^{-1}\Omega(\eta, \tau)$ *in accordance with Lemma* 3.1.8*.*

We now formulate a criterion for the validity of (3.1.48). Applications of this result will be given in Sections 3.2 and 3.3.

Proposition 3.1.12. *Let* $K(\tau)$, $L(\tau)$ *be polynomials of the variable* $\tau \in \mathbb{R}^1$ *with complex coefficients such that* $\operatorname{ord} K(\tau) \leqslant \operatorname{ord} L(\tau)$*, and let all the roots of the polynomial*

$L(\tau)$ lie in the half-plane $\operatorname{Im}\zeta < 0$, $\zeta = \tau + i\sigma$. *Then*

$$\int_0^\infty |K(-i\,d/dt)\,u|^2\,dt \leqslant \Lambda \int_0^\infty |L(-i\,d/dt)\,u|^2\,dt \tag{3.1.48}$$

for all $u \in C_0^\infty(\mathbb{R}^1_+)$. *The sharp constant* Λ *in* (3.1.48) *is equal to* $\sup|K(\tau)/L(\tau)|^2$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^1_+)$, let $f(t) = L(-i\,d/dt)\,u$ for $t \geqslant 0$, $f(t) = 0$ for $t < 0$, and let $v(t) = F^{-1}_{\tau\to t}\,(F_{t\to\tau} f/L(\tau))$. Since the roots of the polynomial $L(\tau)$ lie in the half-plane $\operatorname{Im}\zeta < 0$, we have $u(t) = v(t)$ for $t \geqslant 0$. Therefore,

$$K(-i\,d/dt)\,u = F^{-1}_{\tau\to t}\left(\frac{K(\tau)}{L(\tau)} F_{t\to\tau} f\right), \qquad t \geqslant 0. \tag{3.1.49}$$

Since $\operatorname{ord} K \leqslant \operatorname{ord} L$ and the roots of the polynomial $L(\tau)$ are not real, we conclude that

$$\sup|K(\tau)/L(\tau)| < \infty.$$

It follows from (3.1.49) that for all $u \in C_0^\infty(\mathbb{R}^1)$ the estimate (3.1.48) holds and $\Lambda = \sup|K(\tau)/L(\tau)|^2$.

The opposite inequality for Λ is obtained by applying Lemma 3.1.5 to the estimate (3.1.48). □

3.2 Estimates in a half-space. Necessary and sufficient conditions

In this section we formulate theorems on necessary and sufficient conditions for the validity of the estimates (3.0.1), (3.0.2), and (3.0.1′). Furthermore, we derive a number of corollaries.

3.2.1 Necessary and sufficient conditions for the validity of the estimates (3.0.1), (3.0.2), and (3.0.1′)

Let $R(\xi;\tau)$, $P_j(\xi;\tau)$, and $Q_\alpha(\xi;\tau)$ be the polynomials of τ considered in the Introduction. In this subsection we formulate and prove the validity conditions for the estimates (3.0.1), (3.0.2), and (3.0.1′).

Following Section 3.1, we set

$$\sum_{j=1}^m |P_j(\xi;\tau)|^2 = H_+(\xi;\tau)H_-(\xi;\tau), \tag{3.2.1}$$

where $H_+(\xi;\tau) = \sum_{s=0}^{J} h_s(\xi)\tau^{J-s}$ is a polynomial with the roots lying in the half-plane $\operatorname{Im}\zeta \geqslant 0$, $\zeta = \tau + i\sigma$, and $H_-(\xi;\tau) = \overline{H_+(\xi;\tau)}$. We put also

$$Z = \{\xi \in \mathbb{R}^{n-1} : h_0(\xi) = 0\}.$$

Furthermore, we assume that on the full-measure set $Y \subset \mathbb{R}^{n-1} \setminus Z$ we have $J \geqslant 1$, $\operatorname{mes}_{n-1} Z = 0$;[5] $\operatorname{ord} R(\xi;\tau) \leqslant J$; and $\operatorname{ord} Q_\alpha(\xi;\tau) \leqslant J-1$ $(\alpha = 1,\dots,N)$. For each point $\xi \in Y$ we denote by $\Pi_+(\xi;\tau)$ the greatest common divisor of $H_+(\xi;\tau)$ and $P_1(\xi;\tau),\dots,P_m(\xi;\tau)$. Let the leading coefficient of the polynomial $\Pi_+(\xi;\tau)$ be equal to 1.

Before we give the criteria for the validity of (3.0.1), (3.0.2), and (3.0.1′), we formulate a lemma that follows directly from Lemma 3.1.8.

Lemma 3.2.1. *For each polynomial $D(\xi;\eta,\tau)$ of the variable $\tau \in \mathbb{R}^1$ (depending on a parameter $(\xi;\eta) \in \mathbb{R}^n$) such that $D(\xi;\eta,\tau) \equiv 0 \pmod{\Pi_+(\xi;\tau)}$ and $\operatorname{ord} D \leqslant J-1$ for all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$, there exist uniquely determined polynomials (of τ) $D_j(\xi;\eta,\tau)$, $\operatorname{ord} D_j(\xi;\eta,\tau) \leqslant J-1$ $(j = 1,\dots,m)$, which satisfy conditions (3.0.6)–(3.0.8) for all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$. (Condition (3.0.8) can be omitted in the case $m = 1$).*

We now turn to the main results.

Theorem 3.2.2. *The estimate (3.0.1) holds true if and only if the following conditions are satisfied:*

1. *There exist such functions $\beta_\alpha(\xi;\eta)$ $(\alpha = 1,\dots,N)$ that*

$$\int_{-\infty}^{\infty} \sum_{\alpha=1}^{N} |\beta_\alpha(\xi;\eta)|^2 d\eta < \infty$$

and the congruence

$$\begin{aligned} D(\xi;\eta,\tau) &\overset{\text{def}}{=} [H_+(\xi;\eta)]^{-1}\Omega(\xi;\eta,\tau) \\ &\quad - \sum_{\alpha=1}^{N} \beta_\alpha(\xi;\eta) Q_\alpha(\xi;\tau) \equiv 0 \pmod{\Pi_+(\xi,\tau)} \end{aligned} \tag{3.2.2}$$

is valid for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\eta \in \mathbb{R}^1$. (Here, $\Omega(\xi;\eta,\tau)$ is the polynomial (3.0.3)).

[5] This condition is satisfied, for example, if $h_0(\xi)$ is a polynomial of the variable $\xi \in \mathbb{R}^{n-1}$ (cf. with Remark 1.2.1, Chapter 1).

2. *For almost all* $(\xi;\tau)\in\mathbb{R}^n$ *we have*

$$\int\limits_{-\infty}^{\infty}\inf_{\{\beta_\alpha(\xi;\eta)\}}\left\{\sum_{\alpha=1}^{N}|\beta_\alpha(\xi;\eta)|^2+\int\limits_{-\infty}^{\infty}\frac{\sum_{j=1}^m|D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^m|P_j(\xi;\tau)|^2}d\tau\right\}d\eta \\ +\frac{|R(\xi;\tau)|^2}{\sum_{j=1}^m|P_j(\xi;\tau)|^2}\leqslant\frac{\text{const}}{B(\xi)}, \tag{3.2.3}$$

where $D_j(\xi;\eta,\tau)$ *are polynomials of the variable* $\tau\in\mathbb{R}^1$ *(depending on a parameter* $(\xi;\eta)\in\mathbb{R}^n$*), which satisfy the conditions of Lemma 3.2.1 for all* $\eta\in\mathbb{R}^1$ *and almost all* $\xi\in\mathbb{R}^{n-1}$*, and the infimum is taken over all systems* $\{\beta_\alpha\}$ *figuring in (3.2.2).*

Proof. Necessity. Consider for an arbitrary $A>0$ the following "cut-off" function: $B_A(\xi)=B(\xi)$, if $B(\xi)\leqslant A$, and $B_A(\xi)=A$, if $B(\xi)>A$. In accordance with the definition of the norm $\|\cdot\|_{B^{1/2}}$, the estimate

$$\|R(D)u\|^2_{B_A^{1/2}}\leqslant C\left(\sum_{j=1}^{m}\|P_j(D)u\|^2+\sum_{\alpha=1}^{N}\langle\!\langle Q_\alpha(D)u\rangle\!\rangle^2\right), \\ u\in C_0^\infty(\mathbb{R}^n_+), \tag{3.2.4}$$

follows from (3.0.1) for any $A>0$.

Let $Y\subset\mathbb{R}^{n-1}\setminus Z$ be the full-measure set defined above, and let $\xi\in Y$. Put in (3.2.4) $u(x;t)=h^{(1-n)/2}\varphi(x/h)\,e^{\mathrm{i}x\xi}v(t)$, where $h>0$ is a parameter, $\varphi\in C_0^\infty(\mathbb{R}^{n-1})$, and $v\in C_0^\infty(\mathbb{R}^1_+)$. Since $B_A(\xi)$ is a bounded function, and the coefficients of the polynomials R, P_j, and Q_α are measurable locally bounded functions growing no faster than some power of $|\xi|$ as $|\xi|\to\infty$, we conclude, after passing to the limit as $h\to\infty$ and dividing all terms by $\int_{\mathbb{R}^{n-1}}|\varphi(x)|^2dx$, that the estimate

$$\int\limits_0^\infty|R(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt\leqslant\frac{C}{B_A(\xi)}\Bigg[\int\limits_0^\infty\sum_{j=1}^m\left|P_j(\xi;-\mathrm{i}\,d/dt)\,v\right|^2dt \\ +\sum_{\alpha=1}^N|Q_\alpha(\xi;-\mathrm{i}\,d/dt)\,v|_{t=0}|^2\Bigg], \\ v\in C_0^\infty(\mathbb{R}^1_+), \tag{3.2.5}$$

holds true for almost all $\xi\in\mathbb{R}^{n-1}$. Thus, the necessity of all conditions of Theorem 3.2.2 follows from Theorem 3.1.9.

Sufficiency. According to Theorem 3.1.9, conditions 1 and 2 imply (3.2.5) a.e. in $\mathbb{R}^{n-1}$. We substitute in (3.2.5) the function $v_\xi(t)=\hat{u}(\xi;t)$ with $u\in C_0^\infty(\mathbb{R}^n_+)$. Multiplying both sides of the obtained inequality by $B(\xi)$ and integrating the result over $\mathbb{R}^{n-1}$, we arrive at (3.0.1). ☐

Theorem 3.2.3. *The estimate* (3.0.2) *is valid if and only if condition* 1 *of Theorem* 3.2.2 *is satisfied and the inequality*

$$\int_{-\infty}^{\infty} \inf_{\{\beta_\alpha(\xi;\eta)\}} \left\{ \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} d\tau \right\} d\eta + \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} \leqslant \frac{\text{const}}{B(\xi)} \tag{3.2.6}$$

holds true for almost all $(\xi;\tau) \in \mathbb{R}^n$. *Here, the polynomials* D_j *and the functions* β_α *are the same as in Theorem* 3.2.2.

This theorem is deduced from Theorem 3.1.10 in the same way as Theorem 3.2.2 from Theorem 3.1.1.

In the same way, we derive from Theorem 3.1.9 the followign assertion:

Theorem 3.2.4. *The estimate*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.2.7}$$

holds true if and only if

1. *for almost all* $\xi \in \mathbb{R}^{n-1}$ *and all* $\eta \in \mathbb{R}^1$ *the polynomial* (3.0.3) *satisfies the congruence*
$$\Omega(\xi;\eta,\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)); \tag{3.2.8}$$
2. *the inequality*
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} d\tau\, d\eta + \frac{|R(\xi;\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} \leqslant \frac{\text{const}}{B(\xi)} \tag{3.2.9}$$
holds true for almost all $(\xi;\tau) \in \mathbb{R}^n$. *Here,* $D_j(\xi;\eta,\tau)$ *are polynomials (of* τ*) satisfying for all* $\tau \in \mathbb{R}^1$ *and almost all* $\xi \in \mathbb{R}^{n-1}$ *the conditions of Lemma* 3.2.1 *with* $D(\xi;\eta,\tau) = [H_+(\xi;\eta)]^{-1}\Omega(\xi;\eta,\tau)$.

3.2.2 On the minimal number and algebraic properties of the boundary operators; formulas for $\beta_\alpha(\xi;\eta)$

Let $\mathscr{M}(\xi;\tau)$ be the greatest common divisor of the polynomials $R(\xi;\tau)$ and $\Pi_+(\xi;\tau)$, let the leading coefficients of these polynomials be equal to 1, and let

$$\dot{\Pi}_+(\xi;\tau) = \Pi_+(\xi;\tau)/\mathscr{M}(\xi;\tau); \qquad N(\xi) = \operatorname{ord} \dot{\Pi}_+(\xi;\tau).$$

In this subsection, we show that the number N of the boundary operators $Q_\alpha(D)$, for which (3.0.1) (or (3.0.2)) takes place, cannot be less than $N(\xi)$, and among the polynomials $Q_\alpha(\xi;\tau)$ there are at least $N(\xi)$ linearly independent modulo Π_+. Moreover, if $N = N(\xi)$, then the congruence $Q_\alpha(\xi;\tau) \equiv 0 \ (\mathrm{mod}\, \mathscr{M}(\xi;\tau))$ $(\alpha = 1,\dots,N)$ holds, and the functions $\beta_\alpha(\xi;\eta)$ figuring in (3.2.2) are uniquely determined by this congruence and can be represented in the form (3.2.11).

Corollary 3.2.5. *If the operators $Q_1(D), \dots, Q_N(D)$ satisfy the estimate* (3.0.1) (*or* (3.0.2)), *then the inequality $N \geqslant N(\xi)$ holds, and among the polynomials Q_α there are at least $N(\xi)$ that are linearly independent modulo Π_+ a.e. in $\mathbb{R}^{n-1}$.*

Proof. We denote by $\chi_\varrho(\xi)$ the roots of the polynomials $\dot{\Pi}_+$, Π_+ and represent these polynomials in the form

$$\dot{\Pi}_+(\xi;\tau) = \prod_{\varrho=1}^{s(\xi)} (\tau - \chi_\varrho(\xi))^{k_\varrho(\xi)}, \qquad \Pi_+(\xi;\tau) = \prod_{\varrho=1}^{s_1(\xi)} (\tau - \chi_\varrho(\xi))^{\varkappa_\varrho(\xi)}.$$

Let $s_2(\xi)$ be an integer function such that

a) $\left.\dfrac{\partial^\beta R}{\partial \tau^\beta}\right|_{\tau=\chi_\varrho} = \left.\dfrac{\partial^\beta \mathscr{M}}{\partial \tau^\beta}\right|_{\tau=\chi_\varrho} = 0$, if the following conditions hold:

1. $s_2 + 1 \leqslant \varrho \leqslant s$ and $0 \leqslant \beta \leqslant \varkappa_\varrho - k_\varrho - 1$,

2. $s + 1 \leqslant \varrho \leqslant s_1$ and $0 \leqslant \beta \leqslant \varkappa_\varrho - 1$;

b) $R(\xi;\chi_\varrho) \neq 0$, $\quad \mathscr{M}(\xi;\chi_\varrho) \neq 0$, if $1 \leqslant \varrho \leqslant s_2(\xi)$;

c) $\left.\dfrac{\partial^{\varkappa_\varrho - k_\varrho} R}{\partial \tau^{\varkappa_\varrho - k_\varrho}}\right|_{\tau=\chi_\varrho} \neq 0$, $\left.\dfrac{\partial^{\varkappa_\varrho - k_\varrho} \mathscr{M}}{\partial \tau^{\varkappa_\varrho - k_\varrho}}\right|_{\tau=\chi_\varrho} \neq 0$, if $s_2 + 1 \leqslant \varrho \leqslant s$.

(Here we put for the sake of brevity $\chi_\varrho(\xi) = \chi_\varrho$, $s(\xi) = s$, $s_1(\xi) = s_1$, $s_2(\xi) = s_2$, $k_\varrho(\xi) = k_\varrho$, $\varkappa_\varrho(\xi) = \varkappa_\varrho$. Notice also that condition b) can be omitted for $s_2(\xi) = 0$.)

Suppose that the operators $Q_\alpha(D)$ satisfy (3.0.1) (or (3.0.2)). Assume also that $s_2(\xi) > 0$.[6] Then (3.2.2) implies the validity of the equality

$$\sum_{\alpha=1}^{N} \left.\frac{\partial^\gamma Q_\alpha}{\partial \tau^\gamma}\right|_{\tau=\chi_\varrho(\xi)} \beta_\alpha(\xi;\tau) = \frac{\partial^\gamma}{\partial \tau^\gamma} \left[\frac{R(\xi;\tau)}{\eta - \tau} \right] \Bigg|_{\tau=\chi_\varrho(\xi)} \tag{3.2.10}$$

for $\gamma = 0, \dots, k_\varrho - 1$ in the case $1 \leqslant \varrho \leqslant s_2(\xi)$, as well as for $\gamma = \varkappa_\varrho(\xi) - k_\varrho(\xi), \dots, \varkappa_\varrho(\xi) - 1$ in the case $s_2(\xi) + 1 \leqslant \varrho \leqslant s(\xi)$. From properties b) and c) of the number $s_2(\xi)$ it follows that the right-hand sides of (3.2.10) are linearly independent functions of $\eta \in \mathbb{R}^1$. Since this system is solvable w.r.t. β_α for all $\eta \in \mathbb{R}^1$, we have $\operatorname{rg} \mathfrak{D}(\xi) = N(\xi)$, where $\mathfrak{D}$ is the $N(\xi) \times N$ matrix of the system (3.2.10). Hence, $N \geqslant N(\xi)$. On the other hand, if, for a given $\xi \in \mathbb{R}^{n-1}$, any subsystem of $N(\xi)$ polynomials Q_α is linearly dependent modulo Π_+, then all the minors of order $N(\xi)$ of the matrix $\mathfrak{D}(\xi)$ are obviously equal to zero. □

The above statement can be strengthened in the case $N = N(\xi)$ as follows.

[6] We leave to the reader to perform the obvious necessary changes in all arguments in the case $s_2(\xi) = 0$.

Corollary 3.2.6. *If $N = N(\xi)$ a.e. in $\mathbb{R}^{n-1}$, then conditions* 1 *of Theorems* 3.2.2 *and* 3.2.3 *are satisfied if and only if the polynomials Q_α are linearly independent modulo Π_+ for almost all $\xi \in \mathbb{R}^{n-1}$ and $Q_\alpha \equiv 0 \,(\mathrm{mod}\, \mathscr{M})$, $(\alpha = 1, \dots, N)$. The functions β_α figuring in* (3.2.2) *are uniquely determined by this congruence and satisfy the equations*

$$\beta_\alpha(\xi;\eta) = \frac{G_\alpha(\xi;\eta)}{\Pi_+(\xi;\eta)} \qquad (\alpha = 1, \dots, N), \tag{3.2.11}$$

where $G_\alpha(\xi;\eta)$ are linearly independent polynomials of τ such that $\operatorname{ord} G_\alpha < \operatorname{ord} \Pi_+$ *and* $G_\alpha \equiv 0 \,(\mathrm{mod}\, \mathscr{M})$.

Proof. Using Corollary 3.2.5, we conclude that conditions 1 of Theorems 3.2.2 and 3.2.3 imply the linear independence modulo Π_+ of the polynomials Q_α in the case $N = N(\xi)$. Moreover, the solvability of (3.2.10) is equivalent to the validity of the condition $\det \mathfrak{D}(\xi) \neq 0$, where $\mathfrak{D}$ denotes the matrix of (3.2.10). Calculating the derivatives figuring in the right-hand side of (3.2.10) and multiplying both sides of (3.2.10) by $\Pi_+(\xi;\eta)$, we find that the functions $\beta_\alpha \Pi_+$ satisfy the system of equations with the matrix $\mathfrak{D}(\xi)$ and the right-hand sides $R_{\gamma_\varrho}(\xi;\eta)$, where $R_{\gamma_\varrho}(\xi;\eta)$ are the linearly independent polynomials of η such that $\operatorname{ord} R_{\gamma_\varrho} < \operatorname{ord} \Pi_+$. Solving this system and taking into account the congruence $R_{\gamma_\varrho} \equiv 0 \,(\mathrm{mod}\, \mathscr{M})$, which follows from property a) of the number $s_2(\xi)$,[7] we obtain (3.2.11). Finally, we note that (3.2.2) ensures the validity of the congruence

$$\sum_{\alpha=1}^{N} \beta_\alpha(\xi;\eta) Q_\alpha(\xi;\tau) \equiv 0 \,(\mathrm{mod}\, \mathscr{M}(\xi;\tau)) \tag{3.2.12}$$

for all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$. But then, in view of (3.2.11), we get $Q_\alpha \equiv 0 \,(\mathrm{mod}\, \mathscr{M})$, $(\alpha = 1, \dots, N)$.

Conversely, suppose that $Q_\alpha \equiv 0 \,(\mathrm{mod}\, \mathscr{M})$, $(\alpha = 1, \dots, N)$. Suppose that the polynomials Q_α are linearly independent modulo Π_+, and let

$$\dot{H}_+ = H_+ \mathscr{M}^{-1}, \qquad \dot{Q}_\alpha = Q_\alpha \mathscr{M}^{-1}, \qquad \dot{R} = R\mathscr{M}^{-1},$$

and $\dot{\Omega}(\xi;\eta,\tau) = [\mathscr{M}(\xi;\eta)\mathscr{M}(\xi;\tau)]^{-1}\Omega(\xi;\eta,\tau)$. Hence, for all $\eta \in \mathbb{R}^1$ and for almost all $\xi \in \mathbb{R}^{n-1}$ the congruence

$$[\dot{H}_+(\xi;\eta)]^{-1}\dot{\Omega}(\xi;\eta,\tau) - \sum_{\alpha=1}^{N} \beta_\alpha(\xi;\eta)\dot{Q}_\alpha(\xi;\tau) \equiv 0 \,(\mathrm{mod}\, \Pi_+(\xi;\tau)) \tag{3.2.13}$$

is uniquely solvable and its solution $\{\beta_\alpha\}$ satisfies (3.2.12). But then (2.2.2) is also valid. The relation $\int_{-\infty}^{\infty} \sum_{\alpha=1}^{N} |\beta_\alpha(\xi;\eta)|^2 d\eta < \infty$ follows from (3.2.11). □

[7] See the proof of Corollary 3.2.5.

3.2.3 Estimates for polynomials whose τ-roots lie in the lower complex half-plane

In this subsection, we first establish Proposition 3.2.7, which is related, like Theorem 3.2.4, to the estimates of the type (3.2.7). It can be derived from Proposition 3.1.12 in the same way as Theorem 3.2.2 is derived from Theorem 3.1.9. Then, applying Proposition 3.2.7 and Theorem 3.2.4, we verify inequality (3.2.16), which will be used in Section 3.3.

Proposition 3.2.7. *Let $K(\xi;\tau)$ and $L(\xi;\tau)$ be polynomials of τ with measurable locally bounded coefficients growing no faster than a certain power of $|\xi|$ as $|\xi| \to \infty$, let $\operatorname{ord} K \leqslant \operatorname{ord} L$, and let the τ-roots of the polynomial L lie in the half-plane $\operatorname{Im}\zeta < 0$, $\zeta = \tau + i\sigma$, for almost all $\xi \in \mathbb{R}^{n-1}$. The estimate*

$$\|K(D)u\|^2_{B^{1/2}} \leqslant C\|L(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+) \tag{3.2.14}$$

is true if and only if the inequality

$$B(\xi)|K(\xi;\tau)|^2 \leqslant \mathrm{const}|L(\xi;\tau)|^2 \tag{3.2.15}$$

holds a.e. in $\mathbb{R}^{n-1}$.

Corollary 3.2.8. *Let the polynomials $K(\xi;\tau)$ and $L(\xi;\tau)$ be the same as in Proposition 3.2.7. Then the inequality*

$$\begin{aligned} &\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left|\frac{\bar{L}(\xi;\eta)K(\xi;\tau)-\bar{L}(\xi;\tau)K(\xi;\eta)}{(\eta-\tau)\bar{L}(\xi;\tau)L(\xi;\tau)}\right|^2 d\tau d\eta \\ &\leqslant \mathrm{const}\sup\left|\frac{K(\xi;\tau)}{L(\xi;\tau)}\right|^2 \end{aligned} \tag{3.2.16}$$

holds a.e. in $\mathbb{R}^{n-1}$.

Proof. In accordance with Proposition 3.2.7, condition (3.2.15) implies the estimate (3.2.14), which is a particular case of the estimate (3.2.7) for $m = 1$, $P_1 = L$, and $R = K$. Since the roots of L lie in the half-plane $\operatorname{Im}\zeta < 0$, we have $H_+ = \bar{L}$, $H_- = L$. Therefore, $\Pi_+ = 1$ and the polynomial $D_1(\xi;\eta,\tau)$, satisfying the conditions of Lemma 3.2.1, is given by the formula

$$D_1(\xi;\eta,\tau) = [(\eta-\tau)\bar{L}(\xi;\eta))]^{-1}[\bar{L}(\xi;\eta)K(\xi;\tau) - \bar{L}(\xi;\tau)K(\xi;\eta)]. \tag{3.2.17}$$

Using (3.2.17) and writing down the first term on the left-hand side of (3.2.9), we arrive at (3.2.16). □

A direct proof of (3.2.16), without resorting Theorem 3.2.4, would be apparently much more cumbersome.

3.3 Examples

In this section we consider some applications of theorems from Section 3.2.

3.3.1 The theorem of N. Aronszajn on necessary and sufficient conditions for the coercivity of a system of operators

Let $P_j(\xi;\tau)$, $j = 1, \dots, m$, be homogeneous polynomials in the variable $(\xi;\tau) \in \mathbb{R}^n$ of order J. Following N. Aronszajn [Aro54], we say that a system of operators $P_j(D)$ is *coercive* if the estimate

$$\sum_{|\alpha|=J} \|D^\alpha u\|^2 \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.3.1}$$

holds. In this subsection we show (Proposition 3.3.1) that the well-known result of N. Aronszajn [Aro54] on necessary and sufficient conditions for the coercivity of the system $P_j(D)$ is a consequence of Theorem 3.2.4 established above.

Proposition 3.3.1. *The estimate* (3.3.1) *holds true if and only if for all* $\xi \in \mathbb{R}^{n-1}$ *the polynomials* $P_j(\xi;\tau)$ *have no common roots* $(\xi;z) \neq 0$.

Proof. It is obvious that (3.3.1) is the special case of the estimate (3.2.7) corresponding to the weight $B(\xi) = 1$ and the polynomial $R(\xi;\tau) = (\tau + \mathrm{i}|\xi|)^J$. From the definition of $\Omega(\xi;\eta,\tau)$ (see (3.0.3)) it follows that (3.2.8) is equivalent to the congruence

$$R(\xi;\tau) \equiv 0 \ (\mathrm{mod}\, \Pi_+(\xi;\tau)). \tag{3.3.2}$$

Since $R(\xi;\tau) = (\tau + \mathrm{i}|\xi|)^J$, we obtain that (3.3.2) is valid if and only if $\Pi_+(\xi;\tau) = 1$. It follows from the definition of $\Pi_+(\xi;\tau)$ that the equation $\Pi_+(\xi;\tau) = 1$ is satisfied if and only if for any $\xi \in \mathbb{R}^{n-1}$ the polynomials $P_j(\xi;\tau)$ have no common roots $(\xi;z) \neq 0$ with $\operatorname{Im} z \geqslant 0$. Since these polynomials are homogeneous w.r.t. $(\xi;\tau) \in \mathbb{R}^n$, they have, together with each common root $(\xi;z)$, also the common root $(-\xi;-z)$. Therefore, for all $\xi \in \mathbb{R}^{n-1}$ the polynomials $P_j(\xi;\tau)$ have no common roots $(\xi;z) \neq 0$ with $\operatorname{Im} z \geqslant 0$ if and only if they have no common roots $(\xi;z) \neq 0$ for any $\xi \in \mathbb{R}^{n-1}$.

To complete the proof, we show that condition 2 of Theorem 3.2.4 is automatically satisfied.

Suppose that the polynomials $P_j(\xi;\tau)$ have no common roots $(\xi;z) \neq 0$ for all $\xi \in \mathbb{R}^{n-1}$. Then $\Pi_+(\xi;\tau) = 1$. It is also evident that

$$|(\tau + \mathrm{i}|\xi|^J)|^2 \leqslant \mathrm{const} \sum_{j=1}^{m} |P_j(\xi;\tau)|^2.$$

Therefore (see Theorem 3.1.11), the estimate

$$\sum_{k=0}^{J} \left\| \frac{d^k v}{dt^k} \right\|^2 \leqslant \Lambda(\theta) \sum_{j=1}^{m} \left\| P_j\,(\theta; -\mathrm{i}\, d/dt)\, v \right\|^2 \tag{3.3.3}$$

is satisfied for all $v \in C_0^\infty(\mathbb{R}^1_+)$. Here $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R}^1_+)$ and $\theta \in S^{n-2}$. Let $\Lambda(\theta)$ be the smallest constant in (3.3.3). Since the estimate

$$\left| \sum_{j=1}^{m} \left\| P_j\,(\theta_1; -\mathrm{i}\, d/dt)\, v \right\|^2 - \sum_{j=1}^{m} \left\| P_j\,(\theta_2; -\mathrm{i}\, d/dt)\, v \right\|^2 \right| \leqslant \mathrm{const} \sum_{k=0}^{J} \left\| \frac{d^k v}{dt^k} \right\|^2 |\theta_1 - \theta_2|$$

is obviously fulfilled, the function $1/\Lambda(\theta)$ satisfies the Lipschitz condition on S^{n-2}. On the other hand, the inequality $1/\Lambda(\theta) > 0$ holds for all $\theta \in S^{n-2}$. This means that the function $\Lambda(\theta)$ is bounded from above on S^{n-2}. Thus, (3.2.9) follows from (3.1.47). □

3.3.2 The case $m = 1$, $N = N(\xi)$ in Theorems 3.2.2, 3.2.3, and 3.2.4

In this subsection, we consider estimates of the types (3.0.1), (3.0.2) with *one* operator $\mathscr{P}(D)$ for the case when the number N of the boundary operators $Q_\alpha(D)$ coincides with the order $N(\xi)$ of the polynomial $\dot{\Pi}_+(\xi;\tau)$ (cf. with Corollaries 3.2.5, 3.2.6). It would be shown that the criteria for the validity of such estimates, obtained by the authors in [MG75], follow from Theorems 3.2.2, 3.2.3, and 3.2.4, which were proved in Section 3.2

We introduce the following notation. Let $J \geqslant 1$, and let $\mathscr{P}(\xi;\tau) = p_0(\xi)\tau^J + p_1(\xi)\tau^{J-1} + \cdots + p_J(\xi)$ be a polynomial of τ with measurable locally bounded coefficients that grow no faster than a certain power of $|\xi|$ as $|\xi| \to \infty$. Let R and Q_α $(\alpha = 1, \ldots, N)$ be the same polynomials as in Section 3.2. We set $Z = \{\xi : \xi \in \mathbb{R}^{n-1},\ p_0(\xi) = 0\}$ and assume that $\mathrm{mes}_{n-1} Z = 0$.

For $\xi \in \mathbb{R}^{n-1} \setminus Z$ we define the polynomials (of τ) $\mathscr{P}_+$, $\mathscr{P}_-$, $\mathscr{M}$, and $\dot{\mathscr{P}}_+$ as follows: $\mathscr{P}_+$ a monic polynomial whose τ-roots (counting multiplicities) coincide in the half-plane $\operatorname{Im}\zeta \geqslant 0$, $\zeta = \tau + \mathrm{i}\sigma$, with the τ-roots of the polynomial $\mathscr{P}$; $\mathscr{P}_- = \mathscr{P}/\mathscr{P}_+$; $\mathscr{M}$ is the monic greatest common divisor of R and $\mathscr{P}_+$; and $\dot{\mathscr{P}}_+ = \mathscr{P}_+/\mathscr{M}$. We also consider the polynomials $R_\pm$, $Q_{\alpha\pm}$, which are defined by the following partial fraction decompositions w.r.t. τ:

$$\frac{R}{\mathscr{P}} = \frac{R_+}{\mathscr{P}_+} + \frac{R_-}{\mathscr{P}_-} + c, \qquad c = c(\xi); \qquad \frac{Q_\alpha}{\mathscr{P}} = \frac{Q_{\alpha+}}{\mathscr{P}_+} + \frac{Q_{\alpha-}}{\mathscr{P}_-}. \tag{3.3.4}$$

It is assumed that the condition $N(\xi) = \operatorname{ord} \dot{\mathscr{P}}_+(\xi;\tau) = N$ is fulfilled for all $\xi \in \mathbb{R}^{n-1} \setminus Z$.

