GILBARG-SERRIN EQUATION AND LIPSCHITZ REGULARITY

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Dedicated in memory of James Serrin who passed away ten years ago.

Abstract. We discuss conditions for Lipschitz regularity for a uniformly elliptic equation in divergence form with coefficients that were introduced by Gilbarg & Serrin. In particular, we find cases where Lipschitz regularity holds but the coefficients are not Dini continuous, or do not even have Dini mean oscillation. The form of the coefficients also enables us to obtain specific conditions and examples for which there exists a weak solution that is not Lipschitz continuous.

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1. Introduction and General Theory

The topic of this paper is Lipschitz regularity for weak solutions of a uniformly elliptic equation in divergence form:

$$\partial_j(a_{ij}(x)\,\partial_i u) = 0, \quad \text{for } x \in \Omega, \tag{1}$$

where the coefficients $a_{ij} = a_{ji}$ are bounded, measurable functions in the open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, which contains the closure of the ball $B_\rho = \{x \in \mathbb{R}^d : |x| < \rho\}$. For a restricted class of coefficients a_{ij}, we seek conditions that guarantee that any weak solution $u \in H^{1,2}_{\ell oc}(\Omega)$, i.e. u and ∇u are locally in L^2, must be *Lipschitz continuous at* $x = 0$, i.e. $|u(x) - u(0)| \leq c\,|x|$ holds for $x \in B_\rho$, or *Lipschitz continuous in a neighborhood of* $x = 0$, i.e. $|u(x) - u(y)| \leq c\,|x - y|$ for all $x, y \in B_\rho$. (Here and throughout this paper we let c denote a positive constant whose value may change with each occurence.) Recall that Lipschitz continuous functions are differentiable almost everywhere (by Rademacher's theorem) and so our solution will satisfiy $\nabla u \in L^\infty_{\ell oc}(\Omega)$.

1.1. **Background.** Let us review a little of what is known about the regularity of weak solutions of (1). Any weak solution of (1) is known to be Hölder continuous, i.e. $|u(x) - u(y)| \leq c\,|x - y|^\alpha$ for some $\alpha \in (0, 1)$, at least locally in Ω (cf. De Giorgi [9], Nash [32], Moser [31], Landis [23]). When the coefficients are continuous in Ω, then it is well-known (cf. Agmon, Douglis, Nirenberg [1]) that $\nabla u \in L^p_{\ell oc}(\Omega)$ for $1 < p < \infty$; in fact, this is even true when the coefficients are in VMO (cf. Chiarenza, Frasca, Longo [3], Di Fazio [10]), or in BMO but sufficiently close (depending on pp') to VMO (cf. Maz'ya, Mitrea, Shaposhnikova [28]).[1] Note that continuity of the coefficients is not sufficient to show $\nabla u \in L^\infty_{\ell oc}(\Omega)$ (cf. Jin, Maz'ya, Van Schaftingen [18]). However, if the coefficients are Dini continuous in Ω, then a weak solution is known to be not only Lipschitz continuous but C^1, i.e. continuously differentiable (cf. Hartman, Wintner [17], Burch [2], Taylor [34]). Special attention has been given to pointwise bounds on the gradient of the solutions for linear and nonlinear elliptic equations and systems in [25], [4], [21], [5], [22], [6], [7].

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[1] Here, as usual, BMO stands for bounded mean oscillation and VMO stands for vanishing mean oscillation.

The Dini conditions on the coefficients that we consider use the modulus of continuity ω:

$$|a_{ij}(x) - a_{ij}(y)| \le \omega(|x-y|) \quad \text{for } x, y \in \Omega, \tag{2}$$

where $\omega(r)$ is a continuous, nondecreasing function $[0,\infty) \to [0,\infty)$ with $\omega(0) = 0$. Classically, the coefficients are said to be *Dini continuous* if ω satisfies

$$\int_0^{1/2} \frac{\omega(r)}{r}\, dr < \infty. \tag{3}$$

When $\omega(r) = r^\alpha$ for some $\alpha \in (0,1)$ then this regularity coincides with Hölder continuity. There are weaker moduli of continuity that satisfy the Dini condition. If we let $\omega(r) = |\log r|^{-\gamma}$ for $0 < r < 1/2$ and $\omega(0) = 0$, then ω satisfies (3) if $\gamma > 1$. However, if $\omega(r) = |\log r|^{-\gamma}$ for $0 < \gamma \le 1$, then the coefficients are not Dini continuous, and the usual regularity properties of solutions may not hold.

More recently, generalizations of the Dini condition have been considered. For example, in [25] we considered the "square-Dini condition" on the modulus of continuity ω:

$$\int_0^{1/2} \frac{\omega^2(r)}{r}\, dr < \infty. \tag{4}$$

A function is *square-Dini continuous* if its modulus of continuity ω satisfies (4). Clearly the condition (4) is more general than (3); for example, $\omega(r) = |\log r|^{-\gamma}$ satisfies (4) when $\gamma > 1/2$. The condition (4) alone is not sufficient itself to obtain Lipschitz regularity of solutions, but was used in [25] to reduce such regularity at a point to stability properties of a dynamical system obtained from the coefficients. This dynamical systems approach to studying the properties of solutions of elliptic equations has been used in many other publications: [19], [20], [31], [26], [27]. We shall have more to say about this dynamical system approach below, but for now let us mention another generalization of the Dini condition.

Even more recently, several authors have considered a "Dini mean oscillation" condition on the coefficients. In their comprehensive study of differentiability properties of solutions of quasilinear elliptic equations, Kuusi & Mingione [21] used the notion of Dini mean oscillation in particular to obtain gradient bounds. The same concept was used by Dong & Kim [14] to show that a weak solution of a uniformly elliptic linear equation in divergence form is C^1 if the coefficients have Dini mean oscillation. Let us describe the condition in the following form: a function $a(x)$, defined for $|x| < 4$, has *Dini mean oscillation* if the following modulus function ω_a satisfies the Dini condition (3):

$$\omega_a(r) := \sup_{x \in B_3(0)} \fint_{B_r(x)} |a(y) - \widetilde{a}_{B_r(x)}|\, dy, \quad \text{where } \widetilde{a}_{B_r(x)} := \fint_{B_r(x)} a(y)\, dy, \tag{5}$$

and the slashed integral denotes mean value. There are functions that are not Dini continuous that do have Dini mean oscillation; for example, if $a(x) = (-\log|x|)^{-\gamma}$, then $\omega_a(r) = c\,|\log r|^{-\gamma-1}$, so a has Dini mean oscillation for all $\gamma > 0$. (For such an example, of course, we replace the ball $|x| < 4$ with a smaller one, such as $|x| < 1/2$.) The paper [14] has spawned a series of applications to various elliptic and parabolic equations: [12], [13], [15]. As useful as the concept of Dini mean oscillation has proven to be, however, we will see that there are equations with coefficients that do not have Dini mean oscillation, yet Lipschitz regularity can be obtained by other methods.