Proposition 3.3.2. *The estimate*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(\|\mathscr{P}(D)u\|^2 + \sum_{\alpha=1}^{N} \langle\!\langle Q_\alpha(D)u\rangle\!\rangle^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.3.5}$$

holds if and only if the following conditions are satisfied:

1. *For almost all* $\xi \in \mathbb{R}^{n-1}$ *the polynomials* Q_α *are linearly independent modulo* $\mathscr{P}_+$ *and* $Q_\alpha \equiv 0 \pmod{\mathscr{M}}$ $(\alpha = 1, \dots, N)$*;*
2. *The inequality*

$$\int_{-\infty}^{\infty} \sum_{\alpha=1}^{N} \left|\frac{G_\alpha(\xi;\eta)}{\mathscr{P}_+(\xi;\eta)}\right|^2 d\eta + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\sum_{\alpha=1}^{N} G_\alpha(\xi;\eta)Q_{\alpha-}(\xi;\tau)}{\mathscr{P}_+(\xi;\eta)\mathscr{P}_-(\xi;\tau)}\right|^2 d\tau d\eta + \left|\frac{R(\xi;\tau)}{\mathscr{P}(\xi;\tau)}\right|^2 \leqslant \frac{\mathrm{const}}{B(\xi)} \tag{3.3.6}$$

is fulfilled a.e. in $\mathbb{R}^n$. *Here* G_α *are the polynomials of* η *with* $\operatorname{ord} G_\alpha <$ $\operatorname{ord} \mathscr{P}_+$*, which are uniquely determined a.e. in* $\mathbb{R}^{n-1}$ *by the congruence* $G_\alpha \equiv 0 \pmod{\mathscr{M}}$ $(1 \leqslant \alpha \leqslant N)$ *and by the identity (in* $\tau, \eta \in \mathbb{R}^1$*)*

$$\sum_{\alpha=1}^{N} G_\alpha(\xi;\eta)Q_{\alpha+}(\xi;\tau) = \frac{1}{\eta-\tau}\left[\mathscr{P}_+(\xi;\eta)R_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)R_+(\xi;\eta)\right]. \tag{3.3.7}$$

Proof. We show that this proposition follows from Theorem 3.2.2 for $m = 1$, $P_1 = \mathscr{P}$, and $N = \operatorname{ord} \dot{\mathscr{P}}_+$. Indeed, if $m = 1$ and $P_1 = \mathscr{P}$, then $H_+ = \mathscr{P}_+\overline{\mathscr{P}}_-$, and, consequently, $\Pi_+ = \mathscr{P}_+$ and $\dot{\Pi}_+ = \overline{\mathscr{P}}_+$. By Corollary 3.2.6, we can restrict ourselves to the proof of (3.3.7) and to estimating of the second term on the left-hand side of (3.3.6).

First, we derive (3.3.7). Using the equality $H_+ = \mathscr{P}_+\dot{\mathscr{P}}_-$ and the identities (3.2.11) and (3.3.4), we recast (3.2.2) as

$$\begin{aligned}
&\frac{1}{\mathscr{P}_+(\xi;\eta)\overline{\mathscr{P}}_-(\xi;\eta)}\Bigg\{\mathscr{P}_+(\xi;\tau)\mathscr{P}_+(\xi;\eta)\left[\Omega_1\left(c(\xi) + \frac{R_+(\xi;\eta)}{\mathscr{P}_+(\xi;\eta)}\right) + \Omega_2\right] \\
&+ \mathscr{P}_-(\xi;\tau)\overline{\mathscr{P}}_-(\xi;\eta)(\eta-\tau)^{-1}[\mathscr{P}_+(\xi;\eta)R_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)R_+(\xi;\eta)]\Bigg\} \\
&- \sum_{\alpha=1}^{N} \frac{G_\alpha(\xi;\eta)}{\mathscr{P}_+(\xi;\eta)}[Q_{\alpha+}(\xi;\tau)\mathscr{P}_-(\xi;\tau) - Q_{\alpha-}(\xi;\tau)\mathscr{P}_+(\xi;\tau)] \\
&\equiv 0 \pmod{\mathscr{P}_+(\xi;\tau)},
\end{aligned} \tag{3.3.8}$$

where $\Omega_1 = \Omega_1(\xi;\eta,\tau)$ and $\Omega_2 = \Omega_2(\xi;\eta,\tau)$ are the polynomials of the variables $\eta,\tau \in \mathbb{R}^1$, defined by

$$\Omega_1(\xi;\eta,\tau) = (\eta-\tau)^{-1}[\overline{\mathscr{P}}_-(\xi;\eta)\mathscr{P}_-(\xi;\tau) - \overline{\mathscr{P}}_-(\xi;\tau)\mathscr{P}_-(\xi;\eta)], \tag{3.3.9}$$

$$\Omega_2(\xi;\eta,\tau) = (\eta-\tau)^{-1}[\overline{\mathscr{P}}_-(\xi;\eta)R_-(\xi;\tau) - \overline{\mathscr{P}}_-(\xi;\tau)R_-(\xi;\eta)]. \tag{3.3.10}$$

It is obvious that (3.3.8) is equivalent to the congruence

$$\sum_{\alpha=1}^{N} G_\alpha(\xi;\eta)Q_{\alpha+}(\xi;\tau)$$
$$\equiv (\eta-\tau)^{-1}[\mathscr{P}_+(\xi;\eta)R_+(\xi;\tau) - \mathscr{P}_+(\xi;\tau)R_+(\xi;\eta)] \ (\mathrm{mod}\ \mathscr{P}_+(\xi;\tau)).$$

For all $\eta \in \mathbb{R}^1$ and almost all $\xi \in \mathbb{R}^{n-1}$ both sides of this relation are polynomials of τ with degrees less than ord $\mathscr{P}_+$. This establishes (3.3.7).

Now note that for $m = 1$ and $N = \operatorname{ord} \dot{\mathscr{P}}_+$ the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |D_j(\xi;\eta,\tau)|^2}{\sum_{j=1}^{m} |P_j(\xi;\tau)|^2} d\tau d\eta \leqslant \frac{\mathrm{const}}{B(\xi)},$$

which appears in condition 2 of Theorem 3.2.2, takes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{D(\xi;\eta,\tau)}{\mathscr{P}(\xi;\tau)}\right|^2 d\tau d\eta \leqslant \frac{\mathrm{const}}{B(\xi)}, \tag{3.3.11}$$

where $D(\xi;\eta,\tau)$ is the left-hand side of (3.2.2). After transformation (3.2.2) from the form (3.3.7) to (3.3.8), we obtain that (3.3.11) has the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\Omega_1\left[c(\xi) + \frac{R_+(\xi;\eta)}{\mathscr{P}_+(\xi;\eta)}\right] + \Omega_2}{\overline{\mathscr{P}}_-(\xi;\eta)\mathscr{P}_-(\xi;\tau)} - \frac{\sum_{\alpha=1}^{N} G_\alpha(\xi;\eta)Q_{\alpha-}(\xi;\tau)}{\mathscr{P}_+(\xi;\eta)\mathscr{P}_-(\xi;\tau)} \right|^2 d\tau d\eta$$
$$\leqslant \frac{\mathrm{const}}{B(\xi)}, \tag{3.3.12}$$

where Ω_1 and Ω_2 are the polynomials defined by (3.3.9) and (3.3.10), respectively.

We show that (3.3.12) is equivalent to the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\sum_{\alpha=1}^{N} G_\alpha(\xi;\eta)Q_{\alpha-}(\xi;\tau)}{\mathscr{P}_+(\xi;\eta)\mathscr{P}_-(\xi;\tau)}\right|^2 d\tau d\eta \leqslant \frac{\mathrm{const}}{B(\xi)}, \tag{3.3.13}$$

which enters in condition 2 of the proposition being proved. In accordance with a theorem of Katsnelson ([Kats67], pp. 58–61), there exists a constant $c > 0$ depending only on ord $\mathscr{P} = J$, such that

$$\sup \left| \frac{R_+(\xi;\tau)}{\mathscr{P}_+(\xi;\tau)} \right| \leqslant c \sup \left| \frac{R(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right| \tag{3.3.14}$$

a.e. in $\mathbb{R}^{n-1}$. It follows from (3.3.4) and (3.3.14) that the inequalities

$$\begin{gathered} |c(\xi)| \leqslant \text{const} \sup \left| \frac{R(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right|, \\ \sup \left| \frac{R_-(\xi;\tau)}{\mathscr{P}_-(\xi;\tau)} \right| \leqslant \text{const} \sup \left| \frac{R(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right| \end{gathered} \tag{3.3.15}$$

hold a.e. in $\mathbb{R}^{n-1}$. We set in Corollary 3.2.8 $L = \mathscr{P}_-$, $K = \overline{\mathscr{P}}_-$. If the estimate

$$\left| \frac{R(\xi;\tau)}{\mathscr{P}(\xi;\tau)} \right|^2 \leqslant \frac{\text{const}}{B(\xi)} \tag{3.3.16}$$

holds true, then (3.2.16), (3.3.14), the first of the inequalities (3.3.15), and (3.3.9) yield

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\Omega_1(\xi;\eta,\tau) \left[c(\xi) + \frac{R_+(\xi;\eta)}{\mathscr{P}_+(\xi;\eta)} \right]}{\overline{\mathscr{P}}_-(\xi;\eta)\mathscr{P}_-(\xi;\tau)} \right|^2 d\tau d\eta \leqslant \frac{\text{const}}{B(\xi)}. \tag{3.3.17}$$

Next, we set in Corollary 3.2.3 $L = \mathscr{P}_-$ and $K = R_-$. It follows from (3.2.16), (3.3.16), the second of the inequalities (3.3.15), and (3.3.10) that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\Omega_2(\xi;\eta,\tau)}{\overline{\mathscr{P}}_-(\xi;\eta)\mathscr{P}_-(\xi;\tau)} \right|^2 d\tau d\eta \leqslant \frac{\text{const}}{B(\xi)}. \tag{3.3.18}$$

Using (3.3.17) and (3.3.18), we conclude that (3.3.12) is equivalent to (3.3.13). □

In a similar way, one can show that for $m \neq 1$ and $N = \text{ord}\, \dot{\mathscr{P}}_+$ Theorems 3.2.3 and 3.2.4 imply the following assertions.

Proposition 3.3.3. *The estimate*

$$\begin{gathered} \|R(D)u\|^2_{B^{1/2}} \leqslant C_0 \|P(D)u\|^2, \quad u \in C_0^\infty(\mathbb{R}^n_+), \\ Q_\alpha(D)u(x;0) = 0 \qquad (\alpha = 1,\ldots,N) \end{gathered} \tag{3.3.19}$$

is valid if and only if condition 1 *of Proposition* 3.3.2 *is fulfilled and inequalities* (3.3.16) *and* (3.3.13) *hold a.e. in* $\mathbb{R}^{n-1}$. *Here* G_α *are the same polynomials as in Proposition* 3.3.2.

Proposition 3.3.4. *The estimate*

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\|\mathscr{P}(D)u\|^2, \quad u \in C_0^\infty(\mathbb{R}^n_+) \tag{3.3.20}$$

is valid if and only if inequality (3.3.16) *and the congruence* $R \equiv 0 \pmod{\mathscr{P}_+}$ *are satisfied a.e. in* $\mathbb{R}^{n-1}$.

3.3.3 Examples of estimates for operators of first order with respect to *t*

In this subsection we consider two examples of estimates of the types (3.0.1), (3.0.1′) for operators $P_j(D)$ of order at most one w.r.t. t. The corresponding criteria (Propositions 3.3.5, 3.3.6) are formulated explicitly in the form of necessary and sufficient conditions on the coefficients of the operators figuring in these estimates.

Proposition 3.3.5. *Let* $P(\xi;\tau) = \mathrm{i}\tau - p(\xi)$, $Q(\xi;\tau) = q(\xi)$, *where* $p(\xi)$, $q(\xi)$ *are measurable locally bounded functions that grow no faster than some power of* $|\xi|$ *as* $|\xi| \to \infty$. *The estimate*

$$\left\|\frac{\partial^s u}{\partial t^s}\right\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \|u\|^2 + \langle\!\langle Q(D)u\rangle\!\rangle^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.3.21}$$

with $s = 0, 1$ *holds true if and only if*

$$B(\xi)(1 + |p|^{2s})(1 + \operatorname{Re} p)^{-2} \leqslant \text{const} \tag{3.3.22}$$

for almost all $\xi \in \{\xi \in \mathbb{R}^{n-1} : \operatorname{Re} p(\xi) \geqslant 0\}$ *and*

$$B(\xi)\frac{1 + |p|^{2s}}{(1 + |\operatorname{Re} p|)^2}\left[1 + \frac{1 + |\operatorname{Re} p|}{|q|^2 + (1 + |\operatorname{Re} p|)^{-1}}\right] \leqslant \text{const} \tag{3.3.23}$$

for almost all $\xi \in \{\xi \in \mathbb{R}^{n-1} : \operatorname{Re} p(\xi) < 0\}$.

Proof. The estimate (3.3.21) is a special case of the estimate (3.0.1) (for $m = 2$, $P_1 = P$, $P_2 = 1$, $Q_1 = q$, and $R = \tau^s$ with $s = 0, 1$). Since P_2 and P_1 are relatively prime, we get $\Pi_+ = 1$. This means that condition 1 of Theorem 3.2.2 can be omitted and the infimum in (3.2.3) is taken over all β. We show that (3.2.3) is equivalent to (3.3.22) and (3.3.23).

It can be easily verified that for $P(\xi;\tau) = \mathrm{i}\tau - p(\xi)$ the equality

$$\sup \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} = \frac{(1 + |p|^2)^s}{1 + |\operatorname{Re} p|^2} \qquad (s = 0, 1) \tag{3.3.24}$$

is valid.

The polynomials (of degree zero w.r.t. τ) D_{1s}, D_{2s}, constructed according to Lemma 3.2.1 for $R = \tau^s$, $P_1 = \mathrm{i}\tau - p(\xi)$, $P_2 = 1$, $Q = q(\xi)$, are equal to

$$D_{1s} = -\mathrm{i}\{[H_+(\xi;\eta)]^{-1}\tau_+^s - \beta q\}, \quad D_{2s} = -\mathrm{i}\{[H_+(\xi;\eta)]^{-1}\tau_+^s - \beta q\}(\tau_- + \mathrm{i}p),$$

where $\tau_\pm(\xi)$ are the roots of the polynomial $|P(\xi;\tau)|^2 + 1$, so the equalities $H_\pm(\xi;\tau) = \tau - \tau_\pm(\xi)$ are valid. Therefore, the first term on the left-hand side of (3.2.3) is equal to

$$\int_{-\infty}^{\infty} \inf_\beta \{|\beta|^2 + |a - \beta q|^2 b^2\} d\eta,$$

where $a = [H_+(\xi;\eta)]^{-1}\tau_+^s$ and

$$b^2 = \pi(1 + |\tau_- + \mathrm{i}p|^2)(1 + |\mathrm{Re}\, p|^2)^{-1/2}.$$

Since $\inf_\beta\{|\beta|^2 + |a - \beta q|^2|b|^2\} = |a|^2 b^2(1 + b^2|q|^2)^{-1}$ and the roots of the polynomial $|P(\xi;\tau)|^2 + 1$ are equal to $\tau_\pm = \mathrm{Im}\, p \pm \mathrm{i}(1 + |\mathrm{Re}\, p|^2)^{1/2}$, we see that inequality (3.2.3) can be recast as

$$\frac{(1 + |p|^2)^s}{1 + |\mathrm{Re}\, p|^2}\left[1 + \frac{g(1 + |\mathrm{Re}\, p|^2)^{1/2}}{|q|^2 g + (1 + |\mathrm{Re}\, p|^2)^{1/2}}\right] \leqslant \frac{\mathrm{const}}{B(\xi)}, \tag{3.3.25}$$

where $g = 1 + [(1 + |\mathrm{Re}\, p|^2)^{1/2} - \mathrm{Re}\, p]^2$. A direct check shows that the validity of inequality (3.3.25) a.e. in $\mathbb{R}^{n-1}$ is equivalent to conditions (3.3.22) and (3.3.23). □

We formulate one more result, which follows from Theorem 3.2.4.

Proposition 3.3.6. *Let $P_j(\xi;\tau) = \mathrm{i}\tau - p_j(\xi)$ ($j = 1, \dots, m;\ m \geqslant 1$), where $p_j(\xi)$ are measurable locally bounded functions growing no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Suppose also that $\sum_{j=1}^{m} |p_j(\xi)| \neq 0$ a.e. in $\mathbb{R}^{n-1}$. The estimate*

$$\left\| \frac{\partial^s u}{\partial t^s} \right\|_{B^{1/2}} \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.3.26}$$

with $s = 0, 1$ holds true if and only if the following conditions are satisfied:

1. $\sum_{j,h=1}^{m} |p_j - p_h| \neq 0$ *for almost all* $\xi \in \bigcap_{j=1}^{m} \{\xi : \mathrm{Re}\, p_j(\xi) \leqslant 0\}$.
2. *The inequality*

$$B(\xi) \sum_{j=1}^{m} |p_j|^{2s} \leqslant \mathrm{const}\left[\sum_{j=1}^{m} |\mathrm{Re}\, p_j|^2 + \sum_{j,h=1}^{m} |\mathrm{Im}\,(p_j - p_h)|^2\right] \tag{3.3.27}$$

is valid for almost all $\xi \in \left\{\xi : \sum_{j=1}^{m} \mathrm{Re}\, p_j(\xi) \geqslant 0\right\}$.

3. *The inequality*

$$B(\xi)\sum_{j=1}^{m}|p_j|^{2s} \leqslant \text{const}\sum_{j,h=1}^{m}|p_j - p_h|^2 \tag{3.3.28}$$

is valid for almost all $\xi \in \left\{\xi : \sum_{j=1}^{m} \operatorname{Re} p_j(\xi) < 0\right\}$.

Proof. The estimate (3.3.26) is a special case of the estimate (3.2.7) for $P_j = \mathrm{i}\tau - p_j(\xi)$ and $R = \tau^s$, $s = 0, 1$. We note that condition 1 of Theorem 3.2.4 is satisfied here if and only if $\Pi_+(\xi;\tau) = 1$ a.e. in $\mathbb{R}^n$. The latter condition is equivalent to condition 1 of the proposition to be proved, since the τ-root of the polynomial $\mathrm{i}\tau - p_j$ lies in the half-plane $\operatorname{Im}\zeta \geqslant 0$ if and only if $\operatorname{Re} p_j \leqslant 0$. We show that condition 2 of Theorem 3.2.4 is equivalent to conditions 2 and 3 of Proposition 3.3.6. As in Subsection 2.2.4, we use equations (2.2.37), (2.2.38), and (2.2.39) for the polynomials $H_\pm(\xi;\tau)$, their τ-roots $\tau_\pm(\xi)$ and the function $\alpha(\xi)$.[8]

It is easy to verify that $P_j = \mathrm{i}\tau - p_j(\xi)$ admits the relations

$$\sup \frac{\tau^{2s}}{\sum_{j=1}^{m}|P_j(\xi;\tau)|^2} = m^{1+s}\alpha^{-2}|\tau_+|^{2s}, \qquad s = 0, 1. \tag{3.3.29}$$

In the considered example, the polynomials $D_{js}(\xi;\eta,\tau)$ appearing in (3.2.9) have degree zero w.r.t. τ. Based on Lemma 3.2.1, we find that $D_{js}(\xi;\eta,\tau)$ satisfy the relations

$$\overline{D}_{js} = \frac{\mathrm{i}m^{1/2}\tau_+^s(\mathrm{i}\tau - p_j)}{H_+(\xi;\tau)\sum_{k=1}^{m}(\mathrm{i}\tau_+ - p_k)} \qquad (j = 1,\dots,m;\ s = 0, 1). \tag{3.3.30}$$

From (3.3.29), (3.3.30), and the obvious relations

$$\int_{-\infty}^{\infty}|H_+(\xi;\tau)|^{-2}d\tau = \pi\alpha^{-1}, \qquad |\tau_+|^2 = m^{-1}\sum_{j=1}^{m}|p_j|^2$$

it follows that (3.2.9) is equivalent to the inequality

$$\sum_{j=1}^{m}|p_j|^{2s}\alpha^{-2}\left[1 + \frac{\sum_{j=1}^{m}|\mathrm{i}\tau - p_j|^2}{\left|\sum_{j=1}^{m}(\mathrm{i}\tau_+ - p_j)\right|^2}\right] \leqslant \frac{\text{const}}{B(\xi)}. \tag{3.3.31}$$

We show that the validity of (3.3.31) a.e. in $\mathbb{R}^{n-1}$ is equivalent to conditions 2 and 3 of the proposition being proved. By (2.2.38), the formulas (2.2.42) and (2.2.43)

[8] We note that the references (2.2.37)–(2.2.39) and (2.2.42)–(2.2.44) occuring up to end of this section refer to the formulas of Subsection 2.2.4.

hold true, where $\alpha(\xi)$ and $\beta(\xi)$ are defined by (2.2.39) and (2.2.42), respectively. If $\beta \geqslant 0$, then (2.2.42) and (2.2.43) imply

$$\frac{\sum_{j=1}^{m} |i\tau - p_j|^2}{\left|\sum_{j=1}^{m}(i\tau_+ - p_j)\right|^2} \leqslant \frac{2}{m}.$$

Therefore, using the identity

$$\frac{1}{2}\sum_{j,h=1}^{m}[\mathrm{Im}\,(p_j - p_h)]^2 = m\sum_{j=1}^{m}(\mathrm{Im}\,p_j)^2 - \left(\sum_{j=1}^{m}\mathrm{Im}\,p_j\right)^2,$$

we conclude that on the set $\{\xi \in \mathbb{R}^{n-1} : \beta(\xi) \geqslant 0\}$ inequality (3.3.31) is equivlent to (3.3.27).

Now, suppose $\beta(\xi) < 0$. Then (3.3.31) is equivalent to the inequality

$$B(\xi)\sum_{j=1}^{m}|p_j|^{2s} \leqslant \mathrm{const}(\alpha^2 + \alpha\beta). \tag{3.3.32}$$

From the easily verifiable equality $\alpha^2 - \beta^2 = \frac{1}{2}\sum_{j,h=1}^{m}|p_j - p_h|^2$ (see (2.2.39) and (2.2.44)) it follows that for $\beta < 0$ we have

$$\alpha^2 + \alpha\beta = \alpha^2\left[1 - \left(1 - 2^{-1}\alpha^{-2}\sum_{j,h=1}^{m}|p_j - p_h|^2\right)^{1/2}\right],$$

and (3.3.32) takes the form

$$B(\xi)\sum_{j=1}^{m}|p_j|^{2s}\left[1 - \left(1 - 2^{-1}\alpha^{-2}\sum_{j,h=1}^{m}|p_j - p_h|^2\right)^{1/2}\right] \leqslant \mathrm{const}\sum_{j,h=1}^{m}|p_j - p_h|^2. \tag{3.3.33}$$

The equivalence of (3.3.33) and (3.3.28) is obvious. □

Remark 3.3.7. Obviously, the estimate (3.3.26) is valid if condition1 of Proposition 3.3.6 is satisfied and inequality (3.3.28) holds a.e. in $\mathbb{R}^{n-1}$.

3.4 Notes

The main results of this chapter were established by the authors in [GM80]; some results were announced in [GM75]. Necessary and sufficient conditions for the special case when $m = 1$ and the number N of boundary operators is minimal (see Subsection 3.3.2) were given in the authors' paper [MG75].

Problems of dominance of differential operators were studied by many authors. Without mentioning here the numerous papers on estimates for operators of concrete types as well as on embedding theorems for the Sobolev–Slobodeckij spaces and their generalizations, we would like to point out some works that are more closely related to our topic.

The work of N. Aronszajn [Aro54] established necessary and sufficient conditions for the coercivity of a system of operators in a half-space or in a bounded domain (see Subsection 3.3.1). Here, the estimates of L^2-norms are considered under the assumption that the integration domain has C^1-boundary. A generalization of these results to the case of the L^p-norm, $p > 1$, and to the integration domains with Lipschitz boundary was given by K. T. Smith in [Smi61]. S. Agmon [Agm58] generalized the results of N. Aronszajn to the case of arbitrary integro-differential forms. Criteria for the validity of the estimates for minimal operators in a bounded domain or in $\mathbb{R}^n$ were established by B. Malgrange [Mal56] and L. Hörmander [H55].

Sufficient conditions for the validity of the estimates of the types (3.0.1) and (3.0.2) were given in the papers of M. Schechter [Sch63], [Sch64]. The case $m = 1$ was studied in [Sch64]. As we noted in Chapter 1, the condition established by M. Schechter follows directly from the results of that chapter (see Corollary 1.3.7, Chapter 1).

In [Sch63], the estimate (3.0.1) was studied under the assumption that R, P_j, and Q_α are homogeneous polynomials w.r.t. $(\xi;\tau) \in \mathbb{R}^n$. Here, it is shown that the estimate (3.0.1) holds true if the polynomial R satisfies a condition of the type

$$B(\xi)|R(\xi;\tau)|^2 \leqslant c \sum_{j=1}^{m} |P_j(\xi;\tau)|^2$$

and for each $\xi \in S^{n-2}$ among Q_α there are $N(\xi)$ linearly independent polynomials modulo $\Pi_+(\xi;\tau)$, where $N(\xi) = \operatorname{ord} \Pi_+(\xi;\tau)$.

Finally, we mention the work of K. F. Schubert [Schu71]. There, a special case of inequality (3.0.1′) is examined, namely the estimate

$$\|u\|^2 \leqslant C \sum_{j=1}^{m} \|P_j(D)u\|^2, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{3.4.1}$$

where $P_j(\xi;\tau)$ are polynomials of the variable $(\xi;\tau) \in \mathbb{R}^n$. Here, it was shown that for the case $\operatorname{ord} P_j(\xi;\tau) \leqslant 1$ $(j = 1,\dots,m)$ the estimate (3.4.1) holds true if and only if the condition

$$\sum_{j=1}^{m} |P_j(\xi;z)|^2 \geqslant \text{const} \tag{3.4.2}$$

is fulfilled for all $\xi \in \mathbb{R}^{n-1}$ and all z with $\operatorname{Im} z \geqslant 0$. The condition (3.4.2) is also necessary in the general case. The sufficiency of (3.4.2) was not established in [Schu71] for the general case. One more condition is given in [Schu71] for the polynomials of degree higher than 1. Together with (3.4.2) this condition is sufficient for the validity of (3.4.1).

Chapter 4
Estimates for a maximal operator

4.0 Introduction

In this chapter we study criteria for the validity of the estimates

$$\langle\!\langle R(D)u \rangle\!\rangle^2_{B^{1/2}} \leqslant C(\|P(D)u\|^2 + \|u\|^2), \qquad u \in C_0^\infty(\mathbb{R}^n_+); \tag{4.0.1}$$

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C(\|P(D)u\|^2 + \|u\|^2), \qquad u \in C_0^\infty(\mathbb{R}^n_+). \tag{4.0.2}$$

It is assumed that $R(\xi;\tau)$ and $P(\xi;\tau)$ are polynomials of τ with measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$, and that $\operatorname{ord} P(\xi;\tau) = J \geqslant 1$ a.e. in $\mathbb{R}^{n-1}$. Assume also that the degrees of the polynomials $R(\xi;\tau)$ figuring in (4.0.1) and (4.0.2) satisfy for almost all $\xi \in \mathbb{R}^{n-1}$ the conditions $\operatorname{ord} R \leqslant J-1$ and $\operatorname{ord} R \leqslant J$, respectively. Thus, (4.0.1) is a special case of the estimate (2.0.8) of Chapter 2, and (4.0.2) is a special case of the estimate (3.0.1′) of Chapter 3.

The goal of this chapter is to specify classes of operators P and R for which the necessary and sufficient conditions for the validity of (4.0.1), (4.0.2) following from the results of Chapters 2 and 3 take a much simpler and more explicit form.[1]

In Section 4.2, it is assumed that the polynomial $P(\xi;\tau)$ is quasielliptic of type $l \geqslant 1$ and $R(\xi;\tau) = \tau^s$, where $s = 0,\dots,J-1$ in the estimate (4.0.1) and $s = 0,\dots,J$ in the estimate (4.0.2), respectively. It is shown that (4.0.1) holds true if and only if the inequality

$$B(\xi)(1+\langle\xi\rangle)^{(2s+1)m/J} \leqslant \text{const} \tag{4.0.3}$$

is satisfied a.e. in $\mathbb{R}^{n-1}$, while (4.0.2) remains valid if and only if the inequality

$$B(\xi)(1+\langle\xi\rangle)^{2sm/J} \leqslant \text{const} \tag{4.0.4}$$

is satisfied a.e. in $\mathbb{R}^{n-1}$. (Here, m is an integer and $\langle\cdot\rangle$ is the norm in $\mathbb{R}^{n-1}$ defined by the quasielliptic polynomial $P(\xi;\tau)$).

In Section 4.3, it is assumed that $P(\xi;\tau) = \tau^J + p_1(\xi)\tau^{J-1} + \dots + p_J(\xi)$ is a continuous, positively homogeneous function of degree J w.r.t. $(\xi;\tau)$, and $p_J(\xi) \neq 0$ for all $\xi \neq 0$. It is required that the τ-roots $z_1(\xi),\dots,z_J(\xi)$ of the polynomial P are pairwise distinct for all $\xi \neq 0$, and for each $\varrho = 1,\dots,J$ the function $\operatorname{Im} z_\varrho(\xi)$ either vanishes identically or preserves its sign on the unit sphere $S^{n-2} \subset \mathbb{R}^{n-1}$. It is also assumed that for each $\varrho = 1,\dots,J$ one of the following conditions is fulfilled:

[1] Several complements of the results of Chapters 2 and 3, related to inequalities (4.0.1)–(4.0.2), are presented in Section 4.1.

either $P(\xi;\bar{z}_\varrho(\xi)) \equiv 0$, or $P(\xi;\bar{z}_\varrho(\xi)) \neq 0$ for all $\xi \neq 0$. Under these assumptions, we obtain necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2), where the polynomial R is the same as in Section 4.2. It will be shown that necessary and sufficient conditions for the validity of the estimate (4.0.1) take the form of one of the following inequalities:

$$B(\xi)(1+|\xi|)^{2s+1} \leqslant \text{const}, \tag{4.0.5}$$

provided that at least one of the roots of the polynomial P lies in the half-plane $\operatorname{Im}\zeta > 0$;

$$B(\xi)(1+|\xi|)^{2s+1-J} \leqslant \text{const}, \tag{4.0.6}$$

provided that all the roots of the polynomial P lie in the half-plane $\operatorname{Im}\zeta \leqslant 0$ and at least one of these roots is real;

$$B(\xi)(1+|\xi|)^{2s+1-2J} \leqslant \text{const}, \tag{4.0.7}$$

provided that all the roots of the polynomial P lie in the half-plane $\operatorname{Im}\zeta < 0$. Here and below $\zeta = \tau + i\sigma$. A necessary and sufficient condition for the validity of (4.0.2) is the inequality

$$B(\xi)(1+|\xi|)^{2s} \leqslant \text{const}, \tag{4.0.8}$$

if at least one of the roots of the polynomial P lies in the half-plane $\operatorname{Im}\zeta \geqslant 0$, or the inequality

$$B(\xi)(1+|\xi|)^{2s-2J} \leqslant \text{const}, \tag{4.0.9}$$

if all the roots of the polynomial P lie in the half-plane $\operatorname{Im}\zeta < 0$.

In Section 4.4, we consider some classes of nonhomogeneous polynomials. We show, for example, that if all the τ-roots $z_j(\xi)$ of the polynomial $P(\xi;\tau)$ are real and satisfy the condition $|z_j(\xi) - z_r(\xi)| \geqslant \text{const}$ ($j \neq r$, $j, r = 1, \dots, J$), then the estimate (4.0.1) holds if and only if the inequality

$$B(\xi) \int_{-\infty}^{\infty} \frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1} d\tau \leqslant \text{const} \tag{4.0.10}$$

is valid a.e. in $\mathbb{R}^{n-1}$; moreover the inequality

$$B(\xi) \sup_{\tau\in\mathbb{R}^1} \frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1} \leqslant \text{const} \tag{4.0.11}$$

is a criterion for the validity of the estimate (4.0.2).

In Section 4.5, it is assumed that $P(\xi;\tau) = p_0(\xi)\tau^2 + p_1(\xi)\tau + p_2(\xi)$ and $p_0(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$. Criteria for the validity of the estimates (4.0.1), (4.0.2) are studied for some classes of such polynomials in the case $R(\xi;\tau) = \tau^s$. For example, it will be shown that if $\operatorname{Im} p_k(\xi) \equiv 0$ ($k = 0, 1, 2$) and $s = 0, 1, 2$, then the estimate (4.0.2) holds true if and only if

$$B(\xi)[|p_0| + p_1^2 + |p_0 p_2|]^s \leqslant \text{const}|p_0|^{2s} \quad \text{a.e. in } \mathbb{R}^{n-1}. \tag{4.0.12}$$

In Section 4.6, we discuss in detail the case where $B(\xi) = 1$ and $R(D)$, $P(D)$ are differential polynomials with constant coefficients. In this case, (4.0.2) is equivalent to the embedding $\mathscr{D}(P) \subset \mathscr{D}(R)$, where $\mathscr{D}(P)$ and $\mathscr{D}(R)$ are the domains of the maximal operators generated in $L^2(\mathbb{R}^n_+)$ by the polynomials $P(D)$ and $R(D)$, respectively. This follows from the result of Subsection 4.6.1. It is established there that for a differential polynomial with constant coefficients the maximal operator P in $L^2(\mathbb{R}^n_+)$ is the closure of its restriction to $C_0^\infty(\mathbb{R}^n_+)$. It is well known (see L. Hörmander [H55]) that for maximal operators in a bounded domain $\Omega \subset \mathbb{R}^n$ the embedding $\mathscr{D}(P) \subset \mathscr{D}(R)$ is possible if and only if either $R = aP + b$, where a and b are constants, or R and P are the ordinary differential operators satisfying $\operatorname{ord} R \leqslant \operatorname{ord} P$. In the half-space $\mathbb{R}^n_+$ the embedding $\mathscr{D}(P) \subset \mathscr{D}(R)$ is also possible for non-trivial operators R.

In Subsection 4.6.2, theorems on the trace space of the elements $u \in \mathscr{D}(P)$ are proved. These statements are the strengthening of relevant results of Subsection 2.3.2.

4.1 Preliminary results

In this section we formulate necessary and sufficient conditions for the validity of the estimates (4.0.1) and (4.0.2), which were discussed in the introduction.

Let $H_+(\xi;\tau)$ be a polynomial of τ with roots lying in the half-plane $\operatorname{Im}\zeta > 0$, $\zeta = \tau + i\sigma$, such that

$$|P(\xi;\tau)|^2 + 1 = |H_+(\xi;\tau)|^2 \tag{4.1.1}$$

and $H_-(\xi;\tau) = \overline{H_+(\xi;\tau)}$. We write $H_+(\xi;\tau)$ in the form $H_+(\xi;\tau) = \sum_{l=0}^{J} h_l(\xi)\tau^{J-l}$ and set $Z = \{\xi \in \mathbb{R}^{n-1} : h_0(\xi) = 0\}$. We assume that $\operatorname{mes}_{n-1} Z = 0$.

4.1.1 Results concerning the estimate (4.0.1)

In this subsection we present several assertions about necessary and sufficient conditions for the validity of the estimate (4.0.1), which follow directly from the results of Chapter 2. It is obvious that (4.0.1) is a special case of the estimate (2.2.19) from Chapter 2 corresponding to $m = 2$, $P_1(\xi;\tau) = P(\xi;\tau)$, and $P_2(\xi;\tau) = 1$. Therefore, Corollary 2.2.8 of Chapter 2 and Remark 1.1.11 of Chapter 1 imply the following criterion for the validity of (4.0.1).

Theorem 4.1.1. *The estimate* (4.0.1) *holds true if and only if*

$$\Lambda(\xi) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|T_1(\xi;\tau)|^2 + |T_2(\xi;\tau)|^2}{|P(\xi;\tau)|^2 + 1}\, d\tau \leqslant \frac{\text{const}}{B(\xi)} \tag{4.1.2}$$

for almost all $\xi \in \mathbb{R}^{n-1}$. Here $T_1(\xi;\tau)$ is the quotient and $T_2(\xi;\tau)$ is the remainder of the division of the polynomial (of τ) $R(\xi;\tau)H_-(\xi;\tau)$ by $P(\xi;\tau)$.

In the sequel, we will give other representations for the function $\Lambda(\xi)$ introduced in (4.1.2).

Proposition 4.1.2. *Let $T_2(\xi;\tau)$ and $\Lambda(\xi)$ be the same as in Theorem 4.1.1, and let $S(\xi;\tau)$ be the remainder of the division of the polynomial (of τ) $R\overline{T}_2$ by H_+. Then*

$$\Lambda(\xi) = -\mathrm{Im}\,\frac{s_0(\xi)}{h_0(\xi)}, \tag{4.1.3}$$

where $s_0(\xi)$ and $h_0(\xi)$ are the leading coefficients of the polynomials S and H_+, respectively.

Proof. The identity $RH_- = PT_1 + T_2$ implies the relation

$$\frac{|T_1|^2 + |T_2|^2}{|P|^2+1} = \mathrm{Re}\,[H_+^{-1}(H_+(|T_1|^2 - |R|^2) + 2R\overline{T}_2)]. \tag{4.1.4}$$

Since $\mathrm{ord}\,R < \mathrm{ord}\,P$, we see that the left-hand side of this relation is a proper fraction w.r.t. τ. Therefore, it follows from (4.1.4) that

$$\frac{|T_1|^2 + |T_2|^2}{|P|^2+1} = 2\mathrm{Re}\,\frac{S(\xi;\tau)}{H_+(\xi;\tau)}, \tag{4.1.5}$$

where S is the remainder of the division of $R\overline{T}_2$ by H_+. Integrating both sides of (4.1.5) and taking into account that all τ-roots of the polynomial H_+ lie in the half-plane $\mathrm{Im}\,\zeta > 0$, we arrive at (4.1.3). □

Corollary 4.1.3. *Let $R(\xi;\tau) = 1$, suppose the leading coefficients of the polynomials $P(\xi;\tau)$ and $H_-(\xi;\tau)$ coincide, and let $z_r(\xi)$ and $\zeta_r(\xi)$ $(r = 1,\dots,J)$ be the τ-roots of the polynomials P and H_+, respectively. (The multiplicities of these roots is taken into account). Then the function Λ defined by (4.1.2) is equal to*

$$\Lambda(\xi) = \mathrm{Im}\sum_{r=1}^{J}(z_r(\xi) + \zeta_r(\xi)). \tag{4.1.6}$$

Proof. We write the polynomials P and H_+ as

$$P(\xi;\tau) = \sum_{l=0}^{J} p_l(\xi)\tau^{J-l}, \qquad H_+(\xi;\tau) = \sum_{l=0}^{J} h_l(\xi)\tau^{J-l}.$$

Since $R = 1$ and $h_0 = \bar{p}_0$, we have $T_2 = H_- - P$, $\overline{T}_2 = H_+ - \overline{P}$, and, hence, $s_0 = h_1 - \bar{p}_1$. Therefore, in accordance with (4.1.3), we obtain

$$\Lambda(\xi) = \mathrm{Im}\left(-\frac{h_1}{h_0} - \frac{p_1}{p_0}\right) = \mathrm{Im}\sum_{r=1}^{J}(\zeta_r(\xi) + z_r(\xi)). \qquad \square$$

Proposition 4.1.4. *Let $z_\varrho(\xi)$ be the τ-roots of the polynomial $P(\xi;\tau)$ with multiplicities $\mu_\rho(\xi)$ ($\varrho = 1,\dots,\beta(\xi)$; $\mu_1(\xi) + \cdots + \mu_{\beta(\xi)}(\xi) = J$), let $\zeta_\nu(\xi)$ be the τ-roots of the polynomial $H_+(\xi;\tau)$ with multiplicities $k_\nu(\xi)$ ($\nu = 1,\dots,l(\xi)$; $k_1(\xi) + \cdots + k_{l(\xi)}(\xi) = J$). Suppose that*

$$\mathfrak{G} = \{G_{\varrho s\nu\gamma}(z_\varrho(\xi), \zeta_\nu(\xi)\}$$

with

$$G_{\varrho s\nu\gamma}(z_\varrho(\xi), \zeta_\nu(\xi) = \frac{\mathrm{i}(-1)^s(\gamma+s)!}{(z_\varrho(\xi) - \overline{\zeta}_\nu(\xi))^{s+1}} \tag{4.1.7}$$

is a matrix such that its rows are labeled by the indices ϱ, s and its columns are labeled by the indices ν, γ. These indices take the values: $\varrho = 1,\dots,\beta(\xi)$; $\nu = 1,\dots,l(\xi)$; $s = 0,\dots,\mu_\varrho(\xi)-1$; and $\gamma = 0,\dots,k_\nu(\xi)-1$. Let $\mathfrak{H} = \{H_{\varrho s\nu\gamma}(z_\varrho(\xi), \zeta_\nu(\xi))\}$ denote the $J \times J$ matrix inverse to the matrix $\mathfrak{G}$. Then the function Λ, defined by (4.1.2), is given by

$$\begin{aligned}\Lambda(\xi) = \sum_{\varrho=1}^{\beta(\xi)} \sum_{s=0}^{\mu_\varrho(\xi)-1} \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} &\overline{H_{\varrho s\nu\gamma}(z_\varrho(\xi), \zeta_\nu(\xi))} \\ &\times \overline{R^{(s)}(\xi; z_\varrho(\xi))}\, R^{(\gamma)}(\xi;\zeta_\nu(\xi))\end{aligned} \tag{4.1.8}$$

Proof. We apply Remark 2.1.2, Chapter 2 to the case $m = 2$, $P_1(\tau) = P(\tau)$, $P_2(\tau) = 1$, $\Pi_+(\tau) = \Pi_0(\tau) = \Pi_1(\tau) = 1$, and $D(\tau) = R(\tau)$. In this case, for each fixed $\xi \in \mathbb{R}^{n-1}$ the relation (2.1.16) from Chapter 2 becomes

$$\begin{aligned}\mathfrak{L}_{\nu\gamma}(\xi;\tau) = &\frac{1}{(\tau - \overline{\zeta}_\nu(\xi))^{k_\nu(\xi)-\gamma}} \\ &+ \frac{P(\xi;\tau)}{(\tau - \overline{\zeta}_\nu(\xi))^{k_\nu(\xi)-\gamma}} \sum_{\mu=0}^{k_\nu(\xi)-1-\gamma} \frac{1}{\mu!} \overline{P}^{(\mu)}(\xi; \overline{\zeta}_\nu(\xi))(\tau - \overline{\zeta}_\nu(\xi))^\mu.\end{aligned} \tag{4.1.9}$$

Furthermore, as it was shown in Remark 2.1.2, Chapter 2, the relation

$$R(\tau) = \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} \overline{d^0_{\nu\gamma}}(\xi) \mathfrak{L}_{\nu\gamma}(\xi;\tau) \tag{4.1.10}$$

is valid. We differentiate (4.1.9), use the relation $d^0_{\nu,k_\nu-1-\gamma}(\xi) = \gamma!\varphi^0_{\nu\gamma}(\xi)$, and set $\tau = z_\varrho(\xi)$. Since $P^{(s)}(\xi; z_\varrho(\xi)) = 0$ $(s = 0,\dots,\mu_\varrho(\xi)-1)$, we obtain from (4.1.10) the following system for $\varphi^0_{\nu\gamma}(\xi)$:

$$\begin{aligned}R(s)(\xi; z_\varrho(\xi)) = \mathrm{i}^{-1} \sum_{\nu=1}^{l(\xi)} \sum_{\gamma=0}^{k_\nu(\xi)-1} &\frac{(-1)^{s+1}(\gamma+s)!}{z_\varrho(\xi) - \overline{\zeta}_\nu(\xi))^{s+1}} \varphi^0_{\nu\gamma}(\xi) \\ &(\varrho = 1,\dots,\beta(\xi);\ s = 0,\dots,\mu_\varrho(\xi)-1).\end{aligned} \tag{4.1.11}$$

Replacing $\mathscr{D}$ by R in relation (2.1.53) from Chapter 2 and applying the formula $\Lambda(\xi) = (\mathbf{d}(\xi), \boldsymbol{\varphi}_0(\xi))_k$ which follows from equation (2.1.55) from Chapter 2, we conclude that (4.1.8) follows from (4.1.11). □

Corollary 4.1.5. *Suppose that all assumptions of Proposition 4.1.4 are satisfied. In addition, suppose the* τ*-roots of the polynomials* $P(\xi;\tau)$ *and* $H_+(\xi;\tau)$ *are simple for almost all* $\xi \in \mathbb{R}^{n-1}$*. Then the function* $\Lambda(\xi)$*, defined by (4.1.2), has for a.e. in* $\mathbb{R}^{n-1}$ *the expression*

$$\Lambda(\xi) = \sum_{\nu,\varrho=1}^{J} \overline{\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))}\,\overline{R(\xi; z_\varrho(\xi))}\, R(\xi;\zeta_\nu(\xi)), \tag{4.1.12}$$

where

$$\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi)) = \mathrm{i}\,\frac{\prod_{j=1}^{J}(z_\varrho(\xi) - \overline{\zeta}_j(\xi)) \prod_{\varkappa\neq\varrho}(z_\varkappa(\xi) - \overline{\zeta}_\nu(\xi))}{\prod_{j\neq\varrho}(z_j(\xi) - z_\varrho(\xi)) \prod_{j\neq\nu}(\overline{\zeta}_\nu(\xi) - \overline{\zeta}_j(\xi))}. \tag{4.1.13}$$

Proof. Formula (4.1.12) follows directly from (4.1.8). Using the Cauchy formula for determinants of the type $\det\{(a_\lambda + b_\mu)^{-1}\}$ (see, for example, [PS56], p. 112), we see that the elements $\lambda_{\nu\varrho}$ of the matrix inverse to (4.1.7) are calculated in accordance with (4.1.13). □

By Theorem 4.1.1, each of the newly obtained representations of $\Lambda(\xi)$ corresponds to a specific version of the criterion for the validity of (4.0.1). We formulate some of them below.

Theorem 4.1.1 and Proposition 4.1.2 imply

Corollary 4.1.6. *The estimate (4.0.1) is valid if and only if*

$$\left|\operatorname{Im}\frac{s_0(\xi)}{h_0(\xi)}\right| \leqslant \frac{\text{const}}{B(\xi)} \quad \text{a.e. in } \mathbb{R}^{n-1}. \tag{4.1.14}$$

Here, the functions $s_0(\xi)$ *and* $h_0(\xi)$ *are the same as in Proposition 4.1.2.*

Theorem 4.1.1 and Corollary 4.1.3 imply

Corollary 4.1.7. *Suppose that the leading coefficients of the polynomials* $P(\xi;\tau)$ *and* $H_+(\xi;\tau)$ *coincide a.e. in* $\mathbb{R}^{n-1}$*. The estimate*

$$\langle\!\langle u \rangle\!\rangle^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \|u\|^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+) \tag{4.1.15}$$

is valid if and only if

$$\operatorname{Im}\sum_{\varrho=1}^{J}(z_\varrho(\xi) + \zeta_\varrho(\xi)) \leqslant \frac{\text{const}}{B(\xi)} \quad \text{a.e. in } \mathbb{R}^{n-1}. \tag{4.1.16}$$

Here, $z_\varrho(\xi)$ and $\zeta_\varrho(\xi)$ ($\varrho = 1, \dots, J$) are the τ-roots (counting multiplicities) of the polynomials $P(\xi;\tau)$ and $H_+(\xi;\tau)$, respectively.

Theorem 4.1.1 and Corollary 4.1.5 imply

Corollary 4.1.8. *Suppose that the τ-roots $z_\varrho(\xi)$ and $\zeta_\varrho(\xi)$ of the polynomials $P(\xi;\tau)$ and $H_+(\xi;\tau)$, respectively, are simple a.e. in $\mathbb{R}^{n-1}$. The estimate (4.0.1) is valid if and only if*

$$\sum_{\nu,\varrho=1}^{J} \overline{\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))}\, \overline{R(\xi; z_\varrho(\xi))}\, R(\xi; \zeta_\nu(\xi)) \leqslant \frac{\text{const}}{B(\xi)} \quad \text{a.e. in } \mathbb{R}^{n-1}, \tag{4.1.17}$$

where $\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))$ are the functions defined by (4.1.13).

4.1.2 Results concerning the estimate (4.0.2)

In this subsection we give two statements on necessary and sufficient conditions for the validity of (4.0.2).

Let the polynomials $H_+(\xi;\tau)$ and $H_-(\xi;\tau)$ be the same as in the beginning of Section 4.1, and let

$$\Omega(\xi;\eta,\tau) = (\eta-\tau)^{-1}[H_+(\xi;\eta)R(\xi;\tau) - H_+(\xi;\tau)R(\xi;\eta)], \tag{4.1.18}$$

$\xi \in \mathbb{R}^{n-1}$, $\eta, \tau \in \mathbb{R}^1$.

Theorem 4.1.9. *The estimate (4.0.2) holds if and only if the inequality*

$$\frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\Delta_1(\xi;\eta,\tau)|^2 + |\Delta_2(\xi;\eta,\tau)|^2}{(|P(\xi;\tau)|^2+1)(|P(\xi;\eta)|^2+1)} d\tau\, d\eta \leqslant \frac{\text{const}}{B(\xi)} \tag{4.1.19}$$

is satisfied for almost all $\xi \in \mathbb{R}^{n-1}$ and all $\tau \in \mathbb{R}^1$. Here, $\Delta_1(\xi;\eta,\tau)$ is the quotient and $\Delta_2(\xi;\eta,\tau)$ is the remainder of the division (w.r.t. τ) of $\Omega(\xi;\eta,\tau)H_-(\xi;\tau)$ by $P(\xi;\tau)$ (Ω is the polynomial (4.1.18)).

This result follows directly from Theorem 3.2.4, Chapter 3. It suffices to put there $m = 2$, $P_1(\xi;\tau) = P(\xi;\tau)$ and $P_2(\xi;\tau) = 1$. Since $\Pi_+(\xi;\tau) = 1$, condition (3.2.8) from Chapter 3 can be omitted. It is obvious that $\Delta_j = D_j(\xi;\eta,\tau)H_+(\xi;\eta)$ ($j = 1, 2$), where D_j are the polynomials appearing in inequality (3.2.9) from Chapter 3. Thus, that inequality takes the form (4.1.19).

Remark 4.1.10. A direct verification shows that the polynomials Δ_1, Δ_2 figuring in (4.1.19) can be expressed in terms of the polynomials T_1, T_2, defined in Theorem

4.1.1, as follows:

$$\begin{aligned}\Delta_1(\xi;\eta,\tau) &= (\eta-\tau)^{-1}[(\overline{P}(\xi;\eta)-\overline{P}(\xi;\tau))R(\xi;\eta)\\ &\quad -(T_1(\xi;\eta)-T_1(\xi;\tau))\,H_+(\xi;\eta)],\\ \Delta_2(\xi;\eta,\tau) &= (\eta-\tau)^{-1}[(P(\xi;\tau)T_1(\xi;\eta)+T_2(\xi;\tau))H_+(\xi;\eta)\\ &\quad -\left(P(\xi;\tau)\overline{P}(\xi;\eta)+1\right)R(\xi;\eta)].\end{aligned}$$

The polynomial Δ_2 can also be written in the form

$$\begin{aligned}\Delta_2(\xi;\eta,\tau) &= (\eta-\tau)^{-1}\{(T_1(\xi;\eta)-T_1(\xi;\tau))P(\xi;\tau)H_+(\xi;\eta)\\ &\quad -[R(\xi;\eta)(P(\xi;\tau)\overline{P}(\xi;\eta)+1)-R(\xi;\tau)H_-(\xi;\tau)H_+(\xi;\eta)]\}.\end{aligned}$$

To conclude this subsection we consider the estimate (4.0.2) for the case $R(\xi;\tau)=1$. Let $P_+(\xi;\tau)$ be a polynomial (of τ) with leading coefficient 1, and let its roots coincide (counting multiplicities) with all τ-roots of P in the half-plane $\operatorname{Im}\zeta \geqslant 0$, $\zeta=\tau+\mathrm{i}\sigma$.

Proposition 4.1.11. *The estimate*

$$\|u\|^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2+\|u\|^2\right),\quad u\in C_0^\infty(\mathbb{R}^n_+) \tag{4.1.20}$$

is valid if and only if the following conditions are satisfied:

1. *$B(\xi)\leqslant C$ for almost all $\xi\in\{\xi\in\mathbb{R}^{n-1}:P_+(\xi;\tau)\neq 1\}$;*
2. *$B(\xi)\leqslant C\left(1+|P(\xi;\tau)|^2\right)$ for almost all $\xi\in\{\xi\in\mathbb{R}^{n-1}:P_+(\xi;\tau)=1\}$ and all $\tau\in\mathbb{R}^1$.*

Proof. It is easy to show that (4.1.20) is true if and only if for almost all $\xi\in\mathbb{R}^{n-1}$

$$\int_0^\infty |v(t)|^2dt \leqslant C(B(\xi))^{-1}\left[\int_0^\infty |P\,(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt+\int_0^\infty |v(t)|^2dt\right],\quad v\in C_0^\infty(\mathbb{R}^1_+) \tag{4.1.21}$$

or, equivalently,

$$\left(1-\frac{C}{B(\xi)}\right)\int_0^\infty |v(t)|^2dt \leqslant \frac{C}{B(\xi)}\int_0^\infty |P\,(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt,\quad v\in C_0^\infty(\mathbb{R}^1_+). \tag{4.1.22}$$

The sufficiency of (4.1.21) is proved by substitution $v_\xi(t)=\hat{u}(\xi;t)$; the necessity is shown by the method of localization in ξ (cf. the proof of Theorem 3.2.2, Chapter 3).

Suppose that for some $\xi \in \mathbb{R}^{n-1}$ we have $P_+(\xi;\tau) \neq 1$. In this case, it follows from Proposition 3.3.2 of Chapter 3 that (4.1.22) is impossible, if $C(B(\xi))^{-1} < 1$. On the other hand, the validity of (4.1.22) for $C(B(\xi))^{-1} \geqslant 1$ is obvious.

Consider now such $\xi \in \mathbb{R}^{n-1}$ that $P_+(\xi;\tau) = 1$. Without loss of generality, we can assume that $C(B(\xi))^{-1} \geqslant 1$. However, this means that (4.1.22) is equivalent to the inequality

$$\int_0^\infty |v(t)|^2 dt \leqslant \frac{C}{B(\xi)-C}\int_0^\infty |P\,(\xi;-\mathrm{i}\,d/dt)\,v|^2\,dt, \quad v \in C_0^\infty(\mathbb{R}^1_+). \tag{4.1.23}$$

From Proposition 3.1.12 of Chapter 3 it follows that the exact constant in (4.1.23) is equal to $\sup|P|^{-2}$. Hence, if $P_+(\xi;\tau) = 1$, then (4.1.22) holds true if and only if $|P(\xi;\tau)|^{-2} \leqslant C\,[B(\xi)-C]^{-1}$, or (equivalently) if $B(\xi) \leqslant C\,\left(1+|P(\xi;\tau)|^2\right)$. □

4.2 Quasielliptic polynomials

In this section, we study the criteria, established in Theorems 4.1.1 and 4.1.9, in the case, where $R(\xi;\tau) = \tau^s$ and $P(\xi;\tau)$ is a quasielliptic polynomial w.r.t. the variables ξ, τ.

4.2.1 Polynomials with a generalized-homogeneous principal part

In this subsection, we consider the case $R(\xi;\tau) = \tau^s$ and establish an upper estimate of $\Lambda(\xi)$, defined by (4.1.2), for the polynomials $P(\xi;\tau)$ with generalized homogeneous principal part.

Let us define the notion of the generalized-homogeneous principal part of the polynomial $P(\xi;\tau)$. Suppose that the polynomial

$$P(\xi;\tau) = \sum a_{\boldsymbol{\alpha}} \xi_1^{\alpha_1} \cdots \xi_{n-1}^{\alpha_{n-1}} \tau^{\alpha_n} \tag{4.2.1}$$

of the variables $(\xi;\tau) \in \mathbb{R}^n$ satisfies the following conditions:

1. $m_1,\dots,m_n$ are natural numbers, and $m = \max\limits_{1\leqslant\varrho\leqslant n} m_\varrho$.
2. $\mathbf{q} = (q_1,\dots,q_{n-1},q_n)$, where $q_\varrho = m m_\varrho^{-1} \quad (1 \leqslant \varrho \leqslant m)$.
3. $(\boldsymbol{\alpha},\mathbf{q}) = \alpha_1 q_1 + \cdots + \alpha_{n-1}q_{n-1} + \alpha_n q_n$.
4. The sum on the right-hand side of (4.2.1) runs over all multi-indices $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ such that $(\boldsymbol{\alpha},\mathbf{q}) \leqslant m$.

The polynomial

$$P_0(\xi;\tau) = \sum_{(\boldsymbol{\alpha},\mathbf{q})=m} a_{\boldsymbol{\alpha}} \xi_1^{\alpha_1} \cdots \xi_{n-1}^{\alpha_{n-1}} \tau^{\alpha_n} \tag{4.2.2}$$

is called the *generalized-homogeneous main part* (w.r.t. the weight $\mathbf{q}$) of the polynomial (4.2.1).

We will also write $P(\xi;\tau)$ as $P(\xi;\tau) = \sum_{k=0}^{m_n} p_{m_n-k}(\xi)\tau^k$ and restrict ourselves to the case $p_0(\xi) = 1$. For $\xi \in \mathbb{R}^{n-1}$ we set

$$\langle\xi\rangle^m = \sum_{\varrho=1}^{n-1} |\xi_\varrho|^{m_\varrho}. \tag{4.2.3}$$

Proposition 4.2.1. *Let $R(\xi;\tau) = \tau^s$ $(0 \leqslant s \leqslant m_n - 1)$, and let the polynomial $P(\xi;\tau)$ satisfy conditions* 1–4. *Then the function $\Lambda(\xi)$, defined by* (4.1.2)*, admits the estimate*

$$\Lambda(\xi) \leqslant C\,(1 + \langle\xi\rangle)^{(2s+1)m/m_n} \tag{4.2.4}$$

for all $\xi \in \mathbb{R}^{n-1}$. In particular, if the hyperplane $t = 0$ is not characteristic for the operator $P(D)$, then

$$\Lambda(\xi) \leqslant C\,(1 + |\xi|)^{2s+1}. \tag{4.2.5}$$

Proof. We set $d(\xi) = (1 + \langle\xi\rangle)^{m/m_n}$. From the estimates $|\xi_\varrho| \leqslant \langle\xi\rangle^{q_\varrho}$ $(\varrho = 1, \dots, n-1)$ we obtain for all $\xi \in \mathbb{R}^{n-1}$ the inequalities

$$p_s(\xi) \leqslant c\langle\xi\rangle^{\alpha_1 q_1 + \dots + \alpha_{n-1}q_{n-1}} \leqslant c\langle\xi\rangle^{(\boldsymbol{\alpha},\mathbf{q})-(m_n-s)m/m_n} \leqslant c[d(\xi)]^s, \tag{4.2.6}$$

where $s = 0, 1, \dots, m_n$. The coefficient $g_s(\xi)$ in front of the term τ^s in the polynomial $H(\xi;\tau) = |P(\xi;\tau)|^2 + 1$ is equal to $\sum_{k+t=s} p_k(\xi)\bar{p}_t(\xi)$. This means that $|g_s(\xi)| \leqslant C[d(\xi)]^s$ $(s = 0, 1, \dots, 2m_n)$.

Let τ be a root of the polynomial $H_-(\xi;\tau) = \sum_{s=0}^{m_n} \overline{h_{m_n-s}(\xi)}\tau^s$ with $h_0(\xi) = 1$. Then τ is also the root of the polynomial $H(\xi;\tau)$, and therefore

$$|\tau| \leqslant c\sum_{s=0}^{2m_n} |g_s(\xi)|^{1/s}.$$

Hence, each root τ of the polynomial $H(\xi;\tau)$ satisfies $|\tau| \leqslant c\,d(\xi)$. Since the coefficients $\overline{h_s(\xi)}$ are symmetric functions of the roots of $H_-(\xi;\tau)$, the inequality

$$|\bar{h}_s(\xi)| \leqslant c[d(\xi)]^s \qquad (s = 0, 1, \dots, m_n) \tag{4.2.7}$$

holds for all $\xi \in \mathbb{R}^{n-1}$.

We now consider the relation

$$\tau^s H_-(\xi;\tau) = P(\xi;\tau)T_1(\xi;\tau) + T_2(\xi;\tau), \tag{4.2.8}$$

which determines the polynomials T_1 and T_2. These polynomials can also be expressed as

$$T_1(\xi;\tau) = \tau^s + t_1^{(1)}(\xi)\tau^{s-1} + \dots + t_{s-1}^{(1)}(\xi)\tau + t_s^{(1)}(\xi),$$
$$T_2(\xi;\tau) = t_1^{(2)}(\xi)\tau^{m_n-1} + \dots + t_{m_n-1}^{(2)}(\xi)\tau + t_{m_n}^{(2)}(\xi).$$

Equating the coefficients of like powers of τ in both sides of (4.2.8), we obtain the relations

$$\bar{h}_k(\xi) = \begin{cases} \sum_{\varrho=0}^{k} p_{k-\varrho}(\xi) t_\varrho^{(1)}(\xi) & (k = 0, \dots, s), \\ t_{k-s}^{(2)}(\xi) + \sum_{\varrho=0}^{s} t_\varrho^{(1)}(\xi) p_{k-\varrho}(\xi) & (k = s+1, \dots, m_n), \end{cases}$$

$$t_{k-s}^{(2)}(\xi) + \sum_{\varrho=k-m_n}^{s} t_\varrho^{(1)}(\xi) p_{k-\varrho}(\xi) = 0 \qquad (k = m_n + 1, \dots, m_n + s)$$

(here $t_0^{(1)}(\xi) = 1$). Determining the coefficients of $T_1(\xi;\tau)$ and $T_2(\xi;\tau)$ from these relations and using (4.2.6) and (4.2.7), we see that the inequalities

$$|t_\varrho^{(1)}(\xi)| \leqslant c[d(\xi)]^\varrho \qquad (\varrho = 0, \dots, s), \tag{4.2.9}$$

$$|t_k^{(2)}(\xi)| \leqslant C[d(\xi)]^{k+s} \qquad (k = 1, \dots, m_n) \tag{4.2.10}$$

hold for all $\xi \in \mathbb{R}^{n-1}$. Let $\xi \in \mathbb{R}^{n-1}$, let $c_1 > 0$ be a sufficiently large constant, and let $I_1(\xi) = \{\tau : \tau \in \mathbb{R}^1, |\tau| \leqslant c_1 d(\xi)\}$ and $I_2(\xi) = \mathbb{R}^1 \setminus I_1(\xi)$. Consider the representation

$$\int_{-\infty}^{\infty} \frac{|T_1|^2 + |T_2|^2}{|P|^2 + 1} d\tau = \int_{I_1(\xi)} \frac{|T_1|^2 + |T_2|^2}{|P|^2 + 1} d\tau + \int_{I_2(\xi)} \frac{|T_1|^2 + |T_2|^2}{|P|^2 + 1} d\tau. \tag{4.2.11}$$

The first integral on the right-hand side of (4.2.11) is estimated with the help of (4.2.8) and (4.2.9):

$$\begin{aligned} \int_{I_1(\xi)} \frac{|T_1|^2 + |T_2|^2}{|P|^2 + 1} d\tau &\leqslant c \int_0^{c_1 d(\xi)} \frac{|T_1|^2 + |\tau^s H_- - P T_1|^2}{|P|^2 + 1} d\tau \\ &\leqslant c \int_0^{c_1 d(\xi)} \left(|T_1|^2 + |T_1|^2 |P|^2 + \tau^{2s} |H_-|^2\right) \frac{d\tau}{|H_-|^2} \\ &\leqslant c \int_0^{c_1 d(\xi)} \left(|T_1|^2 + \tau^{2s}\right) d\tau \leqslant c[d(\xi)]^{2s+1}. \end{aligned}$$

The second integral on the right-hand side of (4.2.11) is estimated with the help of (4.2.6) and (4.2.10):

$$\int\limits_{I_2(\xi)} \frac{|T_1|^2 + |T_2|^2}{|P|^2 + 1} d\tau \leqslant c \int\limits_{c_1 d(\xi)}^{\infty} \frac{\tau^{2s} + \tau^{2(m_n - 1)}[d(\xi)]^{2(s+1)}}{\tau^{2m_n} + 1}$$
$$\leqslant c \int\limits_{c_1 d(\xi)}^{\infty} [d(\xi)]^{2(s+1)} \tau^{-2} d\tau = c[d(\xi)]^{2s+1}.$$

Thus, the estimate (4.2.4) is established for all $\xi \in \mathbb{R}^{n-1}$.