1.2. **Dynamical Systems Approach.** Now let us describe the method in [25] of using dynamical systems to study the regularity of of weak solutions of (1) at a given point. The dynamical system is most conveniently formulated when the point in question is $x = 0$ and $a_{ij}(0) = \delta_{ij}$; this can always be achieved by a change of coordinates at 0 since we have assumed the coefficients are symmetric. Under these conditions, let us define the matrix

$$R(r) := \fint_{S^{n-1}} \left(A(r\theta) - nA(r\theta)\theta \otimes \theta\right)\, d\theta, \quad \text{for } 0 < r < 1/2, \tag{6}$$

where $A = (a_{ij})$, $r = |x|$, $\theta = x/|x| \in S^{n-1}$, $A\theta \otimes \theta$ is the outer product of the vectors $A\theta$ and θ, and $d\theta$ denotes standard surface measure on S^{n-1}. Note that $|R(r)| \leq c\,\omega(r)$, where $|\cdot|$ denotes the matrix norm; also note that R need not be symmetric. Let us consider the dynamical system

$$\frac{d\phi}{dt} + R(e^{-t})\,\phi = 0 \quad \text{for } T < t < \infty, \tag{7}$$

where $t = -\log r$ and T is sufficiently large. The stability property that we require is uniform stability: we say that (7) is *uniformly stable as* $t \to \infty$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that any solution of (7) satisfying $|\phi(t_1)| < \delta$ for some $t_1 > T$ satisfies $|\phi(t)| < \varepsilon$ for all $t > t_1$. If the coefficients are square-Dini continuous and the dynamical system (7) is uniformly stable then the main result in [25] (see Theorem 1 below) states that every weak solution of (1) is Lipschitz continuous at $x = 0$. In general, it is not so easy to tell whether (7) is uniformly stable, so several simpler sufficient conditions were given in [25], such as

$$\frac{R(r)}{r} \int_0^r \frac{R(\rho)}{\rho}\, d\rho \in L^1(0, \varepsilon), \tag{8}$$

or

$$\int_{r_1}^{r_2} \frac{\mu(\mathcal{S}(\rho))}{\rho}\, d\rho \leq K < \infty \quad \text{for all } 0 < r_1 < r_2 < \varepsilon, \tag{9}$$
$$\text{where } \mu(\mathcal{S}) = \text{largest eigenvalue of } \mathcal{S} := -(R + R^t)/2.$$

However, if the a_{ij} are radial functions, then $R(r) \equiv 0$, so we have Lipschitz regularity at $x = 0$ only under the condition that the a_{ij} are square-Dini continuous at $x = 0$. More generally, if we let $a^0_{ij}(r)$ denote mean of a_{ij} over the sphere $|x| = r$, then we can uniquely write $a_{ij}(x) = a^0_{ij}(|x|) + a^1_{ij}(x)$; if $a^1_{ij}(x)$ is Dini continuous and $a^0_{ij}(|x|)$ is only square-Dini continuous at $x = 0$, then every solution is Lipschitz continuous at $x = 0$.

Here is one result that we will find useful:

Theorem 1 (Theorem 1 in [25])**.** *Suppose the a_{ij} are bounded, measurable functions in Ω that are square-Dini continuous at $x = 0$. Suppose also that the dynamical system (7) is uniformly stable. Then every weak solution $u \in H^{1,2}(\Omega)$ of (1) is Lipschitz continuous at $x = 0$ and satisfies for ρ sufficiently small*

$$|u(x) - u(0)| \leq \frac{c\,|x|}{\rho} \left(\fint_{|y|<\rho} |u(y)|^2\, dy \right)^{1/2} \quad \text{for } |x| < \rho/2. \tag{10}$$

1.3. **From a Point to a Neighborhood.** Theorem 1 above concerns the Lipschitz continuity at $x = 0$, but under certain conditions, this Lipschitz continuity extends to Ω. Let us consider a simple case where the coefficients are almost everywhere differentiable with an estimate for the growth of $\nabla a_{ij}(x)$ as $|x| \to 0$ and obtain a uniform bound on the gradient of weak solutions; this will also show the solution is C^1.

Theorem 2. *Suppose the a_{ij} are differentiable in $\Omega\backslash\{0\}$ with*

$$|\nabla a_{ij}(x)| \leq \frac{c}{|x|} \quad \text{for } 0 < |x| < 2\rho. \tag{11}$$

Suppose the a_{ij} are square-Dini continuous at $x=0$ and the dynamical system (7) is uniformly stable. Then every weak solution $u \in H^{1,2}(\Omega)$ of (1) is Lipschitz continuous in B_ρ and there is a positive constant c_1 such that

$$|\nabla u(x)| \leq \frac{c_1}{\rho} \fint_{|y|<2\rho} |u(y)|\,dy \quad \text{for } 0<|x|<\rho. \tag{12}$$

Moreover, $u \in C^1(\overline{B_\rho})$.

Proof. We will require standard elliptic interior estimates in an annulus. For $\rho>0$, let us introduce the annulus

$$A_r = \{x \in \mathbb{R}^n : \frac{r}{2} < |x| < 2r\}. \tag{13}$$

We first consider $r=1$. We know $|\nabla a_{ij}(x)| \leq c$ for $x \in A_1$, so if U is an open set such that $\overline{U} \subset A_1$ and $1<p<\infty$, then there is a positive constant c such that

$$\|u\|_{H^{2,p}(U)} \leq c\,\|u\|_{L^1(A_1)}. \tag{14}$$

(The above can be obtained from the standard L^p elliptic estimates: cf. Theorem 15.1" and Remark 1 following Theorem 7.3 in [1]; also (14.3.6) in Chapter 14 of [29].) Taking $p>n$ so that we can use the Sobolev embedding $H^{2,p} \subset C^1$, we obtain the following:

$$|\nabla u(x)| \leq c\,\|u\|_{L^1(A_1)} \quad \text{for } \frac{3}{4} \leq |x| \leq \frac{3}{2}. \tag{15}$$

Applying a dilation to the annulus, we find that if

$$|\nabla a_{ij}(x)| \leq \frac{c}{r} \quad \text{for } x \in A_r, \tag{16}$$

where c is independent of r, then there is a constant c_1 independent of r such that

$$|\nabla u(x)| \leq \frac{c_1}{r} \fint_{A_r} |u(y)|\,dy \quad \text{for } \tfrac{3}{4}r < |x| < \tfrac{3}{2}r. \tag{17}$$

Now, since ρ is fixed, let $r_j = \rho/2^j$ and $A_j = A_{r_j}$ for $j=0,1,\ldots$. For every x satisfying $0<|x|<\rho$, we have $\frac{3}{4}r_j \leq |x| \leq \frac{3}{2}r_j$ for some j, so

$$|\nabla u(x)| \leq \frac{c_1}{r_j} \fint_{A_j} |u(y)|\,dy \leq \frac{c_1}{\rho} \fint_{|y|<2\rho} |u(y)|\,dy. \tag{18}$$

The estimate (12) on the gradient of u follows from (18). Note that (12) implies $|u(x)-u(y)| \leq c\,|x-y|$ for $0<|x|,|y|<\rho$ and (10) implies $|u(x)-u(0)| \leq c|x|$ for $x \in B_\rho$. Hence u is uniformly Lipschitz in B_ρ.

The uniform bound (12) enables us to conclude that $u \in C^1(\overline{B_\rho})$. In fact, we can use a mollifier to obtain coefficients $a^h_{ij}(x)$ that are smooth on $B_{2\rho}$ and $a^h_{ij} \to a_{ij}$ in $L^p(B_\rho)$ as $h \to 0$ for $1 \leq p < \infty$. If we let u^h denote the solution of (1) with the coefficients a^h_{ij} then we know that $u^h \in C^\infty(B_{2\rho})$ and $u^h \to u$ in $H^{1,2}(B_{2\rho})$ (see, for example, pages 150-151 in [31]). Moreover the continuity modulus of the a^h_{ij} does not increase, and the estimate (11) continues to hold with a constant c independent of h near zero, so the estimate (12) applies to u^h. If we consider a sequence $v^m = u^{h_m}$ with $h_m \to 0$ as $m \to \infty$, we know $\|v^\ell - v^m\|_{L^1(B_\rho)} \to 0$ as $\ell, m \to \infty$ and then use (12) to conclude $\|\nabla(v^\ell - v^m)\|_{L^\infty(B_\rho)} \to 0$. By completeness of the Banach space $C^1(\overline{B_\rho})$, this shows $u \in C^1(\overline{B_\rho})$. $\square$

Remark 1. *Instead of assuming differentiability of the coefficients for $x \neq 0$ in Theorem 1, we could have assumed they satisfy $|a_{ij}(x) - a_{ij}(y)| \leq \omega(|x-y|/|x|)$ for $|x-y| < \frac{1}{2}|x|$, where ω satisfies the Dini condition (2). Let us also mention the recent work of di Filippis & Mingione*

[11] *where the condition* (11) *and our additional conditions at* $x = 0$ *are replaced by the gradient of the coefficients belonging to the Lorentz space* $L(n, 1)$.

2. Gilbarg-Serrin Equations

In [16] and [33], Gilbarg and Serrin considered coefficients of the form

$$a_{ij}(r, \theta) = \delta_{ij} + g(r)\, \theta_i \theta_j, \quad \text{where } r = |x| \text{ and } \theta_i = x_i/r. \tag{19}$$

They assumed $g(r)$ is a bounded function satisfying $g(r) > -1 + \varepsilon$, which guarantees uniform ellipticity, and studied solutions of equations involving the associated operators in both divergence and nondivergence form. In this paper, we consider conditions on $g(r)$ that guarantee that a weak solution of (1) is Lipschitz continuous at $x = 0$, or in a neighborhood of $x = 0$, or when there exists a weak solution that is not Lipschitz continuous at $x = 0$. In cases where the solution is Lipschitz continuous in a neighborhood of $x = 0$, we will use Theorem 2 to conclude that the solution is C^1. Let us state several results in this direction.

2.1. Application of the Dynamical Systems Method.

Let us apply the results of the dynamical systems method to (19).

Proposition 1. *Let the coefficients* a_{ij} *be given by* (19)*, where* $g(r)$ *is square-Dini continuous at* 0 *and for some real constant* K

$$\int_{r_1}^{r_2} \frac{g(r)}{r}\, dr \le K < \infty \quad \textit{for} \ \ 0 < r_1 < r_2 < \rho. \tag{20}$$

Moreover, assume $g(r)$ *is differentiable for* $r > 0$ *with derivative satisfying* $|r\, g'(r)| \le c$ *for* $0 < r < \rho$*. Then any weak solution* $u \in H^{1,2}(\Omega)$ *of* (1) *is in* $C^1(\overline{B_\rho})$ *and satisfies the estimate* (12).

Proof. For coefficients of the form (19), it is shown in Section 5 of [25] that the dynamical system (7) reduces to the (scalar) ordinary differential equation

$$\frac{d\phi}{dt} = \frac{n-1}{n}\, \mathrm{g}(t)\, \phi, \tag{21}$$

where $\mathrm{g}(t) = g(e^{-t})$. Now we integrate this ODE and see that it is uniformly stable if and only if there is a real constant K such that

$$\int_s^t \mathrm{g}(\tau)\, d\tau \le K < \infty \quad \text{for } t > s > T.$$

But this condition is equivalent to our assumption (20), so the dynamical system (7) is uniformly stable. The hypothesis $|rg'(r)| \le c$ guarantees the condition (11), so by Theorem 2 we know that $u \in C^1(\overline{B_\rho})$. □

2.2. Application of a Comparison Principle.

We will discuss a comparison principle based upon solutions of the following ordinary differential equation:

$$\frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1}(1 + g(r)) \frac{dv}{dr} \right) - \frac{n-1}{r^2}\, v = 0, \quad \text{for } 0 < r < 1. \tag{22}$$

Proposition 2. *Let* $Z(r)$ *denote a solution of* (22) *satisfying* $Z(1) > 0$ *and the finite energy condition*

$$\int_0^1 \left(|Z'(r)|^2 + r^{-2}|Z(r)|^2 \right) r^{n-1}\, dr < \infty. \tag{23}$$

If $Z(r)$ satisfies $0 < Z(r) \le c\,r$ as $r \to 0$, then every weak solution $u \in H^{1,2}(\Omega)$ of (1) with coefficients given by (19) is Lipschitz continuous at $x = 0$. In addition, if $g(r)$ in (19) is differentiable for $r > 0$ and $|rg'(r)| \le c$ for $0 < r < 2\rho$ then $u \in C^1(\overline{B_\rho})$ and satisfies the estimate (12).

Proof. We let $Z(r)$ be a solution of (22) satisfying $Z(1) > 0$ and the finite energy condition (23). By the non-oscillatory character of (22), $Z(r)$ cannot change sign for $0 < r < 1$, so we have $Z(r) > 0$. Now if $u \in H^{1,2}(B_1)$ is a weak solution of (1) with coefficients a_{ij} given by (19), then we can write the equation as

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}(1+g(r))\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\Delta_\theta u = 0, \tag{24}$$

where Δ_θ denotes the Laplace-Beltrami operator on the sphere S^{n-1}. If we integrate this equation over S^{n-1}, we find that u satisfies

$$⨍_{S^{n-1}} u(r\,\theta)\,d\theta = constant = u(0)$$

where $u(0)$ denotes the value of $u(x)$ at $x = 0$. But since we can subtract a constant from a weak solution of (1), we may assume that $u(r,\theta)$ is orthogonal to 1 on every sphere centered at 0, i.e.

$$\int_{S^{n-1}} u(r\,\theta)\,d\theta = 0 \quad \text{for } 0 < r < 1, \tag{25}$$