Suppose now that the hyperplane $t = 0$ is not characteristic for the operator $P(D)$. We set $m_\varrho = m_n = J$ ($\varrho = 1, \ldots, n-1$) in the conditions 1–4 that define the polynomial (4.2.1). Then the norms $|\xi|$ and $\langle \xi \rangle$ are equivalent in $\mathbb{R}^{n-1}$, and (4.2.5) follows from (4.2.4). □

4.2.2 The estimate (4.2.16) for quasielliptic polynomials of type $l \geqslant 1$

The main result of this subsection is Theorem 4.2.3, which shows that the inequality (4.2.17) is a criterion for the validity of (1.2.16) for quasielliptic polynomials $P(\xi; \tau)$ of type $l \geqslant 1$.

The polynomial (4.2.1) is called a *polynomial of type* l, if for all $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$ the number of the τ-roots of the polynomial $P_0(\xi; \tau)$ lying in the half-plane $\operatorname{Im} \zeta > 0$, $\zeta = \tau + i\sigma$, is equal to l. The polynomial (4.2.1) is called *quasielliptic*, if

$$|P_0(\xi; \tau)| \geqslant c\,(\langle \xi \rangle^m + |\tau|^{m_n}) \tag{4.2.12}$$

for all $(\xi; \tau) \in \mathbb{R}^n$.

Now we show that, for quasielliptic polynomials of type $l \geqslant 1$, the function $\Lambda(\xi)$ admits a lower bound, which is the opposite of (4.2.4).

Proposition 4.2.2. *Let $P(\xi; \tau)$ be a quasielliptic polynomial of type $l \geqslant 1$, and let $R(\xi; \tau) = \tau^s$ ($s = 0, \ldots, m_n - 1$). Then the function $\Lambda(\xi)$, defined by (4.1.2), satisfies the estimate*

$$\Lambda(\xi) \geqslant c\,(1 + \langle \xi \rangle)^{(2s+1)m/m_n} \tag{4.2.13}$$

for all $\xi \in \mathbb{R}^{n-1}$. In particular, if $P(\xi; \tau)$ is a properly elliptic polynomial of even order, then

$$\Lambda(\xi) \geqslant c\,(1 + |\xi|)^{2s+1}. \tag{4.2.14}$$

Proof. If $P(\xi; \tau)$ is a quasielliptic polynomial of type $l \geqslant 1$, then one can easily show that there exists a τ-root $\tau = z(\xi)$ of $P(\xi; \tau)$ such that the estimate

$$\operatorname{Im} z(\xi) \geqslant c\,(1 + \langle \xi \rangle)^{m/m_n} \tag{4.2.15}$$

holds true for large $|\xi|$.

Let $\eta(\xi)$ be a piecewise continuous function in $\mathbb{R}^{n-1}$ which is equal to $z(\xi)$ for large values of $\langle\xi\rangle$ and satisfies $\operatorname{Im}\eta(\xi) \geqslant c$ for all $\xi \in \mathbb{R}^{n-1}$. For each $\xi \in \mathbb{R}^{n-1}$ the function $u(\xi;t) = \exp(\mathrm{i}\eta(\xi)t)$ exponentially tends to zero as $t \to \infty$. Therefore, in accordance with Lemma 2.1.8, Chapter 2, one can approximate u for each fixed $\xi \in \mathbb{R}^{n-1}$ by a sequence $u_\varrho \in C_0^\infty(\mathbb{R}^1_+)$ such that $(u_\varrho - u)^{(s)}\big|_{t=0} = 0$ $(s = 0, \dots, m_n - 1)$ and

$$\lim_{\varrho\to\infty} \int_0^\infty \left[\left| P(\xi; -\mathrm{i}\, d/dt)(u_\varrho - u)\right|^2 + |u_\varrho - u|^2 \right] dt = 0.$$

Since $\Lambda(\xi)$ defined by (4.1.2) is the sharp constant in the inequality

$$|v^{(s)}\big|_{t=0}|^2 \leqslant \Lambda(\xi) \left[\int_0^\infty |P(\xi; -\mathrm{i}\, d/dt)\, v|^2 + |v|^2 \right] dt, \quad v \in C_0^\infty(\mathbb{R}^1_+),$$

the function $u(\xi;t)$ satisfies for all $\xi \in \mathbb{R}^{n-1}$ the inequalities

$$|u^{(s)}(\xi;t)\big|_{t=0}|^2 \leqslant \Lambda(\xi) \int_0^\infty \left[|P(\xi; -\mathrm{i}\, d/dt)\, u(\xi;t)|^2 + |u(\xi;t)|^2 \right] dt$$

$(s = 0, \dots, m_n - 1)$. Hence $\Lambda(\xi) \geqslant c$ for all $\xi \in \mathbb{R}^{n-1}$, and the estimate

$$\Lambda(\xi) \geqslant 2 \operatorname{Im} z(\xi) |z(\xi)|^s \geqslant c\, (1 + \langle\xi\rangle)^{(2s+1)m/m_n}$$

is satisfied for large $\langle\xi\rangle$.

If $P(\xi;\tau)$ is a properly elliptic polynomial of even order J, then $J = 2$, $m_\varrho = m_n = m = J$ $(\varrho = 1, \dots, n-1)$, $l = J/2 \geqslant 1$, while the norms $|\xi|$ and $\langle\xi\rangle$ are equivalent in $\mathbb{R}^{n-1}$, and the estimate $|P_0(\xi;\tau)| \geqslant c\left(|\xi|^2 + \tau^2\right)^{J/2}$ holds for all $\xi \in \mathbb{R}^{n-1}$. This means that in this case (4.2.14) follows from (4.2.13). □

We now turn to the main result of this subsection.

Theorem 4.2.3. *Let $P(\xi;\tau)$ be a quasielliptic polynomial (4.2.1) of type $l \geqslant 1$, and let $P_0(\xi) = 1$. The estimate*

$$\left\langle\!\!\left\langle \frac{\partial^s u}{\partial t^s} \right\rangle\!\!\right\rangle^2_{B^{1/2}} \leqslant C\left(\|P(D)u\|^2 + \|u\|^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{4.2.16}$$

$(s = 0, \dots, m_n - 1)$*, holds true if and only if*

$$B(\xi)\left(1 + \langle\xi\rangle^{(2s+1)m/m_n}\right) \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.2.17}$$

In particular, if $P(\xi;\tau)$ is a properly elliptic polynomial of even order J, then the estimate (4.2.16) with $s = 0, \dots, J-1$ is valid if and only if

$$B(\xi)\left(1 + |\xi|^{2s+1}\right) \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.2.18}$$

This theorem follows immediately from Theorem 4.1.1 and Propositions 4.2.1 and 4.2.2.

Remark 4.2.4. The quasiellipticity condition for the polynomial $P(\xi;\tau)$ and the requirement $l \geqslant 1$ cannot, in general, be omitted in the formulation of Theorem 4.2.3. This is demonstrated by the following examples.

1. Let $P(D) = \dfrac{\partial}{\partial t} + \sum\limits_{\varrho=1}^{n-1} \dfrac{\partial^2}{\partial x_\varrho^2}$ be the backward heat operator, let $s = 0$, and let $R(\xi;\tau) = 1$. The polynomial $P(\xi;\tau) = \mathrm{i}\tau - |\xi|^2$ is quasielliptic, ($m_1 = \cdots = m_{n-1} = 2, m_n = 1, m = 2$), but its root $\tau = -\mathrm{i}|\xi|^2$ (here $P(\xi;\tau) = P_0(\xi;\tau)$) lies in the half-plane $\operatorname{Im}\zeta < 0$, if $|\xi| \neq 0$. Since, $P(\xi;\tau) = \mathrm{i}\tau - |\xi|^2$, we obtain for $|\xi| \to \infty$ the equalities:
$$H_+(\xi;\tau) = -\mathrm{i}\tau - (|\xi|^4 + 1)^{1/2}, \quad H_-(\xi;\tau) = \mathrm{i}\tau - (|\xi|^4 + 1)^{1/2},$$
$$T_1(\xi;\tau) = 1, \quad T_2(\xi;\tau) = |\xi|^2 - (|\xi|^4 + 1)^{1/2},$$
$$\Lambda(\xi) = (|\xi|^4 + 1)^{1/2} - |\xi|^2 = O(|\xi|^{-2}).$$
Thus, $\Lambda(\xi)$ does not satisfy inequality (4.2.13) (in this example we have $|\xi| = \langle\xi\rangle$).

2. Let $P(D) = \mathrm{i}\dfrac{\partial}{\partial t} - \sum\limits_{\varrho=1}^{n-1} \dfrac{\partial^2}{\partial x_\varrho^2}$ be the Schrödinger operator, and let $s = 0$. The polynomial $P(\xi;\tau) = -\tau + |\xi|^2$ is generalized homogeneous, ($m_1 = \cdots = m_{n-1} = 2$, $m_n = 1$, $m = 2$), but not quasielliptic, since its root $\tau = |\xi|^2$ is real. It can be verified directly that $H_+(\xi;\tau) = -\tau + |\xi|^2 + \mathrm{i}$, $H_-(\xi;\tau) = -\tau + |\xi|^2 - \mathrm{i}$, $T_1(\xi;\tau) = 1$, $T_2(\xi;\tau) = \mathrm{i}$, and $\Lambda(\xi) = 1$. Hence, in this example $\Lambda(\xi)$ again does not satisfy (4.2.13) (similarly to the above example, we have $|\xi| = \langle\xi\rangle$).

4.2.3 The estimate (4.2.19) for quasielliptic polynomials of type $l \geqslant 1$

In this subsection, we show that (4.2.19) holds for quasielliptic polynomials $P(\xi;\tau)$ of type $l \geqslant 1$ if and only if (4.2.20) is fulfilled a.e. in $\mathbb{R}^{n-1}$ (see Theorem 4.2.5).

Theorem 4.2.5. *Let $P(\xi;\tau)$ be a quasielliptic polynomial* (4.2.1) *of type $l \geqslant 1$, and let $p_0(\xi) = 1$. The estimate*
$$\left\| \frac{\partial^s u}{\partial t^s} \right\|_{B^{1/2}} \leqslant C \left(\|P(D)u\|^2 + \|u\|^2 \right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{4.2.19}$$
where $s = 0, \dots, m_n$, holds true if and only if
$$B(\xi)\left(1 + \langle\xi\rangle^{2sm/m_n}\right) \leqslant \text{const} \tag{4.2.20}$$

for almost all $\xi \in \mathbb{R}^{n-1}$*. In particular, if* $P(\xi;\tau)$ *is a properly elliptic polynomial of even order* m*, then the estimate* (4.2.19)*, where* $s = 0,\dots,m$*, holds if and only if*

$$B(\xi)\left(1+|\xi|^{2s}\right) \leqslant \text{const} \tag{4.2.21}$$

a.e. in $\mathbb{R}^{n-1}$*.*

Proof. Necessity. Let $u(\xi;t)$ be the function considered in the proof of Proposition 4.2.2. Since it tends exponentially to zero as $t \to \infty$, the estimate

$$B(\xi)\int_0^\infty \left|\frac{\partial^s u(\xi;t)}{\partial t^s}\right|^2 dt \leqslant C\int_0^\infty \left[|P\,(\xi;-\mathrm{i}\,d/dt)\,u(\xi;t)|^2 + |u(\xi;t)|^2\right]dt$$

with $s = 0,\dots,m_n$ holds a.e. in $\mathbb{R}^{n-1}$. This means that $B(\xi) \leqslant$ const a.e. in $\mathbb{R}^{n-1}$, and for large values of $\langle\xi\rangle$ the inequality $B(\xi)|z(\xi)|^{2s} \leqslant$ const holds true. Taking into account (4.2.15), we obtain (4.2.20).

Sufficiency. We estimate the left-hand side of (4.1.19). First, we show that for a quasielliptic polynomial P satisfying conditions 1–4,

$$\sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2+1} \leqslant \text{const}[d(\xi)]^{2(s-m_n)}. \tag{4.2.22}$$

Indeed, we have $\sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{\langle\xi\rangle^{2m}+\tau^{2m_n}+1} \leqslant$ const, if $\langle\xi\rangle \leqslant 1$. Since

$$\sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{\langle\xi\rangle^{2m}+\tau^{2m_n}+1} = \langle\xi\rangle^{2m(s/m_n-1)} \sup_{\tau\in\mathbb{R}^1} \frac{\tau_1^{2s}}{1+\tau_1^{2m_n}+\langle\xi\rangle^{-2m}}$$

and $\sup_{\tau\in\mathbb{R}^1} \dfrac{\tau_1^{2s}}{1+\tau_1^{2m_n}+\langle\xi\rangle^{-2m}} \leqslant$ const for $\langle\xi\rangle > 1$, we see that the inequality

$$\sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{\langle\xi\rangle^{2m}+\tau^{2m_n}+1} \leqslant \text{const}[d(\xi)]^{2(s-m_n)} \tag{4.2.23}$$

is fulfilled for all $\xi \in \mathbb{R}^{n-1}$.

On the other hand, the quasiellipticity of the polynomial P implies the estimate

$$|P(\xi;\tau)| \geqslant \text{const}\left(\langle\xi\rangle^{m}_{m_n} + \tau^{m_n}\right), \tag{4.2.24}$$

provided that $\langle\xi\rangle^m + |\tau|^{m_n} \geqslant c_1$, where c_1 is a constant depending only on the numbers m_ϱ and the coefficients $a_{\boldsymbol{\alpha}}$ of the polynomial P. Since obviously

$$\sup_{\langle\xi\rangle^m+|\tau|^{m_n}\leqslant c_1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2+1} \leqslant \text{const},$$

(4.2.22) follows from (4.2.23) and (4.2.24).

Let us show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\Delta_1(\xi;\eta,\tau)|^2+|\Delta_2(\xi;\eta,\tau)|^2}{(|P(\xi;\eta)|^2+1)(|P(\xi;\tau)|^2+1)}d\tau d\eta \leqslant \mathrm{const}[d(\xi)]^{2s}, \tag{4.2.25}$$

where Δ_1 and Δ_2 are the polynomials defined in Theorem 4.1.9. Let

$$H_+(\xi;\tau)=\sum_{k=0}^{m_n}h_{m_n-k}(\xi)\tau^k, \qquad P(\xi;\tau)=\sum_{k=0}^{m_n}p_{m_n-k}(\xi)\tau^k,$$

and let $\Omega(\xi;\eta,\tau)$ be the polynomial (4.1.18). Dividing $\tau^s\eta^k-\eta^s\tau^k$ by $(\eta-\tau)$, we find that

$$\Omega(\xi;\eta,\tau)=\sum_{j=1}^{m_n}\varphi_j(\xi;\eta,s)\tau^{j-1}, \tag{4.2.26}$$

where $R(\xi;\tau)=\tau^s$ and

$$\varphi_j(\xi;\eta,s)=\begin{cases}-\sum\limits_{k=0}^{j-1}h_{m_n-k}(\xi)\eta^{s+k-j}, & 1\leqslant j\leqslant s,\\ \sum\limits_{k=j}^{m_n}h_{m_n-k}(\xi)\eta^{s+k-j}, & s+1\leqslant j\leqslant m_n.\end{cases} \tag{4.2.27}$$

Let T_{1j}, T_{2j} be the quotient and the reminder of the division of the polynomial (of τ) $\tau^{j-1}H_-(\xi;\tau)$ by $P(\xi;\tau)$. According to Proposition 4.2.1,

$$\int_{-\infty}^{\infty}\frac{|T_{1j}(\xi;\tau)|^2+|T_{2j}(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1}d\tau \leqslant \mathrm{const}[d(\xi)]^{2j-1}. \tag{4.2.28}$$

From the estimates (4.2.6) for $p_s(\xi)$ we get the inequality $|P(\xi;\tau)|^2+1 \geqslant \mathrm{const}(\eta^{2m_n}+1)$ provided that c is a sufficiently large constant and $|\eta|\geqslant cd(\xi)$. Let $I_1(\xi)$ and $I_2(\xi)$ be the intervals defined in the proof of Proposition 4.2.1. It follows from (4.2.22) that

$$\int_{I_1(\xi)}\frac{\eta^{2(s+k-j)}}{|P(\xi;\eta)|^2+1}d\eta \leqslant \mathrm{const}[d(\xi)]^{2(s+k-j-m_n)+1}. \tag{4.2.29}$$

On the other hand, in view of the choice of c we have

$$\begin{aligned}\int_{I_2(\xi)}\frac{\eta^{2(s+k-j)}}{|P(\xi;\eta)|^2+1}d\eta &\leqslant \mathrm{const}\int_{cd(\xi)}\frac{\eta^{2(s+k-j)}}{\eta^{2m_n}+1}d\eta\\ &\leqslant \mathrm{const}[d(\xi)]^{2(s+k-j-m_n)+1}.\end{aligned} \tag{4.2.30}$$

Based on (4.2.26) and (4.2.27), we see that (4.2.25) follows from the estimates (4.2.28)–(4.2.30) and the inequality (4.2.7) for $h_s(\xi)$. It remains only to note that the estimates (4.2.22) and (4.2.25) imply (4.1.8). □

Remark 4.2.6. In the formulation of Theorem 4.2.5, the condition $l \geqslant 1$ cannot, in general, be omitted. For example, condition (4.2.20) for the backward heat operator $P(D) = \dfrac{\partial}{\partial t} + \sum_{\varrho=1}^{n-1} \dfrac{\partial^2}{\partial x_\varrho^2}$ can be written as

$$B(\xi)\left(1 + |\xi|^{4s}\right) \leqslant \text{const} \qquad (s = 0, 1). \tag{4.2.31}$$

The polynomial $P(\xi;\tau) = \mathrm{i}\tau - |\xi|^2$ is quasielliptic, but for $\xi \neq 0$ it has no roots in the half-plane $\operatorname{Im}\zeta > 0$, $\zeta = \tau + \mathrm{i}\sigma$. We claim that the inequality

$$B(\xi)\,(1 + |\xi|)^{4s-4} \leqslant \text{const} \qquad (s = 0, 1) \tag{4.2.32}$$

is necessary and sufficient for the validity of the estimate (4.2.19) for this operator. Indeed, setting $P(\xi;\tau) = \mathrm{i}\tau - |\xi|^2$, we get

$$\sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} = \left(1 + |\xi|^4\right)^{s-1}. \tag{4.2.33}$$

On the other hand, from the definition of Δ_1 and Δ_2 we obtain directly that

$$\Delta_1 = \left(1 + |\xi|^4\right)^{s/2}, \qquad \Delta_2 = \mathrm{i}\left(1 + |\xi|^4\right)^{s/2}\left[|\xi|^2 - \left(1 + |\xi|^4\right)^{1/2}\right]. \tag{4.2.34}$$

Now from (4.2.33) and (4.2.34) it follows that (4.1.8) is equivalent to (4.2.32).

4.3 Homogeneous polynomials with simple roots

In this section we study criteria for the validity of (4.2.16) and (4.2.19) in the case where $P(\xi;\tau) = \tau^J + p_1(\xi)\tau^{J-1} + \cdots + p_J(\xi)$ is a polynomial (of τ) and its coefficients are continuous, positive homogeneous functions of the variable $\xi \in \mathbb{R}^{n-1}$ such that $\deg p_k(\xi) = k$ and $p_J(\xi) \neq 0$ for $\xi \neq 0$.

We assume that the τ-roots $z_1(\xi), \dots, z_J(\xi)$ are pairwise distinct for all $\xi \neq 0$. Suppose that for all $\varrho = 1, \dots, J$ the functions $\operatorname{Im} z_\varrho(\xi)$ either vanish identically, or preserve the sign on the unit sphere S^{n-2}. It is also assumed that for all $\varrho = 1, \dots, J$ one of the following two conditions holds: either $P(\xi; \bar{z}_\varrho(\xi)) \equiv 0$ or $P(\xi; \bar{z}_\varrho(\xi)) \neq 0$ for all $\xi \neq 0$.

We will use the following inequalities, which follow directly from the assumptions made above:

$$c^{-1}|\xi| \leqslant |z_r(\xi)| \leqslant c|\xi|, \qquad c^{-1}|\xi| \leqslant |z_j(\xi) - z_r(\xi)| \leqslant c|\xi|, \\ (j \neq i,\ j, r = 1, \dots, J). \tag{4.3.1}$$

It is also obvious that the τ-roots of the polynomial $P(\xi;\tau)$ can be enumerated in such a way that for all $\xi \neq 0$ the following conditions are satisfied:

$$\begin{array}{llll}
\operatorname{Im} z_r(\xi) = 0 & & (r \leqslant k_1); & \\
\operatorname{Im} z_r(\xi) < 0 & \text{and} & P(\xi; z_r(\xi)) \neq 0 & (k_1 < r \leqslant k_2); \\
\operatorname{Im} z_r(\xi) > 0 & \text{and} & P(\xi; \bar{z}_r(\xi)) \neq 0 & (k_2 < r \leqslant k_3); \\
\operatorname{Im} z_r(\xi) > 0 & \text{and} & P(\xi; \bar{z}_r(\xi)) = 0 & (k_3 < r \leqslant k_4).
\end{array}$$

For $k_4 < r \leqslant J$ we set $z_r(\xi) = \bar{z}_{r-k_4+k_3}(\xi)$.

4.3.1 Asymptotic representations of the τ-roots of the polynomial $H_+(\xi;\tau)$ as $|\xi| \to \infty$

The main results of this subsection, i.e., the criteria for the validity of (4.2.16) and (4.2.19), are established with the help of some estimates of the function $\Lambda(\xi)$, defined by (4.1.2), and the left-hand side of inequality (4.1.19). These estimates are deduced from the following asymptotic representations of the τ-roots of the polynomial $H_+(\xi;\tau)$ as $|\xi| \to \infty$.

Lemma 4.3.1. *Let $P(\xi;\tau)$ be a polynomial (of τ) having all properties listed above. Then each τ-root $z_\varrho(\xi)$ of the polynomial $P(\xi;\tau)$ corresponds to a τ-root $\zeta_\varrho(\xi)$ of the polynomial $H_+(\xi;\tau)$ such that for $|\xi| \to \infty$ the following asymptotic representations hold:*

$$\zeta_\varrho(\xi) = z_\varrho(\xi) + \mathrm{i}c_\varrho(\theta)|\xi|^{1-J} + O(|\xi|^{1-2J}) \quad (\varrho \leqslant k_1); \tag{4.3.2}$$

$$\zeta_\varrho(\xi) = \bar{z}_\varrho(\xi) + c_\varrho(\theta)|\xi|^{1-2J} + O(|\xi|^{1-4J}) \quad (k_1 < \varrho \leqslant k_2); \tag{4.3.3}$$

$$\zeta_\varrho(\xi) = z_\varrho(\xi) + c_\varrho(\theta)|\xi|^{1-2J} + O(|\xi|^{1-4J}) \quad (k_2 < \varrho \leqslant k_3); \tag{4.3.4}$$

$$\zeta_\varrho(\xi) = z_\varrho(\xi) + c_\varrho(\theta)|\xi|^{1-J} + O(|\xi|^{1-2J}) \quad (k_3 < \varrho \leqslant k_4); \tag{4.3.5}$$

$$\zeta_\varrho(\xi) = z_{\varrho-k_4+k_3}(\xi) + c_\varrho(\theta)|\xi|^{1-J} + O(|\xi|^{1-2J}) \; (k_4 < \varrho \leqslant J); \tag{4.3.6}$$

here $c_\varrho(\theta) \neq 0$ are continuous functions on S^{n-2} such that $c_\varrho(\theta) \geqslant \mathrm{const} > 0$ for $\varrho \leqslant k_1$ and $\theta = \xi/|\xi|$.

Proof. Set $\tau'_\varrho(\xi) = \zeta_\varrho(\xi)|\xi|^{-1}$. Since $P(\xi;\tau)$ is a homogeneous function of degree J w.r.t. $(\xi;\tau)$, we have

$$P(\theta;\tau'_\varrho(\xi))\overline{P}(\theta;\tau'_\varrho(\xi)) + |\xi|^{-2J} = 0. \tag{4.3.7}$$

Suppose that $k_1 < \varrho \leqslant k_2$. We expand the first term in (4.3.7) in powers of $(\tau'_\varrho(\xi) - \bar{z}_\varrho(\theta))$. From the continuity of the function P and inequalities (4.3.1) it follows that

$$\left|\frac{\partial[P\overline{P}(\theta;\bar{z}_\varrho(\theta))]}{\partial\tau}\right| = |P(\theta;\bar{z}_\varrho(\theta))| \left|\frac{\partial\overline{P}(\theta;\bar{z}_\varrho(\theta))}{\partial\tau}\right| \geqslant \mathrm{const} > 0.$$

Dividing both parts of (4.3.7) by $\dfrac{\partial[P\overline{P}(\theta;\bar{z}_\varrho(\theta))]}{\partial\tau}$, we obtain (4.3.3).

Let $k_2 < \varrho \leqslant k_3$. The same arguments as in the previous case show that $\left|\dfrac{\partial[P\overline{P}(\theta;z_\varrho(\theta))]}{\partial\tau}\right| \geqslant \text{const} > 0$ for all $\theta \in S^{n-2}$. We expand the first term in (4.3.7) in powers of $(\tau'_\varrho(\xi) - z_\varrho(\theta))$. Dividing both sides of the resulting inequality by $\dfrac{\partial[P\overline{P}(\theta;\bar{z}_\varrho(\theta))]}{\partial\tau}$, we establish (4.3.4).

Next, let $k_3 < \varrho \leqslant k_4$. Since $P(\xi;z_\varrho(\xi)) = P(\xi;\bar{z}_\varrho(\xi)) = 0$, we have

$$\frac{\partial[P\overline{P}(\theta;z_\varrho(\theta))]}{\partial\tau} = 0.$$

From the definition of k_3, k_4 and inequalities (4.3.1) it follows that

$$\left|\frac{\partial^2[P\overline{P}(\theta;z_\varrho(\theta))]}{\partial\tau^2}\right| = 2\left|\frac{\partial P(\theta;z_\varrho(\theta))}{\partial\tau}\right|\left|\frac{\partial\overline{P}(\theta;z_\varrho(\theta))}{\partial\tau}\right| \geqslant \text{const} > 0$$

for all $\theta \in S^{n-2}$. We expand the first term of (4.3.7) in powers of $(\tau'_\varrho(\xi) - z_\varrho(\theta))$ and divide (4.3.7) by $\dfrac{\partial^2[P\overline{P}(\theta;z_\varrho(\theta))]}{\partial\tau^2}$. Taking the square root of both sides of the resulting equality, we establish (4.3.6). Eq. (4.3.2) is proved in the same way.

Finally, let $k_4 < \varrho \leqslant J$. We expand the first term in (4.3.7) in powers of $(\tau'_\varrho(\xi) - \bar{z}_\varrho(\theta))$. Repeating the arguments used in the proof of (4.3.5) (and replacing $z_\varrho(\theta)$ by $\bar{z}_\varrho(\theta)$), we arrive at (4.3.6). □

Remark 4.3.2. From (4.3.2)–(4.3.6) it follows that for large values of $|\xi|$ the τ-roots of the polynomial $H_+(\xi;\tau)$ satisfy the estimates

$$\begin{aligned}
&c^{-1}|\xi| \leqslant |\zeta_\tau(\xi)| \leqslant c|\xi| \qquad (1 \leqslant r \leqslant J);\\
&c^{-1}|\xi| \leqslant |\zeta_j(\xi) - \zeta_r(\xi)| \leqslant c|\xi|\\
&(j \neq i,\ j \notin (k_3,k_4],\ r \notin (k_3,k_4],\ |j-r| \neq k_4 - k_3);\\
&c^{-1}|\xi|^{1-J} \leqslant |\zeta_j(\xi) - \zeta_{j+k_4-k_3}(\xi)| \leqslant c|\xi|^{1-J} \quad (k_3 < j \leqslant k_4).
\end{aligned} \tag{4.3.8}$$

The last of these inequalities is based on the obvious fact that $|c_j(\theta) - c_{j+k_4-k_3}(\theta)| \geqslant \text{const} > 0$ $(k_3 < j \leqslant k_4)$. In particular, it follows from (4.3.8) that the τ-roots $\zeta_r(\xi)$ are pairwise distinct for large values of $|\xi|$.

4.3.2 Necessary and sufficient conditions for the validity of the estimate (4.2.16)

In this subsection we obtain necessary and sufficient conditions for the validity of the estimate (4.2.16) in the case where $P(\xi;\tau)$ is a polynomial possessing the properties listed at the beginning of Section 4.3.

Lemma 4.3.3. *Let $R(\xi;\tau) = \tau^s$, and let $\Lambda(\xi)$ be the function defined by (4.1.2). If the τ-roots of the polynomials P and H_+ are simple, then*

$$\Lambda(\xi) = \sum_{\nu,\varrho=1}^{J} \overline{\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))}\, \overline{[z_\varrho(\xi)]^s}\, [\zeta_\nu(\xi)]^s, \tag{4.3.9}$$

where

$$\lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi)) = \mathrm{i}\, \frac{\prod_{j=1}^{J} (z_\varrho(\xi) - \bar{\zeta}_j(\xi)) \prod_{\varkappa\neq\varrho} (z_\varkappa(\xi) - \bar{\zeta}_\nu(\xi))}{\prod_{j\neq\varrho} (z_j(\xi) - z_\varrho(\xi)) \prod_{j\neq\nu} (\bar{\zeta}_\nu(\xi) - \bar{\zeta}_j(\xi))}. \tag{4.3.10}$$

This lemma follows immediately from Corollary 4.1.5.

We now turn to the study of conditions for the validity of (4.2.16).

Theorem 4.3.4. *Let $P(\xi;\tau) = \tau^J + p_1(\xi)\tau^{J-1} + \cdots + p_J(\xi)$ be a polynomial of τ such that its coefficients and its τ-roots satisfy all requirements listed at the beginning of Section 4.3. The estimate (4.2.16) with $s = 0, \ldots, J-1$ is valid if and only if the following conditions are fulfilled:*

1. *If $\operatorname{Im} z_\varrho(\theta) > 0$ holds for at least one value of ϱ $(1 \leqslant \varrho \leqslant J)$ then the inequality*
$$B(\xi)\,(1+|\xi|)^{2s+1} \leqslant \text{const} \tag{4.3.11}$$
is satisfied a.e. in $\mathbb{R}^{n-1}$.
2. *If $\operatorname{Im} z_\varrho(\theta) = 0$ for at least one value of ϱ $(1 \leqslant \varrho \leqslant J)$ and $\operatorname{Im} z_\varrho(\theta) \leqslant 0$ $(1 \leqslant \varrho \leqslant J)$, then*
$$B(\xi)\,(1+|\xi|)^{2s+1-J} \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.3.12}$$
3. *If $\operatorname{Im} z_\varrho(\theta) < 0$ $(\varrho = 1, \ldots, J)$, then*
$$B(\xi)\,(1+|\xi|)^{2s+1-2J} \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.3.13}$$

Proof. Let $\Lambda(\xi)$ be defined by (4.1.2). Since the hyperplane $t = 0$ is not characteristic for the operator $P(D)$, Proposition 4.2.1 ensures the validity of the inequality $\Lambda(\xi) \leqslant c(1+|\xi|)^{2s+1}$ for almost all $\xi \in \mathbb{R}^{n-1}$. Hence, the function $\Lambda(\xi)$ is locally bounded. Starting from (4.3.9), (4.3.10), and the asymptotic representations (4.3.2)-(4.3.6) we estimate $\Lambda(\xi)$ from above for large values of $|\xi|$.