We use the following comparison principle between Z and u:

Lemma 1. *Let the coefficients a_{ij} be given by (19) and $u \in H^{1,2}(B_1)$ is a weak solution of (1) satisfying (25). Then*

$$\left(⨍_{S^{n-1}} |u(r\,\theta)|^2\,d\theta\right)^{1/2} \le \frac{Z(r)}{Z(1)}\left(⨍_{S^{n-1}} |u(\theta)|^2\,d\theta\right)^{1/2} \quad \textit{for } 0 < r < 1. \tag{26}$$

We will prove Lemma 1 below, but to complete the proof of Proposition 2, recall by De Giorgi [9] that a weak solution of (1) satisfies the estimate

$$\sup_{|x|<\delta} |u(x)| \le c\left(⨍_{B_{2\delta}} |u(y)|^2\,dy\right)^{1/2} \quad \text{for } 0 < \delta \le 1/2. \tag{27}$$

We may combine this with (26) and the assumption $Z(r) \le c\,r$ to conclude that $u(x)$ is Lipschitz continuous at $x = 0$. If g is differentiable for $0 < r < 2\rho$, then we can show u is Lipschitz continuous in $B_\rho(0)$, and if $|rg'(r)| \le c$ then we can obtain the uniform bound on $\nabla u(x)$ for $x \in B_\rho(0)\backslash\{0\}$, exactly as in the proof of Theorem 1. $\square$

Remark 2. *The differentiability at $x = 0$ of the solution $Z(r)\Theta_1(\theta)$ of (1) means $Z(r)\Theta_1(\theta) = V \cdot x + o(r)$ where V is a constant vector. Multiplying both sides by Θ_1^{-1} and integrating over the unit sphere we obtain $Z(r) = c\,r + o(r)$ for some constant c.*

Proof of Lemma 1. Considering u as a weak solution of (24) in the ball B_ρ for $0 < \rho < 1$, we have

$$\int_{B_\rho(0)} \left((1+g(r))\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}|\nabla_\theta u|^2\right) dx = (1+g(\rho))\rho^{n-1}\int_{S^{n-1}} u(\rho\,\theta)\frac{\partial u}{\partial r}(\rho\,\theta)\,d\theta. \tag{28}$$

Using (25), minimizing the integration over the sphere S^{n-1} on the left hand side, and rewriting the expression on the right hand side, we find

$$\int_{S^{n-1}}\int_0^\rho \left((1+g(r))\left|\frac{\partial u}{\partial r}\right|^2 + \frac{n-1}{r^2}|u|^2\right) r^{n-1}drd\theta \le (1+g(\rho))\frac{\rho^{n-1}}{2}\frac{d}{d\rho}\int_{S^{n-1}} (u(\rho\,\theta))^2 d\theta. \tag{29}$$

Now consider the variational functional associated with (22):

$$J(v) = \int_0^\rho \left((1+g(r))|v'(r)|^2 + \frac{n-1}{r^2}|v(r)|^2 \right) r^{n-1}\, dr \tag{30a}$$

with the boundary condition

$$v(\rho) = \left(\fint_{S^{n-1}} |u(\rho\,\theta)|^2\, d\theta \right)^{1/2}. \tag{30b}$$

If we let v_* denote the minimizer for (30), then we can express it as a scalar multiple of $Z(r)$ by

$$v_*(r) = \frac{Z(r)}{Z(\rho)} \left(\fint_{S^{n-1}} |u(\rho\,\theta)|^2\, d\theta \right)^{1/2}, \tag{31}$$

and the minimum value of J is

$$J(v_*) = (1+g(\rho))\rho^{n-1} \frac{Z'(\rho)}{Z(\rho)} \fint_{S^{n-1}} |u(\rho\,\theta)|^2\, d\theta. \tag{32}$$

Now if we fix $\theta \in S^{n-1}$ and evaluate J on $v(r) = u(r,\theta)$, we obtain

$$\int_0^\rho \left((1+g(r)) \left| \frac{\partial u}{\partial r} \right|^2 + \frac{n-1}{r^2}\, |u|^2 \right) r^{n-1} dr \geq J(v_*).$$

If we integrate both sides over S^{n-1} and use (29) and (32), we obtain

$$\frac{1}{2}\frac{d}{d\rho}\|u(\rho)\|^2 \geq \frac{Z'(\rho)}{Z(\rho)}\|u(\rho)\|^2 \quad \text{where } \|u(\rho)\|^2 := \int_{S^{n-1}} |u(\rho\,\theta)|^2\, d\theta. \tag{33}$$

But (33) implies

$$\frac{d}{d\rho} \log \frac{\|u(\rho)\|}{Z(\rho)} \geq 0,$$

which implies that $\|u(\rho)\|/Z(\rho)$ is increasing, so (26) follows. □

2.3. **Conditions for Loss of Lipschitz Regularity.** Propositions 1 and 2 provide conditions under which Lipschitz regularity holds. Now let us consider conditions under which a weak solution exists which is *not* Lipschitz continuous.

Proposition 3. *Let the coefficients a_{ij} be given by* (19)*, where $g(r)$ is square-Dini continuous at* 0 *and*

$$\limsup_{r\to 0} \int_r^{1/2} \frac{g(\rho)}{\rho}\, d\rho = +\infty. \tag{34}$$

Also assume $g(r)$ has finite total variation, i.e. $\int_0^{1/2} |g'(r)|\, dr < \infty$. Then there exists a weak solution u of (1) *in B_ρ that is not Lipschitz continuous at $x=0$. If we also have $g(r) > 0$ for $r < \varepsilon$, then we do not require $g(r)$ to be square-Dini continuous for the conclusion to hold.*

Proof. We want to find a weak solution $u \in H^{1,2}(B_\rho)$ of (1) that is bounded but not Lipschitz continuous. Without loss of generality, we may assume $\rho = 1/2$ and $u(0) = 0$. We shall look for a solution in the form $u(r,\theta) = v(r)\,\Theta_1(\theta)$, where $v(0) = 0$ and Θ_1 is an eigenfunction for the first eigenvalue $\lambda = n-1$ for the Laplace-Beltrami operator on the unit sphere. We find that $v(r)$ should satisfy the ordinary differential equation (22). With a change of independent variable to $t = -\log r$, the equation for v can be written as

$$\frac{d}{dt}\left((1+\mathrm{g}) \frac{dv}{dt} \right) - (n-2)(1+\mathrm{g})\frac{dv}{dt} - (n-1)v = 0, \quad \text{for } 0 < t < \infty, \tag{35}$$

where $\mathrm{g}(t) = g(e^{-t})$, and this can be written as a 1st-order system

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 & (1+\mathrm{g})^{-1} \\ n-1 & n-2 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} \quad \text{where } w = (1+\mathrm{g})\frac{dv}{dt}. \tag{36}$$

Let M_{g} denote the 2×2 matrix on the right hand side of (36). For $\mathrm{g}=0$, M_0 has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = n-1$ (corresponding to the solutions r and r^{1-n} of (22) when $g=0$). If we let E denote the matrix of eigenvectors for M_0, then we can rewrite (36) as

$$\frac{d}{dt}V = (A_0 + B(\mathrm{g}))\,V, \quad \text{where } \begin{bmatrix} v \\ w \end{bmatrix} = E\,V \tag{37a}$$

and

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & n-1 \end{bmatrix}, \quad B(\mathrm{g}) = \frac{\mathrm{g}}{n(1+\mathrm{g})}\begin{bmatrix} n-1 & -(n-1)^2 \\ 1 & -(n-1) \end{bmatrix}. \tag{37b}$$

The eigenvalues of $A_0 + B(\mathrm{g})$ may be denoted $\Lambda_1(\mathrm{g})$ and $\Lambda_2(\mathrm{g})$, where $\Lambda_1(\mathrm{g}) \to -1$ and $\Lambda_2(\mathrm{g}) \to n-1$ as $\mathrm{g}\to 0$. We can calculate $\Lambda' := d\Lambda/d\mathrm{g}$ and evaluate $\Lambda'(0)$ to find

$$\Lambda_1'(0) = \frac{n-1}{n}. \tag{38}$$

Consequently

$$\Lambda_1(\mathrm{g}) = -1 + \frac{n-1}{n}\mathrm{g} + O(|\mathrm{g}|^2) \quad \text{as } \mathrm{g}\to 0. \tag{39}$$

Similarly, we can use $\Lambda(0) = n-1$ and obtain

$$\Lambda_2(\mathrm{g}) = n-1 + \frac{n-1}{n}\mathrm{g} + O(|\mathrm{g}|^2) \quad \text{as } \mathrm{g}\to 0. \tag{40}$$

At this point we have not used the fact that $g(r)$ is square-Dini continuous at $r=0$, so we do not assert that $\int_0^\infty |\mathrm{g}(t)|^2\,dt < \infty$. Nevertheless, because we have assumed that $g(r)$ has finite total variation for $0 < r < 1/2$, we know that the matrix function $B(t) := B(\mathrm{g}(t))$ has finite total variation for $t > \log 2$, so we can use the asymptotic theory of linear systems of ordinary differential equations. Denoting the eigenvalues as $\lambda_1(t) := \Lambda_1(\mathrm{g}(t))$ and $\lambda_2(t) := \Lambda_2(\mathrm{g}(t))$, we have

$$\lambda_1(t) = -1 + \frac{n-1}{n}\,\mathrm{g}(t) + O(|\mathrm{g}(t)|^2) \quad \text{as } t\to\infty. \tag{41}$$

We can apply Theorem 8.1 in [8] (see also Theorem 16.3.1 in [19]) to conclude that there exists a solution $V_1(t)$ with the asymptotic behavior

$$\lim_{t\to\infty} V_1(t)\,\exp\left[-\int_0^t \lambda_1(\tau)\,d\tau\right] = V_0 \quad \text{as } t\to\infty, \tag{42}$$

where $V_0 = (1,0)^t$ is the eigenvector for the eigenvalue $\Lambda_1(0) = -1$.

Now we shall invoke the sqaure-Dini assumption $\int_0^\infty |\mathrm{g}(t)|^2\,dt < \infty$. This mean that (37) can be written

$$\frac{d}{dt}V = (A_0 + \mathrm{g}(t)\,B_0)\,V + C(\mathrm{g}(t))\,V, \tag{43a}$$

where

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & n-1 \end{bmatrix}, \quad B_0 := \frac{1}{n}\begin{bmatrix} n-1 & -(n-1)^2 \\ 1 & -(n-1) \end{bmatrix}, \quad \text{and} \quad \int_0^\infty |C(\mathrm{g}(t))|\,dt < \infty. \tag{43b}$$

Now Theorem 8.1 in [8] shows that (42) can be sharpened somewhat to

$$V_1(t) = V_0 \exp\left[-t + \frac{n-1}{n}\int_0^t \mathrm{g}(\tau)\,d\tau\right](1+o(1)) \quad \text{as } t\to\infty. \tag{44}$$

If we transform back to our original dependent variables v, w using $(v,w)^t = E^{-1}V$ and return to the original independent variable $r = e^{-t}$, then we find there is a solution $v(r)$ of (22) with the asymptotic behavior

$$v(r) = c\,r \exp\left[\frac{n-1}{n}\int_r^{1/2} \frac{g(\rho)}{\rho}\,d\rho\right](1+o(1)) \quad \text{as } r\to 0. \tag{45}$$

But the condition (34) shows that this solution of (22) is not Lipschitz at $r=0$. In other words, we have found a bounded solution $u(r,\theta) = v(r)\Theta_1(\theta)$ of (1) that is not Lipschitz at $r = 0$.

If we do not assume that $g(r)$ is square-Dini at $r = 0$, but we know $g(r) > 0$ for $r < \varepsilon$, then we still can conclude

$$\frac{n-1}{n}\int_0^t \left(\mathrm{g}(\tau) + O(|\mathrm{g}(\tau)|^2)\right)\,d\tau \to +\infty \quad \text{as } t\to\infty,$$

which shows that our solution $v(r)$ is not Lipschitz continuous at $r = 0$. □

Remark 3. *Propositions 2 and 3 both depend on the asymptotic behavior of the positive finite energy solution $Z(r)$ of (22) in such a way that $Z(r) \le c\,r$ becomes a necessary and sufficient condition for every weak solution of (1) to be Lipschitz continuous at $x = 0$.*

Remark 4. *As we shall see in the following Examples, it is possible to find functions $g(r)$ that do not have Dini mean oscillation, yet satisfy the conditions of Proposition 1, so all weak solutions are Lipschitz continuous in a neighborhood of $x = 0$. Proposition 3, on the other hand, can be used to find examples of coefficients a_{ij} that do not satisfy the Dini mean oscillation condition: if $g(r)$ satisfies the square-Dini condition and (34), then (1) has weak solutions $u \in H^{1,2}(\Omega)$ that are not Lipschitz continuous, so by the results of [14] the coefficients (19) cannot have Dini mean oscillation.*

2.4. **Examples.** In this section we consider several specific functions $g(r)$ and what our preceding results tell us about Lipschitz regularity at $x = 0$ or in a neighborhood of $x = 0$. We frequently find that Proposition 2 enables us to extend results beyond what Proposition 1 is able to provide due to the condition that $g(r)$ be square-Dini continuous at $r = 0$. But both Propositions 1 and 2 apply in some cases that do not have Dini mean oscillation.