First, we observe that inequalities (4.3.1) and (4.3.8) imply the estimates

$$\prod_{j\neq\varrho} (z_j - z_\varrho) \prod_{j\neq\nu} (\bar{\zeta}_\nu - \bar{\zeta}_j) \geqslant \begin{cases} c|\xi|^{2J-2}, & \text{if } \quad \nu \leqslant k_3, \\ c|\xi|^{J-2}, & \text{if } \quad k_3 < \nu \leqslant J. \end{cases} \tag{4.3.14}$$

From the same inequalities and the representations (4.3.2)–(4.3.6) it follows that

$$|z_\varrho(\xi) - \bar{\zeta}_\varrho(\xi)| \leqslant \begin{cases} c|\xi|^{1-J}, & \text{if } \varrho \leqslant k_1 \text{ or } k_4 \leqslant \varrho \leqslant J, \\ c|\xi|^{1-2J}, & \text{if } k_1 < \varrho \leqslant k_2, \\ c|\xi|, & \text{if } k_2 < \varrho \leqslant k_4, \end{cases} \tag{4.3.15}$$

and

$$\left|\prod_{j \neq \varrho} (z_\varrho - \bar{\zeta}_j)\right| \leqslant \begin{cases} c|\xi|^{J-1}, & \text{if } \varrho \leqslant k_4, \\ c|\xi|^{-1}, & \text{if } k_4 < \varrho \leqslant J. \end{cases} \tag{4.3.16}$$

Therefore, for $\nu \neq \varrho$ we have

$$\left|\prod_{j=1}^{J} (z_\varrho - \bar{\zeta}_j)\right| \leqslant \begin{cases} c, & \text{if } \varrho \leqslant k_1, \\ c|\xi|^{-J}, & \text{if } k_1 < \varrho \leqslant k_2 \text{ or } k_4 \leqslant \varrho \leqslant J, \\ c|\xi|^{J}, & \text{if } k_2 < \varrho \leqslant k_4, \end{cases} \tag{4.3.17}$$

$$\left|\prod_{\varkappa \neq \varrho} (z_\varkappa - \bar{\zeta}_\varrho)\right| \leqslant \begin{cases} c|\xi|^{J-1}, & \text{if } \varrho \leqslant k_3 \text{ or } k_4 < \varrho \leqslant J, \\ c|\xi|^{-1}, & \text{if } k_3 < \varrho \leqslant k_4, \end{cases} \tag{4.3.18}$$

$$\left|\prod_{\varkappa \neq \varrho} (z_\varkappa - \bar{\zeta}_\nu)\right| \leqslant \begin{cases} c|\xi|^{-1}, & \text{if } \nu \leqslant k_1 \text{ or } k_3 \leqslant \nu \leqslant J, \\ c|\xi|^{-1-J}, & \text{if } k_1 < \nu \leqslant k_2, \\ c|\xi|^{J-1}, & \text{if } k_2 < \nu \leqslant k_3. \end{cases} \tag{4.3.19}$$

Let $\lambda_{\nu\varrho}(\xi) = \lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))$ be the functions given by (4.3.10). Taking into account inequalities (4.3.14), (4.3.17)-(4.3.19), we obtain:

$$|\lambda_{\varrho\varrho}(\xi)| \leqslant \begin{cases} c|\xi|^{1-J}, & \text{if } \varrho \leqslant k_1 \text{ or } k_4 < \varrho \leqslant J, \\ c|\xi|^{1-2J}, & \text{if } k_1 < \varrho \leqslant k_2, \\ c|\xi|, & \text{if } k_2 < \varrho \leqslant k_4; \end{cases} \tag{4.3.20}$$

If $\nu \neq \varrho$, then we have

$$|\lambda_{\nu\varrho}(\xi)| \leqslant \begin{cases} c|\xi|^{1-J}, & \text{if } \varrho \leqslant k_1, k_2 < \nu \leqslant J \text{ or } \nu \leqslant k_1,\ k_2 < \varrho \leqslant k_4; \\ c|\xi|^{1-2J}, & \text{if } \varrho, \nu \leqslant k_1 \text{ or } k_1 < \varrho \leqslant k_2,\ k_2 < \nu \leqslant J \\ & \text{or } k_2 < \varrho \leqslant k_4,\ k_1 < \nu \leqslant k_2 \text{ or } k_4 < \varrho \leqslant J, \\ & k_2 < \nu \leqslant J; \\ c|\xi|^{1-3J}, & \text{if } \varrho \leqslant k_1,\ k_1 < \nu \leqslant k_2 \text{ or } k_1 < \varrho \leqslant k_2, \\ & \nu \leqslant k_1 \text{ or } k_4 < \varrho \leqslant J,\ \nu \leqslant k_1; \\ c|\xi|^{1-4J}, & \text{if } k_1 < \varrho,\ \nu \leqslant k_2 \text{ or } k_4 < \varrho \leqslant J, \\ & k_1 < \nu \leqslant k_2; \\ c|\xi|, & \text{if } k_2 < \varrho \leqslant k_4,\ k_2 < \nu \leqslant J. \end{cases} \tag{4.3.21}$$

The estimates (4.3.20)–(4.3.21) combined with (4.3.1) and (4.3.8) imply the following assertions.

Suppose $\operatorname{Im} z_\varrho(\theta) > 0$ for at least one ϱ $(1 < \varrho \leqslant J)$. Then

$$\Lambda(\xi) \leqslant c(1 + |\xi|)^{2s+1}. \tag{4.3.22}$$

Next, suppose $\operatorname{Im} z_\varrho(\theta) \leqslant 0$ for $\varrho = 1, \dots, J$, and $\operatorname{Im} z_\varrho(\theta) = 0$ for at least one ϱ $(1 \leqslant \varrho \leqslant J)$. Then

$$\Lambda(\xi) \leqslant c(1 + |\xi|)^{2s+1-J}. \tag{4.3.23}$$

Finally, let $\operatorname{Im} z_\varrho(\theta) < 0$ for $\varrho = 1, \dots, J$. Then

$$\Lambda(\xi) \leqslant c(1 + |\xi|)^{2s+1-2J}. \tag{4.3.24}$$

We show that in all these cases the function $\Lambda(\xi)$ satisfies the opposites of the respective estimates (4.3.22)–(4.3.24), respectively.

Suppose that $v(\xi; t) = \exp(\mathrm{i}\eta(\xi)t)$, where $\eta(\xi)$ is a function satisfying $\operatorname{Im} \eta(\xi) \geqslant \text{const} > 0$ for all $\xi \in \mathbb{R}^{n-1}$. Since $v(\xi; t)$ tends exponentially to zero as $t \to \infty$ for all $\xi \in \mathbb{R}^{n-1}$, it satisfies the inequality

$$\left| \frac{\partial^s v(\xi; t)}{\partial t^s} \bigg|_{t=0} \right|^2 \leqslant \Lambda(\xi) \int\limits_0^\infty \Big[|P\,(\xi; -\mathrm{i}\, d/dt)\, v(\xi; t)|^2 + |v(\xi; t)|^2 \Big]\, dt.$$

This yields the estimate

$$2|\eta(\xi)|^{2s} \operatorname{Im} \eta(\xi) \leqslant \Lambda(\xi)[1 + |P(\xi; \eta(\xi))|^2]. \tag{4.3.25}$$

The definition of $\eta(\xi)$ for large values of $|\xi|$ is given in accordance with the distribution of the τ-roots of the polynomial $P(\xi; \tau)$.

Suppose that $\operatorname{Im} z_\varrho(\theta) > 0$ for at least one value of ϱ $(1 \leqslant \varrho \leqslant J)$. Setting $\eta(\xi) = z_\varrho(\xi)$, we find in view of (4.3.25) that $2|z_\varrho(\xi)|^{2s} \operatorname{Im} z_\varrho(\xi) \leqslant \Lambda(\xi)$. Then, the first estimate in (4.3.1) and the homogeneity of the function $\operatorname{Im} z_\varrho(\xi)$ yield the inequality

$$\Lambda(\xi) \geqslant c(1 + |\xi|)^{2s+1}. \tag{4.3.26}$$

Next, suppose that $\operatorname{Im} \zeta_\varrho(\theta) = 0$ for at least one value of ϱ $(1 \leqslant \varrho \leqslant J)$ and $\operatorname{Im} \zeta_\varrho(\theta) \leqslant 0$ $(\varrho = 1, \dots, J)$. We denote by $\zeta_\varrho(\xi)$ the τ-root of the polynomial $H_+(\xi; \tau)$ which corresponds (see (4.3.2)) to the root $z_\varrho(\xi)$ satisfying $\operatorname{Im} z_\varrho(\xi) = 0$, and put $\eta(\xi) = \zeta_\varrho(\xi)$. It follows immediately from (4.3.2)–(4.3.6) that $|\zeta_\varrho(\xi) - z_j(\xi)| \leqslant c|\xi|$ for $j \neq \varrho$, $|\zeta_\varrho(\xi) - z_\varrho(\xi)| \leqslant c|\xi|^{1-J}$, and $\operatorname{Im} \zeta_\varrho(\xi) \geqslant c|\xi|^{1-J}$. Hence, $|P(\xi; \zeta_\varrho(\xi))| = \left| \prod_{j=1}^J (\zeta_\varrho(\xi) - z_j(\xi) \right| \leqslant c$. Moreover, it follows from (4.3.8) and (4.3.25) that

$$\Lambda(\xi) \geqslant c(1 + |\xi|)^{2s+1-J}. \tag{4.3.27}$$

Finally, let $\operatorname{Im} z_\varrho(\theta) < 0$ for all $\varrho = 1, \dots, J$. We define $\eta(\xi)$ by the equality $\eta(\xi) = \zeta_\varrho(\xi)$, where $\zeta_\varrho(\xi)$ is an arbitrary τ-root of the polynomial $H_+(\xi; \tau)$. From

(4.3.3) it follows that $\operatorname{Im}\zeta_\varrho(\xi) \geqslant C|\xi|$ and $|P(\xi;\zeta_\varrho(\xi))| \leqslant C(1+|\xi|)^J$. Taking into account (4.3.8), we obtain from (4.3.25) the estimate

$$\Lambda(\xi) \geqslant C(1+|\xi|)^{2s+1-2J}. \tag{4.3.28}$$

If $|\xi|$ is a bounded quantity, then the validity of (4.3.26)–(4.3.28) follows from (4.3.25) and the inequality $\operatorname{Im}\eta(\xi) \geqslant \text{const} > 0$. Thus, all assertions of Theorem 4.3.4 follow from Theorem 4.1.1 and the estimates (4.3.22)–(4.3.24) and (4.3.26)–(4.3.28). □

4.3.3 Necessary and sufficient conditions for the validity of the estimate (4.2.19)

In this subsection we establish criteria for the validity of the estimate (4.2.19) in our particular class of polynomials $P(\xi;\tau)$.

Theorem 4.3.5. *Let the polynomial $P(\xi;\tau)$ be as in Theorem 4.3.4. The estimate (4.2.19) with $s=0,\dots,J$ is valid if and only if the following conditions are satisfied:*

1. *If $\operatorname{Im} z_\varrho(\theta) \geqslant 0$ for at least one value of ϱ $(1\leqslant\varrho\leqslant J)$, then*

$$B(\xi)\,(1+|\xi|)^{2s} \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.3.29}$$

2. *If $\operatorname{Im} z_\varrho(\theta) < 0$ for all ϱ $(\varrho = 1,\dots,J)$, then*

$$B(\xi)\,(1+|\xi|)^{2(s-J)} \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.3.30}$$

Proof. Necessity. Let $\eta(\xi)$ be the function defined in the proof of the estimates (4.3.26)–(4.3.28). Substituting the function $v(\xi;t) = \exp(\mathrm{i}\eta(\xi)t)$ into the inequality

$$B(\xi)\int_0^\infty \left|\frac{\partial^s v(\xi;t)}{\partial t^s}\right|^2 dt \leqslant C\int_0^\infty \left[|P(\xi;-\mathrm{i}\,d/dt)\,v(\xi;t)|^2 + |v(\xi;t)|^2\right] dt,$$

we obtain

$$B(\xi)|\eta(\xi)|^{2s} \leqslant C[|P(\xi;\eta(\xi))|^2 + 1]. \tag{4.3.31}$$

From (4.3.31) and the condition $\operatorname{Im}\eta(\xi) \geqslant \text{const} > 0$ it follows that $B(\xi) \leqslant \text{const}$ a.e. in $\mathbb{R}^{n-1}$, provided $|\xi|$ is a bounded quantity. Inequalities (4.3.29) and (4.3.30) are deduced from (4.3.31) for large $|\xi|$ in exactly the same way as inequalities (4.3.26)–(4.3.28) were obtained above.

Sufficiency. We estimate the left-hand side of (4.1.19). First, we show that if $\operatorname{Im} z_\varrho(\theta) = 0$ for at least one ϱ $(1\leqslant\varrho\leqslant J)$, then

$$c_1(1+|\xi|)^{2s} \leqslant \sup_{\tau\in\mathbb{R}^1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2+1} \leqslant c_2(1+|\xi|)^{2s}, \tag{4.3.32}$$

while if $\operatorname{Im} z_\varrho(\theta) \neq 0$ for all $\varrho = 1, \dots, J$, then

$$\sup_{\tau \in \mathbb{R}^1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} \leqslant c_3 (1 + |\xi|)^{2(s-J)}. \tag{4.3.33}$$

Let c be a sufficiently large constant, and let $I_1(\xi) = \{\tau \in \mathbb{R}^1 : |\tau| \leqslant c(1 + |\xi|)\}$, $I_2(\xi) = \mathbb{R}^1 \setminus I_1(\xi)$. Since $p_k(\xi) \leqslant \text{const}(1 + |\xi|)^k$ ($k = 1, \dots, J$), the estimate $|P(\xi;\tau)|^2 \geqslant \text{const}\, \tau^{2J}$ holds on the set $I_2(\xi)$, provided c is sufficiently large. Therefore,

$$\sup_{\tau \in I_2(\xi)} \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} \leqslant \text{const}. \tag{4.3.34}$$

Suppose that for a given ϱ ($1 \leqslant \varrho \leqslant J$) we have $\operatorname{Im} z_\varrho(\theta) = 0$. Then, it follows from the definition of $I_1(\xi)$ and the inequality $|P(\xi;\tau)|^2 \geqslant 0$ that

$$\sup_{\tau \in I_1(\xi)} \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} \leqslant \text{const}(1 + |\xi|)^{2s}. \tag{4.3.35}$$

Taking into account (4.3.34) and (4.3.35), we obtain the upper bound of (4.3.32). To prove the lower bound, it suffices to put $\tau = z_\varrho(\xi)$ and $\operatorname{Im} z_\varrho(\theta) = 0$.

Let $\operatorname{Im} z_\varrho(\theta) \neq 0$ for all $\varrho = 1, \dots, J$. Then, obviously $|P(\xi;\tau)|^2 \geqslant \text{const}(\tau^2 + |\xi|^2)^J$. Therefore, the estimate (4.3.33) is a particular case of (4.2.22) for $m = m_n = J$, $\langle \xi \rangle = |\xi|$.

We estimate the second term on the left-hand side of (4.1.19). It suffices to establish such estimate for large values of $|\xi|$. Indeed, if $|\xi|$ is bounded and $B(\xi) \leqslant \text{const}$, then the inequality

$$B(\xi) \int_0^\infty \left| \frac{d^s u}{dt^s} \right|^2 dt \leqslant C \int_0^\infty \left[|P\,(\xi; -\mathrm{i}\, d/dt)\, u(t)|^2 + |u(t)|^2 \right] dt, \; u \in C_0^\infty(\mathbb{R}^1_+)$$

is trivial. We denote by $E(\xi)$ the second term on the left-hand side of (4.1.19). Note that, for large values of $|\xi|$, not only the τ-roots of the polynomial $P(\xi;\tau)$, but also the τ-roots of the polynomial $H_+(\xi;\tau)$ are pairwise distinct. Using the definition of the polynomials Δ_1 and Δ_2 that figure in (4.1.19), and applying Corollary 4.1.5 we obtain

$$E(\xi) = \int_{-\infty}^{\infty} \sum_{\nu,\varrho=1}^{J} \overline{\lambda_{\nu\varrho}(\xi)}\, \frac{\overline{\Omega(\xi;\eta, z_\varrho(\xi))}\, \Omega(\xi;\eta,\zeta_\nu(\xi))}{|P(\xi;\eta)|^2 + 1} d\eta, \tag{4.3.36}$$

where $\lambda_{\nu\varrho}(\xi) = \lambda_{\nu\varrho}(z_\varrho(\xi), \zeta_\nu(\xi))$ are the functions (4.3.10) and $\Omega(\xi;\eta,\tau) = (\eta - \tau)^{-1}[H_+(\xi;\eta)\tau^s - H_+(\xi;\tau)\eta^s]$. Using the equalities

$$\frac{\Omega(\xi;\eta,z_\varrho(\xi))}{H_+(\xi;\eta)} = \sum_{k=1}^{J} \frac{[\zeta_k(\xi)]^s \prod_{j\neq k}(z_\varrho(\xi)-\zeta_j(\xi))}{\prod_{j\neq k}(\zeta_k(\xi)-\zeta_j(\xi))(\eta-\zeta_k(\xi))}$$

$$\frac{\Omega(\xi;\eta,\zeta_\nu(\xi))}{H_+(\xi;\eta)} = \frac{[\zeta(\xi)]^s}{\eta-\zeta_\nu(\xi)}$$

we conclude after calculation of the integral on the right-hand side of (4.3.36) that

$$E(\xi) = 2\pi \mathrm{i} \sum_{\nu,\varrho,k=1}^{J} \overline{\lambda_{\nu\varrho}(\xi)}\, \Omega_{\nu\varrho k}(\xi)[\zeta_\nu(\xi)]^s[\bar{\zeta}_k(\xi)]^s, \tag{4.3.37}$$

where

$$\Omega_{\nu\varrho k}(\xi) = \frac{\prod_{j\neq k}(\bar{z}_\varrho(\xi)-\bar{\zeta}_j(\xi))}{(\zeta_\nu(\xi)-\bar{\zeta}_k(\xi))\prod_{j\neq k}(\bar{\zeta}_k(\xi)-\bar{\zeta}_j(\xi))}. \tag{4.3.38}$$

Let k_1, k_2, k_3, k_4 be the natural numbers defined at the beginning of this section. Using the asymptotic representations (4.3.2)–(4.3.6) and the estimates (4.3.8), we obtain for large $|\xi|$ the following inequalities:

$$|\zeta_\nu(\xi)\bar{\zeta}_k(\xi)| \geqslant c|\xi|, \qquad \text{if } \nu \neq k, \tag{4.3.39}$$

$$|\zeta_\nu(\xi)-\bar{\zeta}_\nu(\xi)| \geqslant \begin{cases} c|\xi|^{1-J}, & \text{if } \nu \leqslant k_1, \\ c|\xi|, & \text{if } k_1 < \nu \leqslant J, \end{cases} \tag{4.3.40}$$

$$\Big|\prod_{j\neq k}(\bar{\zeta}_k(\xi)-\bar{\zeta}_j(\xi))\Big| \geqslant \begin{cases} c|\xi|^{J-1}, & \text{if } k \leqslant k_3, \\ c|\xi|^{-1}, & \text{if } k_3 < k \leqslant J. \end{cases} \tag{4.3.41}$$

Combining (4.3.39)–(4.3.41), we see that

$$|\zeta_\nu(\xi)-\bar{\zeta}_k(\xi)|\Big|\prod_{j\neq k}(\bar{\zeta}_k(\xi)-\bar{\zeta}_j(\xi))\Big| \geqslant \begin{cases} c, & \text{if } \nu = k \leqslant k_1 \\ & \text{or } k_3 < \nu \leqslant J, \\ c|\xi|^J, & \text{if } k_1 < \nu = k \leqslant k_3 \\ & \text{or } k \leqslant k_3 \text{ and } \nu \neq k. \end{cases} \tag{4.3.42}$$

Let us estimate the numerator of the right-hand side of (4.3.38). From equations (4.3.2)–(4.3.6) and the estimates (4.3.8) we obtain for $\varrho = k$ the inequality

$$\Big|\prod_{j\neq\varrho}(\bar{z}_\varrho(\xi)-\bar{\zeta}_j(\xi))\Big| \leqslant \begin{cases} c|\xi|^{J-1}, & \text{if } \varrho \leqslant k_3 \text{ or } k_4 < \varrho \leqslant J, \\ c|\xi|^{-1}, & \text{if } k_3 < \varrho \leqslant k_4, \end{cases} \tag{4.3.43}$$

and for $\varrho \neq k$ the inequality

$$\Big|\prod_{j\neq k}(\bar{z}_\varrho(\xi)-\bar{\zeta}_j(\xi))\Big| \leqslant \begin{cases} c|\xi|^{-1}, & \text{if } \varrho \leqslant k_1, \\ c|\xi|^{J-1}, & \text{if } \varrho \leqslant k_3 \text{ or } k_4 < \varrho \leqslant J, \\ c|\xi|^{-1-J}, & \text{if } k_2 < \varrho \leqslant k_3, \\ c|\xi|^{-1-2J}, & \text{if } k_3 < \varrho \leqslant k_4. \end{cases} \tag{4.3.44}$$

The estimates (4.3.42)–(4.3.44), (4.3.20), and (4.3.21) immediately yield

$$E(\xi) \leqslant \mathrm{const}(1+|\xi|)^{2s}, \tag{4.3.45}$$

if $\operatorname{Im} z_\varrho(\theta) \geqslant 0$ for at least one ϱ, and

$$E(\xi) \leqslant \mathrm{const}(1+|\xi|)^{2s-2J}, \tag{4.3.46}$$

if $\operatorname{Im} z_\varrho(\theta) < 0$ for all ϱ ($\varrho = 1,\dots,J$).

Thus, the sufficiency of conditions of Theorem 4.3.5 follows from Theorem 4.1.9 and the estimates (4.3.32), (4.3.33), (4.3.45), and (4.3.46). □

4.4 Some classes of nonhomogeneous polynomials with simple roots

Let $P(\xi;\tau) = \tau^J + p_1(\xi)\tau^{J-1} + \dots + p_J(\xi)$ be a polynomial of τ with measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. We assume that its τ-roots $z_1(\xi),\dots,z_J(\xi)$ satisfy a.e. in $\mathbb{R}^{n-1}$ the condition

$$|z_j(\xi)-z_r(\xi)| \geqslant \mathrm{const} > 0 \quad (j \neq r,\ j,r = 1,\dots,J). \tag{4.4.1}$$

In this section the criteria from Theorems 4.1.1 and 4.1.9 are studied for the three classes of polynomials defined by the conditions

$$\operatorname{Im} z_j(\xi) \equiv 0, \quad \operatorname{Im} z_j(\xi) \leqslant \mathrm{const} < 0, \quad \operatorname{Im} z_j(\xi) \geqslant \mathrm{const} > 0 \tag{4.4.2}$$

($j = 1,\dots,J$), respectively.

Instead of (1.0.1) and (1.0.2) we will consider the estimates

$$\langle\!\langle R(D)u\rangle\!\rangle^2_{B^{1/2}} \leqslant C\left(N^2\|P(D)u\|^2 + \|u\|^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{4.4.3}$$

$$\|R(D)u\|^2_{B^{1/2}} \leqslant C\left(N^2\|P(D)u\|^2 + \|u\|^2\right), \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{4.4.4}$$

where $N > 0$ is a sufficiently large constant, depending only on J and on the constants from the estimates (1.4.1)–(1.4.2). It is obvious that the estimate (4.4.3) is equivalent to (4.0.1), while the estimate (4.4.4) is equivalent to (4.0.2). We also note that, w.r.t. (4.4.3)–(4.4.4), $H_+(\xi;\tau)$ is a polynomial of τ such that its τ-roots lie in the half-plane $\operatorname{Im}\zeta > 0$, $\zeta = \tau + i\sigma$, and $|H_+(\xi;\tau)|^2 = N^2|P(\xi;\tau)|^2 + 1$. This notation will be used throughout the whole Section 4.4.

4.4.1 A formula for the function $\Lambda(\xi)$

In this subsection, equation (4.4.5) will be established for the function $\Lambda(\xi)$, defined in (4.1.2) by the polynomial NP. Furthermore, from (4.4.5) we will derive the asymptotic representations of $\Lambda(\xi)$ as $N \to \infty$. On the basis of these representations, criteria for the validity of (4.0.1) and (4.0.2) will be obtained.

Lemma 4.4.1. *Let $P(\xi;\tau)$ and $H_+(\xi;\tau)$ be the polynomials with simple τ-roots $z_j = z_j(\xi)$ and $\zeta_j = \zeta_j(\xi)$. Let $\Lambda(\xi)$ be the function defined in* (4.1.2) *by the polynomial NP. Then*

$$\Lambda(\xi) = \mathrm{i}N^{-2} \sum_{j,\varrho,\nu=1}^{J} R(\xi;z_j)\overline{R(\xi;z_\varrho)} \times \frac{\prod_{k=1}^{J}(\zeta_k - \bar{z}_\varrho)}{(\zeta_\nu-\zeta_j)(\zeta_\nu-\bar{z}_\varrho)\prod_{k\neq j}(z_j-z_k)\prod_{k\neq\varrho}(\bar{z}_k-\bar{z}_\varrho)\prod_{k\neq\nu}(\zeta_\nu-\zeta_k)}. \tag{4.4.5}$$

Proof. Using Corollary 4.1.5, we obtain the equation

$$\Lambda(\xi) = \mathrm{i}\sum_{\varrho,\nu=1}^{J} R(\xi;\zeta_\nu)R(\xi;z_\varrho)\frac{\prod_{k=1}^{J}(\bar{z}_\varrho-\zeta_k)\prod_{k\neq\varrho}(\bar{z}_k-\zeta_\nu)}{\prod_{k\neq\varrho}(\bar{z}_k-\bar{z}_\varrho)\prod_{k\neq\nu}(\zeta_\nu-\zeta_k)}. \tag{4.4.6}$$

Applying the Lagrange interpolation formula, we get

$$R(\xi;\zeta_\nu) = \sum_{j=1}^{J}\frac{P(\xi;\zeta_\nu)R(\xi;z_j)}{(\zeta_\nu-z_j)\dfrac{\partial P(\xi;z_j)}{\partial\tau}}. \tag{4.4.7}$$

Substituting the right-hand side of (4.4.7) in (4.4.6) and using the equality $P(\xi;\zeta_\nu)\overline{P}(\xi;\zeta_\nu) = -N^{-2}$, we can finally convert (4.4.6) to the form (4.4.5). □

4.4.2 Asymptotic representations as $N \to \infty$ for the τ-roots $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$

Lemma 4.4.2. *Suppose that the τ-roots $z_j(\xi)$ of the polynomial $P(\xi;\tau)$ satisfy* (4.4.1) *and $\operatorname{Im} z_j(\xi) \equiv 0$ ($j = 1,\dots,J$). Then each τ-root $z_j(\xi)$ corresponds to a τ-root $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ such that the uniform w.r.t. ξ asymptotic equality*

$$\zeta_j(\xi) = z_j(\xi) + \frac{\mathrm{i}N^{-1}}{\left|\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|}[1+O(N^{-1})] \qquad (j=1,\dots,J) \tag{4.4.8}$$

is satisfied as $N \to \infty$.

Proof. By hypothesis, the coefficients of the polynomial $P(\xi;\tau)$ are real. Therefore, the τ-roots of the polynomial $H_+(\xi;\tau)$ must satisfy one of the relations $P(\xi;\zeta(\xi)) \mp \mathrm{i}N^{-1} = 0$, or, what is the same, one of the relations

$$\zeta - z_j(\xi) = \pm \frac{\mathrm{i}N^{-1}}{\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}} - \sum_{k=2}^{J} \frac{(\zeta - z_j(\xi))^k \dfrac{\partial^k P(\xi;z_j(\xi))}{\partial\tau^k}}{k!\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}}. \tag{4.4.9}$$

Let $c_1 > 0$ be a constant such that the estimate

$$\left|\frac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|^{-k} \left|\frac{\partial^k P(\xi;z_j(\xi))}{\partial\tau^k}\right| \leqslant c_1$$

holds for almost all $\xi \in \mathbb{R}^{n-1}$.[2] One can assume that the number N satisfies the inequality $N\log(1+c_1^{-1}) > 2$. Let C be a number from the interval $(2, N\log(1+c_1^{-1}))$. We set $Z = \zeta - z_j(\xi)$ and rewrite (1.4.9) as $Z = f(Z)$. Since $C > 2$, the function f maps the disc

$$\left\{ Z \in \mathbb{C}^1 : |Z| \leqslant \left|\frac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|^{-1} CN^{-1} \right\}$$

into itself. On the other hand, the inequality

$$|f'(Z)| \leqslant \sum_{k=2}^{J} \frac{C^{k-1}N^{-(k-1)}}{(k-1)!} c_1 < c_1(e^{CN^{-1}} - 1) < 1$$

holds true for all Z belonging to this disc. Therefore, for almost all $\xi \in \mathbb{R}^{n-1}$ the equation $Z = f(Z)$ has a unique solution in this disc. Thus, we found a unique τ-root $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ which satisfies the inequality

$$|\zeta_j(\xi) - z_j(\xi)| \leqslant \left|\frac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|^{-1} CN^{-1}.$$

The asymptotic representation (4.4.8) of the root $\zeta_j(\xi)$ (here $\operatorname{Im}\zeta_j(\xi) > 0$) follows from equation (4.4.9) and the obvious estimate

$$\begin{aligned} &\left| \sum_{k=2}^{J} \frac{(\zeta_j(\xi) - z_j(\xi))^k \dfrac{\partial^k P(\xi;z_j(\xi))}{\partial\tau^k}}{k!\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}} \right| \\ &\leqslant \sum_{k=2}^{J} \frac{C^k N^{-k}}{k!} \left|\frac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|^{-k-1} \left|\frac{\partial^k P(\xi;z_j(\xi))}{\partial\tau^k}\right| \\ &\leqslant \mathrm{const} \left|\frac{\partial P(\xi;z_j(\xi))}{\partial\tau}\right|^{-1} N^{-2}. \end{aligned}$$

□

[2] Existence of such constant obviously follows from (4.4.1).

Lemma 4.4.3. *Let the τ-roots $z_j(\xi)$ of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(j = 1,\dots,J)$ a.e. in $\mathbb{R}^{n-1}$. Then each τ-root $z_j(\xi)$ corresponds to a τ-root $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ such that the uniform w.r.t. ξ asymptotic equality*

$$\zeta_j(\xi) = \bar z_j(\xi) - \frac{N^{-2}}{P(\xi;\bar z_j(\xi))\dfrac{\partial \overline{P}(\xi;\overline{z}_j(\xi))}{\partial\tau}}[1+O(N^{-2})] \quad (j = 1,\dots,J) \tag{4.4.10}$$

is satisfied as $N \to \infty$.