Example 1. Let us consider (19) with

$$g(r) = |\log r|^{-\gamma} \quad \text{for } \gamma > 0. \tag{46}$$

For $\gamma > 1$, $g(r)$ is Dini continuous at $r = 0$, so we know weak solutions of (1) are Lipschitz continuous (in fact, C^1). For $0 < \gamma \le 1$, $g(r)$ is not Dini continuous: in fact

$$\int_0^{1/2} \frac{g(r)}{r}\,dr = \int_0^{1/2} \frac{|\log r|^{-\gamma}}{r}\,dr = +\infty,$$

so condition (34) holds. Moreover, $g(r)$ has finite total variation:

$$\int_0^{1/2} |g'(r)|\,dr = \gamma \int_0^{1/2} \frac{|\log r|^{-\gamma-1}}{r}\,dr < \infty.$$

Now for $1/2 < \gamma \le 1$, $g(r)$ is square-Dini continuous, so we can use Proposition 3 to conclude there exists a weak solution of (1) that is not Lipschitz continuous at $x = 0$. In fact, since $g(r) > 0$ for $0 < r < 1/2$ we can use Proposition 3 to extend this conclusion to $0 < \gamma \le 1$. **Summary**

for (46): *For $\gamma > 1$, all weak solutions of* (1) *are Lipschitz continuous; but for $0 < \gamma \leq 1$, there exists a weak solution of* (1) *that is not Lipschitz continuous at $x = 0$.*

Now let us consider (19) with

$$g(r) = -|\log r|^{-\gamma} \quad \text{for } \gamma > 0. \tag{47}$$

For $\gamma > 1$, as with (46), $g(r)$ is Dini continuous at $r = 0$, so we know weak solutions are Lipschitz continuous; but for all $\gamma > 0$ we have $|rg'(r)| \leq c$ and

$$-\infty \leq \int_0^{1/2} \frac{g(r)}{r}\, dr = -\int_0^{1/2} \frac{|\log r|^{-\gamma}}{r} < 0.$$

For $1/2 < \gamma \leq 1$, $g(r)$ is square-Dini continuous and the condition (20) is trivially satisfied with $K = 0$, so we could use Proposition 1 to conclude that all weak solutions of (1) are Lipschitz continuous. However, we can instead use Proposition 2 to handle all values $0 < \gamma \leq 1$. We need to find the solution $Z(r)$ of (22) with the desired properties. From (41) and (42) we see that there is a solution $V(t)$ of (37) such that $|V(t)| = o(e^{-t})$, which means we have our solution $v(t) = Z(r)$ satisfying $|Z(r)| \leq c\, r$. If we rescale so that $Z(1) = 1$, then the non-oscillatory character of (22) shows $Z(r)$ remains positive. We can also confirm that Z satisfies the finite energy condition (23), so we can apply Proposition 2 to reach our conclusion. **Summary for** (47): *For $\gamma > 0$, all weak solutions of* (1) *are Lipschitz continuous.*

Note: For $0 < \gamma \leq 1$, these results for (46) and (47) cannot be obtained using Dini mean oscillation for a_{ij} because that condition is satisfied only for $\gamma > 1$: this can be confirmed by direct calculation at $x = 0$ (see the Appendix), but for (46) it also follows from Remark 4.

Example 2. Let us consider an example where $g(r)$ oscillates as $r \to 0$:

$$g(r) = \frac{\sin(|\log r|)}{|\log r|^{\beta}}, \quad \text{where } \beta > 0. \tag{48}$$

For $\beta > 1/2$, this $g(r)$ is square-Dini continuous and the condition $|rg'(r)| \leq c$ in Proposition 1 is satisfied. Also note that

$$\int_0^{e^{-1}} \frac{g(r)}{r}\, dr = \int_1^{\infty} \frac{\sin\tau}{\tau^{\beta}}\, d\tau \quad \text{converges} \tag{49}$$

since $\tau^{-\beta}$ is strictly decreasing and $\sin\tau$ alternates sign on intervals $[k\pi, (k+1)\pi]$ for $k = 0, 1, \ldots$. In fact, for any interval $[t_1, t_2] \subset (2\pi, \infty)$ we have

$$\int_{t_1}^{t_2} \frac{\sin\tau}{\tau^{\beta}}\, d\tau \leq \int_{2\pi}^{3\pi} \frac{\sin\tau}{\tau^{\beta}}\, d\tau < \infty, \tag{50}$$

and this confirms condition (20). Consequently, for $\beta > 1/2$ all conditions of Proposition 1 are satisfied, and we conclude that all weak solutions of (1) are Lipschitz continuous. To obtain this conclusion for all $\beta > 0$, let us appeal to Proposition 2. Using asymptotic analysis as above with (49), we can find a finite energy solution $Z(t)$ of (22) satisfying $0 < Z(r) \leq cr$ for $0 < r < 1$, so Proposition 2 shows that all weak solutions are Lipschitz continuous.

Note: These results for $0 < \beta \leq 1$ in (48) cannot be obtained using Dini mean oscillation since one can show by direct calculation (see the Appendix) that the coefficients a_{ij} have Dini mean oscillation only for $\beta > 1$.

Example 3. Consider the following function:

$$g(r) = \frac{-C_1\, \sin(|\log r|) - C_2 \cos(|\log r|)}{A + \sin(|\log r|) - \cos(|\log r|)} \tag{51}$$

where

$$C_1 = \frac{(n-1)^2}{(n-1)^2+1} + 1, \quad C_2 = \frac{n-1}{(n-1)^2+1} - 1,$$

and $A > 1$ is chosen sufficiently large. This complicated expression for $g(r)$ arises because we actually want the following function $Z(r)$ to be a solution of (22):

$$Z(r) = r\ (A + \sin(|\log r|)). \tag{52}$$

Using $Z(r)$ in (22), we can work backwards to find $g(r)$ and confirm (51). This means that, if we use this $g(r)$ in our coefficients a_{ij} as in (19), then by Proposiiton 2 any weak solution $u \in H^{1,2}(B_{1/2}(0))$ of (1) must be Lipschitz continuous in $B_{1/2}(0)$. This is in spite of the fact that the coefficients are not square-Dini continuous or Dini mean oscillation (see Appendix); in fact, g does not even vanish as $r \to 0$, and we claim that

$$\int_0^{1/2} \frac{g(r)}{r}\, dr = +\infty. \tag{53}$$

This is interesting since the condition (53) is associated with the existence of a weak solution that is not Lipschitz continuous (cf. Proposition 3). However, since $g(r)$ does not even vanish as $r \to 0$, any intuition from the above asymptotic analysis is not relevant for this example.