Proof. Consider the equation

$$\begin{aligned}\zeta - \bar z_j(\xi) = {} & \frac{-N^{-2}}{P(\xi;\overline{z}_j(\xi))\dfrac{\partial \overline{P}(\xi;\overline{z}_j(\xi))}{\partial\tau}} \\ & - \sum_{k=2}^{2J} \frac{(\zeta - \bar z_j(\xi))^k \dfrac{\partial^k[P\overline{P}(\xi;\overline{z}_j(\xi))]}{\partial\tau^k}}{k!\,P(\xi;\overline{z}_j(\xi))\dfrac{\partial\overline{P}(\xi;\overline{z}_j(\xi))}{\partial\tau}}\end{aligned} \tag{4.4.11}$$

and note that, in the studied case, the τ-roots of the polynomial $H_+(\xi;\tau)$ satisfy this equation. From (4.4.1) and the inequalities $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(j = 1,\dots,J)$ it follows that

$$\left|\frac{\partial^k[P\overline{P}(\xi;\bar z_j(\xi))]}{\partial\tau^k}\right| \left|P(\xi;\bar z_j(\xi))\frac{\partial\overline{P}(\xi;\bar z_j(\xi))}{\partial\tau}\right|^{-k} \leqslant \text{const} \quad (k = 2,\dots,J)$$

a.e. in $\mathbb{R}^{n-1}$. Therefore (cf. the proof of Lemma 4.4.2), the right-hand side of (4.4.11) represents a contracting mapping of the disc

$$\left\{\zeta : |\zeta - \bar z_j(\xi)| \leqslant \left|\frac{\partial P(\xi;\bar z_j(\xi))}{\partial\tau}P(\xi;\bar z_j(\xi))\right|^{-1} CN^{-2}\right\}$$

to itself, and the unique fixed point of this mapping determines a τ-root $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ for which (4.4.10) holds. □

Lemma 4.4.4. *Let the τ-roots $z_j(\xi)$ of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(j = 1,\dots,J)$ a.e. in $\mathbb{R}^{n-1}$. Then each τ-root $z_j(\xi)$ corresponds to a τ-root $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ such that the uniform w.r.t. ξ asymptotic equality*

$$\zeta_j(\xi) = z_j(\xi) - \frac{N^{-2}}{\overline{P}(\xi;z_j(\xi))\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}}[1+O(N^{-2})] \quad (j = 1,\dots,J) \tag{4.4.12}$$

is satisfied as $N \to \infty$.

Proof. Noting that the τ-roots of the polynomial $H_+(\xi;\tau)$ satisfy the equation

$$\zeta - z_j(\xi) = \frac{-N^{-2}}{\overline{P}(\xi;z_j(\xi))\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}} - \sum_{k=2}^{2J} \frac{(\zeta - z_j(\xi))^k \dfrac{\partial^k[P\overline{P}(\xi;z_j(\xi))]}{\partial\tau^k}}{k!\overline{P}(\xi;z_j(\xi))\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}},$$

we can repeat the proof of Lemma 4.4.3 with appropriate modifications. □

Remark 4.4.5. It follows from (4.4.8), (4.4.10), and (4.4.12) that for sufficiently large N the τ-roots of the polynomial $H_+(\xi;\tau)$ are paarwise distinct a.e. in $\mathbb{R}^{n-1}$, if the conditions of one of Lemmas 4.4.2–4.4.4 are satisfied.

4.4.3 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for polynomials P with the real τ-roots

Proposition 4.4.6. *Let the τ-roots of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let the equalities $\operatorname{Im} z_j(\xi) \equiv 0$ ($j = 1,\dots,J$) hold. Then the function $\Lambda(\xi)$, defined by (4.1.2) for the polynomial NP, admits for $N \to \infty$ the asymptotic representations*

$$\Lambda(\xi) = N^{-1} \sum_{\varrho=1}^{J} \frac{R(\xi;z_\varrho(\xi))|^2}{\left|\dfrac{\partial P(\xi;z_\varrho(\xi))}{\partial\tau}\right|}[1 + O(N^{-1})], \tag{4.4.13}$$

$$\Lambda(\xi) = \frac{N^{-2}}{2\pi} \int_{-\infty}^{\infty} \frac{|R(\xi;\tau)|^2}{N^2|P(\xi;\tau)|^2 + 1} d\tau[1 + O(N^{-1})], \tag{4.4.14}$$

that are uniform w.r.t. ξ.

Proof. First, we establish (4.4.13). Using (4.4.8), (4.4.1), and the relations $\operatorname{Im} z_j(\xi) \equiv 0$ ($j = 1,\dots,J$), we obtain

$$\prod_{k=1}^{J}(\zeta_k - \bar{z}_\varrho) = \prod_{k=1}^{J}(\zeta_k - z_\varrho)$$
$$= \frac{(-1)^{J-1}\mathrm{i}[N^{-1} + O(N^{-2})]}{\left|\dfrac{\partial P(\xi;z_j)}{\partial\tau}\right|}\left[\frac{\partial P(\xi;z_j)}{\partial\tau} + O(N^{-1})\right],$$
$$\prod_{k\neq j}(z_j - z_k) = \frac{\partial P(\xi;z_j)}{\partial\tau},$$
$$\prod_{k\neq\varrho}(\bar{z}_j - \bar{z}_\varrho) = \prod_{k\neq\varrho}(z_k - z_\varrho) = (-1)^{J-1}\frac{\partial P(\xi;z_\varrho)}{\partial\tau}.$$

Hence we can transform (4.4.5) into the form

$$\Lambda(\xi) = -N^{-2}\sum_{j,\varrho=1}^{J}\frac{R(\xi;z_j)\overline{R}(\xi;z_\varrho)[N^{-1}+O(N^{-2})]}{\dfrac{\partial P(\xi;z_j)}{\partial\tau}\dfrac{\partial P(\xi;z_\varrho)}{\partial\tau}\left|\dfrac{\partial P(\xi;z_\varrho)}{\partial\tau}\right|}$$

$$\times\left(\frac{\partial P(\xi;z_\varrho)}{\partial\tau}+O(N^{-1})\right)\sum_{\nu=1}^{J}\left(z_\nu - z_j + \frac{\mathrm{i}[N^{-1}+O(N^{-2})]}{\left|\dfrac{\partial P(\xi;z_\nu)}{\partial\tau}\right|}\right)^{-1}$$

$$\times\left(z_\nu - z_\varrho + \frac{\mathrm{i}[N^{-1}+O(N^{-2})]}{\left|\dfrac{\partial P(\xi;z_\nu)}{\partial\tau}\right|}\right)^{-1}\times\left(\frac{\partial P(\xi;z_\nu)}{\partial\tau}+O(N^{-1})\right)^{-1}. \tag{4.4.15}$$

We denote by $\Lambda_1(\xi)$ the group of the terms on the right-hand side of (4.4.15) for the values $1\leqslant j=\nu=\varrho\leqslant J$. It is clear that

$$\Lambda_1(\xi) = N^{-1}\sum_{\varrho=1}^{J}\frac{|R(\xi;z_\varrho(\xi))|^2}{\left|\dfrac{\partial P(\xi;z_\varrho(\xi))}{\partial\tau}\right|}(1+O(N^{-1})). \tag{4.4.16}$$

On the other hand, it follows from (4.4.1) and (4.4.15) that $\Lambda(\xi)-\Lambda_1(\xi)$ satisfies the estimate

$$|\Lambda(\xi)-\Lambda_1(\xi)|\leqslant CN^{-2}\sum_{j\neq\varrho}\left|\frac{R(\xi;z_j(\xi))\overline{R}(\xi;z_\varrho(\xi))}{\dfrac{\partial P(\xi;z_j(\xi))}{\partial\tau}\dfrac{\partial P(\xi;z_\varrho(\xi))}{\partial\tau}}\right|. \tag{4.4.17}$$

Relation (4.4.13) now follows from (4.4.16) and (4.4.17).

Let us proceed to the proof of (4.4.14). Using the residue theorem and (4.4.7), we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|R(\xi;\tau)|^2}{N^2|P(\xi;\tau)|^2+1}d\tau$$

$$= -\mathrm{i}N^{-2}\sum_{\varrho,j,\nu=1}^{J}\frac{R(\xi;z_j)\overline{R}(\xi;z_\varrho)}{(\zeta_\nu-\zeta_j)(\zeta_\nu-\zeta_\varrho)\prod\limits_{k\neq j}(z_j-z_k)\prod\limits_{k\neq\varrho}(z_\varrho-z_k)} \tag{4.4.18}$$

$$\times\frac{1}{\prod\limits_{k\neq\nu}(\zeta_\nu-\zeta_k)\prod\limits_{k=1}^{J}(\zeta_\nu-\bar{\zeta}_k)}.$$

Each term on the right-hand side of (4.4.18) differs from the related term from the right-hand side of (4.4.5) by the factor

$$a_{\varrho\nu} = (-1)^J \prod_{k=1}^{J} (\zeta_\nu - \bar{\zeta}_k)^{-1} \prod_{k=1}^{J} (\zeta_k - z_\varrho)^{-1}.$$

Taking into account (4.4.8), we can write $a_{\varrho\nu}$ as

$$a_{\varrho\nu} = \frac{\left|\dfrac{\partial P(\xi; z_\nu)}{\partial \tau}\right|}{(N^{-1} + O(N^{-2}))\left(\dfrac{\partial P(\xi; z_\nu)}{\partial \tau} + O(N^{-1})\right)} \times \frac{\left|\dfrac{\partial P(\xi; z_\varrho)}{\partial \tau}\right|}{(N^{-1} + O(N^{-2}))\left(\dfrac{\partial P(\xi; z_\varrho)}{\partial \tau} + O(N^{-1})\right)}.$$

Hence, $a_{\varrho\varrho} = N^2(1 + O(N^{-1}))$, and for $\nu \neq \rho$ we have $|a_{\rho\nu}| = N^2(1 + O(N^{-1}))$. Combination of this equation with (4.4.16) and (4.4.17) gives (4.4.14). □

4.4.4 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with real τ-roots

As an immediate consequence of Theorem 4.1.1 and Proposition 4.4.6 we obtain

Theorem 4.4.7. *Let the τ-roots of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let* $\operatorname{Im} z_j(\xi) = 0$ $(j = 1, \dots, J)$*. The estimate (4.0.1) holds true if and only if one of the equivalent inequalities*

$$B(\xi) \sum_{\varrho=1}^{J} \frac{|R(\xi; z_\varrho(\xi))|^2}{\left|\dfrac{\partial P(\xi; z_\varrho(\xi))}{\partial \tau}\right|} \leqslant \text{const}, \tag{4.4.19}$$

$$B(\xi) \int_{-\infty}^{\infty} \frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2 + 1} d\tau \leqslant \text{const} \tag{4.4.20}$$

is satisfied a.e. in $\mathbb{R}^{n-1}$.

Further we study, under the same assumptions, a criterion for the validity of the estimate (4.0.2).

Theorem 4.4.8. *Let $P(\xi;\tau)$ be the same polynomial as in Theorem 4.4.7. The estimate (4.0.2) holds if and only if*

$$B(\xi) \sup_{\tau\in\mathbb{R}^1} \frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1} \leqslant \text{const} \quad \textit{for almost all } \xi \in \mathbb{R}^{n-1}. \tag{4.4.21}$$

Proof. The *necessity* of (4.4.21) follows from Theorem 4.1.9.

Let us prove the *sufficiency*. Let $E(\xi)$ be the second term on the left-hand side of (4.1.19), which corresponds to the polynomial NP. From the definition of the polynomials Δ_1 and Δ_2 figuring in (4.1.19) and equation (4.4.14) it follows that

$$E(\xi) = \frac{N^{-1}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \frac{\Omega(\xi;\eta,\tau)}{H_+(\xi;\eta)\overline{H_+(\xi;\tau)}} \right|^2 d\tau d\eta [1+O(N^{-1})], \tag{4.4.22}$$

with $\Omega(\xi;\eta,\tau) = (\eta-\tau)^{-1}[H_+(\xi;\eta)R(\xi;\tau) - H_+(\xi;\tau)R(\xi;\eta)]$.

We set in Corollary 3.2.8, Chapter 3 $K(\xi;\tau) = R(\xi;\tau)$ and $L(\xi;\tau) = \overline{H_+(\xi;\tau)}$. Then (4.2.16) can be rewritten as

$$N(1+O(N^{-1}))E(\xi) \leqslant \text{const} \sup_{\tau\in\mathbb{R}^1} \frac{|R(\xi;\tau)|^2}{|NP(\xi;\tau)|^2+1},$$

which shows that (4.1.19) is equivalent to (4.4.21). □

4.4.5 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta < 0$

Proposition 4.4.9. *Let the τ-roots of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let* $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(1 \leqslant j \leqslant J)$. *Then the function $\Lambda(\xi)$, defined by (4.1.2) for the polynomial NP, admits for $N \to \infty$ the asymptotic representation*

$$\Lambda(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{R(\xi;\tau)}{NP(\xi;\tau)} \right|^2 d\tau (1+O(N^{-2})), \tag{4.4.23}$$

which is uniform w.r.t. ξ.

Proof. From (4.4.10), (4.4.1), and the inequalities $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(j = 1, \dots, J)$ it follows that

$$\prod_{k=1}^{J} (\zeta_k - \bar{z}_\varrho) = \frac{(-1)^J [N^{-2} + O(N^{-4})]}{P(\xi; \bar{z}_\varrho) \left| \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau} \right|} \left(\frac{\partial \overline{P}(\xi; \bar{z}_\varrho}{\partial \tau} + O(N^{-2}) \right),$$

$$\prod_{k \neq \nu} (\zeta_\nu - \zeta_\varrho) = \frac{\partial \overline{P}(\xi; \bar{z}_\nu)}{\partial \tau} + O(N^{-2}).$$

It is also clear that

$$\prod_{k \neq j} (z_j - z_k) = \frac{\partial P(\xi; z_j)}{\partial \tau}, \qquad \prod_{k \neq \varrho} (\bar{z}_k - \bar{z}_\varrho) = (-1)^{J-1} \frac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}.$$

Therefore, (4.4.5) can be transformed to the form

$$\Lambda(\xi) = \frac{\mathrm{i}}{N^2} \sum_{\varrho, j=1}^{J} \frac{R(\xi; z_j) \overline{R(\xi; z_\varrho)} \left(\dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau} + O(N^{-2}) \right) (N^{-2} + O(N^{-4}))}{\dfrac{\partial P(\xi; z_j)}{\partial \tau} \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau} P(\xi; \bar{z}_\varrho) \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}}$$
$$\times \sum_{\nu=1}^{J} \left(\bar{z}_\nu - z_j - \frac{N^{-2} + O(N^{-4})}{P(\xi; \bar{z}_\nu) \dfrac{\partial \overline{P}(\xi; \bar{z}_\nu)}{\partial \tau}} \right)^{-1}$$
$$\times \left(\bar{z}_\nu - \bar{z}_\varrho - \frac{N^{-2} + O(N^{-4})}{P(\xi; \bar{z}_\nu) \dfrac{\partial \overline{P}(\xi; \bar{z}_\nu)}{\partial \tau}} \right)^{-1} \left(\frac{\partial \overline{P}(\xi; \bar{z}_\nu)}{\partial \tau} + O(N^{-2}) \right)^{-1}. \tag{4.4.24}$$

Denote by $\Lambda_1(\xi)$ the group of terms on the right-hand side of (4.4.24) for those $1 \leqslant \nu = \varrho \leqslant J$ holds. It is clear that

$$\Lambda_1(\xi) = \sum_{j, \varrho=1}^{J} \frac{R(\xi; z_j) \overline{R}(\xi; \bar{z}_\varrho)(1 + O(N^{-2}))}{(\bar{z}_\varrho - z_j) \dfrac{\partial P(\xi; z_j)}{\partial \tau} \dfrac{\overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}} \tag{4.4.25}$$

and

$$\Lambda(\xi) - \Lambda_1(\xi) = \mathrm{i} N^{-4} \sum_{j, \varrho=1}^{J} \alpha_{\varrho j} \frac{R(\xi; z_j) \overline{R}(\xi; \bar{z}_\varrho)(1 + O(N^{-2}))}{(\bar{z}_\varrho - z_j) \dfrac{\partial P(\xi; z_j)}{\partial \tau} \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}}, \tag{4.4.26}$$

where

$$\alpha_{\varrho j} = [P(\xi; \bar{z}_\varrho)]^{-1} \sum_{\nu \neq \varrho} \left[\frac{\partial \overline{P}(\xi; z_\nu)}{\partial \tau} \right]^{-1} [(z_\nu - z_j)^{-1} - (\bar{z}_\nu - \bar{z}_\varrho)^{-1}]. \quad (4.4.27)$$

Since the τ-roots $z_j(\xi)$ satisfy (4.4.1) and $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(j = 1, \dots, J)$, we have $|\alpha_{\varrho j}| \leqslant \text{const}$ $(\varrho, j = 1, \dots, J)$. Therefore, relations (4.4.25) and (4.4.26) imply

$$\Lambda(\xi) = \mathrm{i} N^{-2} \sum_{j,\varrho=1}^{J} \frac{R(\xi; z_j)\overline{R}(\xi, \bar{z}_\varrho)(1 + O(N^{-2}))}{\dfrac{\partial P(\xi; z_j)}{\partial \tau} \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau} (\bar{z}_\varrho - z_j)}. \quad (4.4.28)$$

In view of

$$\sum_{j,\varrho=1}^{J} \frac{\overline{R}(\xi; \bar{z}_\varrho) R(\xi; z_j)}{\dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau} \dfrac{\partial P(\xi; z_j)}{\partial \tau} (\bar{z}_\varrho - z_j)} = \sum_{\varrho=1}^{J} \frac{\overline{R}(\xi; \bar{z}_\varrho) R(\xi; \bar{z}_\varrho)}{P(\xi; \bar{z}_\varrho) \dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \left| \frac{R(\xi; \tau)}{P(\xi; \tau)} \right|^2 d\tau,$$

we can transform (4.4.28) to the form (4.4.23). □

4.4.6 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im} \zeta < 0$

From Theorem 4.1.1 and Proposition 4.4.9 we deduce

Theorem 4.4.10. *Let the τ-roots of the polynomial $P(\xi; \tau)$ satisfy (4.4.1), and let $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ $(1 \leqslant j \leqslant J)$. The estimate (4.0.1) holds true if and only if inequality (4.4.20) is satisfied a.e. in $\mathbb{R}^{n-1}$.*

Indeed, from (4.4.23) and the conditions $\operatorname{Im} z_j(\xi) \leqslant \text{const} < 0$ it follows that (4.1.2) is equivalent to (4.4.20).

Under the same assumptions, one can formulate a criterion for the validity of the estimate (4.0.2) as follows:

Theorem 4.4.11. *Let $P(\xi; \tau)$ be the same polynomial as in Theorem 4.4.10. The estimate (4.0.2) holds true if and only if inequality (4.4.21) is satisfied a.e. in $\mathbb{R}^{n-1}$.*

This theorem is derived from Theorem 4.4.10 in the same way as Theorem 4.4.8 is derived from Theorem 4.4.7.

4.4.7 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta > 0$

Proposition 4.4.12. *Let the τ-roots $z_j(\xi)$ of the polynomial $P(\xi;\tau)$ satisfy (4.4.1), and let $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(1 \leqslant j \leqslant J)$. Then the function $\Lambda(\xi)$, defined by (4.1.2) for the polynomial NP, admits for $N \to \infty$ the asymptotic representation*

$$\Lambda(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{A(\xi;\tau)}{P(\xi;\tau)} \right|^2 d\tau (1 + O(N^{-2})), \tag{4.4.29}$$

which is uniform w.r.t. ξ. Here $A(\xi;\tau)$ is the remainder of the division (w.r.t. τ) of the polynomial $R\overline{P}$ by P.

Proof. From (4.4.12), (4.4.1), and the inequalities $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(j = 1, \dots, J)$ it follows that $\prod_{k=1}^{J}(\zeta_k - \overline{z}_\rho) = (-1)^J P(\xi;\overline{z}_\varrho) + O(N^{-2})$ and $\prod_{k\neq\nu}(\zeta_\nu - \zeta_k) = \dfrac{\partial P(\xi;z_\nu)}{\partial\tau} + O(N^{-2})$. Therefore, one can rewrite (4.4.5) as

$$\begin{aligned}\Lambda(\xi) = &-\frac{\mathrm{i}}{N^2} \sum_{\varrho,j=1}^{J} \frac{R(\xi;z_j)\overline{R(\xi;z_\varrho)}(P(\xi;\overline{z}_\varrho) + O(N^{-2}))}{\dfrac{\partial P(\xi;z_j)}{\partial\tau}\dfrac{\partial\overline{P}(\xi;\overline{z}_\varrho)}{\partial\tau}} \\ &\times \sum_{\nu=1}^{J} \left(z_\nu - z_j - \frac{N^{-2} + O(N^{-4})}{\overline{P}(\xi;z_\nu)\dfrac{\partial\overline{P}(\xi;z_\nu)}{\partial\tau}} \right)^{-1} \\ &\times \left(z_\nu - \overline{z}_\varrho - \frac{N^{-2} + O(N^{-4})}{\overline{P}(\xi;z_\nu)\dfrac{\partial P(\xi;z_\nu)}{\partial\tau}} \right)^{-1} \left(\frac{\partial P(\xi;z_\nu)}{\partial\tau} + O(N^{-2}) \right)^{-1}.\end{aligned} \tag{4.4.30}$$

Denote by $\Lambda_1(\xi)$ the group of terms on the right-hand side of (4.4.30) for those $1 \leqslant \nu = j \leqslant J$ holds. It is obvious that

$$\Lambda_1(\xi) = \mathrm{i} \sum_{\varrho,j=1}^{J} \frac{R(\xi;z_j)\overline{P}(\xi;z_j)\overline{R(\xi;z_\varrho)}P(\xi;\overline{z}_\varrho)}{(z_j - \overline{z}_\varrho)\dfrac{\partial P(\xi;z_j)}{\partial\tau}\dfrac{\partial\overline{P}(\xi;\overline{z}_\varrho)}{\partial\tau}}(1 + O(N^{-2})) \tag{4.4.31}$$

and

$$\begin{aligned}\Lambda(\xi) - \Lambda_1(\xi) = \mathrm{i}N^{-2} \sum_{\varrho,j=1}^{J} \beta_{\varrho j} &\frac{R(\xi;z_j)\overline{P}(\xi;z_j)\overline{R(\xi;z_\varrho)}P(\xi;\overline{z}_\varrho)}{(z_j - \overline{z}_\varrho)\dfrac{\partial P(\xi;z_j)}{\partial\tau}\dfrac{\partial\overline{P}(\xi;\overline{z}_\varrho)}{\partial\tau}} \\ &\times (1 + O(N^{-2})),\end{aligned} \tag{4.4.32}$$

where

$$\beta_{\varrho j} = [\overline{P}(\xi; z_j)]^{-1} \sum_{\nu \neq j} \left[\frac{\partial P(\xi; z_\nu)}{\partial \tau} \right]^{-1} [(z_\nu - z_j)^{-1} - (z_\nu - \bar{z}_\varrho)^{-1}]. \tag{4.4.33}$$

Inequalities $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(1 \leqslant j \leqslant J)$ and condition (4.4.1) imply the estimates $|\beta_{\varrho j}| \leqslant \text{const}$ $(\varrho, j = 1, \dots, J)$. Therefore, in view of (4.4.31) and (4.4.32) we obtain

$$\begin{aligned} \Lambda(\xi) &= \mathrm{i} \sum_{\varrho, j=1}^{J} \frac{R(\xi; z_j)\overline{P}(\xi; z_j)\overline{R(\xi; z_\varrho)}P(\xi; \bar{z}_\varrho)}{(z_j - \bar{z}_\varrho)\dfrac{\partial P(\xi; z_j)}{\partial \tau}\dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}}(1 + O(N^{-2})) \\ &= \mathrm{i} \sum_{\varrho, j=1}^{J} \frac{A(\xi; z_j)\overline{A}(\xi; \bar{z}_\varrho)}{(z_j - \bar{z}_\varrho)\dfrac{\partial P(\xi; z_j)}{\partial \tau}\dfrac{\partial \overline{P}(\xi; \bar{z}_\varrho)}{\partial \tau}}(1 + O(N^{-2})), \end{aligned} \tag{4.4.34}$$

where $A(\xi; \tau)$ is the remainder of the division (w.r.t. τ) of the polynomial $R\overline{P}$ by P. It remains only to note that (4.4.34) can be transformed to the form (4.4.29) in the same way as (4.4.28) was transformed to the form (4.4.23). (When using the residue theorem one should take into account that $\operatorname{Im} \bar{z}_\varrho < 0$). □

4.4.8 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im} \zeta > 0$

Theorem 4.1.1 and Proposition 4.4.12 imply the following assertion.

Theorem 4.4.13. *Let the τ-roots $z_j(\xi)$ of the polynomial $P(\xi; \tau)$ satisfy* (4.4.1)*, and let* $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(1 \leqslant j \leqslant J)$*. The estimate* (4.0.1) *holds true if and only if t*

$$B(\xi) \int_{-\infty}^{\infty} \frac{|A(\xi; \tau)|^2}{|P(\xi; \tau)|^2 + 1} d\tau \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.4.35}$$

Here $A(\xi; \tau)$ is the remainder of the division (*w.r.t.* τ) *of the polynomial $R\overline{P}$ by P.*

Indeed, from the condition $\operatorname{Im} z_j(\xi) \geqslant \text{const} > 0$ $(1 \leqslant j \leqslant J)$ and equation (4.4.29) it follows that inequality (4.1.2) is equivalent to (4.4.35).

We now turn to establishing a criterion for the validity of (4.0.2).

Theorem 4.4.14. *Let $P(\xi;\tau)$ and $A(\xi;\tau)$ be the same polynomials as in Theorem 4.4.13. The estimate (4.0.2) holds true if and only if the inequality*

$$\sup_{\tau\in\mathbb{R}^1}\frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\mathscr{E}(\xi;\eta,\tau)|^2}{(|P(\xi;\eta)|^2+1)(|P(\xi;\tau)|^2+1)}d\tau d\eta \leqslant \frac{\text{const}}{B(\xi)} \tag{4.4.36}$$

with

$$\mathscr{E}(\xi;\eta,\tau)=(\eta-\tau)^{-1}[H_+(\xi;\eta)A(\xi;\tau)-H_+(\xi;\tau)A(\xi;\eta)] \tag{4.4.37}$$

is satisfied a.e. in $\mathbb{R}^{n-1}$.

Proof. Denote by $E(\xi)$ the second summand of the left side of (4.1.19) which corresponds to the polynomial NP. From the definition of the polynomials Δ_1, Δ_2 figuring in (4.1.19) and equation (4.4.29) it follows that

$$E(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\mathscr{E}(\xi;\eta,\tau)|^2}{(|NP(\xi;\eta)|^2+1)|P(\xi;\tau)|^2}d\tau d\eta \times(1+O(N^{-2})). \tag{4.4.38}$$

Since $\operatorname{Im} z_j(\xi)\geqslant \text{const}>0$ $(1\leqslant j\leqslant J)$, (4.4.38) implies the equivalence of (4.1.19) and (4.4.36). □

A more simply formulated sufficient condition for the validity of (4.0.2) gives

Theorem 4.4.15. *Let $P(\xi;\tau)$ and $A(\xi;\tau)$ be the same polynomials as in Theorem 4.4.13. If*

$$B(\xi)\sup_{\tau\in\mathbb{R}^1}\frac{|A(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1}\leqslant \text{const}\quad \textit{a.e. in } \mathbb{R}^{n-1}, \tag{4.4.39}$$

then the estimate (4.0.2) holds.

Proof. We put in Corollary 3.2.8 of Chapter 3 $K(\xi;\tau)=A(\xi;\tau)$ and $L(\xi;\tau)=\overline{H_+(\xi;\tau)}$, and suppose that $\mathscr{E}(\xi;\eta,\tau)$ is the polynomial (4.4.37). Then (4.2.16) takes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\mathscr{E}(\xi;\eta,\tau)|^2}{(|P(\xi;\eta)|^2+1)(|P(\xi;\tau)|^2+1)}d\tau d\eta\leqslant \text{const}\sup_{\tau\in\mathbb{R}^1}\frac{|A(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1}. \tag{4.4.40}$$

On the other hand, it follows from the definition of $A(\xi;\tau)$ that $\dfrac{R}{P}-\dfrac{Q}{\overline{P}}=\dfrac{A}{P\overline{P}}$, where $Q=Q(\xi;\tau)$ is the quotient of the division (w.r.t. τ) of the polynomial $R\overline{P}$ by

P. By assumption, the roots of the polynomials P and $\overline{P}$ lie in the half-planes $\operatorname{Im}\zeta > 0$ and $\operatorname{Im}\zeta < 0$, $\zeta = \tau + i\sigma$, respectively. Therefore, according to Katsnelson's theorem (see Remark 3.3.7, Chapter 3), there exists a constant $c > 0$, depending only on ord $P = J$, such that

$$\sup_{\tau\in\mathbb{R}^1}\left|\frac{R}{P}\right| \leqslant c \sup_{\tau\in\mathbb{R}^1}\frac{|A|}{P\overline{P}} \quad \text{a.e. in } \mathbb{R}^{n-1}. \tag{4.4.41}$$

Thus, we obtain

$$\sup_{\tau\in\mathbb{R}^1}\left|\frac{R}{P}\right|^2 \leqslant \text{const} \sup_{\tau\in\mathbb{R}^1}\left|\frac{A}{P\overline{P}}\right|^2 \leqslant \text{const} \sup_{\tau\in\mathbb{R}^1}\frac{|A|^2}{|P|^2+1}.$$

From here, taking into account (4.4.40), one can see that condition (4.4.39) is sufficient for the validity of (4.0.2). □

4.5 Second-order polynomials of τ

In this section we establish criteria for the validity of the estimates (4.2.16), (4.2.19) in the case, where $P(\xi;\tau) = p_0(\xi)\tau^2 + p_1(\xi)\tau + p_2(\xi)$ is a polynomial of the second order w.r.t. τ with measurable coefficients, that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster that some power of $|\xi|$ as $|\xi| \to \infty$. We assume that $p_0(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$. We restrict ourselves to the following three cases:

1. $p_1(\xi) \equiv 0$,
2. $\operatorname{Im} p_k(\xi) \equiv 0 \quad (k = 0, 1, 2)$,
3. $\operatorname{Re} p_1(\xi) \equiv 0, \quad \operatorname{Im} p_k(\xi) \equiv 0 \quad (k = 0, 2)$.

The proofs of the main results are based on Theorems 4.1.1, 4.1.9, Proposition 4.1.2 and Lemmas 4.5.1, 4.5.2.

4.5.1 Preliminary results

In this subsection, we add to the results of Section 4.1 two more necessary and sufficient conditions for the validity of the estimates (4.0.1) and (4.0.2). These assertions (Lemmas 4.5.1, 4.5.2) will be used later to treat of the estimates (4.2.16), (4.2.19) for the second-order (w.r.t. τ) operator $P(\xi;\tau)$.

Lemma 4.5.1. *Let $H_+(\xi;\tau)$ be a polynomial with the simple τ-roots $\zeta_1(\xi),\ldots,\zeta_J(\xi)$, and let $\mathfrak{B} = \{P_{\varrho\nu}(\zeta_\varrho(\xi),\zeta_\nu(\xi))\}$ be a positive definite $J\times J$ matrix with the entries*

$$P_{\varrho\nu}(\zeta_\varrho(\xi),\zeta_\nu(\xi)) = i\,\frac{P(\xi;\zeta_\varrho(\xi))\overline{P(\xi;\zeta_\nu(\xi))}+1}{\zeta_\varrho(\xi)-\overline{\zeta}_\nu(\xi)} \qquad (\varrho,\nu = 1,\ldots,J). \tag{4.5.1}$$

Furthermore, let $\mathfrak{B}^{-1} = \{\mathscr{P}_{\varrho\nu}(\xi)\}$. *The estimate* (4.0.1) *holds if and only if*

$$B(\xi) \sum_{\varrho,\nu=1}^{J} \overline{\mathscr{P}_{\varrho\nu}(\xi)}\, \overline{R(\xi;\zeta_\varrho(\xi))}\, R(\xi;\zeta_\nu(\xi)) \leqslant \text{const} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.5.2}$$

Proof. Let $\Lambda(\xi)$ be defined by (4.1.2). For any fixed $\xi \in \mathbb{R}^{n-1}$ we can compute $\Lambda(\xi)$ by the formula (2.1.55), Chapter 2. Solving equation (2.1.56), Chapter 2, we find that in the studied case

$$\varphi_0(\xi) = (\overline{\varphi_\nu^0}(\xi)) = \left(\sum_{\varrho=1}^{J} \mathscr{P}_{\varrho\nu}(\xi) R(\xi;\zeta_\varrho(\xi)) \right). \tag{4.5.3}$$

Using the formula (2.1.55), Chapter 2, we obtain

$$\Lambda(\xi) = \sum_{\varrho,\nu=1}^{J} \overline{\mathscr{P}_{\varrho\nu}(\xi)}\, \overline{R(\xi;\zeta_\varrho(\xi))}\, R(\xi;\zeta_\nu(\xi)).$$

Thus, condition (4.1.2) takes the form (4.5.2). □

Lemma 4.5.2. *Let the assumptions of Lemma* 4.5.1 *be satisfied. The estimate* (4.0.2) *holds if and only if*

$$\frac{|R(\xi;\tau)|^2}{|P(\xi;\tau)|^2+1} + \sum_{\varrho,\nu=1}^{J} \overline{\mathscr{P}_{\varrho\nu}(\xi)} \frac{\overline{R(\xi;\zeta_\varrho(\xi))}\, R(\xi;\zeta_\nu(\xi))}{i(\overline{\zeta}_\varrho(\xi) - \zeta_\nu(\xi))} \leqslant \frac{\text{const}}{B(\xi)} \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.5.4}$$

Proof. Let $\Delta_1(\xi;\eta,\tau)$ and $\Delta_2(\xi;\eta,\tau)$ be the same as in Theorem 4.1.9. Applying (4.5.3) to the polynomial (of τ) $\Omega(\xi;\eta,\tau)$ defined by (4.1.18), and using the definition (4.1.2) of $\Lambda(\xi)$, we find that

$$\int_{-\infty}^{\infty} \frac{|\Delta_1(\xi;\eta,\tau)|^2 + |\Delta_2(\xi;\eta,\tau)|^2}{|P(\xi;\tau)|^2+1} d\tau = 2\pi \sum_{\varrho,\nu=1}^{J} \overline{\mathscr{P}_{\varrho\nu}(\xi)}\, \overline{\Omega(\xi;\eta,\zeta_\varrho(\xi))} \times \Omega(\xi;\eta,\zeta_\nu(\xi)).$$

On the other hand, it is easy to see that

$$[H_+(\xi;\eta)]^{-1}\, \Omega(\xi;\eta,\zeta_\nu(\xi)) = (\eta - \zeta_\nu(\xi))^{-1} R(\xi;\zeta_\nu(\xi)) \quad (1 \leqslant \nu \leqslant J)$$

and

$$\int_{-\infty}^{\infty} (\eta - \overline{\zeta}_\varrho(\xi))^{-1} (\eta - \zeta_\nu(\xi))^{-1} d\eta = 2\pi i (\zeta_\nu(\xi) - \overline{\zeta}_\varrho(\xi))^{-1}.$$

Thus, inequalities (4.5.4) and (4.1.19) are equivalent. □

4.5.2 The case $p_1(\xi) \equiv 0$

In this subsection, we specify necessary and sufficient conditions for the validity of the estimates (4.2.16), (4.2.19) in the case where $P(\xi;\tau) = p_0(\xi)\tau^2 + p_2(\xi)$.

Theorem 4.5.3. *Let $P(\xi;\tau) = p_0(\xi)\tau^2 + p_2(\xi)$, let $p_0(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$, and let $p_0(\xi)$, $p_2(\xi)$ be measurable functions that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. The estimate (4.2.16) with $s = 0, 1$ holds true if and only if*

$$B(\xi)(1 + |p_2|)^{s+1/2} \leqslant \text{const}|p_0|^{s+1/2} \tag{4.5.5}$$

for almost all $\xi \in \{\xi : \text{Re}\,(p_0\bar{p}_2) \geqslant 0\}$ and

$$B(\xi)[1 + |\text{Im}\,(|p_0|^{-1}p_0\bar{p}_2)|](1 + |p_2|)^{s-1/2} \leqslant \text{const}|p_0|^{s+1/2} \tag{4.5.6}$$

for almost all $\xi \in \{\xi : \text{Re}\,(p_0\bar{p}_2) < 0\}$.

Proof. Let $\zeta_1(\xi), \zeta_2(\xi)$ be the τ-roots of the polynomial $H_+(\xi;\tau)$. Since

$$|H_+(\xi;\tau)|^2 = |p_0|^2\tau^4 + 2\text{Re}\,(p_0\bar{p}_2)\tau^2 + |p_2|^2 + 1,$$

we have

$$\left.\begin{aligned} &\zeta_{1,2}^2 = \alpha \pm \beta \mathrm{i}, \quad 0 < \arg\zeta_1 < \frac{\pi}{2}, \quad \frac{\pi}{2} < \arg\zeta_2 < \pi, \\ &\alpha = -|p_0|^{-2}\text{Re}\,(p_0\bar{p}_2), \quad \beta = |p_0|^{-2}\{|p_0|^2 + |\text{Im}\,(p_0\bar{p}_2)|^2\}^{1/2} \end{aligned}\right\}. \tag{4.5.7}$$

This means that $\zeta_1(\xi) \neq \zeta_2(\xi)$, and so we can apply Lemma 4.5.1. It is easy to see that $\mathscr{P}_{11} = \dfrac{2\text{Im}\,\zeta_1}{|P(\xi;\zeta_1)|^2 + 1}$, $\mathscr{P}_{22} = \dfrac{2\text{Im}\,\zeta_2}{|P(\xi;\zeta_2)|^2}$ and $\mathscr{P}_{12} = \mathscr{P}_{21} = 0$ are the entries of the matrix $\mathfrak{B}^{-1}$. Therefore, setting $R(\xi;\tau) = \tau^s$, we can write (4.5.2) in the form

$$B(\xi)\left[\frac{\text{Im}\,\zeta_1|\zeta_1|^{2s}}{|P(\xi;\zeta_1)|^2 + 1} + \frac{\text{Im}\,\zeta_2|\zeta_2|^{2s}}{|P(\xi;\zeta_2)|^2 + 1}\right] \leqslant \text{const}. \tag{4.5.8}$$

It follows from (4.5.7) that $|\zeta_1|^{2s} = |\zeta_2|^{2s} = (\alpha^2 + \beta^2)^{s/2} = |p_0|^{-s}(1 + |p_2|^2)^{s/2}$ and $|P(\xi;\zeta_1)| \cdot |P(\xi;\zeta_2)| = 1$. Hence, we get

$$\frac{|\zeta_1|^{2s}}{|P(\xi;\zeta_1)|^2 + 1} + \frac{|\zeta_2|^{2s}}{|P(\xi;\zeta_2)|^2 + 1} = |p_0|^{-s}(1 + |p_2|^2)^{s/2}. \tag{4.5.9}$$

Calculating $\text{Im}\,\zeta_1$ and $\text{Im}\,\zeta_2$ on the basis of (4.5.7), we get

$$\begin{aligned} \text{Im}\,\zeta_1 = \text{Im}\,\zeta_2 &= \left\{\frac{1}{2}[(\alpha^2 + \beta^2)^{1/2} - \alpha]\right\}^{1/2} \\ &= 2^{-1/2}|p_0|^{-1}[|p_0|(1 + |p_2|^2)^{1/2} + \text{Re}\,(p_0\bar{p}_2)]^{1/2}. \end{aligned} \tag{4.5.10}$$

From (4.5.9) and (4.5.10) it follows that (4.5.8) can be written in the form

$$B(\xi)|p_0|^{-s-1}(1+|p_2|^2)^{s/2}[|p_0|(1+|p_2|^2)^{1/2}+\mathrm{Re}\,(p_0\bar{p}_2)]^{1/2} \leqslant \text{const}. \tag{4.5.11}$$

Let $\mathrm{Re}\,(p_0\bar{p}_2) \geqslant 0$. Then

$$c_1|p_0|(1+|p_2|) \leqslant |p_0|(1+|p_2|^2)^{1/2}+\mathrm{Re}\,(p_0\bar{p}_2) \leqslant c_2|p_0|(1+|p_2|),$$

and (4.5.11) is equivalent to (4.5.5).

Let $\mathrm{Re}\,(p_0\bar{p}_2) < 0$. Then

$$|p_0|(1+|p_2|^2)^{1/2}+\mathrm{Re}\,(p_0\bar{p}_2) = \frac{|p_0|^2+|\mathrm{Im}\,(p_0\bar{p}_2)|^2}{|p_0|(1+|p_2|^2)^{1/2}-\mathrm{Re}\,(p_0\bar{p}_2)},$$

$$c_1|p_0|(1+|p_2|) \leqslant |p_0|(1+|p_2|^2)^{1/2}-\mathrm{Re}\,(p_0\bar{p}_2) \leqslant c_2|p_0|(1+|p_2|),$$

and (4.5.11) is equivalent to (4.5.6) □

Theorem 4.5.4. *Let the polynomial $P(\xi;\tau)$ be the same as in Theorem 4.5.3. The estimate (4.2.19) with $s=0,1,2$ holds if and only if the inequality*

$$B(\xi)(1+|p_2|)^s \leqslant \text{const}|p_0|^s \quad \textit{a.e. in } \mathbb{R}^{n-1}. \tag{4.5.12}$$

Proof. Set $R(\xi;\tau)=\tau^s$ in the estimate (4.5.4). Calculating the entries of the matrix $\mathfrak{B}^{-1}$ (cf. the proof of Theorem 4.5.3), we can write (4.5.4) in the form

$$\frac{\tau^{2s}}{|P(\xi;\tau)|^2+1}+\frac{|\zeta_1|^{2s}}{|P(\xi;\zeta_1)|^2+1}+\frac{|\zeta_2|^{2s}}{|P(\xi;\zeta_2)|^2+1} \leqslant \frac{\text{const}}{B(\xi)}. \tag{4.5.13}$$

It can be verified directly that

$$\sup_{\tau\in\mathbb{R}^1}\frac{1}{|P(\xi;\tau)|^2+1} = \begin{cases}(|p_0|^2+1)^{-1}, & \text{if } \mathrm{Re}\,(p_0\bar{p}_2)>0,\\ [|p_2|^2|\mathrm{Im}\,(p_2/p_0)|^2+1]^{-1}, & \text{if } \mathrm{Re}\,(p_0\bar{p}_2)\leqslant 0,\end{cases} \tag{4.5.14}$$

$$\sup_{\tau\in\mathbb{R}^1}\frac{\tau^2}{|P(\xi;\tau)|^2+1} = \frac{(1+|p_2|^2)^{1/2}}{|p_0|}\left[\left|p_0\frac{(1+|p_2|^2)^{1/2}}{|p_0|}+p_2\right|^2+1\right]^{-1}, \tag{4.5.15}$$

$$\sup_{\tau\in\mathbb{R}^1}\frac{\tau^4}{|P(\xi;\tau)|^2+1} = \begin{cases}|p_0|^{-2}, & \text{if } \mathrm{Re}\,(p_0\bar{p}_2)\geqslant 0,\\ (|p_2|^2+1)|p_0|^{-2}[1+|p_0|^2|\mathrm{Im}\,(p_2/p_0)|^2]^{-1}, & \\ \quad\text{if } \mathrm{Re}\,(p_0\bar{p}_2)<0.\end{cases} \tag{4.5.16}$$

Equations (4.5.9) and (4.5.14)–(4.5.16) imply the equivalence of (4.5.13) and (4.5.12). □

4.5.3 The case $\operatorname{Im} p_k(\xi) \equiv 0$ $(k = 0, 1, 2)$

In the next two theorems, we study conditions for the validity of the estimates (4.2.16), (4.2.19) for polynomials of the second order w.r.t. τ with real coefficients.

Theorem 4.5.5. *Let $P(\xi;\tau) = p_0(\xi)\tau^2 + 2p_1(\xi)\tau + p_2(\xi)$ be a polynomial with real measurable coefficients that are locally bounded in $\mathbb{R}^{n-1}$ and grow no faster than some power of $|\xi|$ as $|\xi| \to \infty$. Suppose that $r = p_0^{-1}(p_0p_2 - p_1^2)$ and $p_0(\xi) \neq 0$ a.e. in $\mathbb{R}^{n-1}$. The estimate (4.2.16) with $s = 0, 1$ holds true if and only if the inequality*

$$B(\xi)(p_1^2 + |p_0| + |p_0p_2|)^s \leqslant \mathrm{const}|p_0|^{2s+1/2}(1+|r|)^{-1/2} \tag{4.5.17}$$

is satisfied for almost all $\xi \in \{\xi : p_0p_2 - p_1^2 \geqslant 0\}$ and the inequality

$$B(\xi)(p_1^2 + |p_0| + |p_0p_2|)^s \leqslant \mathrm{const}|p_0|^{2s+1/2}(1+|r|)^{1/2} \tag{4.5.18}$$

is satisfied for almost all $\xi \in \{\xi : p_0p_2 - p_1^2 < 0\}$.

Proof. Represent the polynomial $P(\xi;\tau)$ in the form $P(\xi;\tau) = p_0(\tau+q)^2 + r$, where $q = p_0^{-1}p_1$ and $r = p_0^{-1}(p_0p_2 - p_1^2)$. Let $\zeta_1(\xi), \zeta_2(\xi)$ be the τ-roots of the polynomial $H_+(\xi;\tau)$. Since $|H_+(\xi;\tau)|^2 = |p_0|^2(\tau+q)^4 + 2p_0r(\tau+q)^2 + r^2 + 1$, we have

$$\begin{aligned}&(\zeta_{1,2}+q)^2 = \alpha \pm \beta \mathrm{i},\ 0 < \arg(\zeta_1+q) < \frac{\pi}{2},\ \frac{\pi}{2} < \arg(\zeta_2+q) < \pi;\\ &\alpha = -p_0^{-1}r, \quad \beta = |p_0|^{-1}.\end{aligned} \tag{4.5.19}$$

Thus, we obtain the relation $\zeta_1 \neq \zeta_2$, and we can apply Lemma 4.5.1. Since the coefficients of the polynomial $P(\xi;\tau)$ are real, we see that $[P(\xi;\zeta_1)]^2 = [P(\xi;\zeta_2)]^2 = -1$.

On the other hand, $P(\xi;\zeta_2) = p_0(\alpha - \beta \mathrm{i})^2 + r = P(\xi;\bar{\zeta}_1)$ and, similarly, $P(\xi;\zeta_1) = P(\xi;\bar{\zeta}_2)$. Therefore, $P(\xi;\zeta_1)\overline{P(\xi;\zeta_2)} + 1 = 0$ and $P(\xi;\zeta_2)\overline{P(\xi;\zeta_1)} + 1 = 0$. Hence, the entries of the matrix $\mathfrak{B}^{-1}$ are $\mathscr{P}_{11} = \operatorname{Im}\zeta_1$, $\mathscr{P}_{22} = \operatorname{Im}\zeta_2$ and $\mathscr{P}_{12} = \mathscr{P}_{21} = 0$. Since $R(\xi;\tau) = \tau^s$, we conclude that (4.5.2) and the inequality

$$B(\xi)[\operatorname{Im}\zeta_1|\zeta_1|^{2s} + \operatorname{Im}\zeta_2|\zeta_2|^{2s}] \leqslant \mathrm{const} \tag{4.5.20}$$

are equivalent.

Relations (4.5.19) yield

$$\operatorname{Im}\zeta_1 = \operatorname{Im}\zeta_2 = 2^{-1/2}(|p_0|^{-1}(1+r^2)^{1/2} + p_0^{-1}r)^{1/2}, \tag{4.5.21}$$

$$|\zeta_1|^2 + |\zeta_2|^2 = 2p_0^{-2}\{p_1^2 + [p_0^2 + (p_0p_2 - p_1^2)^2]^{1/2}\}. \tag{4.5.22}$$

This means that

$$c^{-1}(|\zeta_1|^2 + |\zeta_2|^2) \leqslant (p_1^2 + |p_0| + |p_0p_2|)|p_0|^{-2} \leqslant c(|\zeta_1|^2 + |\zeta_2|^2), \tag{4.5.23}$$

$$c^{-1}\operatorname{Im}\zeta_j \leqslant |p_0|^{-1/2}(1+|r|)^{1/2} \leqslant c\operatorname{Im}\zeta_j, \quad \text{if} \quad p_0p_2 - p_1^2 \geqslant 0, \tag{4.5.24}$$

$$c^{-1}\operatorname{Im}\zeta_j \leqslant |p_0|^{-1/2}(1+|r|)^{-1/2} \leqslant c\operatorname{Im}\zeta_j, \quad \text{if} \quad p_0p_2 - p_1^2 < 0. \tag{4.5.25}$$

It follows from (4.5.23)–(4.5.25) that inequality (4.5.20) is equivalent to (4.5.17) for $p_0 p_2 - p_1^2 \geqslant 0$, and to (4.5.18) for $p_0 p_2 - p_1^2 < 0$. □

Theorem 4.5.6. *Let a polynomial $P(\xi; \tau)$ be the same as in Theorem 4.5.5. The estimate (4.2.19) with $s = 0, 1, 2$ holds true if and only if the inequality*

$$B(\xi)[|p_0| + p_1^2 + |p_0 p_2|]^s \leqslant \text{const}\, p_0^{2s} \quad \text{a.e. in } \mathbb{R}^{n-1}. \tag{4.5.26}$$

Proof. Set $R(\xi; \tau) = \tau^s$ in the estimate (4.5.4). Then one can write (cf. the proof of Theorem 4.5.5) inequality (4.5.4) in the form

$$\frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} + (|\zeta_1|^2 + |\zeta_2|^2)^s \leqslant \frac{\text{const}}{B(\xi)}. \tag{4.5.27}$$

A direct check shows that

$$\sup_{\tau \in \mathbb{R}^1} \frac{1}{|P(\xi;\tau)|^2 + 1} = \begin{cases} (r^2+1)^{-1}, & \text{if } p_0 p_2 - p_1^2 \geqslant 0, \\ 1, & \text{if } p_0 p_2 - p_1^2 < 0; \end{cases} \tag{4.5.28}$$

$$\sup_{\tau \in \mathbb{R}^1} \frac{(\tau+q)^2}{|P(\xi;\tau)|^2 + 1} = |p_0|^{-1}(1+r^2)^{1/2}[|p_0|p_0|^{-1}(1+r^2)^{1/2} + r|^2 + 1]^{-1}; \tag{4.5.29}$$

$$\sup_{\tau \in \mathbb{R}^1} \frac{(\tau+q)^4}{|P(\xi;\tau)|^2 + 1} = \begin{cases} p_0^{-2}, & \text{if } p_0 p_2 - p_1^2 \geqslant 0, \\ p_0^{-2}(1+r^2), & \text{if } p_0 p_2 - p_1^2 < 0. \end{cases} \tag{4.5.30}$$

(In (4.5.28)–(4.5.30) we have $q = p_0^{-1} p_1$ and $r = p_0^{-1}(p_0 p_2 - p_1^2)$).

On the other hand, one can write (4.5.22) as

$$|\zeta_1|^2 + |\zeta_2|^2 = 2[q^2 + |p_0|^{-1}(1+r^2)^{1/2}]. \tag{4.5.31}$$

Combining (4.5.27)–(4.5.30) and (4.5.31), we find that

$$\sup_{\tau \in \mathbb{R}^1} \frac{\tau^{2s}}{|P(\xi;\tau)|^2 + 1} \leqslant \text{const}(|\zeta_1|^2 + |\zeta_2|^2)^s. \tag{4.5.32}$$

This means that (4.5.27) is equivalent to the inequality

$$B(\xi)(|\zeta_1|^2 + |\zeta_2|^2)^s \leqslant \text{const}. \tag{4.5.33}$$

□

4.5.4 The estimate (4.2.16) in the case Re $p_1(\xi) \equiv 0$, Im $p_k(\xi) \equiv 0$ ($k = 0, 2$)

Finally, we consider a class of the second-order polynomials w.r.t. τ, for which the criterion from Lemma 4.5.1 allows a more explicit formulation.

Theorem 4.5.7. *Let* $P(\xi;\tau) = p_0(\xi)\tau^2 + \mathrm{i}p_1(\xi)\tau + p_2(\xi)$, *let* $p_k(\xi)$ $(k = 0, 1, 2)$ *be real measurable functions that are locally bounded in* $\mathbb{R}^{n-1}$ *and grow no faster than some power of* $|\xi|$ *as* $|\xi| \to \infty$, *and let* $p_0(\xi) \neq 0$ *a.e. in* $\mathbb{R}^{n-1}$. *The estimate* (4.2.4) *with* $s = 0$ *holds if and only if the inequality*

$$B(\xi) \leqslant \mathrm{const}\,[|p_0|^{1/2}(1 + |p_2|)^{-1/2} + |p_1|(1 + |p_2|)^{-1}] \tag{4.5.34}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \geqslant 0,\ p_0 p_2 \geqslant 0\}$; *the inequality*

$$B(\xi) \leqslant \mathrm{const}\,[|p_0|^{1/2}(1 + |p_2|)^{1/2} + |p_1|^{-1}(1 + |p_2|)] \tag{4.5.35}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \geqslant 0,\ p_0 p_2 < 0\}$; *the inequality*

$$B(\xi)[|p_0|^{1/2}(1 + |p_2|)^{1/2} + |p_1|] \leqslant \mathrm{const}\,|p_0| \tag{4.5.36}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \leqslant 0,\ p_0 p_2 \geqslant 0\}$; *and the inequality*

$$B(\xi)[|p_0|^{1/2}(1 + |p_2|)^{-1/2} + |p_1|] \leqslant \mathrm{const}\,|p_0| \tag{4.5.37}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \leqslant 0,\ p_0 p_2 < 0\}$.

Proof. Let $\Lambda(\xi)$ be defined by (4.1.2). We show that

$$\Lambda(\xi) = |p_0|^{-1}[p_1^2 + 2|p_0|(1 + p_2^2)^{1/2} + 2p_0 p_2]^{1/2} - p_0^{-1} p_1. \tag{4.5.38}$$

Indeed, since the estimate (4.2.4) is considered for $s = 0$ ($R(\xi;\tau) = 1$), one can use Corollary 4.1.3 to calculate $\Lambda(\xi)$. Let $z_1(\xi)$, $z_2(\xi)$, $\zeta_1(\xi)$, $\zeta_2(\xi)$ be the τ-roots of the polynomials $P(\xi;\tau)$ and $H_+(\xi;\tau)$, respectively. It is obvious that $\mathrm{Im}\,(z_1 + z_2) = -p_0^{-1} p_1$. Therefore, equation (4.1.6) can be written in the form

$$\Lambda(\xi) = -p_0^{-1} p_1 + \mathrm{Im}\,\zeta_1(\xi) + \mathrm{Im}\,\zeta_2(\xi). \tag{4.5.39}$$

Calculating $\mathrm{Im}\,\zeta_1(\xi) + \mathrm{Im}\,\zeta_2(\xi)$, we consider two cases.

Let $p_1^4 + 4p_1^2 p_0 p_2 - 4p_0^2 < 0$. Then the relation

$$|H_+(\xi;\tau)|^2 = p_0\tau^4 + (p_1^2 + 2p_0 p_2)\tau^2 + p_2^2 + 1 \tag{4.5.40}$$

yields

$$\begin{gathered} \zeta_{1,2}^2 = \alpha \pm \beta\mathrm{i}, \quad 0 < \arg\zeta_1 < \frac{\pi}{2}, \quad \frac{\pi}{2} < \arg\zeta_2 < \pi, \\ \alpha = -2^{-1} p_0^{-2}(2p_0 p_2 + p_1^2), \quad \beta = 2^{-1} p_0^{-2}(4p_0^2 - 4p_1^2 p_0 p_2 - p_1^4)^{1/2}. \end{gathered} \tag{4.5.41}$$

Therefore,

$$\begin{aligned} (\mathrm{Im}\,\zeta_1)^2 = (\mathrm{Im}\,\zeta_2)^2 &= 2^{-1}[(\alpha^2 + \beta^2)^{1/2} - \alpha] \\ &= 4^{-1} p_0^{-2}[p_1^2 + 2|p_0|(1 + p_2^2)^{1/2} + 2p_0 p_2]. \end{aligned} \tag{4.5.42}$$

Thus, (4.5.42) and (4.5.39) yield (4.5.38).

Let $p_1^4 + 4p_1^2 p_0 p_2 - 4p_0^2 \geqslant 0$. Then

$$\zeta_{1,2}^2 = \alpha \pm b, \quad b = 2^{-1} p_0^{-2}(p_1^4 + 4p_1^2 p_0 p_2 - 4p_0^2)^{1/2}, \tag{4.5.43}$$

where α is defined by (4.5.41). It is obvious that $\alpha < 0$ and $0 \leqslant b < |\alpha|$. Hence $\zeta_1 = |\alpha + b|^{1/2}\mathrm{i}$ and $\zeta_2 = |\alpha - b|^{1/2}\mathrm{i}$, and (4.5.39) can be recast as

$$[\Lambda(\xi) + p_0^{-1} p_1]^2 = |\alpha + b| + |\alpha - b| + 2(\alpha^2 - b^2)^{1/2}. \tag{4.5.44}$$

From (4.5.41) and (4.5.43) we get

$$\begin{aligned} |\alpha + b| + |\alpha - b| = |2\alpha| = p_0^{-2}(2p_0 p_2 + p_1^2), \\ \alpha^2 - b^2 = p_0^{-2}(p_2^2 + 1). \end{aligned} \tag{4.5.45}$$

Combining (4.5.44) and (4.5.45), we obtain (4.5.38). A direct check shows that the following upper and lower bounds for $\Lambda(\xi)$ follow from (4.5.38):

$$c^{-1}\Lambda = (1 + |p_2|)[|p_1| + |p_0|^{1/2}(1 + |p_2|)^{1/2}] \leqslant c\Lambda, \tag{4.5.46}$$

provided that $p_0^{-1} p_1 \geqslant 0$ and $p_0 p_2 \geqslant 0$;

$$c^{-1}\Lambda \leqslant [|p_0|^{1/2}(1 + |p_2|)^{1/2} + |p_1|(1 + |p_2|)]^{-1} \leqslant c\Lambda, \tag{4.5.47}$$

provided that $p_0^{-1} p_1 \geqslant 0$ and $p_0 p_2 < 0$;

$$c^{-1}\Lambda \leqslant [|p_0|^{1/2}(1 + |p_2|)^{1/2} + |p_1|]|p_0|^{-1} \leqslant c\Lambda, \tag{4.5.48}$$

provided that $p_0^{-1} p_1 \leqslant 0$ and $p_0 p_2 \geqslant 0$;

$$c^{-1}\Lambda \leqslant [|p_0|^{1/2}(1 + |p_2|)^{-1/2} + |p_1|]|p_0|^{-1} \leqslant c\Lambda, \tag{4.5.49}$$

provided that $p_0^{-1} p_1 \leqslant 0$ and $p_0 p_2 < 0$.

Thus, the assertions of the theorem to be proved follow from Theorem 4.1.1 and the estimates (4.5.46)–(4.5.49). □

Theorem 4.5.8. *Let the polynomial $P(\xi;\tau)$ be the same as in Theorem 4.5.7. The estimate (4.2.4) with $s = 1$ holds true if and only if the inequality*

$$B(\xi)[p_1^2 + |p_0|(1 + |p_2|)^3] \leqslant \mathrm{const}\,|p_0|[|p_1|^3 + |p_0|^{3/2}(1 + |p_2|)^{3/2}] \tag{4.5.50}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \geqslant 0,\ p_0 p_2 \geqslant 0\}$; the inequality

$$B(\xi) \leqslant \mathrm{const}\,|p_0|[|p_1| + |p_0|^{1/2}(1 + |p_2|)^{-1/2}] \tag{4.5.51}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \geqslant 0,\ p_0 p_2 < 0\}$; the inequality

$$B(\xi)[|p_1|^3 + |p_0|^{3/2}(1 + |p_2|)^{3/2}] \leqslant \mathrm{const}\,|p_0|^3 \tag{4.5.52}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \leqslant 0,\ p_0 p_2 \geqslant 0\}$; the inequality

$$B(\xi)[|p_1| + |p_0|^{1/2}(1 + |p_2|)^{-1/2}][p_1^2 + (1 + |p_2|)|p_0|] \leqslant \mathrm{const}\,|p_0|^3 \tag{4.5.53}$$

is satisfied for almost all $\xi \in \{\xi : p_0 p_1 \leqslant 0,\ p_0 p_2 < 0\}$.

Proof. Let $\Lambda(\xi)$ be defined by (4.1.2). We show that

$$\Lambda = |p_0|^{-3}[p_1^2 + 2|p_0|(1+p_2^2)^{1/2} + 2p_0p_2]^{1/2}[p_1^2 + |p_0|(1+p_2^2)^{1/2}] - p_0^{-3}p_1[p_1^2 + 2|p_0|(1+p_2^2)^{1/2} + p_0p_2]. \tag{4.5.54}$$

To this end we use Proposition 4.1.2. Let $\zeta_1(\xi)$, $\zeta_2(\xi)$ be the τ-roots of the polynomial $H_+(\xi;\tau)$, so that $H_+(\xi;\tau) = p_0(\xi)\tau^2 - p_0(\xi)(\zeta_1+\zeta_2)\tau + p_0(\xi)\zeta_1\zeta_2$ and $H_-(\xi;\tau) = \overline{H_+(\xi;\tau)}$. Then we find from the representation $\tau H_- = PT_1 + T_2$ that

$$\overline{T}_2(\xi;\tau) = -[p_0\zeta_1\zeta_2 + \mathrm{i}p_1(\zeta_1+\zeta_2) + p_2 + p_0^{-1}p_1^2]\tau + p_2(\zeta_1+\zeta_2) - \mathrm{i}p_0^{-1}p_1p_2. \tag{4.5.55}$$

Let $S(\xi;\tau)$ be the remainder of the division of the polynomial $\tau\overline{T}_2$ by H_+. It follows immediately from (4.5.55) that the leading coefficient of the polynomial S is equal to

$$-\mathrm{i}p_0^{-1}p_1p_2 + p_0\zeta_1\zeta_2(\zeta_1+\zeta_2) - \mathrm{i}p_1(\zeta_1+\zeta_2)^2 - (\zeta_1+\zeta_2)p_0^{-1}p_1^2.$$

Hence, in accordance with Proposition 4.1.2,

$$\Lambda(\xi) = \mathrm{Im}\,[\mathrm{i}p_0^{-2}p_1p_2 - \zeta_1\zeta_2(\zeta_1+\zeta_2) + \mathrm{i}p_0^{-1}p_1(\zeta_1+\zeta_2)^2 + p_0^{-2}p_1^2(\zeta_1+\zeta_2)]. \tag{4.5.56}$$

Consider two cases. Let $p_1^4 + 4p_1^2p_0p_2 - 4p_0^2 < 0$. Then (4.5.41) yields

$$\left.\begin{aligned} &\zeta_1+\zeta_2 = 2\mathrm{i}\,\mathrm{Im}\,\zeta_1 = \mathrm{i}|p_0|^{-1}[p_1^2 + 2|p_0|(1+p_2^2)^{1/2} + 2p_0p_2]^{1/2},\\ &\zeta_1\zeta_2 = -|\zeta_1|^2 = -|p_0|^{-1}(1+p_2^2)^{1/2}. \end{aligned}\right\} \tag{4.5.57}$$

Thus, (4.5.56) and (4.5.57) imply (4.5.54).

Let $p_1^4 + 4p_1^2p_0p_2 - 4p_0^2 \geqslant 0$. Then (4.5.57) also follows from (4.5.43). Hence (4.5.54) is again true in this case.

Suppose now that $p_0^{-1}p_1 \leqslant 0$. Then (4.5.54) yields the following upper and lover bounds for the function $\Lambda(\xi)$:

$$c^{-1}\Lambda \leqslant |p_0|^{-3}[|p_1|^3 + |p_0|^{3/2}(1+|p_2|)^{3/2}] \leqslant c\Lambda, \tag{4.5.58}$$

if $p_0p_2 \geqslant 0$; and

$$c^{-1}\Lambda \leqslant |p_0|^{-3}[|p_1| + |p_0|^{1/2}(1+|p_2|)^{-1/2}][p_1^2 + |p_0|(1+|p_2|)] \leqslant c\Lambda, \tag{4.5.59}$$

if $p_0p_2 < 0$.