To confirm (53), let us compute

$$\int_0^1 \frac{g(r)}{r}\, dr = \int_0^\infty \mathrm{g}(t)\, dt = \int_0^\infty \frac{-C_1\ \sin t - C_2 \cos t}{A + \sin t - \cos t}\, dt$$

By periodicity, it suffices to compute

$$\begin{aligned}\int_0^{2\pi} \frac{-C_1\ \sin t - C_2 \cos t}{A + \sin t - \cos t}\, dt &= \frac{1}{A}\int_0^{2\pi} \frac{-C_1\ \sin t - C_2 \cos t}{1 + \frac{1}{A}(\sin t - \cos t)}\, dt \\ &= A^{-2}\int_0^{2\pi} (\sin t - \cos t)\, [C_1 \sin t + C_2 \cos t]\ dt + O(A^{-3})\end{aligned},$$

provided A is sufficiently large. By orthogonality, the cross terms vanish and we are left with computing

$$\int_0^{2\pi} \left[C_1 \sin^2 t - C_2 \cos^2 t\right]\, dt = \int_0^{2\pi} \frac{C_1 - C_2}{2}\, dt,$$

where we have used the half-angle trigonometric identities for $\sin^2$ and $\cos^2$ as well as $\int_0^{2\pi} \cos 2t\, dt = 0 = \int_0^{2\pi} \sin 2t\, dt$. But it is clear that $C_1 - C_2 > 0$, so we obtain

$$\int_0^\infty \frac{-C_1\ \sin t - C_2 \cos t}{A + \sin t - \cos t}\, dt = +\infty.$$

Appendix A. Computations of mean oscillation at $x = 0$

All of the functions that we consider are smooth except at $x = 0$, so we will restrict our attention to the mean oscillation at $x = 0$, i.e. over balls of radius r centered at $x = 0$. However, if we want to show for certain examples that Dini mean oscillation over a domain fails, it suffices to show that it fails at $x = 0$.

First let us confirm that $a(x) = |\log r|^{-\gamma}$ has Dini mean oscillation at $x = 0$ for all $\gamma > 0$. Since $a(x) = a(r)$ is radial, we easily compute

$$\begin{aligned} \widetilde{a}(r) := \widetilde{a}_{B_r(0)} &= \fint_{B_r(0)} a(x)\,dx = \frac{n}{r^n}\int_0^r |\log\rho|^{-\gamma}\,\rho^{n-1}\,d\rho \\ &= |\log r|^{-\gamma} - \gamma\,|\log r|^{-\gamma-1} + \frac{\gamma(\gamma+1)n}{r^n}\int_0^r |\log\rho|^{-\gamma-2}\,\rho^{n-1}\,d\rho. \end{aligned} \tag{54}$$

So $a(x) - \widetilde{a}(r) = \gamma\,|\log r|^{-\gamma-1} + O(|\log r|^{-\gamma-2})$ and hence the mean oscillation is

$$\omega_a(r) = \fint_{B_r(0)} |a(x) - \widetilde{a}(r)|\,dx = \gamma\,|\log r|^{-\gamma-1} + O(|\log r|^{-\gamma-2}). \tag{55}$$

Since $\omega_a(r)$ satisfies (2), this confirms that $a(x)$ has Dini mean oscillation at $x = 0$ for all $\gamma > 0$.

In the above calculation, there was no angular oscillation, and this contributed to a satisfying the Dini mean oscillation condition for $\gamma > 0$. The situation is quite different for functions with angular dependence. In order to estimate the mean oscillation for the coefficients a_{ij} in (19), let us introduce the matrix

$$\Theta = (\theta_i\theta_j)_{i,j=1,\dots,n}\,, \tag{56}$$

and denote its mean value over the unit sphere S^{n-1} by $\widetilde{\Theta}$. Then a calculation shows

$$\widetilde{\Theta} = \frac{1}{n}\,(\delta_{ij})_{i,j=1,\dots,n}\,. \tag{57}$$

Now let us multiply the matrix Θ by a scalar function $g(r)$ that is smooth for $0 < r < 1/2$. If we denote by $\widetilde{g\Theta}$ the mean value of $g\,\Theta$ over the ball $B_r(0)$, then we have

$$\widetilde{g\Theta} = \widetilde{g}\,\widetilde{\Theta}, \quad \text{where } \widetilde{g}(r) = \frac{n}{r^n}\int_0^r g(\rho)\rho^{n-1}\,d\rho.$$

Now we may estimate the matrix norm of $g\,\Theta - \widetilde{g\,\Theta}$ from below by

$$\begin{aligned} \|g\,\Theta - \widetilde{g\,\Theta}\| &\geq \|g\,(\Theta - \widetilde{\Theta})\| - \|(g - \widetilde{g})\,\widetilde{\Theta}\| \\ &= |g(r)|\,\|\Theta - \widetilde{\Theta}\| - |g(r) - \widetilde{g}(r)|\,\|\widetilde{\Theta}\|. \end{aligned} \tag{58}$$

If we can show $|g(r) - \widetilde{g}(r)| = o(|g(r)|)$ as $r \to 0$, then we know that the matrix $g(r)\Theta$ has Dini mean oscillation at $x = 0$ if and only if the function $|g(r)|$ itself satisfies the Dini condition at $r = 0$.

For example, consider the coefficients in Example 1, which we can write in matrix form as

$$A(x) = I + |\log r|^{-\gamma}\Theta, \quad \text{where } I \text{ denotes the identity matrix.} \tag{59}$$

The mean value of I is itself, so we have

$$A(x) - \widetilde{A}(r) = g(r)\,\Theta - \widetilde{g}(r)\,\widetilde{\Theta} \quad \text{where } g(r) = |\log r|^{-\gamma}.$$

As we found in (54), $|g(r) - \widetilde{g}(r)| = o(|g(r)|)$ as $r \to 0$, so by (58), $A(x)$ has Dini mean oscillation at $x = 0$ if and only if $g(r) = |\log r|^{-\gamma}$ satisfies the Dini condition. So Example 1 has Dini mean oscillation at $x = 0$ for $\gamma > 1$, but not for $0 < \gamma \leq 1$.

Next let us consider the coefficients in Example 2. Again we need only concern ourselves with the matrix

$$g(r)\Theta \quad \text{where } g(r) = \frac{\sin(|\log r|^{\alpha})}{|\log r|^{\beta}}.$$

For $0 < r < 1$ we use integration by parts and a geometric series to evaluate

$$\begin{aligned}\widetilde{g}(r) &= \frac{n}{r^{n-1}} \int_0^r \frac{\sin(-\log\rho)}{(-\log\rho)^\beta} \rho^{n-1}\, d\rho = ne^{nt} \int_t^\infty \frac{\sin\tau}{\tau^\beta} e^{-n\tau} d\tau \\ &= \frac{\sin t}{t^\beta} + e^{nt} \int_t^\infty \frac{\cos\tau}{\tau^\beta} e^{-n\tau} d\tau = \frac{\sin t}{t^\beta} + \frac{\cos t}{nt^\beta} - \frac{1}{n} e^{nt} \int_t^\infty \frac{\sin\tau}{\tau^\beta} e^{-n\tau} d\tau \\ &= \frac{n}{n^2+1} \left(\frac{n\sin t + \cos t}{t^\beta} \right)\end{aligned}$$

Consequently,

$$g(r) - \widetilde{g}(r) = \frac{1}{n^2+1} \frac{\sin(|\log r|) - n\cos(|\log r|)}{|\log r|^\beta}. \tag{60}$$

Now we can estimate

$$⨍_{S^{n-1}} |g(r)\Theta - \widetilde{g}(r)\widetilde{\Theta}|\, ds \geq \left| ⨍_{S^{n-1}} (g(r)\Theta - \widetilde{g}(r)\widetilde{\Theta})\, ds \right| = |g(r) - \widetilde{g}(r)|\, |\widetilde{\Theta}|.$$

But if $0 < \beta \leq 1$, then

$$\int_0^{1/2} \frac{|g(r) - \widetilde{g}(r)|}{r} dr = \frac{1}{n^2+1} \int_{\log 2}^\infty \frac{|\sin t - n\cos t|}{t^\beta} dt = \infty$$

since the numerator is periodic. This shows that the coefficients in Example 2 have Dini mean oscillation at $x = 0$ only for $\beta > 1$.

Finally, we consider the coefficients in Example 3. Recall

$$g(r) = \frac{-C_1\sin(|\log r|) - C_2\cos(|\log r|)}{A + \sin(|\log r|) - \cos(|\log r|)} = A^{-1}(-C_1\sin(|\log r|) - C_2\cos(|\log r|)) + O(A^{-2})$$

for large A. We can use this to approximate $\widetilde{g}(r)$, the mean value over the ball $B_r(0)$, and then express the answer in terms of $t = -\log r$:

$$\widetilde{g}(e^{-t}) = -\frac{n}{A(n^2+1)} (C_1(n\sin t - \cos t) + C_2(n\cos t - \sin t)) + O(A^{-2}).$$

Hence

$$g(e^{-t}) - \widetilde{g}(e^{-t}) = -\frac{1}{A(n^2+1)} [(C_1 + nC_2)\sin t + (C_2 + nC_1)\cos t] + O(A^{-2}).$$

But

$$\int_{\log r}^\infty |(C_1 + nC_2)\sin t + (C_2 + nC_1)\cos t|\, dt = +\infty,$$

since the integrand is periodic, so $|g(r) - \widetilde{g}(r)|$ does not satisfy the Dini condition. Hence Example 3 does not have Dini mean oscillation.

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References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math., **12** (1959), 623-727.

[2] C. Burch, *The Dini condition and regularity of weak solutions of elliptic equations*, J. Diff. Eq. **30** (1978), 308-323.

[3] F. Chiarenza, M. Frasca and P. Longo, *Solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **336** (1993), 841–853.

[4] A. Cianchi, V. Maz'ya, *Global Lipschitz regularity for a class of quasilinear elliptic* Commun. PDE **36**, (2011), 100–133.
[5] A. Cianchi, V. Maz'ya, *Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems* Eur. Math. Soc. (JEMS), (2014), **16**, p.571-595
[6] A. Cianchi, V. Maz'ya, *Global boundedness of the gradient for a class of nonlinear elliptic systems* Arch. Ration. Mech. Anal., **212** (2014), p.129-177.
[7] A. Cianchi, V. Maz'ya, *Global gradient estimates in elliptic problems under minimal data and domain regularity*, Commun. Pure Appl. Anal. **14** (2015), no. 1, 285–311.
[8] E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
[9] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degil integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3) **3** (1957), 25-43.
[10] G. Di Fazio, *Estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital A(7), **10** (1996), 409-420.
[11] C. Di Fiippis, G. Mingione, *Lipschitz bounds and nonautonomous integrals*, Arch. Rat. Mech. Anal, doi: 10.1007/s00205-021-01698-5
[12] H. Dong, L. Escauriaza, S. Kim, *On C^1, C^2, and weak type* $(1,1)$ *estimates for linear elliptic operators, Part II*, Math. Ann. **370** (2018), no. 1-2, 447-489.
[13] H. Dong, L. Escauriaza, S. Kim, *On $C^{1/2,1}$, $C^{1,2}$, and $C^{0,0}$ estimates for linear parabolic operators*, J. Evol. Equ. (2021) https://doi.org/10.1007/s00028-021-00729-8
[14] H. Dong, S. Kim, *On C^1, C^2, and weak type* $(1,1)$ *estimates for linear elliptic operators*, Comm. Partial Differential Equations, **42** (2017), 417-435.
[15] H. Dong, J. Lee, S. Kim, *On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients*, Indiana Univ. Math. J. **69** (2020), no. 6, 1815–1853.
[16] D. Gilbarg, J. Serrin, *On isolated singularities of solutions of second-order elliptic equations,* J. Analyse Math. **4** (1955/56), 309-340.
[17] P. Hartman, A. Wintner, *On uniform Dini conditions in the theory of linear partial differential equations of elliptic type,* American J. Math. **77** (1955), 329-354.
[18] T. Jin, V.G. Maz'ya, J. Van Schaftingen, *Patholgical solutions to elliptic problems in divergence form with continuous coefficients*, Comptes rendus. Mathématique, **47** (2009), p.773-778.
[19] V.A. Kozlov, V.G. Maz'ya, *Differential Equations with Operator Coefficients*, Springer-Verlag, New York, 1999.
[20] V.A. Kozlov, V.G. Maz'ya, *Asymptotic formula for solutions to the Dirichlet problem of elliptic equations with discontinuous coefficients near the boundary*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), **2** (2003), 551-600.
[21] T. Kuusi, G. Mingione, *Universal potential estimates*, J. Funct. Anal. **262** (2012), 4205-4269.
[22] T. Kuusi, G. Mingione, *A nonlinear Stein theorem*, Calc. Var. **51** (2014), 45–86.
[23] E. M. Landis, *Second order equations of elliptic and parabolic type*, Translations of Mathematical Monographs, 171. American Mathematical Society, Providence, RI, 1998.
[24] V.G. Maz'ya, *On weak solutions of the Dirichlet and Neumann problems*, Trans. Moscow Math. Soc. **20** (1969), 135-172.
[25] V.G. Maz'ya, R.McOwen, *Differentiablilty of solutions to second-order elliptic equations via dynamical systems*, J. Differential Equations, **250** (2010), 1137-1168.
[26] V.G. Maz'ya, R.McOwen, *Second-order differentiability for solutions of elliptic equations in the plane*, J. Mathematical Sciences, 191 (2013), 243-253.
[27] V.G. Maz'ya, R.McOwen, *Differentiability of solutions to the Neumann problem with low-regularity data via dynamical systems*, *Operator Theory: Advances and Applications*, 261 (2017), 343-385.
[28] V.G. Maz'ya, M. Mitrea, T. Shaposhnikova, *The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients*, J. Anal. Math. **110** (2010), 167–239.
[29] V.G. Maz'ya, T. Shaposhnikova, *Theory of Sobolev Multipliers*, Springer, 2009.
[30] R. McOwen, *On elliptic operators in nondivergence and double divergence form*, Operator Theory: Advances and Applications, 193 (2009), 159-169.
[31] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.
[32] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931-954.
[33] J. Serrin, *Pathological solutions of elliptic differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 385–387.
[34] M. Taylor, *Tools for PDE*, Mathematical Surveys and Monographs **81**, AMS, 2000.

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