Let $p_0^{-1}p_1 \geqslant 0$. Then (4.5.54) can be transformed as follows:

$$\begin{aligned}\Lambda = {} & |p_0|^{-1}[p_1^2 + 2|p_0|(1+p_2^2)^{3/2} + 2p_0p_2(1+p_2^2)]\\ & \times\{[p_1^2 + 2|p_0|(1+p_2^2)^{1/2} + 2p_0p_2]^{1/2}[p_1^2 + |p_0|(1+p_2^2)^{1/2}]\\ & + |p_1|[p_1^2 + 2|p_0|(1+p_2^2)^{1/2} + p_0p_2]\}^{-1}.\end{aligned} \tag{4.5.60}$$

From (4.5.60) it follows that

$$c^{-1}\Lambda \leqslant \frac{p_1^2 + |p_0|(1+p_2)^3}{|p_0|(|p_1|^3 + |p_0|^{3/2}(1+|p_2|))^{3/2}} \leqslant c\Lambda, \tag{4.5.61}$$

if $p_0 p_1 \geqslant 0$; and

$$c^{-1}\Lambda \leqslant |p_0|^{-1}[|p_1| + |p_0|^{1/2}(1+|p_2|)^{-1/2}]^{-1} \leqslant c\Lambda, \tag{4.5.62}$$

if $p_0 p_2 \geqslant 0$.

Thus, the statements of the theorem to be proved follow from Theorem 4.1.1 and the estimates (4.5.58), (4.5.61) and (4.5.62). □

4.6 On the space of traces of functions belonging to the domain of the maximal operator

In this section we return to the questions considered in Section 2.3 and study them for the case $N = 2$, $P_1(\xi;\tau) = P(\xi;\tau)$, and $P_2(\xi;\tau) = 1$. As in Section 2.3, we assume that $R(\xi;\tau)$ and $P(\xi;\tau)$ are polynomials of the variable $(\xi;\tau) \in \mathbb{R}^n$, and, consequently, $R(D)$ and $P(D)$ are differential operators with constant coefficients.

Following Proposition 2.3.4, Chapter 2, we define an open set $\Xi \subset \mathbb{R}^{n-1}$, $\mathrm{mes}_{n-1}(\mathbb{R}^{n-1} \setminus \Xi) = 0$ with the following properties:

1. The orders of the polynomials $R(\xi;\tau)$ and $H_+(\xi;\tau)$ are constant for all $\xi \in \Xi$.
2. The roots $\tau = \zeta(\xi)$ of the polynomial $H_+(\xi;\tau)$ are analytic, and their multiplicities are constant in each component Ξ_α of the set Ξ.

Denote by P the maximal operator, defined in $L^2(\mathbb{R}^n_+)$ by the differential polynomial $P(D)$. Its domain is

$$\mathscr{D}(P) = \{u : u \in L^2(\mathbb{R}^n_+);\ P(D)u \in L^2(\mathbb{R}^n_+)\},$$

and $P(D)$ is understood in the sense of distributions.

In Subsection 4.6.1 we show that P is the closure of its restriction to $C_0^\infty(\mathbb{R}^n_+)$. Thus, in the case where $P(\xi;\tau)$, $R(\xi;\tau)$ are polynomials of the variables $(\xi;\tau) \in \mathbb{R}^n$ and $B(\xi) = 1$, all results of this chapter related to the estimate (4.0.2) can be considered as criteria for the embedding $\mathscr{D}(P) \subset \mathscr{D}(R)$ of the domains of the maximal operators P and R.

The result of Subsection 4.6.2 is a strengthening of the result of Subsection 2.3.2. Here we show that the "trace space" $R(D)u\big|_{t=0}$ of the elements $u \in \mathscr{D}(P)$ coincides with the closure of the linear span of the set of functions $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ satisfying $\langle\!\langle \varphi \rangle\!\rangle^2_{\Lambda^{-1/2}} = \int\limits_{\mathbb{R}^{n-1}} \frac{|\hat\varphi(\xi)|^2}{\Lambda(\xi)} d\xi < \infty$ in the topology given by the norm $\langle\!\langle\cdot\rangle\!\rangle_{\Lambda^{-1/2}}$. $\Lambda(\xi)$ is the function defined by formula (4.1.2).

4.6.1 The maximal operator as closure of its restriction on the set of functions infinitely differentiable up to the boundary

Proposition 4.6.1. *For the differential polynomial $P(D)$ with constant coefficients, the maximal operator P in $L^2(\mathbb{R}^n_+)$ coincides with the closure of its restriction on $C_0^\infty(\mathbb{R}^n_+)$.*

Proof. Let $u(x;t) \in \mathscr{D}(P)$, $g(x;t) = P(D)u(x;t)$, and let $\omega_\delta(z)$ be an infinitely differentiable function supported in the interval $0 < z < \delta$ and such that $\int_{\mathbb{R}^1_+} \omega_\delta(z)dz = 1$. We set

$$u_\delta(x;z) = \int_{\mathbb{R}^1_+} \omega_\delta(z-t)u(x;t)dt, \quad u^\delta(x;z) = \int_{\mathbb{R}^1_+} \overline{\omega_\delta(t-z)}u(x;t)dt.$$

Let $\varphi(x;t)$ be an arbitrary infinitely differentiable function with compact support in the half-space $\{(x;t) : x \in \mathbb{R}^{n-1},\ t > 0\}$. It is obvious that the function $\varphi_\delta(x;z)$ has the same properties. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n_+} g(x;z)\overline{\varphi_\delta(x;z)}dxdz &= \int_{\mathbb{R}^n_+} u(x;z)\overline{\overline{P(D)}\varphi_\delta(x;z)}dxdz \\
&= \int_{\mathbb{R}^n_+} u(x;z)\overline{P(D_x;D_z)\int_{\mathbb{R}^n_+} \omega_\delta(z-t)\varphi(x;t)dt}dxdz \\
&= \int_{\mathbb{R}^n_+} u(x;z)\overline{\int_{\mathbb{R}^1_+} \omega_\delta(z-t)\overline{P(D_x;D_t)}\varphi(x;t)dt}dxdz \\
&= \int_{\mathbb{R}^n_+} \overline{\overline{P(D)}\varphi(x;t)} \int_{\mathbb{R}^1_+} \overline{\omega_\delta(z-t)}u(x;z)dzdxdt \\
&= \int_{\mathbb{R}^n_+} \overline{\overline{P(D)}\varphi(x;t)}u^\delta(x;t)dxdt.
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^n_+} g(x;z)\overline{\varphi_\delta(x;z)}dxdz = \int_{\mathbb{R}^n_+} \overline{\varphi(x;t)}g^\delta(x;t)dxdt.$$

Hence $u^\delta(x;t) \in \mathscr{D}(P)$ and $P(D)u^\delta = g^\delta$. It follows from the properties of the kernel $\omega_\delta(y;z)$ that the functions $u^\delta(x;t)$ are infinitely differentiable and

$$\lim_{\delta\to 0}\left\{\|u^\delta - u\|^2_{L^2(\mathbb{R}^n_+)} + \|P(D)(u^\delta - u)\|^2_{L^2(\mathbb{R}^n_+)}\right\} = 0.$$

Let $\hat{u}^\delta(\xi;t)$ be the Fourier transform of $u^\delta(x;t)$. Define a sequence of infinitely differentiable cut-off functions $\chi_k(\xi)$ by

$$\chi_k(\xi) = \begin{cases} 0, & \text{if } |\xi| \geqslant 2k, \\ 0, & \text{if } \xi \in H_k, \\ 1, & \text{if } |\xi| \leqslant k \quad \text{and} \quad \xi \notin H_k \end{cases}$$

(here, as in Section 2.3, H_k is a neighborhood (in $\mathbb{R}^{n-1}$) of the closed set $H = \mathbb{R}^{n-1} \setminus \Xi$ such that $\text{mes}_{n-1} H_k < \dfrac{1}{k}$ $(k = 1, 2, \dots)$). Set $\hat{u}_k^\delta(\xi;t) = \chi_k(\xi)\hat{u}_\delta(\xi;t)$. Then

$$\lim_{k\to+\infty}\Big\{\|u_k^\delta(\xi;t) - \hat{u}^\delta(\xi;t)\|^2_{L^2(\mathbb{R}^n_+)} + \Big\|P\,(\xi; -\mathrm{i}\,d/dt)\Big(\hat{u}_k^\delta(\xi;t) - \hat{u}^\delta(\xi;t)\Big)\Big\|^2_{L^2(\mathbb{R}^n_+)}\Big\} = 0.$$

Let $\xi \in \mathbb{R}^{n-1}$ be a point from support Z_k of the function $\chi_k(\xi)$. Since $P(\xi; -\mathrm{i}\,d/dt)$ is an ordinary differential operator of order J, the norm

$$\Big\{\|\hat{u}^\delta(\xi;t)\|^2_{L^2(\mathbb{R}^1_+)} + \Big\|P\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}^\delta(\xi;t)\Big\|^2_{L^2(\mathbb{R}^1_+)}\Big\}^{1/2}$$

(uniform w.r.t. Z_k) is equivalent to the norm $\|\hat{u}^\delta(\xi;t)\|_{W_2^J(\mathbb{R}^1_+)}$. Now define a sequence of infinitely differentiable cut-off functions $\psi_r(t)$ as follows:

$$\psi_r(t) = \begin{cases} 0, & \text{if } t \geqslant 2r, \\ 1, & \text{if } 0 \leqslant t \leqslant r \quad (r = 1, 2, \dots). \end{cases}$$

Since the set of compactly supported functions is dense in $W_2^J(\mathbb{R}^1_+)$, the equality

$$\lim_{r\to\infty}\|\hat{u}^\delta(\xi;t) - \hat{u}^\delta(\xi;t)\psi_r(t)\|^2_{W_2^J(\mathbb{R}^1_+)} = 0$$

holds true (uniformly w.r.t. $\xi \in Z_k$). Setting $\hat{v}_{kr}^\delta(\xi;t) = \hat{u}_k^\delta(\xi;t)\psi_r(t)$ $(k, r = 1, 2, \dots)$, we arrive at

$$\lim_{k,r\to\infty}\Big\{\|\hat{v}_{kr}^\delta(\xi;t) - \hat{u}^\delta(\xi;t)\|^2_{L^2(\mathbb{R}^n_+)} + \Big\|P\,(\xi; -\mathrm{i}\,d/dt)\Big(\hat{v}_{kr}^\delta(\xi;t) - \hat{u}(\xi;t)\Big)\Big\|^2_{L^2(\mathbb{R}^n_+)}\Big\} = 0.$$

Let $v_{kr}^\delta(x;t)$ be the inverse Fourier transform of $\hat{v}_{kr}^\delta(\xi;t)$ w.r.t. ξ. This function is infinitely differentiable w.r.t. x and t and compactly supported w.r.t. t. Moreover, as a function of x, it converges to zero (together with all its derivatives) faster than any power of $|x|^{-1}$ as $|x| \to \infty$.

Now define a sequence of infinitely differentiable cut-off functions $\eta_s(x)$ by the formula

$$\eta_s(x) = \begin{cases} 0, & \text{if } |x| \geqslant 2s, \\ 1, & \text{if } |x| \leqslant s \quad (s = 1, 2, \dots) \end{cases}$$

and set $w^\delta_{krs}(x;t) = v^\delta_{kr}(x;t)\eta_s(x)$. Using Parseval's identity and the properties of the functions $v^\delta_{kr}(x;t)$, we obtain

$$\lim_{k,r,s\to\infty} \{\|w^\delta_{krs}(x;t) - u^\delta(x;t)\|^2_{L^2(\mathbb{R}^n_+)} + \|P(D)\left(w^\delta_{krs}(x;t) - u^\delta(x;t)\right)\|^2_{L^2(\mathbb{R}^n_+)}\} = 0.$$

Therefore, $u^\delta_{krs}(x;t) \in C^\infty_0(\mathbb{R}^n_+)$ and

$$\lim_{\substack{k,f,s\to\infty \\ \delta\to 0}} \{\|w^\delta_{krs} - u\|^2_{L^2(\mathbb{R}^n_+)} + \|P(D)\left(w^\delta_{krs} - u\right)\|^2_{L^2(\mathbb{R}^n_+)}\} = 0. \qquad \square$$

4.6.2 Description of the "trace space"

In this subsection, we formulate two theorems (an embedding theorem, and a continuation theorem), which provide a complete description of the trace space $R(D)u\big|_{t=0} = 0$ of the elements of the domain of the maximal operator.

Theorem 4.6.2. *Let $\Lambda(\xi)$ be defined by (4.1.2). Then the estimate*

$$\int\limits_{\mathbb{R}^{n-1}} |R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0}|^2 \frac{d\xi}{\Lambda(\xi)} \leqslant \|P(D)u\|^2 + \|u\|^2 \tag{4.6.1}$$

holds for all $u \in C^\infty_0(\mathbb{R}^n_+)$.

This statement follows directly from Theorem 2.3.7 and Remark 2.1.4, Chapter 2.

Remark 4.6.3. Using Theorem 4.6.2 and Proposition 4.6.1, one can give a meaning to the expression $R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}|_{t=0}$ for any function $u(x;t) \in \mathscr{D}(P)$. Namely, approximating $u(x;t) \in \mathscr{D}(P)$ by a sequence $u_k(x;t) \in C^\infty_0(\mathbb{R}^n_+)$ such that $\lim_{k\to\infty}(\|u_k - u\|^2 + \|P(D)(u_k - u)\|^2) = 0$, we find that

$$\lim_{k,r\to\infty} \int\limits_{\mathbb{R}^{n-1}} |R\,(\xi; -\mathrm{i}\,d/dt)\,(\hat{u}_k(\xi;t) - \hat{u}_r(\xi;t))|_{t=0}|^2 \frac{d\xi}{\Lambda(\xi)} = 0.$$

Therefore, the expression $R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0}$ can be defined for $u \in \mathscr{D}(P)$ by means of the condition

$$\lim_{k\to\infty} \int\limits_{\mathbb{R}^{n-1}} |R\,(\xi; -\mathrm{i}\,d/dt)\,u_k(\xi;t)|_{t=0} - R\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0}|^2 \frac{d\xi}{\Lambda(\xi)} = 0.$$

Understanding the expression $R\,(\xi;-\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0}$ for $u \in \mathscr{D}(P)$ in this sense, one can strengthen the formulation of Theorem 4.6.2 as follows: the estimate (4.6.1) holds true for all $u(x;t) \in \mathscr{D}(P)$, if $\Lambda(\xi)$ is defined by (4.1.2).

Now we formulate a continuation theorem. Its statement is sharper than the corresponding result from Section 2.3.

Theorem 4.6.4. *Suppose that the function $\Lambda(\xi)$ is defined a.e. in $\mathbb{R}^{n-1}$ by formula* (4.1.2)*. Then for any function $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ such that*

$$\langle\!\langle \varphi \rangle\!\rangle^2_{\Lambda^{-1/2}} = \int\limits_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{\Lambda(\xi)}\,d\xi < \infty, \tag{4.6.2}$$

there exists a function $u \in \mathscr{D}(P)$ satisfying the following conditions:

$$R\,(\xi;-\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)|_{t=0} = \hat{\varphi}(\xi), \tag{4.6.3}$$

$$\|P(D)u\|^2 + \|u\|^2 = \langle\!\langle \varphi \rangle\!\rangle^2_{\Lambda^{-1/2}}. \tag{4.6.4}$$

Proof. Following the proof of Theorem 2.3.8, Chapter 2, we first determine the function φ^0_ν a.e. on the set Ξ from the relations

$$R^{(\sigma)}(\xi;\zeta_\varrho(\xi)) = \sum_{\nu=1}^{J} \sum_{\gamma=0}^{k_\nu(\xi)-1} P_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi))\overline{\varphi^0_{\nu\gamma}}(\xi)$$
$$(\varrho = 1,\dots,J, \quad \sigma = 0,\dots,k_\varrho(\xi)-1),$$

where

$$P_{\varrho\sigma\nu\gamma}(\xi;\zeta_\varrho(\xi),\zeta_\nu(\xi)) = \mathrm{i}\Bigg[\frac{(-1)^\sigma(\gamma+\sigma)!}{(\zeta_\varrho(\xi)-\overline{\zeta}_\nu(\xi))^{\gamma+\sigma+1}}$$
$$+\sum_{g=0}^{\gamma}\sum_{h=0}^{\sigma} \frac{(-1)^{\sigma-h}\,C^g_\gamma\,C^h_\sigma(\gamma-g+\sigma-h)!}{(\zeta_\varrho(\xi)-\overline{\zeta}_\nu(\xi))^{\gamma-g+\sigma-h+1}} P^{(h)}(\xi;\zeta_\varrho(\xi))\overline{P}^{(g)}(\xi;\overline{\zeta}_\nu(\xi))\Bigg].$$

Then

$$\Lambda(\xi) = \sum_{\nu=1}^{J}\sum_{\gamma=0}^{k_\nu(\xi)-1} R^{(\gamma)}(\xi;\zeta_\nu(\xi))\varphi^0_{\nu\gamma}(\xi).$$

Set

$$\varphi_{\nu\gamma}(\xi) = \frac{\hat{\varphi}(\xi)\varphi^0_{\nu\gamma}(\xi)}{\Lambda(\xi)} \qquad (\nu = 1,\dots,J, \quad \gamma = 0,\dots,k_\nu(\xi)-1).$$

It may be assumed (cf. Proposition 2.3.6, Chapter 2) that the functions $\Lambda(\xi)$ and $\varphi_{\nu\gamma}(\xi)$ are infinitely differentiable in each component $\Xi_\alpha \subset \Xi$. Consider the function

$$\hat{u}(\xi;t) = \sum_{\nu=1}^{J} \sum_{\gamma=0}^{k_\nu(\xi)-1} \varphi_{\nu\gamma}(\xi)(\mathrm{i}t)^\gamma \exp(\mathrm{i}\zeta_\nu(\xi)t)$$

(here $\xi \in \Xi$, $t \in \mathbb{R}^1_+$), and put $\hat{v}(\xi;t) = P\left(\xi; \frac{1}{\mathrm{i}}\frac{d}{dt}\right)\hat{u}(\xi;t)$. Arguing in the same way as in the derivation of the second assertion in the proof of Theorem 2.3.8, Chapter 2, we obtain

$$\int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty \left(|\hat{v}(\xi;t)|^2 + |\hat{u}(\xi;t)|^2\right) dt = \langle\!\langle \varphi \rangle\!\rangle^2_{\Lambda^{-1/2}}. \tag{4.6.5}$$

Hence, by (4.6.2), we conclude that $\hat{u}(\xi;t), \hat{v}(\xi;t) \in L^2(\mathbb{R}^n_+)$.

Let $\eta(x;t)$ be an infinitely differentiable function with compact support in the half-space $\{(x;t) : x \in \mathbb{R}^{n-1}, t > 0\}$, and let $\hat{\eta}(\xi;t)$ be its Fourier transform w.r.t. x. Integrating by parts with respect to t, we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty \hat{u}(\xi;t)\overline{P(\xi;-\mathrm{i}\,d/dt)\hat{\eta}(\xi;t)}dt \\
&= \int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty P(\xi;-\mathrm{i}\,d/dt)\,\hat{u}(\xi;t)\overline{\hat{\eta}(\xi;t)}dt = \int\limits_{\mathbb{R}^{n-1}} d\xi \int\limits_0^\infty \hat{v}(\xi;t)\overline{\hat{\eta}(\xi;t)}dt.
\end{aligned}$$

Let $u(\xi;t)$ and $v(\xi;t)$ be the inverse Fourier transforms of the functions $\hat{u}(\xi;t)$ and $\hat{v}(\xi;t)$ w.r.t. ξ, respectively. Then, by the Parseval identity, we have $u, v \in L^2(\mathbb{R}^n_+)$ and

$$\int\limits_{\mathbb{R}^n_+} u(x;t)\overline{P(D)}\eta(x;t)dxdt = \int\limits_{\mathbb{R}^n_+} v(x;t)\overline{P(D)}\eta(x;t)dxdt.$$

Thus, $u \in \mathscr{D}(P)$, the equality $P(D)u = v$ holds in the sense of distributions, and (4.6.4) follows from (4.6.5). Relation (4.6.3) is obtained in the same way as the first statement in the proof of Theorem 2.3.8, Chapter 2. □

Finally, we give two results concerning the trace space of functions belonging to the domain of the maximal operator for some differential polynomials of concrete types. Propositions 4.2.1, 4.2.2 and Theorems 4.6.2, 4.6.4 imply

Corollary 4.6.5. *Let $P(D)$ be a partial differential operator of order $J \geqslant 1$ with constant coefficients such that the relations*

$$P(\xi;\tau) = \sum_{k=0}^{J} p_{J-k}(\xi)\tau^k, \qquad p_0(\xi) \neq 0$$

hold for all $\xi \in \mathbb{R}^{n-1}$ and $0 \leqslant s \leqslant J-1$.

1. *If $P(\xi;\tau)$ is a polynomial with generalized homogeneous principal part (4.2.2), then the mapping*

$$u \longmapsto \frac{\partial^s u}{\partial t^s}\bigg|_{t=0}, \tag{4.6.6}$$

acting from $C_0^\infty(\mathbb{R}^n_+)$ to $C_0^\infty(\mathbb{R}^{n-1})$, can be extended to a continuous homomorphism of the space $\mathscr{D}(P)$ into the closure of $C_0^\infty(\mathbb{R}^{n-1})$ in the topology given by the norm $\left(\int_{\mathbb{R}^{n-1}} \frac{|\hat{\varphi}(\xi)|^2}{(1+\langle\xi\rangle)^{(2s+1)m/J}} d\xi\right)^{1/2}$ (in other words, to the space dual to the Slobodeckii space

$$W^{\mathbf{l}}_{x,2}(\partial\mathbb{R}^n_+), \qquad \mathbf{l} = \left(\frac{m_1}{J}\left(s+\frac{1}{2}\right), \dots, \frac{m_{n-1}}{J}\left(s+\frac{1}{2}\right)\right),$$

see [Slo58]).[3]

2. *If $P(\xi;\tau)$ is a quasielliptic polynomial of type $l \geqslant 1$, then the homomorphism* (4.6.6) *defined in item* 1 *is surjective.*

3. *If the hyperplane $t = 0$ is not characteristic for the operator $P(D)$, then the mapping* (4.6.6) *acting from $C_0^\infty(\mathbb{R}^n_+)$ to $C_0^\infty(\mathbb{R}^{n-1})$ can be extended to a continuous homomorphism of the space $\mathscr{D}(P)$ into the space $\mathscr{H}_{-s-1/2}(\partial\mathbb{R}^n_+)$.*

4. *If $P(\xi;\tau)$ is a properly elliptic polynomial of even order, then the homomorphism* (4.6.6) *constructed in item* 3 *is surjective.*

From the estimates (4.3.23), (4.3.27), established in the proof of Theorem 4.3.4, and Theorems 4.6.2, 4.6.4, we deduce the following assertion.

Corollary 4.6.6. *Let $P(\xi;\tau) = \tau^J + p_1(\xi)\tau^{J-1} + \dots + p_J(\xi)$ be a homogeneous, hyperbolic in the sense of Petrovsky polynomial of order $J \geqslant 1$, and let $R(\xi;\tau) = \tau^s$ ($s = 0,\dots,J-1$). Then the mapping* (4.6.6) *acting from $C_0^\infty(\mathbb{R}^n_+)$ to $C_0^\infty(\mathbb{R}^{n-1})$ can be extended to a continuous surjective homomorphism from the space $\mathscr{D}(P)$ onto $\mathscr{H}_{-s-1/2+J/2}(\partial\mathbb{R}^n_+)$.*

[3] Here, $m_1, \dots, m_{n-1}, m_n = J, m$ are the integers defined at the beginning of Section 4.2, and $\langle\cdot\rangle$ is the norm (4.2.3).

4.7 Notes

The results of this chapter were established by the authors in the papers [MG79] and [GM74]. Some of these results were announced in [GM72] and [GM75].

The question of whether a maximal operator is the closure of its restriction to the set of functions that are infinitely differentiable up to the boundary of a domain, was studied in the works of many authors (M. S. Birman [Bir53], F. Browder [Bro58], L. Hörmander [H58], [H61], L. P. Nizhnik[Niz59], J. Peetre[Pee59]) under various assumptions on the type of the operator, its coefficients, and on the domain in $\mathbb{R}^n$, where the functions u are given. Among the other works on this topic, we would like to mention the paper by C. Baiocchi [Bai69], where a representative bibliography is provided. The result of Subsection 4.6.1 was established by the authors ([GM74], Lemma 12).

In connection with the issues discussed in Section 4.6, and, in particular, in connection with the results of Corollary 4.6.5, we mention the work of Ch. Goulauic and P. Grisvard [GouGri70]. In this paper, for differential operators $P(x; D)$ in a domain $\Omega \subset \mathbb{R}^n$ it is proved that the corresponding traces on $\partial\Omega$ of the elements $u \in \mathscr{D}(P)$ belong to the space $\mathscr{H}_{-s-1/2}(\partial\Omega)$ provided that the coefficients of $P(x; D)$ and the boundary $\partial\Omega$ are sufficiently smooth and satisfy some other conditions (in particular, it is assumed that $\partial\Omega$ is in all points not characteristic for the polynomial P).

In the case of an elliptic polynomial P, item 4 of Corollary 4.6.5 evidently follows from the theorem on the complete collection of homeomorphisms for elliptic operators (see, for example, [Roi71], p. 225–256, and references therein).

Notation

$\mathbb{R}^n_+$	upper half-space $\{(x,t) : x \in \mathbb{R}^{n-1}, t \geqslant 0\}$ of the space $\mathbb{R}^n$.
$\partial\mathbb{R}^n_+$	boundary of the half-space $\mathbb{R}^n_+$ (hyperplane $t = 0$).
S^{n-2}	unit sphere in $\mathbb{R}^{n-1}$.
dx	Lebesgue measure in $\mathbb{R}^n$.
$\mathrm{mes}_n E$	n-dimensional Lebesgue measure of a set E.
$C_0^\infty(\mathbb{R}^n)$	space of infinitely differentiable functions with compact support in $\mathbb{R}^n$.
$C_0^\infty(0,+\infty)$	space of infinitely differentiable functions with compact support in $(0,+\infty)$.
$C_0^\infty(\mathbb{R}^n_+)$	space of restrictions of functions from $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}^n_+$.
$\mathbf{C}_0^\infty(\mathbb{R}^n_+)$ ($\mathbf{C}_0^\infty(\mathbb{R}^{n-1})$)	space of m-dimensional vector-valued functions with components belonging to $C_0^\infty(\mathbb{R}^n_+)$ ($C_0^\infty(\mathbb{R}^{n-1})$).
$\mathbb{C}^m$	complex m-dimensional unitary space.
$\lvert\cdot\rvert$	norm in $\mathbb{C}^m$.
$\lVert\cdot\rVert$	norm in $L^2(\mathbb{R}^n_+)$ or in the direct product of m copies of $L^2(\mathbb{R}^n_+)$.
$\langle\!\langle\cdot\rangle\!\rangle$	norm in $L^2(\partial\mathbb{R}^n_+)$ or in the direct product of N copies of $L^2(\partial\mathbb{R}^n_+)$.
$\hat{u}(\xi,t)$	Fourier transform of the function $u(x,t) \in C_0^\infty(\mathbb{R}^n_+)$ (or the vector-valued function $u(x,t) \in \mathbb{C}_0^\infty(\mathbb{R}^n_+)$) w.r.t. the x-variable: $\hat{u}(\xi,t) = (2\pi)^{(1-n)/2} \int_{\mathbb{R}^{n-1}} e^{-ix\cdot\xi} u(x,t)\,dx,$ $x\cdot\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_{n-1}\xi_{n-1}.$
B	measurable function on $\mathbb{R}^{n-1}$, which is positive almost everywhere.[1]
$\lVert\cdot\rVert_{B^{1/2}}$	norm defined by $\lVert u\rVert^2_{B^{1/2}} = \int_{\mathbb{R}^{n-1}} \int_0^\infty B(\xi)\lvert\hat{u}(\xi;t)\rvert^2\,dt\,d\xi;$

[1] The requirement of positivity almost everywhere the function B is introduced in this book only for reasons of simplicity. Instead, we can assume that $B(\xi) \geqslant 0$. Then, in all propositions, the assumptions involving B must be satisfied almost everywhere on the set $\{\xi : B(\xi) > 0\}$.

$\langle\!\langle \cdot \rangle\!\rangle_{B^{1/2}}$	norm defined by $\langle\!\langle u \rangle\!\rangle^2_{B^{1/2}} = \int_{\mathbb{R}^{n-1}} B(\xi)\lvert\hat{u}(\xi;0)\rvert^2 d\xi;$
$P(D)$	operator: differential operator with respect to t and pseudo-differential operator with respect to x, defined in $\mathbf{C}_0^\infty(\mathbb{R}^n_+)$ (or in $C_0^\infty(\mathbb{R}^n_+)$) by a Fourier integral representation formula $P(D)u = (2\pi)^{(1-n)/2} \int_{\mathbb{R}^{n-1}} e^{\mathrm{i}x\cdot\xi} P\,(\xi; -\mathrm{i}\,d/dt)\,\hat{u}(\xi;t) d\xi.$
$\operatorname{supp} u$	(closed) support of a function u.
$\boldsymbol{\alpha}$	multiindex, i.e., an n-tuple of nonnegative integers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$; the sum $\sum_{k=1}^{n} \alpha_k$ will be denoted by $\lvert\boldsymbol{\alpha}\rvert$.
$D^{\boldsymbol{\alpha}}$	differential operator $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_\rho = -\mathrm{i}\,\partial/\partial x_\rho$, $\rho = 1, \ldots, n-1$, and $D_n = -\mathrm{i}\,\partial/\partial t$.
$\overline{\mathfrak{M}}$	complex conjugate matrix $\overline{\mathfrak{M}} = \{\overline{M}_{ij}\}$ of a matrix $\mathfrak{M} = \{M_{ij}\}$ with complex entries.
$\mathfrak{M}^*$	conjugate transpose of a matrix $\mathfrak{M}$ ($\mathfrak{M}^* = \overline{\mathfrak{M}^T}$, where $\mathfrak{M}^T$ denotes the transpose of $\mathfrak{M}$).
$\mathfrak{M}^{-1}$	inverse of a matrix $\mathfrak{M}$.
$\mathfrak{M}^{\mathrm{c}}$	square matrix whose rows consist of the algebraic complements of the column entries of a square matrix $\mathfrak{M}$.
$\operatorname{tr}\mathfrak{M}$	trace of a matrix $\mathfrak{M}$.
$\det\mathfrak{M}$	determinant of a matrix $\mathfrak{M}$.
$\operatorname{rg}\mathfrak{G}$	rank of a matrix $\mathfrak{G}$.
$\ker\mathfrak{G}$	nullspace (kernel) of a matrix $\mathfrak{G}$.
I	identity matrix of order $m \times m$.
$\delta_{jk} = \begin{cases} 0, \text{ if } j \neq k, \\ 1, \text{ if } j = k, \end{cases}$	Kronecker symbol.
$C_s^\gamma = \dfrac{s!}{\gamma!(s-\gamma)!}$	number of γ-combinations from a given set of s element (binomial coefficient).
C, C_0, C'	various positive constants which appear in estimates and do not depend on the (vector)-functions u.

$\deg f(\xi, \tau)$	degree of a homogeneous function f of the variables $(\xi, \tau) \in \mathbb{R}^n$.
$\operatorname{ord} R(\xi, \tau)$	order of a polynomial R, with coefficients depending on $\xi \in \mathbb{R}^{n-1}$, with respect to the variable τ.
$R^{(j)}(\xi, \tau)$	partial derivative of order j with respect to τ.
$\arg \zeta$	argument of a complex number ζ.
$\operatorname{sgn} \xi$	the sign function, equal to 1 if $\xi > 0$, 0 if $\xi = 0$, and -1 if $\xi < 0$.
$\emptyset$	empty set.

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